Spivak, *Physics for Mathematicians: Mechanics I*

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Isometries

If (S,ρ) and (T,δ) are metric spaces, then a map $f\colon S \to T$ is an ***isometry*** if

$$\delta\bigl(f(x), f(y)\bigr) = \rho(x,y)$$

for all $x,y \in S$.

Let X and Y be normed vector spaces, and let $A \subseteq X$ be a subset. Since e.g. the norm on X induces a metric ρ_X given by $\rho_X(\boldsymbol{x},\boldsymbol{x}') = \|\boldsymbol{x}-\boldsymbol{x}'\|_X$, a (not necessarily linear) map $\varphi\colon A \to Y$ is an isometry if

$$\|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}')\|_Y = \|\boldsymbol{x}-\boldsymbol{x}'\|_X$$

for all $\boldsymbol{x},\boldsymbol{x}' \in A$. If X and Y are also inner product spaces, then we have the following:

Lemma 5.1

Let X,Y be inner product spaces, and let $A \subseteq X$ be a subset containing $\mathbf{0}$. Let $\varphi\colon A \to Y$ be a map such that $\varphi(\mathbf{0}) = \mathbf{0}$. Then φ is an isometry if and only if

$$\langle \varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')\rangle_Y = \langle \boldsymbol{x},\boldsymbol{x}'\rangle_X \tag{5.1}$$

for all $\boldsymbol{x},\boldsymbol{x}' \in A$.

Proof. Notice the identities

$$\|\boldsymbol{x}-\boldsymbol{x}'\|_X^2 = \|\boldsymbol{x}\|_X^2 - 2\langle \boldsymbol{x},\boldsymbol{x}'\rangle_X + \|\boldsymbol{x}'\|_X^2$$

and

$$\|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}')\|_Y^2 = \|\varphi(\boldsymbol{x})\|_Y^2 - 2\langle \varphi(\boldsymbol{x}),\varphi(\boldsymbol{x}')\rangle_Y + \|\varphi(\boldsymbol{x}')\|_Y^2.$$

First assume that (5.1) holds. Substituting $\boldsymbol{x}'$ for $\boldsymbol{x}$ we find that $\|\varphi(\boldsymbol{x})\|_Y = \|\boldsymbol{x}\|_X$, and we similarly have $\|\varphi(\boldsymbol{x}')\|_Y = \|\boldsymbol{x}'\|_X$. The above identities then imply that φ is an isometry.

If conversely φ is an isometry, notice that

$$\|\varphi(\boldsymbol{x})\|_Y = \|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{0})\|_Y = \|\boldsymbol{x} - \boldsymbol{0}\|_X = \|\boldsymbol{x}\|_X,$$

and we similarly have $\|\varphi(\boldsymbol{x}')\|_Y = \|\boldsymbol{x}'\|_X$. The above identities then imply (5.1). □

PROPOSITION 5.2

Every isometry $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is on the form

$$\varphi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

for a unique $d \times d$ matrix A and vector $\boldsymbol{b} \in \mathbb{R}^d$. Furthermore, A is orthogonal.

In particular, any isometry that preserves the origin is linear.

PROOF. We first show that $\varphi(\boldsymbol{x}) = \varphi_0(\boldsymbol{x}) + \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^d$, where $\varphi_0\colon \mathbb{R}^d \to \mathbb{R}^d$ is an isometry fixing the origin. Simply let $\boldsymbol{b} = \varphi(\boldsymbol{0})$ and $\varphi_0(\boldsymbol{x}) = \varphi(\boldsymbol{x}) - \boldsymbol{b}$. Then

$$\varphi_0(\boldsymbol{0}) = \varphi(\boldsymbol{0}) - \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0},$$

and for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ we have

$$\|\varphi_0(\boldsymbol{x}) - \varphi_0(\boldsymbol{y})\| = \|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\|,$$

so φ_0 is an isometry.

Next we show that φ_0 is linear; since it is an isometry, this will imply the existence of an orthogonal matrix A as above. For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d$ we have by Lemma 5.1

$$\begin{aligned}
\langle \varphi_0(\boldsymbol{x}+\boldsymbol{y}), \varphi_0(\boldsymbol{z}) \rangle &= \langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z} \rangle \\
&= \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle \\
&= \langle \varphi_0(\boldsymbol{x}), \varphi_0(\boldsymbol{z}) \rangle + \langle \varphi_0(\boldsymbol{y}), \varphi_0(\boldsymbol{z}) \rangle \\
&= \langle \varphi_0(\boldsymbol{x}) + \varphi_0(\boldsymbol{y}), \varphi_0(\boldsymbol{z}) \rangle.
\end{aligned}$$

Since φ_0 preserves orthogonality, it maps an orthogonal basis into an orthogonal set. Since φ_0 is also injective (since it is an isometry), this set is in fact a basis. Replacing $\boldsymbol{z}$ by the elements in such a basis, it follows that

$\langle\varphi_0(\boldsymbol{x}+\boldsymbol{y}),\boldsymbol{w}\rangle = \langle\varphi_0(\boldsymbol{x})+\varphi_0(\boldsymbol{y}),\boldsymbol{w}\rangle$ for all $\boldsymbol{w}\in\mathbb{R}^d$. Hence $\varphi_0(\boldsymbol{x}+\boldsymbol{y}) = \varphi_0(\boldsymbol{x})+\varphi_0(\boldsymbol{y})$. We similarly find that, for $\alpha\in\mathbb{R}$,

$$\begin{aligned}\langle\varphi_0(\alpha\boldsymbol{x}),\varphi_0(\boldsymbol{z})\rangle &= \langle\alpha\boldsymbol{x},\boldsymbol{z}\rangle\\ &= \alpha\langle\boldsymbol{x},\boldsymbol{z}\rangle\\ &= \alpha\langle\varphi_0(\boldsymbol{x}),\varphi_0(\boldsymbol{z})\rangle\\ &= \langle\alpha\varphi_0(\boldsymbol{x}),\varphi_0(\boldsymbol{z})\rangle,\end{aligned}$$

implying that $\varphi_0(\alpha\boldsymbol{x}) = \alpha\varphi_0(\boldsymbol{x})$ Thus φ_0 is linear.

To show uniqueness, assume that $A\boldsymbol{x}+\boldsymbol{b} = \varphi(\boldsymbol{x}) = A'\boldsymbol{x}+\boldsymbol{b}'$ for all $\boldsymbol{x}\in\mathbb{R}^d$. Then $\boldsymbol{b} = \varphi(\boldsymbol{0}) = \boldsymbol{b}'$, so $A\boldsymbol{x} = A'\boldsymbol{x}$. Since this holds for all $\boldsymbol{x}$, it follows that $A = A'$ as desired. □

Consider points $\boldsymbol{b}_1,\dots,\boldsymbol{b}_K\in\mathbb{R}^3$ and let $\boldsymbol{b} = (\boldsymbol{b}_1,\dots,\boldsymbol{b}_K)$. Let further $\boldsymbol{c} = (\boldsymbol{c}_1,\dots,\boldsymbol{c}_K)$ be a rigid motion of $\boldsymbol{b}$. That is, there is a one-parameter family of isometries $\varphi_t\colon\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_K\}\to\mathbb{R}^3$ such that $\varphi_t(\boldsymbol{b}_i) = \boldsymbol{c}_i(t)$. However, to apply Proposition 5.2 to φ_t it must be defined on $\mathbb{R}^3$. A natural question is then whether it is possible to extend an isometry from a subset of $\mathbb{R}^3$ to all $\mathbb{R}^3$. The answer is affirmative, as we shall see below.

Another natural question is whether such an extension is unique. If the points $\boldsymbol{b}_1,\dots,\boldsymbol{b}_K$ do not lie in a plane, then by renumbering we may assume that $\boldsymbol{b}_2-\boldsymbol{b}_1$, $\boldsymbol{b}_3-\boldsymbol{b}_1$, and $\boldsymbol{b}_4-\boldsymbol{b}_1$ are linearly independent and hence span $\mathbb{R}^3$. The proof below will show that this implies that an extension of φ_t is unique.

Lemma 5.3

Let $A\subseteq\mathbb{R}^d$ and let $\varphi\colon A\to\mathbb{R}^d$ be an isometry. Then φ extends to an isometry $\tilde{\varphi}\colon\mathbb{R}^d\to\mathbb{R}^d$.

Furthermore, we have the following uniqueness results:

(i) *If A contains $d+1$ affinely independent elements, then φ has a unique isometric extension to $\mathbb{R}^d$.*

(ii) *If A contains d affinely independent elements, then φ has a unique isometric extension to $\mathbb{R}^d$ that is also orientation preserving.*[1]

Proof. By translation we may assume that $\boldsymbol{0}\in A$ (if A is empty then the claim is obvious) and that $\varphi(\boldsymbol{0}) = \boldsymbol{0}$. First extend φ by linearity to $\operatorname{span} A$. This is well-defined, since if $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in A$ and $\alpha_1,\dots,\alpha_n\in\mathbb{R}$, then by Lemma 5.1 we

[1] By this we mean that if $\tilde{\varphi}(\boldsymbol{x}) = A\boldsymbol{x}+\boldsymbol{b}$ in accordance with Proposition 5.2, then $A\in\mathrm{SO}(d)$.

have

$$\Big\|\sum_{i=1}^{n}\alpha_i\varphi(\boldsymbol{x}_i)\Big\|^2 = \sum_{i,j=1}^{n}\alpha_i\alpha_j\langle\varphi(\boldsymbol{x}_i),\varphi(\boldsymbol{x}_j)\rangle = \sum_{i,j=1}^{n}\alpha_i\alpha_j\langle\boldsymbol{x}_i,\boldsymbol{x}_j\rangle = \Big\|\sum_{i=1}^{n}\alpha_i\boldsymbol{x}_i\Big\|^2.$$

This also shows that φ extends to an isometry on $\operatorname{span} A$, and in particular φ is linear by Proposition 5.2.

Since φ is linear it maps $\operatorname{span} A$ to a subspace $\varphi(\operatorname{span} A)$ of $\mathbb{R}^d$ with the same dimension. Let U and V be the orthogonal complements of $\operatorname{span} A$ and $\varphi(\operatorname{span} A)$ respectively, and let $\psi\colon U \to V$ be an isometry. Letting $\tilde{\varphi} = \varphi \oplus \psi$, it follows easily by Pythagoras' theorem that $\tilde{\varphi}$ is also an isometry as desired.

For the first uniqueness claim, notice that if A contains a collection of $d+1$ affinely independent elements and $\boldsymbol{a}$ is one element in such a collection, then $A - \boldsymbol{a}$ contains d linearly independent vectors. Hence its span, which is contained in $\operatorname{span} A$, is d-dimensional.

To prove the second uniqueness claim, notice that an argument similar to the above shows that $\operatorname{span} A$ has dimension at least $d-1$. Then notice that if $\dim U = 1$, then there are exactly two isometries $U \to V$, only one of which makes $\tilde{\varphi}$ orientation preserving. □

Motion relative to the centre of mass

In the section 'The inertial tensor', Spivak considers a rigid motion $\boldsymbol{c} = (\boldsymbol{c}_1,\dots,\boldsymbol{c}_K)$ of a collection of points $\boldsymbol{b}_1,\dots,\boldsymbol{b}_K$. These are given by $\boldsymbol{c}_i(t) = B(t)\boldsymbol{b}_i + \boldsymbol{w}(t)$ for some $B\colon \mathbb{R} \to \mathrm{SO}(3)$ and $\boldsymbol{w}\colon \mathbb{R} \to \mathbb{R}^3$.

Spivak then claims that we may choose $\boldsymbol{w}$ to be the centre of mass $\boldsymbol{C}$ given by

$$\boldsymbol{C} = \frac{1}{M}\sum_{i=1}^{K} m_i\boldsymbol{c}_i, \quad \text{where} \quad M = \sum_{i=1}^{K} m_i.$$

Given the uniqueness part of Proposition 5.2, it is not immediately clear that this is possible, even if it is intuitively reasonable. Inserting the rigid motion in the expression for $\boldsymbol{C}$, we get

$$\begin{aligned}\boldsymbol{C}(t) &= \frac{1}{M}\sum_{i=1}^{K} m_i\big(B(t)\boldsymbol{b}_i + \boldsymbol{w}(t)\big) = \frac{1}{M}\sum_{i=1}^{K} m_i B(t)\boldsymbol{b}_i + \boldsymbol{w}(t)\\ &= B(t)\frac{\sum_{i=1}^{K} m_i\boldsymbol{c}_i(0)}{M} + \boldsymbol{w}(t) = B(t)\boldsymbol{C}(0) + \boldsymbol{w}(t).\end{aligned}$$

That is, the centre of mass has the exact same time evolution as each of the individual particles. If we choose our coordinate system such that $\boldsymbol{C}(0) = \boldsymbol{0}$, we thus get $\boldsymbol{C}(t) = \boldsymbol{w}(t)$.

Notice that this does not mean that we consider the motion as seen from the centre of mass sytem. We simply choose an inertial system in which $\boldsymbol{C}(0) = \boldsymbol{0}$.

König's theorems

REMARK 5.4: König's theorem for angular momenta.
For $i = 1, \dots, K$ consider a particle with mass m_i and position vector $\boldsymbol{c}_i$ with respect to an origin O at rest in an inertial frame. Furthermore, let $\boldsymbol{p}_i$ be its momentum. If $\boldsymbol{C}$ is the centre of mass of all n particles, then we let $\overline{\boldsymbol{c}} = \boldsymbol{c} - \boldsymbol{C}$ be the relative position of the ith particle, and let $\overline{\boldsymbol{p}} = m_i \overline{\boldsymbol{c}}_i'$. Denote by M the total mass of the particles, and let $\boldsymbol{P}$ be the total momentum (equivalently $\boldsymbol{P} = M\boldsymbol{C}'$). The angular momentum of the ith particle with respect to the centre of mass is then

$$\begin{aligned}\overline{\boldsymbol{l}}_i &= \overline{\boldsymbol{c}}_i \times \overline{\boldsymbol{p}}_i = (\boldsymbol{c}_i - \boldsymbol{C}) \times (\boldsymbol{p}_i - m_i \boldsymbol{C}') \\ &= \boldsymbol{l}_i - m_i \boldsymbol{c}_i \times \boldsymbol{C}' - \boldsymbol{C} \times \boldsymbol{p}_i + m_i \boldsymbol{C} \times \boldsymbol{C}'.\end{aligned}$$

Therefore, the total angular momentum with respect to the centre of mass is

$$\begin{aligned}\overline{\boldsymbol{L}} &= \sum_{i=1}^{K} \overline{\boldsymbol{l}}_i = \boldsymbol{L} - M\boldsymbol{C} \times \boldsymbol{C}' - \boldsymbol{C} \times \boldsymbol{P} + M\boldsymbol{C} \times \boldsymbol{C}' \\ &= \boldsymbol{L} - \boldsymbol{C} \times \boldsymbol{P} = \boldsymbol{L} - \boldsymbol{L}_{\mathrm{CM}},\end{aligned}$$

where $\boldsymbol{L}_{\mathrm{CM}} = \boldsymbol{C} \times \boldsymbol{P}$ is the angular momentum of the centre of mass. Hence

$$\boldsymbol{L} = \overline{\boldsymbol{L}} + \boldsymbol{L}_{\mathrm{CM}},$$

which is *König's theorem* for angular momenta. ⌟

REMARK 5.5: Decomposition of torque.
Next let each particle i be acted upon by an external force $\boldsymbol{F}_i$, and let $\boldsymbol{F}$ be the total force on the system. If the total external torque relative to the origin O is $\boldsymbol{\tau}$, then the external torque relative to the centre of mass is

$$\begin{aligned}\overline{\boldsymbol{\tau}} &= \sum_{i=1}^{K} \overline{\boldsymbol{c}}_i \times \boldsymbol{F}_i = \sum_{i=1}^{K} (\boldsymbol{c}_i - \boldsymbol{C}) \times \boldsymbol{F}_i \\ &= \sum_{i=1}^{K} \boldsymbol{\tau}_i - \boldsymbol{C} \times \boldsymbol{F} = \boldsymbol{\tau} - \boldsymbol{\tau}_{\mathrm{CM}},\end{aligned}$$

where $\boldsymbol{\tau}_{\mathrm{CM}} = \boldsymbol{C} \times \boldsymbol{F}$ is the torque on the centre of mass. That is,

$$\boldsymbol{\tau} = \overline{\boldsymbol{\tau}} + \boldsymbol{\tau}_{\mathrm{CM}}.$$

Now notice that

$$\boldsymbol{L}'_{\text{CM}} = (\boldsymbol{C}' \times \boldsymbol{P}) + (\boldsymbol{C} \times \boldsymbol{P}') = \boldsymbol{C} \times \boldsymbol{F} = \boldsymbol{\tau}_{\text{CM}},$$

since $\boldsymbol{C}'$ and $\boldsymbol{P} = M\boldsymbol{C}'$ are parallel. König's theorem thus implies that

$$\overline{\boldsymbol{L}}' = \boldsymbol{L}' - \boldsymbol{L}'_{\text{CM}} = \boldsymbol{\tau} - \boldsymbol{\tau}_{\text{CM}} = \overline{\boldsymbol{\tau}}.$$

That is, relative to the centre of mass, the external torque is the time derivative of the angular momentum, even if the centre of mass frame is not inertial. ⌟

Remark 5.6: König's theorem for kinetic energy.
Notice that

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{K} m_i \|\boldsymbol{v}_i\|^2 = \frac{1}{2}\sum_{i=1}^{K} m_i \|\overline{\boldsymbol{v}}_i + \boldsymbol{C}'\|^2 \\ &= \frac{1}{2}\sum_{i=1}^{K} m_i \|\overline{\boldsymbol{v}}_i\|^2 + \frac{1}{2}\sum_{i=1}^{K} m_i \|\boldsymbol{C}'\|^2 + \boldsymbol{C}' \cdot \sum_{i=1}^{K} m_i \overline{\boldsymbol{v}}_i. \end{aligned}$$

But the last term vanishes since $\sum_{i=1}^{K} m_i \overline{\boldsymbol{v}}_i$ is the time derivative of $\sum_{i=1}^{K} m_i \overline{\boldsymbol{c}}_i$, which is zero. Hence

$$T = \overline{T} + T_{\text{CM}},$$

which is *König's theorem* for kinetic energy. ⌟