Roman, *Advanced Linear Algebra*

Danny Nygård Hansen

20th November 2022

1 • Vector Spaces

EXERCISE 1.11

Show that if S is a subspace of a vector space V, then $\dim S \leq \dim V$. Furthermore, if $\dim S = \dim V < \infty$ then $S = V$.

SOLUTION. Let $\mathcal{B}$ be a basis for S. Then this is linearly independent as a subset of V, hence is contained in a basis $\mathcal{B}'$ for V by Theorem 1.9. Then $\mathcal{B} \subseteq \mathcal{B}'$, so it follows that $\dim S \leq \dim V$.

Now assume that $\dim S = \dim V < \infty$. Then $|\mathcal{B}| = |\mathcal{B}'|$, but since each basis is finite and one is contained in the other, we must have $\mathcal{B} = \mathcal{B}'$. Hence $S = V$. □

EXERCISE 1.12

Suppose that $V = U \oplus S_1 = U \oplus S_2$. What can you say about the relationship between S_1 and S_2? What can you say if $S_1 \subseteq S_2$?

SOLUTION. By Theorem 3.6, all complements of U are isomorphic, so we always have $S_1 \cong S_2$. Assume that $S_1 \subseteq S_2$, and let $s \in S_2$. Then $s = u + s'$ for some $u \in U$ and $s' \in S_1$. But then s' also lies in S_2, so since the sum $U \oplus S_2$ is direct we have $u = 0$. □

2 • Linear Transformations

EXERCISE 2.15

Suppose that $T \in \mathcal{L}(V, W)$.

(a) Given $L \in \mathcal{L}(U, W)$, show that there exists an $R \in \mathcal{L}(V, U)$ with $T = LR$ if and only if $\operatorname{im} T \subseteq \operatorname{im} L$:

$$\begin{tikzcd} V \arrow[r, dashed, "R"'] \arrow[rr, bend left, "T"] & U \arrow[r, "L"'] & W \end{tikzcd}$$

(b) Given $R \in \mathcal{L}(V, U)$, show that there exists an $L \in \mathcal{L}(U, W)$ with $T = LR$ if and only if $\ker R \subseteq \ker T$:

$$\begin{tikzcd} V \arrow[r, "R"'] \arrow[rr, bend left, "T"] & U \arrow[r, dashed, "L"'] & W \end{tikzcd}$$

In particular, both monomorphisms and epimorphisms split.

Solution. (a) Write $W = \ker L \oplus M$ for some subspace $M \subseteq W$. Then the restriction $L|_M : M \to \operatorname{im} L$ is bijective, so let $R = (L|_M)^{-1} T$, which is well-defined since $\operatorname{im} T \subseteq \operatorname{im} L$.

(b) Write $V = \ker R \oplus M$ for some subspace $M \subseteq V$. Then $R|_M : M \to \operatorname{im} R$ is bijective. Writing $U = \operatorname{im} R \oplus N$ for some subspace $N \subseteq U$, let $L = T \circ [(R|_M)^{-1}, 0]$. For $v \in \ker R \subseteq \ker T$ we have

$$LRv = L(0) = 0 = Tv,$$

and for $v \in M$ we have

$$LRv = T(R|_M)^{-1} Rv = Tv,$$

as required. □

Exercise 2.22

Let $T \in \mathcal{L}(V)$. If $TS = ST$ for all $S \in \mathcal{L}(V)$, show that $T = \alpha \operatorname{id}_V$ for some $\alpha \in \mathbb{F}$. I.e., the centre of the ring $\mathcal{L}(V)$, with multiplication given by function composition, is the subspace $\langle \operatorname{id}_V \rangle$.

Solution. This is obvious if $\dim V \in \{0, 1\}$, so assume that $\dim V \geq 2$.

First let $v \in V \setminus \{0\}$ and write $V = \langle v \rangle \oplus U$ for some subspace U, and define $S \in \mathcal{L}(V)$ by letting $Sv = v$ and $Su = 0$ for $u \in U$. If T and S commute, then

$$STv = TSv = Tv.$$

Hence $Tv \in \langle v \rangle$ (which includes the possibility that $Tv = 0$).

Next assume that $v, w \in V$ are linearly independent, write $V = \langle v, w \rangle \oplus U_2$ for some subspace U_2, and define S by letting $Sv = w$, $Sw = v$ and $Su = 0$. Let $\alpha \in \mathbb{F}$ be such that $Tv = \alpha v$. Then

$$Tw = TSv = STv = \alpha Sv = \alpha w.$$

Hence $Tv = \alpha v$ for all $v \in V$ as desired. □

3 • The Isomorphism Theorems

EXERCISE 3.18

Let S be a subspace of V. Prove that $(V/S)^* \cong S^0$.

SOLUTION. Let $\pi \colon V \to V/S$ be the quotient map, such that for every $\varphi \in (V/S)^*$ we have

$$\begin{tikzcd} V \arrow[rr, "\varphi\circ\pi"] \arrow[dr, "\pi"'] & & \mathbb{F} \\ & V/S \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Consider the map $(V/S)^* \to V^*$ given by $\varphi \mapsto \varphi \circ \pi$. This is injective by the universal property of quotients. Also by this property, a functional $\psi \in V^*$ factors through π if and only if $S \subseteq \ker \psi$, i.e. if $\psi \in S^0$. Hence the image of the above map is precisely S^0. □

8 • Eigenvalues and Eigenvectors

EXERCISE 8.6

An operator $T \in \mathcal{L}(V)$ is *nilpotent* if $T^n = 0$ for some positive $n \in \mathbb{N}$.

(a) Show that if T is nilpotent, then the spectrum of T is $\{0\}$.

(b) Find a non-nilpotent operator T with spectrum $\{0\}$.

SOLUTION. (a) Let $\lambda \in \mathbb{F}$ be an eigenvalue of T, and let $v \in V$ be a corresponding eigenvector. Then

$$0 = T^n v = T^{n-1} T v = \lambda T^{n-1} v = \cdots = \lambda^n v.$$

Hence $\lambda^n = 0$, since otherwise $v = \lambda^{-n} 0 = 0$. But then $\lambda = 0$.

(b) Let $S \in \mathcal{L}(\mathbb{R}^2)$ be rotation by $\pi/2$ radians, and let $0 \in \mathcal{L}(\mathbb{R})$ be the trivial map. Then $T = S \oplus 0 \in \mathcal{L}(\mathbb{R}^3)$ has spectrum $\{0\}$ but is clearly not nilpotent. □

EXERCISE 8.9

An *involution* is a linear operator $S \in \mathcal{L}(V)$ for which $S^2 = \mathrm{id}_V$. If T is idempotent, what can you say about $2T - \mathrm{id}_V$? Construct a one-to-one correspondence between the set of idempotents on V and the set of involutions.

SOLUTION. Note that by 2 we mean $1+1$, where 1 is the multiplicative identity of $\mathbb{F}$. Since $2^2 = (1+1)^2 = 1+1+1+1 = 4$, we have $(2T)^2 = 2^2T^2 = 4T^2$ as expected.

Notice that

$$\begin{aligned}(2T - \mathrm{id}_V)^2 &= (2T)^2 + \mathrm{id}_V^2 - 2 \cdot 2T \,\mathrm{id}_V \\ &= 4T^2 + \mathrm{id}_V - 4T \\ &= 4T + \mathrm{id}_V - 4T \\ &= \mathrm{id}_V,\end{aligned}$$

since T is idempotent. Hence the map $T \mapsto 2T - \mathrm{id}_V$ sends idempotents to involutions.

Let 2^{-1} be the multiplicative inverse of $2 = 1+1$ in $\mathbb{F}$, and similarly for 4^{-1}. If $S \in \mathcal{L}(V)$ is an involution, then

$$\begin{aligned}2^{-1}(S + \mathrm{id}_V) \circ 2^{-1}(S + \mathrm{id}_V) &= 4^{-1}(S^2 + S + S + \mathrm{id}_V) \\ &= 4^{-1}(\mathrm{id}_V + S + S + \mathrm{id}_V) \\ &= 2^{-1}(S + \mathrm{id}_V),\end{aligned}$$

so $2^{-1}(S + \mathrm{id}_V)$ is idempotent. And the map $S \mapsto 2^{-1}(S + \mathrm{id}_V)$ is clearly an inverse to the above map, so these give a bijection between the idempotents and involutions on $\mathcal{V}$. □

EXERCISE 8.20

Let $T \colon \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ be a function with the following properties: For all matrices $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$,

(a) $T(\alpha A) = \alpha T(A)$,

(b) $T(A + B) = T(A) + T(B)$, and

(c) $T(AB) = T(BA)$.

Show that there exists a $\beta \in \mathbb{F}$ such that $T = \beta \,\mathrm{tr}$.

SOLUTION. The first two properties say that T is linear, so it suffices to prove the claim on a basis for $\mathcal{M}_n(\mathbb{F})$. Furthermore, the third property implies that T is invariant under similarity, in particular under change of basis.

Let E_{ij} be the matrix whose (i,j)-th entry is 1 and all other entries are 0. Notice that $E_{ij}e_k = \delta_{jk}e_i$, so that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. The matrices E_{ii} and E_{jj} are similar, so $\beta := T(E_{ii}) = T(E_{jj})$. If $i \neq j$, then notice that $E_{ii}E_{ij} = E_{ij}$ but $E_{ij}E_{ii} = 0$. The third property thus implies that

$$T(E_{ij}) = T(E_{ii}E_{ij}) = T(E_{ij}E_{ii}) = T(0) = 0.$$

Hence $T(E_{ij}) = \beta\delta_{ij} = \beta \operatorname{tr} E_{ij}$ as desired. □

EXERCISE 8.21

A pair of linear operators $T, S \in \mathcal{L}(V)$ (with $\dim V < \infty$) is *simultaneously diagonalisable* if there is an ordered basis $\mathcal{V}$ for V such that ${}_{\mathcal{V}}[T]_{\mathcal{V}}$ and ${}_{\mathcal{V}}[S]_{\mathcal{V}}$ are both diagonal. Prove that two diagonalisable operators T and S are simultaneously diagonalisable if and only if they commute.

SOLUTION. First assume that T and S are simultaneously diagonalisable, and write $\mathcal{V} = (v_1, \dots, v_n)$. Then each v_i is an eigenvector for both T and S, so let $Tv_i = \lambda_i v_i$ and $Sv_i = \mu_i v_i$. Then

$$TSv_i = \mu_i Tv_i = \mu_i\lambda_i v_i = \lambda_i\mu_i v_i = \lambda_i Sv_i = STv_i,$$

so T and S commute. Alternatively, we may simply notice that the matrix representations of T and S commute (since they are diagonal), so

$${}_{\mathcal{V}}[TS]_{\mathcal{V}} = {}_{\mathcal{V}}[T]_{\mathcal{V}} \cdot {}_{\mathcal{V}}[S]_{\mathcal{V}} = {}_{\mathcal{V}}[S]_{\mathcal{V}} \cdot {}_{\mathcal{V}}[T]_{\mathcal{V}} = {}_{\mathcal{V}}[ST]_{\mathcal{V}},$$

and hence $TS = ST$.

In order to prove the converse we will need a couple of lemmas, starting with:

> *Assume that the subspace U is invariant under $T \in \mathcal{L}(V)$. If $v_1, \dots, v_k \in V$ are eigenvectors of T corresponding to distinct eigenvalues and $v_1 + \cdots + v_k \in U$, then all v_i lie in U.*

We prove this claim by induction. For $k = 1$ this is obvious, so assume that it holds for $k-1$. Then let λ_i be the eigenvalue corresponding to v_i, and put $u = v_1 + \cdots + v_k$. Notice that

$$Tu - \lambda_1 u = (\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_k - \lambda_1)v_k.$$

The left-hand side lies in U. Hence each summand on the right-hand side lies in U by induction, and since the eigenvalues are distinct so do $v_2, \dots, v_k$. Since U is a subspace, v_1 does as well.

> *If $T \in \mathcal{L}(V)$ is diagonalisable and the subspace U is invariant under T, then $T|_U \in \mathcal{L}(U)$ is also diagonalisable.*

Each $u \in U$ is a finite sum of eigenvectors of T corresponding to distinct eigenvalues. Hence each eigenvector also lies in U, so

$$U = \bigoplus_{\lambda \in \operatorname{Spec} T} U \cap E_T(\lambda) = \bigoplus_{\lambda \in \operatorname{Spec} T|_U} E_{T|_U}(\lambda),$$

where the last equality follows since $U \cap E_T(\lambda)$ is precisely the set of eigenvectors of T corresponding to λ that lie in U, i.e. the eigenvectors of $T|_U$ corresponding to λ.

Finally assume that $TS = ST$. If $v \in E_T(\lambda)$, then

$$TSv = STv = \lambda Sv,$$

so also $Sv \in E_T(\lambda)$. In other words, every eigenspace of T is invariant under S. By the lemma above, S restricted to $E_T(\lambda)$ is thus diagonalisable, hence has a basis $\mathcal{V}_\lambda$ of eigenvectors of S. But these are also eigenvectors of T. Then $\mathcal{V} = \bigcup_{\lambda \in \operatorname{Spec} T} \mathcal{V}_\lambda$ is a basis for V consisting of simultaneous eigenvectors of T and S. □

10 • Structure Theory for Normal Operators

EXERCISE 10.5

Prove that if $\|Tv\| = \|T^*v\|$ for all $v \in V$, where V is complex, then T is normal.

SOLUTION. The assumption implies that

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2 = \|T^*v\|^2 = \langle T^*v, T^*v\rangle = \langle TT^*v, v\rangle,$$

so the claim follows from Theorem 9.2(2).

(In fact, this also holds if V is real. For then

$$\begin{aligned}
\|T^{\mathbb{C}}(v + \mathrm{i}\,u)\|^2 &= \|Tv\|^2 + \|Tu\|^2 \\
&= \|T^*v\|^2 + \|T^*u\|^2 \\
&= \|(T^*)^{\mathbb{C}}(v + \mathrm{i}\,u)\|^2 \\
&= \|(T^{\mathbb{C}})^*(v + \mathrm{i}\,u)\|^2,
\end{aligned}$$

so $T^{\mathbb{C}}$ is normal, and hence T is normal.) □