Roman, *Advanced Linear Algebra*

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1 • Vector Spaces

EXERCISE 1.11

Show that if S is a subspace of a vector space V, then $\dim S \leq \dim V$. Furthermore, if $\dim S = \dim V < \infty$ then $S = V$.

SOLUTION. Let $\mathcal{B}$ be a basis for S. Then this is linearly independent as a subset of V, hence is contained in a basis $\mathcal{B}'$ for V by Theorem 1.9. Then $\mathcal{B} \subseteq \mathcal{B}'$, so it follows that $\dim S \leq \dim V$.

Now assume that $\dim S = \dim V < \infty$. Then $|\mathcal{B}| = |\mathcal{B}'|$, but since each basis is finite and one is contained in the other, we must have $\mathcal{B} = \mathcal{B}'$. Hence $S = V$. $\square$

EXERCISE 1.12

Suppose that $V = U \oplus S_1 = U \oplus S_2$. What can you say about the relationship between S_1 and S_2? What can you say if $S_1 \subseteq S_2$?

SOLUTION. By Theorem 3.6, all complements of U are isomorphic, so we always have $S_1 \cong S_2$. Assume that $S_1 \subseteq S_2$, and let $s \in S_2$. Then $s = u + s'$ for some $u \in U$ and $s' \in S_1$. But then s' also lies in S_2, so since the sum $U \oplus S_2$ is direct we have $u = 0$. $\square$

2 • Linear Transformations

EXERCISE 2.15

Suppose that $T \in \mathcal{L}(V, W)$.

(a) Given $L \in \mathcal{L}(U, W)$, show that there exists an $R \in \mathcal{L}(V, U)$ with $T = LR$ if and only if $\operatorname{im} T \subseteq \operatorname{im} L$:

$$\begin{tikzcd} V \arrow[r, dashed, "R"'] \arrow[rr, bend left, "T"] & U \arrow[r, "L"'] & W \end{tikzcd}$$

(b) Given $R \in \mathcal{L}(V,U)$, show that there exists an $L \in \mathcal{L}(U,W)$ with $T = LR$ if and only if $\ker R \subseteq \ker T$:

$$\begin{tikzcd} V \arrow[r, "R"'] \arrow[rr, bend left, "T"] & U \arrow[r, dashed, "L"'] & W \end{tikzcd}$$

In particular, both monomorphisms and epimorphisms split.

Solution. (a) Write $W = \ker L \oplus M$ for some subspace $M \subseteq W$. Then the restriction $L|_M \colon M \to \operatorname{im} L$ is bijective, so let $R = (L|_M)^{-1} T$, which is well-defined since $\operatorname{im} T \subseteq \operatorname{im} L$.

(b) Write $V = \ker R \oplus M$ for some subspace $M \subseteq V$. Then $R|_M \colon M \to \operatorname{im} R$ is bijective. Writing $U = \operatorname{im} R \oplus N$ for some subspace $N \subseteq U$, let $L = T \circ [(R|_M)^{-1}, 0]$. For $v \in \ker R \subseteq \ker T$ we have

$$LRv = L(0) = 0 = Tv,$$

and for $v \in M$ we have

$$LRv = T(R|_M)^{-1} Rv = Tv,$$

as required. □

Exercise 2.22

Let $T \in \mathcal{L}(V)$. If $TS = ST$ for all $S \in \mathcal{L}(V)$, show that $T = \alpha \operatorname{id}_V$ for some $\alpha \in \mathbb{F}$. I.e., the centre of the ring $\mathcal{L}(V)$, with multiplication given by function composition, is the subspace $\langle \operatorname{id}_V \rangle$.

Solution. This is obvious if $\dim V \in \{0,1\}$, so assume that $\dim V \geq 2$.

First let $v \in V \setminus \{0\}$ and write $V = \langle v \rangle \oplus U$ for some subspace U, and define $S \in \mathcal{L}(V)$ by letting $Sv = v$ and $Su = 0$ for $u \in U$. If T and S commute, then

$$STv = TSv = Tv.$$

Hence $Tv \in \langle v \rangle$ (which includes the possibility that $Tv = 0$).

Next assume that $v, w \in V$ are linearly independent, write $V = \langle v, w\rangle \oplus U_2$ for some subspace U_2, and define S by letting $Sv = w$, $Sw = v$ and $Su = 0$. Let $\alpha \in \mathbb{F}$ be such that $Tv = \alpha v$. Then

$$Tw = TSv = STv = \alpha Sv = \alpha w.$$

Hence $Tv = \alpha v$ for all $v \in V$ as desired. □

3 • The Isomorphism Theorems

EXERCISE 3.18

Let S be a subspace of V. Prove that $(V/S)^* \cong S^0$.

SOLUTION. Let $\pi\colon V \to V/S$ be the quotient map, such that for every $\varphi \in (V/S)^*$ we have

$$\begin{tikzcd} V \arrow[rr, "\varphi\circ\pi"] \arrow[dr, "\pi"'] & & \mathbb{F} \\ & V/S \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Consider the map $(V/S)^* \to V^*$ given by $\varphi \mapsto \varphi \circ \pi$. This is injective by the universal property of quotients. Also by this property, a functional $\psi \in V^*$ factors through π if and only if $S \subseteq \ker\psi$, i.e. if $\psi \in S^0$. Hence the image of the above map is precisely S^0. □

8 • Eigenvalues and Eigenvectors

EXERCISE 8.21

A pair of linear operators $T, S \in \mathcal{L}(V)$ (with $\dim V < \infty$) is *simultaneously diagonalisable* if there is an ordered basis $\mathcal{V}$ for V such that ${}_{\mathcal{V}}[T]_{\mathcal{V}}$ and ${}_{\mathcal{V}}[S]_{\mathcal{V}}$ are both diagonal. Prove that two diagonalisable operators T and S are simultaneously diagonalisable if and only if they commute.

SOLUTION. First assume that T and S are simultaneously diagonalisable, and write $\mathcal{V} = (v_1, \dots, v_n)$. Then each v_i is an eigenvector for both T and S, so let $Tv_i = \lambda_i v_i$ and $Sv_i = \mu_i v_i$. Then

$$TSv_i = \mu_i Tv_i = \mu_i\lambda_i v_i = \lambda_i\mu_i v_i = \lambda_i Sv_i = STv_i,$$

so T and S commute. Alternatively, we may simply notice that the matrix representations of T and S commute (since they are diagonal), so

$${}_{\mathcal{V}}[TS]_{\mathcal{V}} = {}_{\mathcal{V}}[T]_{\mathcal{V}} \cdot {}_{\mathcal{V}}[S]_{\mathcal{V}} = {}_{\mathcal{V}}[S]_{\mathcal{V}} \cdot {}_{\mathcal{V}}[T]_{\mathcal{V}} = {}_{\mathcal{V}}[ST]_{\mathcal{V}},$$

and hence $TS = ST$.

In order to prove the converse we will need a couple of lemmas, starting with:

> *Assume that the subspace U is invariant under $T \in \mathcal{L}(V)$. If $v_1, \dots, v_k \in V$ are eigenvectors of T corresponding to distinct eigenvalues and $v_1 + \cdots + v_k \in U$, then all v_i lie in U.*

We prove this claim by induction. For $k = 1$ this is obvious, so assume that it holds for $k - 1$. Then let λ_i be the eigenvalue corresponding to v_i, and put $u = v_1 + \cdots + v_k$. Notice that

$$Tu - \lambda_1 u = (\lambda_2 - \lambda_1)v_2 + \cdots + (\lambda_k - \lambda_1)v_k.$$

The left-hand side lies in U. Hence each summand on the right-hand side lies in U by induction, and since the eigenvalues are distinct so do $v_2, \dots, v_k$. Since U is a subspace, v_1 does as well.

> *If $T \in \mathcal{L}(V)$ is diagonalisable and the subspace U is invariant under T, then $T|_U \in \mathcal{L}(U)$ is also diagonalisable.*

Each $u \in U$ is a finite sum of eigenvectors of T corresponding to distinct eigenvalues. Hence each eigenvector also lies in U, so

$$U = \bigoplus_{\lambda \in \operatorname{Spec} T} U \cap E_T(\lambda) = \bigoplus_{\lambda \in \operatorname{Spec} T|_U} E_{T|_U}(\lambda),$$

where the last equality follows since $U \cap E_T(\lambda)$ is precisely the set of eigenvectors of T corresponding to λ that lie in U, i.e. the eigenvectors of $T|_U$ corresponding to λ.

Finally assume that $TS = ST$. If $v \in E_T(\lambda)$, then

$$TSv = STv = \lambda Sv,$$

so also $Sv \in E_T(\lambda)$. In other words, every eigenspace of T is invariant under S. By the lemma above, S restricted to $E_T(\lambda)$ is thus diagonalisable, hence has a basis $\mathcal{V}_\lambda$ of eigenvectors of S. But these are also eigenvectors of T. Then $\mathcal{V} = \bigcup_{\lambda \in \operatorname{Spec} T} \mathcal{V}_\lambda$ is a basis for V consisting of simultaneous eigenvectors of T and S. $\square$