Radon Measures

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1 • Introduction

2 • General properties of measures

We assume that the reader is familiar with abstract measure spaces and topological spaces. Below we fix terminology and prove some elementary results.

2.1. Essential measures

If $\mathcal{J} \subseteq 2^X$ and $\mu\colon \mathcal{J} \to [0,\infty]$ such that $\emptyset \in \mathcal{J}$ and $\mu(\emptyset) = 0$, then μ gives rise to an outer measure μ^* on X by

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(B_n) \;\middle|\; (B_n)_{n\in\mathbb{N}} \subseteq \mathcal{J} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} B_n \right\}.$$

In the case that μ is a measure and $\mathcal{J}$ is a σ-algebra, we may rephrase this as

$$\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{J} \text{ and } A \subseteq B\}.$$

In this case we also have $\mu^*(A) = \mu(A)$ if $A \in \mathcal{J}$.

Definition 2.1: *Essential measures*

Let $(X, \mathcal{E}, M)$ be a measure space. We say that a map $m\colon \mathcal{B}(X) \to [0,\infty]$ is the *essential measure* associated with M if

$$m(A) = \sup\{M^*(B) \mid B \subseteq A \text{ and } M^*(B) < \infty\},$$

for all $A \in \mathcal{E}$.

LEMMA 2.2

If $(X,\mathcal{E},M)$ is a measure space and m is the essential measure associated with M, then $m(A) = M(A)$ when $M(A) < \infty$ or $m(A) = \infty$, and

$$m(A) = \sup\{M(B) \mid B \in \mathcal{E}, B \subseteq A \text{ and } M(B) < \infty\}, \tag{2.1}$$

for all $A \in \mathcal{E}$. Furthermore, m is a measure on $\mathcal{E}$.

PROOF. Let $A \in \mathcal{E}$, and assume that $M(A) < \infty$. Then $M^*(A) = M(A) < \infty$ by the definition of the outer measure M^*, so $m(A) = M^*(A)$ by the definition of m. With this, (2.1) is obvious when $M(A) < \infty$.

Now assume that $m(A) = \infty$. Then for any $R > 0$ there exists a $B \subseteq A$ such that $M^*(B) \geq R$. Now let $C \in \mathcal{E}$ such that $B \subseteq C$ and $M(C) < \infty$. Then $B \subseteq A \cap C \subseteq A$, so

$$R \leq M^*(B) \leq M(A \cap C) < \infty.$$

Since R was arbitrary, M can take on arbitrarily large but finite values on subsets of A, so (2.1) follows. It also follows that $M(A) = \infty$.

Finally we show that m is a Borel measure on X. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{E}$, and let $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$ be such that $M(B) < \infty$. If we let $B_n = B \cap A_n$, then

$$M(B) = M\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n=1}^{\infty} M(B_n) \leq \sum_{n=1}^{\infty} m(A_n),$$

where the inequality follows since $B_n \subseteq A_n$ and $M(B_n) < \infty$. This inequality holds for all $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$, so taking the supremum over such B yields

$$m\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \sum_{n=1}^{\infty} m(A_n).$$

We prove the opposite inequality. If the left-hand side is infinite this is obvious, so assume that it is finite. For $\varepsilon > 0$ there is then a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $C_n \subseteq A_n$ and $M(C_n) < \infty$ such that $m(A_n) \leq M(C_n) + \varepsilon/2^n$. It follows that

$$\sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} M(C_n) + \varepsilon \leq \sum_{n=1}^{\infty} m(A_n) + \varepsilon,$$

and since ε was arbitrary, the inequality follows. [Something's missing here?]☐

2.2. *Borel measures on Hausdorff spaces*

Below we let X denote a Hausdorff topological space. A *Borel measure* on X is a measure on the Borel σ-algebra $\mathcal{B}(X)$ of X. A Borel measure μ on X is called *outer regular* on a set $B \in \mathcal{B}(X)$ if

$$\mu(B) = \inf\{\mu(U) \mid U \supseteq B, U \text{ open}\},$$

and *inner regular* on B if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B, K \text{ compact}\}.$$

If μ is outer (inner) regular on all Borel sets, then we call it *outer (inner) regular*. Furthermore, if μ is both outer and inner regular, then it is simply called *regular*.

A Borel measure μ on X is called *locally finite* if every point has a neighbourhood U with $\mu(U) < \infty$. We have the following characterisation of local finiteness:

PROPOSITION 2.3

If a Borel measure on X is locally finite, then it is finite on all compact sets. The converse is also true if X is locally compact.

PROOF. Let μ be a locally finite Borel measure on X, and let $K \subseteq X$ be compact. Every $x \in K$ has an open neighbourhood U_x with $\mu(U_x) < \infty$. The collection $\{U_x \mid x \in K\}$ is an open cover of K, so it has a finite subcover, say $U_{x_1}, \dots, U_{x_n}$. But then

$$\mu(K) \leq \mu\Big(\bigcup_{i=1}^{n} U_{x_i}\Big) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty,$$

as desired.

Conversely, suppose that X is locally compact and that μ is a Borel measure on X that is finite on compact sets. Then every point has a compact neighbourhood, so every point has a neighbourhood on which μ is finite. Hence μ is locally finite. ☐

Above we defined

3 • Radon measures

Let X be a Hausdorff topological space.

DEFINITION 3.1: *Radon measures, R_1*

A Radon measure on X is a pair of measures (M, m) on $\mathcal{B}(X)$ such that

(i) m is the essential measure associated with M,

(ii) M is locally finite and outer regular,

(iii) m is inner regular, and

(iv) $m(B) = M(B)$ for $B \in \mathcal{B}(X)$ if B is open or $M(B) < \infty$.

Definition 3.2: *Radon measures, R_2*

A Radon measure on X is a measure M on $\mathcal{B}(X)$ that is locally finite, outer regular, and inner regular on open sets.

Definition 3.3: *Radon measures, R_3*

A Radon measure on X is a measure m on $\mathcal{B}(X)$ that is locally finite and inner regular.

Theorem 3.4

(i) *Let (M, m) be an R_1-Radon measure on X. Then M is an R_2-Radon measure, and m is an R_3-Radon measure.*

(ii) *Let M be an R_2-Radon measure on X. If m is the essential measure associated with M, then (M, m) is an R_1-Radon measure.*

(iii) *Let m be an R_3-Radon measure on X, and define a map $M: \mathcal{B}(X) \to [0, \infty]$ by*

$$M(A) = \inf\{m(U) \mid U \text{ open and } A \subseteq U\}.$$

Then (M, m) is an R_1-Radon measure on X.

Proof.

Put this in different terms:

Definition 3.5: *Radon measures*

An *outer Radon measure* on X is a Borel measure on X that is locally finite, outer regular, and inner regular on open sets.

An *inner Radon measure* on X is a Borel measure on X that is locally finite and inner regular.

We denote the set of outer Radon measures on X by $\mathcal{M}^+(X)$ and the set of inner Radon measures by $\mathcal{M}^-(X)$. Define maps $E: \mathcal{M}^+(X) \to \mathcal{M}^-(X)$ and $P: \mathcal{M}^-(X) \to \mathcal{M}^+(X)$, where E maps an outer Radon measure to the corresponding essential measure, and P maps an inner Radon measure to the corresponding principal measure.

Proposition 3.6

Every outer Radon measure is inner regular on all its σ-finite sets.

This is Folland (2007, Proposition 7.5), though Folland only considers Radon measures on locally compact spaces. In fact the claim holds in any Hausdorff space, as the proof below demonstrates.

PROOF. Let μ be an outer Radon measure on X, and let $A \in \mathcal{B}(X)$. First assume that $\mu(A) < \infty$ and let $\varepsilon > 0$. By outer regularity there is an open set $U \supseteq A$ such that $\mu(U) < \mu(A) + \varepsilon$, and by inner regularity on U there is a compact set $K \subseteq U$ with $\mu(U) < \mu(K) + \varepsilon$. Furthermore, since $\mu(U \setminus A) < \varepsilon$ there exists an open set $V \supseteq U \setminus A$ such that $\mu(V) < \varepsilon$. Now let $F = K \setminus V$ and notice that F is compact and that $F \subseteq A$. It follows that

$$\mu(F) = \mu(K) - \mu(K \cap V) > \mu(A) - \varepsilon - \mu(V) > \mu(A) - 2\varepsilon.$$

Hence μ is inner regular on A.

Now assume that $\mu(A) = \infty$ and that there exists an increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X)$ with $\mu(A_n) < \infty$, and whose union is A. It follows by continuity of μ that $\mu(A_n) \to \infty$ as $n \to \infty$, so for any $R > 0$ there is an $n \in \mathbb{N}$ such that $\mu(A_n) > R$. By inner regularity on A_n there exists a compact set $K \subset A_n$ such that $\mu(K) > R$. Hence μ is also inner regular on A. $\square$

THEOREM 3.7

The maps E and P are well-defined and each other's inverses.

PROOF. *Well-definition of E*: Let μ be an outer Radon measure on X, and let ν be the associated essential measure. If $A \in \mathcal{B}(X)$ and $\mu(A) < \infty$, then μ is inner regular on A by Proposition 3.6. But since μ and ν agree on sets with finite μ-measure, ν is also inner regular on A.

Similarly, every point in X has a neighbourhood with finite μ-measure, so this neighbourhood also has finite ν-measure. Thus ν is an inner Radon measure.

Well-definition of P: Let ν be an inner Radon measure on X with associated principal measure μ. We first show that μ and ν agree on μ-finite sets, so let $A \in \mathcal{B}(X)$ with $\mu(A) < \infty$. Clearly $\nu(A) \leq \mu(A)$, so we prove the other inequality. Let $U \supseteq A$ be an open set with $\nu(U) < \infty$. Then also $\nu(U \setminus A) < \infty$, so for $\varepsilon > 0$ there exists a compact set $K \subseteq U \setminus A$ with $\nu(U \setminus A) \leq \nu(K) + \varepsilon$ by inner regularity. Then $V = U \setminus K$ is an open set containing A, so

$$\mu(A) \leq \nu(V) = \nu(A) + \nu(U \setminus A) - \nu(K) \leq \nu(A) + \varepsilon.$$

Since ε was arbitrary, it follows that $\mu(A) \leq \nu(A)$.

Next we show that μ is σ-additive, so let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoints sets in $\mathcal{B}(X)$. For $\varepsilon > 0$ there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets with

$A_n \subseteq U_n$ such that $\nu(U_n) \leq \mu(A_n) + \varepsilon$. It follows that

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \nu\Big(\bigcup_{n\in\mathbb{N}} U_n\Big) = \sum_{n=1}^{\infty} \nu(U_n) \leq \sum_{n=1}^{\infty} \mu(A_n) + \varepsilon.$$

So μ is countably subadditive since ε was arbitrary. The opposite inequality is obvious if $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \infty$, and if not then the sets A_n also have finite μ-measure. Hence

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Thus μ is in fact a measure.

Finally, μ is clearly locally finite since ν is, and since μ and ν agree on μ-finite sets, outer regularity follows easily from the definition of μ.

$P \circ E = \mathrm{id}$: Let μ be an outer Radon measure, $\nu = E(\mu)$ its corresponding essential measure, and $\mu' = P(\nu)$ the principal measure associated with ν. We must show that $\mu = \mu'$.

Let $A \in \mathcal{B}(X)$. Since μ is outer regular,

$$\mu(A) = \inf\{\mu(U) \mid U \text{ open and } A \subseteq U\}.$$

Comparing this with the definition of μ' and recalling that $\nu \leq \mu$, we find that $\mu'(A) \leq \mu(A)$.

For the opposite inequality, let $\varepsilon > 0$ and let $U \supseteq A$ be an open set such that $\nu(U) \leq \mu'(A) + \varepsilon$. Because $\nu(U) = \mu(U)$ by Lemma 3.11, we have

$$\mu'(A) + \varepsilon \geq \nu(U) = \mu(U) \geq \mu(A),$$

so it follows that $\mu'(A) \geq \mu(A)$ since ε was arbitrary.

$E \circ P = \mathrm{id}$: Conversely, let ν be an inner Radon measure and let $\mu = P(\nu)$ and $\nu' = E(\mu)$. Let $A \in \mathcal{B}(X)$ and notice that

$$\begin{aligned}
\nu(A) &= \sup\{\nu(K) \mid K \text{ compact and } K \subseteq A\} \\
&= \sup\{\mu(K) \mid K \text{ compact and } K \subseteq A\} \\
&= \sup\{\mu(B) \mid B \in \mathcal{B}(X),\, B \subseteq A \text{ and } \mu(B) < \infty\} \\
&= \nu'(A).
\end{aligned}$$

The second and third equalities follow since μ is locally finite, hence finite on compact sets by Proposition 2.3, so $\nu(K) = \mu(K) < \infty$. ☐

Now let X be a locally compact Hausdorff space, and let $C_c(X)$ denote the space of continuous complex-valued functions on X. A linear functional I on $C_c(X)$ is said to be *positive* if $I(f) \geq 0$ when $f \geq 0$. A Borel measure μ on X is called a *representing measure* for I if $I(f) = \int f \,\mathrm{d}\mu$ for all $f \in C_c(X)$.

THEOREM 3.8: *The Riesz Representation Theorem*

Every positive linear functional on $C_c(X)$ has a unique R_2-Radon representing measure.

PROOF. Folland (2007, Theorem 7.2). □

PROPOSITION 3.9

Let (M,m) be an R_1-Radon measure on X, and let I be a positive linear functional on $C_c(X)$. Then M is a representing measure for I if and only if m is. [Uniqueness?]

PROOF. This amounts to showing that

$$\int f \,\mathrm{d}m = \int f \,\mathrm{d}M$$

for all $f \in C_c(X)$. Pick one such f, and let $K = \operatorname{supp} f$. Since K is compact and both m and M are locally finite, Lemma 2.2 implies that m and M agree when restricted to K. The claim follows. □

Don't know where to put this, but just to write it down:

DEFINITION 3.10: *Principal measures*

Let m be a measure on a topological space X. The *principal measure*[1] associated with m is the map $M\colon \mathcal{B}(X) \to [0,\infty]$ given by

$$M(A) = \inf\{m(U) \mid U \text{ open and } A \subseteq U\}.$$

[Is this a measure in general?]

LEMMA 3.11

Let M be an R_2-Radon measure on X, and let m be the essential measure associated with M. Then $m(U) = M(U)$ for all open $U \subseteq X$.

PROOF. Let $U \subseteq X$ be open. Since $m \leq M$ in general, we only need to show that $m(U) \geq M(U)$. Since M is inner regular on U, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of U such that $M(U) = \lim_{n\to\infty} M(K_n)$. Furthermore, $M(K_n) < \infty$ because M is locally finite, so since m and M agree when M is finite it follows that

$$M(U) = \lim_{n\to\infty} M(K_n) = \lim_{n\to\infty} m(K_n) \leq m(U),$$

as desired. □

[1] I do not believe this is standard terminology. Bauer (2001) uses the term 'principal measure' for the R_2-Radon measure determined by a positive linear functional, as in Riesz' representation theorem.

PROPOSITION 3.12

Let M be an R_2-Radon measure on X, let m be the essential measure associated with M, and let M' be the principal measure associated with m. Then $M = M'$. In particular, every R_2-Radon measure is the principal measure of some R_3-Radon measure.

PROOF. Let $A \in \mathcal{B}(X)$. Since M is outer regular,

$$M(A) = \inf\{M(U) \mid U \text{ open and } A \subseteq U\}.$$

Comparing this with the definition of M' and recalling that $m \leq M$, we find that $M'(A) \leq M(A)$.

For the opposite inequality, let $\varepsilon > 0$ and let $U \supseteq A$ be an open set such that $m(U) \leq M'(A) + \varepsilon$. Because $m(U) = M(U)$ by [lemma], we have

$$M'(A) + \varepsilon \geq m(U) = M(U) \geq M(A),$$

so it follows that $M'(A) \geq M(A)$ since ε was arbitrary. □

References

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