Radon Measures

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1 • Introduction

2 • General properties of measures

We assume that the reader is familiar with abstract measure spaces and topological spaces. Below we fix terminology and prove some elementary results.

Proposition 2.1

Let $(X,\mathcal{E},\mu)$ be a measure space, and define $\mu_0\colon \mathcal{E} \to [0,\infty]$ by

$$\mu_0(A) = \sup\{\mu(B) \mid B \in \mathcal{E}, B \subseteq A, \mu(B) < \infty\}.$$

Then μ_0 is a semifinite measure on $(X,\mathcal{E})$ called the semifinite part *of μ. Also, $\mu_0 \leq \mu$, and for $A \in \mathcal{E}$ with $\mu(A) < \infty$ we have $\mu_0(A) = \mu(A)$. If μ is already semifinite, then $\mu_0 = \mu$.*

Furthermore, there is a measure ν on $(X,\mathcal{E})$ only assuming the values 0 and ∞, such that $\mu = \mu_0 + \nu$.

Proof. Clearly $\mu_0 \leq \mu$ with equality for μ-finite sets. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{E}$, and let $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$ be such that $\mu(B) < \infty$. If we let $B_n = B \cap A_n$, then

$$\mu(B) = \mu\Bigl(\bigcup_{n\in\mathbb{N}} B_n\Bigr) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu_0(A_n),$$

where the inequality follows since $B_n \subseteq A_n$ and $\mu(B_n) < \infty$. This inequality holds for all $B \subseteq \bigcup_{n\in\mathbb{N}} A_n$, so taking the supremum over such B yields

$$\mu_0\Bigl(\bigcup_{n\in\mathbb{N}} A_n\Bigr) \leq \sum_{n=1}^{\infty} \mu_0(A_n).$$

We prove the opposite inequality. If the left-hand side is infinite this is obvious, so assume that it is finite. For $\varepsilon > 0$ there is then a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{E}$ with $C_n \subseteq A_n$ and $\mu(C_n) < \infty$ such that $\mu_0(A_n) \leq \mu(C_n) + \varepsilon/2^n$. By continuity of μ we have

$$\mu\Big(\bigcup_{n\in\mathbb{N}} C_n\Big) = \lim_{k\to\infty} \mu\Big(\bigcup_{n\leq k} C_n\Big) \leq \mu_0\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) < \infty,$$

since $\mu(C_n) \leq \mu_0(A_n)$. Notice that the C_n are also pairwise disjoint, so it follows that

$$\sum_{n=1}^{\infty} \mu_0(A_n) - \varepsilon \leq \sum_{n=1}^{\infty} \mu(C_n) = \mu\Big(\bigcup_{n\in\mathbb{N}} C_n\Big) \leq \mu_0\Big(\bigcup_{n\in\mathbb{N}} A_n\Big),$$

and since ε was arbitrary, the inequality follows.

Next, it is obvious that μ_0 is semifinite, since if $\mu_0(A) = \infty$ then by definition there is a set $B \subseteq A$ with $0 < \mu(B) < \infty$. But then also $0 < \mu_0(B) < \infty$.

Now assume that μ is already semifinite, and consider $A \in \mathcal{E}$ with $\mu(A) = \infty$. Then for any $R > 0$ there is a subset $B \subseteq A$ with $R < \mu(B) < \infty$. It follows that $\mu_0(A) = \infty$, so $\mu_0 = \mu$.

TODO final claim. □

Different non-semifinite measures can have the same semifinite part, so the map $\mu \mapsto \mu_0$ is not generally invertible.

LEMMA 2.2

Let $(X, \mathcal{E}, \mu)$ be a measure space. For any $A \in \mathcal{E}$ there is a $B \in \mathcal{E}$ with $B \subseteq A$ such that $\mu_0(A) = \mu(B)$.

PROOF. This is obvious if $\mu_0(A) = \infty$, so assume that $\mu_0(A) < \infty$. By definition of μ_0 there is a sequence (B_n) of measurable subsets of A such that $\mu(B_n) \leq \mu_0(A) \leq \mu(B_n) + \frac{1}{n}$ and $\mu(B_n) < \infty$. Letting $C_n = B_1 \cup \cdots \cup B_n$ we also have $\mu(C_n) < \infty$, and since C_n is also a subset of A we have $\mu(C_n) \leq \mu_0(A) \leq \mu(C_n) + \frac{1}{n}$. The claim follows by letting $B = \bigcup_{n\in\mathbb{N}} C_n$. □

If $\mathcal{J} \subseteq 2^X$ and $\mu\colon \mathcal{J} \to [0,\infty]$ such that $\emptyset \in \mathcal{J}$ and $\mu(\emptyset) = 0$, then μ gives rise to an outer measure μ^* on X by

$$\mu^*(A) = \inf\Big\{\sum_{n=1}^{\infty} \mu(B_n) \;\Big|\; (B_n)_{n\in\mathbb{N}} \subseteq \mathcal{J} \text{ and } A \subseteq \bigcup_{n\in\mathbb{N}} B_n\Big\}.$$

Assuming that $\mathcal{J}$ is closed under countable unions (e.g., $\mathcal{J}$ is a topology or a σ-algebra) and that μ is countably sub-additive, we may rephrase this as

$$\mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{J} \text{ and } A \subseteq B\}.$$

If μ is also increasing, then we also have $\mu^*(A) = \mu(A)$ for all $A \in \mathcal{J}$. Finally we write $\mu^+ = \mu^*|_{\mathcal{J}}$. If $\mathcal{J}$ is a σ-algebra (and especially if μ is a measure) we may thus consider whether μ^+ is (also) a measure.

2.1. Borel measures

Below we let X denote a Hausdorff topological space with Borel algebra $\mathcal{B}(X)$. A measure on $(X, \mathcal{B}(X))$ is called a *Borel measure* on X. A Borel measure μ on X is *outer regular* on a set $A \in \mathcal{B}(X)$ if

$$\mu(A) = \inf\{\mu(U) \mid U \supseteq A, U \text{ open}\},$$

and *(strongly) inner regular* on A if

$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}.$$

We also say that μ is *weakly inner regular* on A if

$$\mu(A) = \sup\{\mu(F) \mid F \subseteq A, F \text{ closed}\}.$$

Strong inner regularity of course implies weak inner regularity. If μ is outer (resp. inner, weakly inner) regular on all Borel sets, then we call it outer (resp. inner, weakly inner) regular. Furthermore, if μ is both outer and inner regular, then it is simply called *regular*.

Lemma 2.3

Let μ be a Borel measure. Then μ is inner regular on all σ-compact sets, and weakly inner regular on all F_σ-sets. If μ is finite, then it is outer regular on all G_δ-sets.

Lemma 2.4

If μ is outer (resp. inner, weakly inner) regular on all its finite sets, then it is outer (resp. inner, weakly inner) regular on all its σ-finite sets.

If A is σ-finite and μ is outer regular on A, then given $\varepsilon > 0$ there is an open $U \supseteq A$ such that $\mu(U \setminus A) < \varepsilon$.

Proof. First assume that μ is outer regular, let A be σ-finite, and let $\varepsilon > 0$. Then there are sets A_n with finite measure such that $A = \bigcup_{n\in\mathbb{N}} A_n$, as well as open sets $U_n \supseteq A_n$ such that $\mu(U_n \setminus A_n) < \varepsilon/2^n$. Letting $U = \bigcup_{n\in\mathbb{N}} U_n$ we have

$$\mu(U \setminus A) \leq \mu\Bigl(\bigcup_{n\in\mathbb{N}} (U_n \setminus A_n)\Bigr) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus A_n) \leq \varepsilon,$$

as desired.

Instead assume that μ is inner (resp. weakly inner) regular. We may assume that $\mu(A) = \infty$ so that $\mu(A_n) \to \infty$. The claim then follows since this yields an unbounded sequence of compact (resp. closed) subsets of the A_n. $\square$

Lemma 2.5

A σ-finite Borel measure is outer regular if and only if it is weakly inner regular. In this case, for each $A \in \mathcal{B}(X)$ and $\varepsilon > 0$ there is an open set U and a closed set F such that $F \subseteq A \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

Proof. Let μ be outer regular and consider $A \in \mathcal{B}(X)$. Since A and A^c are σ-finite, Lemma 2.4 yields open sets set $U \supseteq A$ and $V \supseteq A^c$ such that $\mu(U \setminus A) < \frac{\varepsilon}{2}$ and $\mu(V \setminus A^c) < \frac{\varepsilon}{2}$. Letting $F = V^c$ this implies that

$$\mu(U \setminus F) = \mu(U \setminus A) + \mu(A \setminus F) = \mu(U \setminus A) + \mu(V \setminus A^c) < \varepsilon.$$

This also implies that $\mu(A \setminus F) < \varepsilon$, showing that μ is weakly inner regular. The converse is similar. □

Lemma 2.6

Let μ be a finite Borel measure on a G_δ-space X. Then μ is outer regular and weakly inner regular.

Proof. Let $\mathcal{A}$ be the subcollection of $\mathcal{B}(X)$ consisting of sets on which μ is both outer regular and weakly inner regular. Since μ is finite, a Borel set A lies in $\mathcal{A}$ if and only if given $\varepsilon > 0$ there is an open set U and a closed set F such that $F \subseteq A \subseteq U$ and $\mu(U \setminus F) < \varepsilon$. Clearly $\mathcal{A}$ is thus closed under complementation.

Since every open set is an F_σ-set, Lemma 2.3 implies that μ is weakly inner regular on all open sets. Clearly μ is outer regular on all open sets, so $\mathcal{A}$ contains all open sets.

Finally let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$, and let $\varepsilon > 0$. For each n choose an open U_n and a closed F_n such that $F_n \subseteq A_n \subseteq U_n$ and $\mu(U_n \setminus F_n) < \varepsilon/2^n$. Letting $F = \bigcup_{n\in\mathbb{N}} F_n$ and $U = \bigcup_{n\in\mathbb{N}} U_n$ we thus have $F \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq U$, and

$$\mu(U \setminus F) \leq \mu\Big(\bigcup_{n\in\mathbb{N}} (U_n \setminus F_n)\Big) \leq \sum_{n=1}^{\infty} \mu(U_n \setminus F_n) = \varepsilon.$$

Finally, continuity of μ from above[1] shows that there is an $n \in \mathbb{N}$ such that

$$\mu\Big(U \setminus \bigcup_{k=1}^{n} F_n\Big) < \varepsilon.$$

Thus $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$, so $\mathcal{A}$ is a σ-algebra containing all open sets, hence it contains $\mathcal{B}(X)$ as desired. □

[1] Here we use that μ is finite. Note that the rest of the proof goes through if μ is only σ-finite.

REMARK 2.7. Note that Lemma 2.6 does not hold for arbitrary σ-finite measures. For instance, let τ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let $\mu(A) = \tau(A \cap \mathbb{Q})$. Then μ is a σ-finite Borel measure, but e.g. $\mu(\{0\}) = 1$, while the measure of any open set is infinite. ⌟

LEMMA 2.8

Let μ be a semifinite Borel measure on X. If μ is (weakly) inner regular on all μ-finite sets, then μ is (weakly) inner regular.

PROOF. Let $A \in \mathcal{B}(X)$ be a Borel set with $\mu(A) = \infty$. For any $R > 0$ there is a Borel set $B \subseteq A$ with $R < \mu(B) < \infty$. Since μ is inner regular on B, there is a compact set $K \subseteq B$ with $R < \mu(K)$. Since R was arbitrary, this shows that μ is inner regular on A.

The proof is the same for weak inner regularity. □

Furthermore, we say that a Borel measure μ on X is *tight* if for every $\varepsilon > 0$ there is a compact set K such that $\mu(X \setminus K) < \varepsilon$. This property is especially interesting when μ is finite. We have the following relationships between tightness and inner regularity:

LEMMA 2.9

Let μ be a Borel measure on X.

(1) *If μ is tight, then μ is inner regular on X.*

(2) *If μ is finite and inner regular on X, then μ is tight.*

(3) *If μ is tight and weakly inner regular on a set $A \in \mathcal{B}(X)$ with finite μ-measure, then μ is (strongly) inner regular on A.*

PROOF. The first two claims are obvious. For the third, assume that μ is tight and weakly inner regular on A. Let $\varepsilon > 0$ and choose a closed set $F \subseteq A$ and a compact set $K \subseteq X$ such that $\mu(A \setminus F) < \varepsilon$ and $\mu(X \setminus K) < \varepsilon$. Then

$$A \setminus (K \cap F) = (A \setminus K) \cup (A \setminus F) \subseteq (X \setminus K) \cup (A \setminus F),$$

and hence $\mu(A \setminus (K \cap F)) \leq 2\varepsilon$. □

A Borel measure μ on X is called *locally finite* if every point has a neighbourhood U with $\mu(U) < \infty$. This makes sense on non-Hausdorff spaces but we do not consider local finiteness in this generality. We have the following characterisation of local finiteness:

LEMMA 2.10

If a Borel measure on X is locally finite, then it is finite on all compact sets. The converse is also true if X is locally compact.

PROOF. Let μ be a locally finite Borel measure on X, and let $K \subseteq X$ be compact. Every $x \in K$ has an open neighbourhood U_x with $\mu(U_x) < \infty$. The collection $\{U_x \mid x \in K\}$ is an open cover of K, so it has a finite subcover, say $U_{x_1}, \ldots, U_{x_n}$. But then

$$\mu(K) \leq \mu\Bigl(\bigcup_{i=1}^{n} U_{x_i}\Bigr) \leq \sum_{i=1}^{n} \mu(U_{x_i}) < \infty,$$

as desired.

Conversely, suppose that X is locally compact and that μ is a Borel measure on X that is finite on compact sets. Then every point has a compact neighbourhood, so every point has a neighbourhood on which μ is finite. Hence μ is locally finite. □

PROPOSITION 2.11

Let μ be a Borel measure on X that is finite on compact sets. If μ is inner regular on $A \in \mathcal{B}(X)$, then μ_0 is also inner regular on A and $\mu(A) = \mu_0(A)$.

Since every Borel measure is inner regular on compact sets, this in particular says that μ and μ_0 agree on compact sets, though this follows directly from finiteness of μ on compacta.

PROOF. The first claim is obvious since μ and μ_0 agree on compact sets because μ is finite on compacta. For the second claim, since $\mu_0 \leq \mu$ in general we only need to show that $\mu_0(A) \geq \mu(A)$. Since μ is inner regular on A, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of A such that $\mu(A) = \lim_{n\to\infty} \mu(K_n)$. It follows that

$$\mu(A) = \lim_{n\to\infty} \mu(K_n) = \lim_{n\to\infty} \mu_0(K_n) \leq \mu_0(A),$$

as desired. □

While general measures can be finite, semifinite or σ-finite, Borel measures can also be locally finite or finite on compacta. There are of course relationships between these properties, and we list some of them below.

PROPOSITION 2.12

A Borel measure μ on X is σ-finite if one of the following is satisfied:

(1) *X is Lindelöf and μ is locally finite.*

(2) *X is σ-compact and μ is finite on compacta.*

2.2. Borel measures on Polish spaces

THEOREM 2.13

Every locally finite Borel measure on a Polish space is regular.

PROOF. We first prove that if μ is a finite Borel measure on a Polish space X, then μ is regular. By Lemma 2.6 and Lemma 2.9 it suffices to show that μ is tight. Let $\varepsilon > 0$. Since X is Lindelöf there is for every $k \in \mathbb{N}$ a sequence $(B_n^k)_{n\in\mathbb{N}}$ of open balls of radius $1/k$ covering X. By continuity of μ there exists for each k an $N_k \in \mathbb{N}$ such that

$$\mu(X) \leq \mu\Big(\bigcup_{n\leq N_k} B_n^k\Big) + \frac{\varepsilon}{2^k},$$

or in other words such that the complement of $\bigcup_{n\leq N_k} B_n^k$ has measure less than $\varepsilon/2^k$. Now let

$$A = \bigcap_{k\in\mathbb{N}} \bigcup_{n\leq N_k} B_n^k.$$

It is easy to see that A is totally bounded. It then follows that A is relatively compact since X is complete. Furthermore,

$$\mu(X \setminus \overline{A}) \leq \mu(X \setminus A) \leq \sum_{n=1}^{\infty} \mu\Big(X \setminus \bigcup_{n\leq N_k} B_n^k\Big) \leq \varepsilon.$$

This proves the claim.

Next let μ be locally finite, and let $(V_n)_{n\in\mathbb{N}}$ be an open covering of X such that $V_n \uparrow X$.[2] For $n \in \mathbb{N}$ define a finite measure μ_n on X by $\mu_n(A) = \mu(A \cap V_n)$. Let $A \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\mu(A) \leq \mu_n(A) + \varepsilon$. Since μ_n is inner regular by the above, there is a compact set $K \subseteq A$ such that $\mu_n(A) \leq \mu_n(K) + \varepsilon$. In total,

$$\mu(A) \leq \mu_n(K) + 2\varepsilon \leq \mu(K) + 2\varepsilon,$$

so μ is inner regular on A since ε was arbitrary.

Next, for $n \in \mathbb{N}$ there exists an open set $U_n \supseteq A$ such that $\mu_n(U_n \setminus A) \leq \varepsilon/2^n$ since μ_n is outer regular by the above. If we let $U = \bigcup_{n\in\mathbb{N}} U_n \cap V_n$, then

$$A = A \cap \bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} A \cap V_n \subseteq U,$$

[2] The existence of such a covering follows since X is Lindelöf and μ is locally finite.

since $A \subseteq U_n$. Furthermore,

$$U \setminus A = \bigcup_{n \in \mathbb{N}} (U_n \setminus A) \cap V_n,$$

which implies that

$$\mu(U \setminus A) \leq \sum_{n=1}^{\infty} \mu_n(U_n \setminus A) \leq \varepsilon,$$

so μ is outer regular on A. □

3 • Radon measures

3.1. Definitions and basic properties

Radon measures are usually only defined on *locally compact* Hausdorff spaces, but this extra assumption is for many purposes superfluous and overly strict: Indeed, another natural setting for Radon measures is that of Polish spaces (or more general metrisable spaces), particularly in probability theory. Thus we consider a general Hausdorff topological space X below.

Definition 3.1: *Radon measures*

An *outer Radon measure* on X is a Borel measure on X that is locally finite, outer regular, and inner regular on open sets.

An *inner Radon measure* on X is a Borel measure on X that is locally finite and inner regular.

Schwartz (1973) gives three different (but equivalent, as he shows) definitions of Radon measures. His definition R_2 is what we call an outer Radon measure, and his R_3 is our inner Radon measure. We will return to his R_1 definition in Definition 3.9.

We begin by deriving some properties of Radon measures.

Proposition 3.2

Every outer Radon measure is inner regular on all its σ-finite sets.

This is Proposition 7.5 in Folland (2007), though Folland only considers Radon measures on locally compact spaces. In fact the claim holds in any Hausdorff space, as the proof below demonstrates.

Proof. Let μ be an outer Radon measure on X, and let $A \in \mathcal{B}(X)$. First assume that $\mu(A) < \infty$ and let $\varepsilon > 0$. By outer regularity there is an open set $U \supseteq A$ such that $\mu(U) < \mu(A) + \varepsilon$, and by inner regularity on U there is a compact set

$K \subseteq U$ with $\mu(U) < \mu(K) + \varepsilon$. Furthermore, since $\mu(U \setminus A) < \varepsilon$ there exists an open set $V \supseteq U \setminus A$ such that $\mu(V) < \varepsilon$. Now let $F = K \setminus V$ and notice that F is compact and that $F \subseteq A$. It follows that

$$\mu(F) = \mu(K) - \mu(K \cap V) > \mu(A) - \varepsilon - \mu(V) > \mu(A) - 2\varepsilon.$$

Hence μ is inner regular on A.

Now assume that $\mu(A) = \infty$ and that there exists an increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X)$ with $\mu(A_n) < \infty$, and whose union is A. It follows by continuity of μ that $\mu(A_n) \to \infty$ as $n \to \infty$, so for any $R > 0$ there is an $n \in \mathbb{N}$ such that $\mu(A_n) > R$. By inner regularity on A_n there exists a compact set $K \subset A_n$ such that $\mu(K) > R$. Hence μ is also inner regular on A. □

Proposition 3.3

Every inner Radon measure is semifinite.

Proof. Let ν be an inner Radon measure, and let A be a Borel set with $\nu(A) = \infty$. By local finiteness, ν is finite on compacta. Hence, by inner regularity on A, A has compact subsets of arbitrarily large finite ν-measure. In particular, A has a compact subset K with $0 < \nu(K) < \infty$ as claimed. □

Proposition 3.4

If μ is an outer Radon measure, then μ_0 is an inner Radon measure.

Proof. Clearly μ_0 is locally finite. By Lemma 2.8 it thus suffices to show that μ_0 is inner regular on every μ_0-finite set, so let A be such a set. By Lemma 2.2 there is a Borel set $B \subseteq A$ such that $\mu_0(A) = \mu(B)$. But by Proposition 3.2 μ is regular on B, and since μ and μ_0 agree on compacta, it follows that μ_0 is inner regular on A. □

Proposition 3.5

If ν is inner regular on all ν-finite sets, then ν^+ is an outer regular Borel measure. Also, $\nu \le \nu^+$, and for $A \in \mathcal{B}(X)$ with $\nu^+(A) < \infty$ we have $\nu(A) = \nu^+(A)$.

Recall that ν^+ is the outer measure ν^* restricted to $\mathcal{B}(X)$.

Proof. We first show that ν^+ and ν agree on ν^+-finite sets, so let $A \in \mathcal{B}(X)$ with $\nu^+(A) < \infty$. Clearly $\nu(A) \le \nu^+(A)$, so we prove the other inequality. Let $U \supseteq A$ be an open set with $\nu(U) < \infty$. Then also $\nu(U \setminus A) < \infty$, so for $\varepsilon > 0$ there exists a compact set $K \subseteq U \setminus A$ with $\nu(U \setminus A) \leq \nu(K) + \varepsilon$ by inner regularity. Then $V = U \setminus K$ is an open set containing A, so

$$\nu^+(A) \leq \nu(V) = \nu(A) + \nu(U \setminus A) - \nu(K) \leq \nu(A) + \varepsilon.$$

Since ε was arbitrary, it follows that $\nu^+(A) \leq \nu(A)$.

We next show that every open set is ν^*-measurable. That is, for $U \subseteq X$ open and $E \subseteq X$ with $\nu^*(E) < \infty$ we show that

$$\nu^*(E) \geq \nu^*(E \cap U) + \nu^*(E \cap U^c).$$

This is clear when E is open (indeed, when E is Borel), since then ν^* can be replaced first with ν^+ and then with ν. For arbitrary E there is an open $V \supseteq E$ such that $\nu^*(V) = \nu(V) \leq \nu^*(E) + \varepsilon$, which implies that

$$\begin{aligned}\nu^*(E) + \varepsilon \geq \nu^*(V) &\geq \nu^*(V \cap U) + \nu^*(V \cap U^c) \\ &\geq \nu^*(E \cap U) + \nu^*(E \cap U^c).\end{aligned}$$

Since ε was arbitrary, it follows that U is ν^*-measurable. Carathéodory's theorem then implies that ν^+ is a measure.

Alternatively, we can prove directly that ν^+ is countably additive: Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoints sets in $\mathcal{B}(X)$. For $\varepsilon > 0$ there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets with $A_n \subseteq U_n$ such that $\nu(U_n) \leq \nu^+(A_n) + \varepsilon/2^n$. It follows that

$$\nu^+\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \nu\Big(\bigcup_{n\in\mathbb{N}} U_n\Big) \leq \sum_{n=1}^{\infty} \nu(U_n) \leq \sum_{n=1}^{\infty} \nu^+(A_n) + \varepsilon.$$

So ν^+ is countably subadditive since ε was arbitrary. The opposite inequality is obvious if $\nu^+(\bigcup_{n\in\mathbb{N}} A_n) = \infty$, and if not then the sets A_n also have finite ν^+-measure. Hence

$$\nu^+\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \nu^+(A_n).$$

Thus ν^+ is countably additive.

Finally, the definition of ν^+ immediately implies that it is outer regular. □

Corollary 3.6

If ν is an inner Radon measure, then ν^+ is an outer Radon measure.

Proof. It follows from Proposition 3.5 that ν^+ is outer regular, and ν^+ is locally finite since ν and ν^+ agree on all sets of finite ν^+-measure. Finally, ν and ν^+ agree on all open sets by definition of ν^+, so ν^+ is inner regular on open sets. □

We denote the set of outer Radon measures on X by $\mathcal{M}^+(X)$ and the set of inner Radon measures by $\mathcal{M}^-(X)$. The above results imply that there are maps $\mathcal{M}^+(X) \to \mathcal{M}^-(X)$ and $\mathcal{M}^-(X) \to \mathcal{M}^+(X)$ given by $\mu \mapsto \mu_0$ and $\nu \mapsto \nu^+$ respectively. These have the following fundamental property:

THEOREM 3.7

The maps $\mu \mapsto \mu_0$ and $\nu \mapsto \nu^+$ are each other's inverses.

PROOF. $(\mu_0)^+ = \mu$: Let μ be an outer Radon measure, and let $A \in \mathcal{B}(X)$. Since μ is outer regular,

$$\mu(A) = \inf\{\mu(U) \mid U \text{ open and } A \subseteq U\}.$$

Comparing this with the definition of $(\mu_0)^+$ and recalling that $\mu_0 \leq \mu$, we find that $(\mu_0)^+(A) \leq \mu(A)$.

For the opposite inequality, let $\varepsilon > 0$ and let $U \supseteq A$ be an open set such that $\mu_0(U) \leq (\mu_0)^+(A) + \varepsilon$. Because $\mu_0(U) = \mu(U)$ by Proposition 2.11, we have

$$(\mu_0)^+(A) + \varepsilon \geq \mu_0(U) = \mu(U) \geq \mu(A),$$

so it follows that $(\mu_0)^+(A) \geq \mu(A)$ since ε was arbitrary.

$(\nu^+)_0 = \nu$: Conversely, let ν be an inner Radon measure and let $A \in \mathcal{B}(X)$. Notice that

$$\begin{aligned}
\nu(A) &= \sup\{\nu(K) \mid K \text{ compact and } K \subseteq A\} \\
&= \sup\{\nu^+(K) \mid K \text{ compact and } K \subseteq A\} \\
&= \sup\{\nu^+(B) \mid B \in \mathcal{B}(X),\, B \subseteq A \text{ and } \nu^+(B) < \infty\} \\
&= (\nu^+)_0(A).
\end{aligned}$$

The first equality follows by inner regularity of ν. The second equality follows since ν^+ is locally finite, hence finite on compact sets by Lemma 2.10, so $\nu(K) = \nu^+(K)$ by Proposition 3.5. The third follows similarly, where we also use that ν^+ is outer Radon along with Proposition 3.2, to ensure that the third supremum is no larger than the second. The final equality follows by definition of $(\nu^+)_0$. □

PROPOSITION 3.8

If μ is an outer Radon measure that is also inner regular, then $\mu_0 = \mu$. Conversely, if ν is an inner Radon measure that is also outer regular, then $\nu^+ = \nu$.

PROOF. Note that μ is also an inner Radon measure, so by Proposition 3.3 it is semifinite. Hence Proposition 2.1 implies that $\mu_0 = \mu$.

For the converse, ν is already an outer Radon measure, so by the above we have $\nu = \nu_0$. But then it follows that $\nu^+ = (\nu_0)^+ = \nu$. □

Definition 3.9: *Radon pairs*

Let μ and ν be outer and inner Radon measures respectively, such that $\nu = \mu_0$ (or equivalently $\mu = \nu^+$). The pair (μ, ν) is then called a *Radon pair*[3].

A Radon pair is what Schwartz (1973) describes in his R_1 definition of Radon measures.

3.2. Locally compact Hausdorff spaces

Now let X be a locally compact Hausdorff space, and let $C_c(X)$ denote the space of continuous complex-valued functions on X with compact support. A linear functional I on $C_c(X)$ is said to be *positive* if $I(f) \geq 0$ when $f \geq 0$. A Borel measure μ on X is called a *representing measure* for I if $I(f) = \int f \,\mathrm{d}\mu$ for all $f \in C_c(X)$.

Theorem 3.10: *The Riesz Representation Theorem*

Every positive linear functional on $C_c(X)$ has a unique outer Radon representing measure.

Proof. Folland (2007, Theorem 7.2). □

Proposition 3.11

Let (μ, ν) be a Radon pair on X, and let I be a positive linear functional on $C_c(X)$. Then μ is a representing measure for I if and only if ν is. In particular, I has a unique inner Radon representing measure.

Proof. This amounts to showing that

$$\int f \,\mathrm{d}\nu = \int f \,\mathrm{d}\mu$$

for all $f \in C_c(X)$. Pick one such f, and let $K = \operatorname{supp} f$. Since K is compact and both ν and μ are locally finite, **??** implies that ν and μ agree when restricted to K. The claim follows.

The final claim follows since μ is the unique outer Radon representing measure, and Theorem 3.7 furnishes a bijection between inner and outer Radon measures on X. □

- LCH where every open is σ-compact: locally finite Borel => regular. (Folland 7.8, Cohn 7.2.3 ish, Bauer 29.6).

[3] Similar to principal measures, I do not believe this is standard terminology. In fact, as far as I know I have just made it up!

- First countable Hausdorff: inner regular and finite on compacta => locally finite (Bauer 25.4)
- Inner Radon on Polish => outer regular (Bauer 26.4)
- μ and μ_0 agree on sigma-finites (Bauer 28.5), add to first prop above if I haven't already.
- Agreeing on open sets (Bauer 28.7) + compact sets?
- Characteristaion of representing measures (Bauer 29.2-4)
- sigma-comapct LCH: representing measures coincide (Bauer 29.6)
- sigma-compact LCH: Inner Radon => outer regular (Bauer 29.7)
- Representing measures for bounded forms (Bauer 29.9ff)
- LCH: Finite inner Radon => outer regular (Bauer 29.11)
- Uniqueness of representing measure on scLCH (Bauer 29.13)
- Density of $C_c(X)$ in $L^p(\mu)$ for regular μ - only need inner regularity on open sets (Bauer 29.14 + remark after)
- Regularity of completions (Cohn exercise 7.2.2)
- Can use measures whose σ-algebras just include Borel, not equal it? See Cohn p. 189ff.
- Cohn exercise 7.2.3 interesting
- Cohn ex 7.2.5
- Cohn ex 7.2.6
- Cohn Baire? ex 7.2.8-11
- Cohn 7.4.1 + remarks after. + ex. 7.4.3

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