Radon Measures

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1 • Introduction

2 • General properties of measures

We assume that the reader is familiar with abstract measure spaces and topological spaces. Below we fix terminology and prove some elementary results.

Below we let X denote a Hausdorff topological space. A *Borel measure* on X is a measure on the Borel σ-algebra $\mathcal{B}(X)$ of X. A Borel measure μ on X is called *outer regular* on a set $B \in \mathcal{B}(X)$ if

$$\mu(B) = \inf\{\mu(U) \mid U \supseteq B, U \text{ open}\},$$

and *inner regular* on B if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B, K \text{ compact}\}.$$

If μ is outer (inner) regular on all Borel sets, then we call it *outer (inner) regular*. If μ is both outer and inner regular, then it is simply called *regular*.

A Borel measure μ on X is called *locally finite* if every point has a neighbourhood U with $\mu(U) < \infty$. We have the following characterisation of local finiteness:

Proposition 2.1

If a Borel measure on X is locally finite, then it is finite on all compact sets. The converse is also true if X is locally compact.

PROOF. Let μ be a locally finite Borel measure on X, and let $K \subseteq X$ be compact. Every $x \in K$ has an open neighbourhood U_x with $\mu(U_x) < \infty$. The collection $\{U_x \mid x \in K\}$ is an open cover of K, so it has a finite subcover, say $U_{x_1}, \dots, U_{x_n}$. But then

$$\mu(K) \leq \mu\Big(\bigcup_{i=1}^{n} U_{x_i}\Big) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty,$$

as desired.

Conversely, suppose that X is locally compact and that μ is a Borel measure that is finite on compact sets. Then every point has a compact neighbourhood, so every point has a neighbourhood on which μ is finite. Hence μ is locally finite. □

If $\mathcal{J} \subseteq 2^X$ and $\mu\colon \mathcal{J} \to [0,\infty]$ such that $\emptyset \in \mathcal{J}$ and $\mu(\emptyset) = 0$, then μ gives rise to an outer measure μ^* on X by

$$\mu^*(A) = \inf\Big\{\sum_{n=1}^{\infty} \mu(B_n) \,\Big|\, (B_n)_{n\in\mathbb{N}} \subseteq \mathcal{J} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} B_n\Big\}.$$

In the case that μ is a measure and $\mathcal{J}$ is a σ-algebra, we may rephrase this as

$$\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{J} \text{ and } A \subseteq B\}.$$

In this case we also have $\mu^*(A) = \mu(A)$ if $A \in \mathcal{J}$.

DEFINITION 2.2

Let m and M be Borel measures on X. We say that m is the *essential measure* associated with M if

$$m(A) = \sup\{M^*(B) \mid B \subseteq A \text{ and } M^*(B) < \infty\},$$

for all $A \in \mathcal{B}(X)$.

LEMMA 2.3

If m and M are Borel measures on X and m is the essential measure associated with M, then $m(A) = M(A)$ when $M(A) < \infty$ or $m(A) = \infty$, and

$$m(A) = \sup\{M(B) \mid B \in \mathcal{B}(X), B \subseteq A \text{ and } M(B) < \infty\}, \tag{2.1}$$

for all $A \in \mathcal{B}(X)$.

PROOF. Let $A \in \mathcal{B}(X)$, and assume that $M(A) < \infty$. Then $m(A) = M^*(A) = M(A) < \infty$. For $\varepsilon > 0$ there exists a $B \subseteq A$ such that

$$m(A) \leq M^*(B) + \varepsilon \leq M(A) + \varepsilon,$$

and since ε was arbitrary, it follows that $m(A) \leq M(A)$. Conversely,

$$m(A) \geq M^*(A) = M(A),$$

so in total $m(A) = M(A)$. With this (2.1) is obvious when $M(A) < \infty$.

Now assume that $m(A) = \infty$. Then for any $R > 0$ there exists a $B \subseteq A$ such that $M^*(B) \geq R$. Now let $C \in \mathcal{B}(X)$ such that $B \subseteq C$ and $M(C) < \infty$. Then $B \subseteq A \cap C \subseteq A$, so

$$R \leq M^*(B) \leq M(A \cap C) \leq m(A).$$

Since R was arbitrary, M can take on arbitrarily large but finite values on subsets of A, so (2.1) follows. It also follows that $M(A) = \infty$. □

3 • Radon measures

Let X be a Hausdorff topological space.

Definition 3.1: *Radon measures, R_1*

A Radon measure on X is a pair of measures (M, m) on $\mathcal{B}(X)$ such that

(i) m is the essential measure associated with M,

(ii) M is locally finite and outer regular,

(iii) m is inner regular, and

(iv) $m(B) = M(B)$ for $B \in \mathcal{B}(X)$ if B is open or $M(B) < \infty$.

Definition 3.2: *Radon measures, R_2*

A Radon measure on X is a measure M on $\mathcal{B}(X)$ that is locally finite, outer regular, and inner regular on open sets.

Definition 3.3: *Radon measures, R_3*

A Radon measure on X is a measure m on $\mathcal{B}(X)$ that is locally finite and inner regular.

Now let X be a locally compact Hausdorff space, and let $C_c(X)$ denote the space of continuous complex-valued functions on X. A linear functional I on $C_c(X)$ is said to be *positive* if $I(f) \geq 0$ when $f \geq 0$. A Borel measure μ on X is called a *representing measure* for I if $I(f) = \int f \, \mathrm{d}\mu$ for all $f \in C_c(X)$.

THEOREM 3.4: *The Riesz Representation Theorem*

Every positive linear functional on $C_c(X)$ has a unique R_2-Radon representing measure.

PROOF. Folland (2007, Theorem 7.2) □

PROPOSITION 3.5

Let I be a positive linear functional on X, and let M be the R_2-Radon representing measure for I. If m is the essential measure associated with M, then m is also a representing measure for I. [Uniqueness?]

PROOF. This amounts to showing that

$$\int f \, \mathrm{d}m = \int f \, \mathrm{d}M$$

for all $f \in C_c(X)$. Pick one such f, and let $K = \operatorname{supp} f$. Since K is compact and both m and M are locally finite, Lemma 2.3 implies that m and M agree when restricted to K. The claim follows. □

References

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