Radon Measures

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9th November 2021

1 • Introduction

2 • General properties of measures

We assume that the reader is familiar with abstract measure spaces and topological spaces. Below we fix terminology and prove some elementary results.

2.1. Essential measures

If $\mathcal{J} \subseteq 2^X$ and $\mu \colon \mathcal{J} \to [0,\infty]$ such that $\emptyset \in \mathcal{J}$ and $\mu(\emptyset) = 0$, then μ gives rise to an outer measure μ^* on X by

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(B_n) \;\middle|\; (B_n)_{n\in\mathbb{N}} \subseteq \mathcal{J} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} B_n \right\}.$$

In the case that μ is a measure and $\mathcal{J}$ is a σ-algebra, we may rephrase this as

$$\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{J} \text{ and } A \subseteq B\}.$$

In this case we also have $\mu^*(A) = \mu(A)$ if $A \in \mathcal{J}$.

Definition 2.1: *Essential measures*

Let $(X,\mathcal{E},M)$ be a measure space. We say that a map $m \colon \mathcal{B}(X) \to [0,\infty]$ is the *essential measure* associated with M if

$$m(A) = \sup\{M^*(B) \mid B \subseteq A \text{ and } M^*(B) < \infty\},$$

for all $A \in \mathcal{E}$.

Lemma 2.2

If $(X,\mathcal{E},M)$ is a measure space and m is the essential measure associated with M, then $m(A) = M(A)$ when $M(A) < \infty$ or $m(A) = \infty$, and

$$m(A) = \sup\{M(B) \mid B \in \mathcal{E}, B \subseteq A \text{ and } M(B) < \infty\}, \tag{2.1}$$

for all $A \in \mathcal{E}$. Furthermore, m is a measure on $\mathcal{E}$.

Proof. Let $A \in \mathcal{E}$, and assume that $M(A) < \infty$. Then $M^*(A) = M(A) < \infty$ by the definition of the outer measure M^*, so $m(A) = M^*(A)$ by definition on m. With this, (2.1) is obvious when $M(A) < \infty$.

Now assume that $m(A) = \infty$. Then for any $R > 0$ there exists a $B \subseteq A$ such that $M^*(B) \geq R$. Now let $C \in \mathcal{E}$ such that $B \subseteq C$ and $M(C) < \infty$. Then $B \subseteq A \cap C \subseteq A$, so

$$R \leq M^*(B) \leq M(A \cap C) < \infty.$$

Since R was arbitrary, M can take on arbitrarily large but finite values on subsets of A, so (2.1) follows. It also follows that $M(A) = \infty$.

Finally we show that m is a Borel measure on X. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{E}$, and let $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$ be such that $M(B) < \infty$. If we let $B_n = B \cap A_n$, then

$$M(B) = M\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n=1}^{\infty} M(B_n) \leq \sum_{n=1}^{\infty} m(A_n),$$

where the inequality follows since $B_n \subseteq A_n$ and $M(B_n) < \infty$. This inequality holds for all $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$, so taking the supremum over such B yields

$$m\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \sum_{n=1}^{\infty} m(A_n).$$

We prove the opposite inequality. If the left-hand side is infinite this is obvious, so assume that it is finite. For $\varepsilon > 0$ there is then a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $C_n \subseteq A_n$ and $M(C_n) < \infty$ such that $m(A_n) \leq M(C_n) + \varepsilon/2^n$. It follows that

$$\sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} M(C_n) + \varepsilon \leq \sum_{n=1}^{\infty} m(A_n) + \varepsilon,$$

and since ε was arbitrary, the inequality follows. □

2.2. Borel measures on Hausdorff spaces

Below we let X denote a Hausdorff topological space. A *Borel measure* on X is a measure on the Borel σ-algebra $\mathcal{B}(X)$ of X. A Borel measure μ on X is called *outer regular* on a set $B \in \mathcal{B}(X)$ if

$$\mu(B) = \inf\{\mu(U) \mid U \supseteq B, U \text{ open}\},$$

and *inner regular* on B if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B, K \text{ compact}\}.$$

If μ is outer (inner) regular on all Borel sets, then we call it *outer (inner) regular*. Furthermore, if μ is both outer and inner regular, then it is simply called *regular*.

A Borel measure μ on X is called *locally finite* if every point has a neighbourhood U with $\mu(U) < \infty$. We have the following characterisation of local finiteness:

PROPOSITION 2.3

If a Borel measure on X is locally finite, then it is finite on all compact sets. The converse is also true if X is locally compact.

PROOF. Let μ be a locally finite Borel measure on X, and let $K \subseteq X$ be compact. Every $x \in K$ has an open neighbourhood U_x with $\mu(U_x) < \infty$. The collection $\{U_x \mid x \in K\}$ is an open cover of K, so it has a finite subcover, say $U_{x_1}, \dots, U_{x_n}$. But then

$$\mu(K) \leq \mu\biggl(\bigcup_{i=1}^{n} U_{x_i}\biggr) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty,$$

as desired.

Conversely, suppose that X is locally compact and that μ is a Borel measure that is finite on compact sets. Then every point has a compact neighbourhood, so every point has a neighbourhood on which μ is finite. Hence μ is locally finite. □

3 • Radon measures

Let X be a Hausdorff topological space.

DEFINITION 3.1: *Radon measures, R_1*

A Radon measure on X is a pair of measures (M, m) on $\mathcal{B}(X)$ such that

(i) m is the essential measure associated with M,

(ii) M is locally finite and outer regular,

(iii) m is inner regular, and

(iv) $m(B) = M(B)$ for $B \in \mathcal{B}(X)$ if B is open or $M(B) < \infty$.

Definition 3.2: *Radon measures, R_2*

A Radon measure on X is a measure M on $\mathcal{B}(X)$ that is locally finite, outer regular, and inner regular on open sets.

Definition 3.3: *Radon measures, R_3*

A Radon measure on X is a measure m on $\mathcal{B}(X)$ that is locally finite and inner regular.

Theorem 3.4

(i) *Let (M, n) be an R_1-Radon measure on X. Then M is an R_2-Radon measure, and m is an R_3-Radon measure.*

(ii) *Let M be an R_2-Radon measure on X. Then*

Now let X be a locally compact Hausdorff space, and let $C_c(X)$ denote the space of continuous complex-valued functions on X. A linear functional I on $C_c(X)$ is said to be *positive* if $I(f) \geq 0$ when $f \geq 0$. A Borel measure μ on X is called a *representing measure* for I if $I(f) = \int f \,\mathrm{d}\mu$ for all $f \in C_c(X)$.

Theorem 3.5: *The Riesz Representation Theorem*

Every positive linear functional on $C_c(X)$ has a unique R_2-Radon representing measure.

Proof. Folland (2007, Theorem 7.2) □

Proposition 3.6

Let I be a positive linear functional on X, and let M be the R_2-Radon representing measure for I. If m is the essential measure associated with M, then m is also a representing measure for I. [Uniqueness?]

Proof. This amounts to showing that

$$\int f \,\mathrm{d}m = \int f \,\mathrm{d}M$$

for all $f \in C_c(X)$. Pick one such f, and let $K = \operatorname{supp} f$. Since K is compact and both m and M are locally finite, Lemma 2.2 implies that m and M agree when restricted to K. The claim follows. □

References

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