Spivak, *Physics for Mathematicians: Mechanics I*

Danny Nygård Hansen

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5 • Rigid Bodies

Isometries

If (S,ρ) and (T,δ) are metric spaces, then a map $f\colon S \to T$ is an ***isometry*** if

$$\delta\bigl(f(x), f(y)\bigr) = \rho(x,y)$$

for all $x,y \in S$.

Let X and Y be normed vector spaces, and let $A \subseteq X$ be a subset. Since e.g. the norm on X induces a metric ρ_X given by $\rho_X(\boldsymbol{x},\boldsymbol{x}') = \|\boldsymbol{x}-\boldsymbol{x}'\|_X$, a (not necessarily linear) map $\varphi\colon A \to Y$ is an isometry if

$$\|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}')\|_Y = \|\boldsymbol{x}-\boldsymbol{x}'\|_X$$

for all $\boldsymbol{x},\boldsymbol{x}' \in A$. If X and Y are also inner product spaces, then we have the following:

Lemma 5.1

Let X, Y be inner product spaces, and let $A \subseteq X$ be a subset containing $\mathbf{0}$. Let $\varphi\colon A \to Y$ be a map such that $\varphi(\mathbf{0}) = \mathbf{0}$. Then φ is an isometry if and only if

$$\langle \varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')\rangle_Y = \langle \boldsymbol{x}, \boldsymbol{x}'\rangle_X \tag{5.1}$$

for all $\boldsymbol{x},\boldsymbol{x}' \in A$.

Proof. Notice the following identity:

$$\begin{aligned}
\|\varphi(\boldsymbol{x})\|_Y^2 - 2\langle \varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')\rangle_Y + \|\varphi(\boldsymbol{x}')\|_Y^2 &= \|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}')\|_Y^2 \\
&= \|\boldsymbol{x}-\boldsymbol{x}'\|_X^2 \\
&= \|\boldsymbol{x}\|_X^2 - 2\langle \boldsymbol{x}, \boldsymbol{x}'\rangle_X + \|\boldsymbol{x}'\|_X^2
\end{aligned}$$

implies that $\langle \varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')\rangle_Y = \langle \boldsymbol{x}, \boldsymbol{x}'\rangle_X$ as desired.

If (5.1) holds, then letting $\boldsymbol{x}' = \boldsymbol{0}$ so that $\varphi(\boldsymbol{x}') = \boldsymbol{0}$, this implies that $\|\varphi(\boldsymbol{x})\|_Y = \|\boldsymbol{x}\|_X$. If conversely φ is an isometry, notice that

$$\|\varphi(\boldsymbol{x})\|_Y = \|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{0})\|_Y = \|\boldsymbol{x} - \boldsymbol{0}\|_X = \|\boldsymbol{x}\|.$$

The above calculation then implies (5.1). □

Proposition 5.2

Every isometry $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is on the form

$$\varphi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

for a unique $d \times d$ matrix A and vector $\boldsymbol{b} \in \mathbb{R}^d$. Furthermore, A is orthogonal.

In particular, any isometry that preserves the origin is linear.

Proof. We first show that $\varphi(\boldsymbol{x}) = \varphi_0(\boldsymbol{x}) + \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^d$, where $\varphi_0\colon \mathbb{R}^d \to \mathbb{R}^d$ is an isometry fixing the origin. Simply let $\boldsymbol{b} = \varphi(\boldsymbol{0})$ and $\varphi_0(\boldsymbol{x}) = \varphi(\boldsymbol{x}) - \boldsymbol{b}$. Then

$$\varphi_0(\boldsymbol{0}) = \varphi(\boldsymbol{0}) - \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0},$$

and for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ we have

$$\|\varphi_0(\boldsymbol{x}) - \varphi_0(\boldsymbol{y})\| = \|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\|,$$

so φ_0 is an isometry.

Next we show that φ_0 is linear; since it is an isometry, this will imply the existence of an orthogonal matrix A as above. For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d$ we have by Lemma 5.1

$$\begin{aligned}
\langle \varphi_0(\boldsymbol{x}+\boldsymbol{y}), \varphi_0(\boldsymbol{z})\rangle &= \langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}\rangle \\
&= \langle \boldsymbol{x}, \boldsymbol{z}\rangle + \langle \boldsymbol{y}, \boldsymbol{z}\rangle \\
&= \langle \varphi_0(\boldsymbol{x}), \varphi_0(\boldsymbol{z})\rangle + \langle \varphi_0(\boldsymbol{y}), \varphi_0(\boldsymbol{z})\rangle \\
&= \langle \varphi_0(\boldsymbol{x}) + \varphi_0(\boldsymbol{y}), \varphi_0(\boldsymbol{z})\rangle.
\end{aligned}$$

Since φ_0 preserves orthogonality, it maps an orthogonal basis into an orthogonal set. Since φ_0 is also injective (since it is an isometry), this set is in fact a basis. Replacing $\boldsymbol{z}$ by the elements in such a basis, it follows that $\langle \varphi_0(\boldsymbol{x}+\boldsymbol{y}), \boldsymbol{w}\rangle = \langle \varphi_0(\boldsymbol{x})+\varphi_0(\boldsymbol{y}), \boldsymbol{w}\rangle$ for all $\boldsymbol{w} \in \mathbb{R}^d$. Hence $\varphi_0(\boldsymbol{x}+\boldsymbol{y}) = \varphi_0(\boldsymbol{x})+\varphi_0(\boldsymbol{y})$. We similarly find that, for $\alpha \in \mathbb{R}$,

$$\begin{aligned}
\langle \varphi_0(\alpha\boldsymbol{x}), \varphi_0(\boldsymbol{z})\rangle &= \langle \alpha\boldsymbol{x}, \boldsymbol{z}\rangle \\
&= \alpha\langle \boldsymbol{x}, \boldsymbol{z}\rangle \\
&= \alpha\langle \varphi_0(\boldsymbol{x}), \varphi_0(\boldsymbol{z})\rangle \\
&= \langle \alpha\varphi_0(\boldsymbol{x}), \varphi_0(\boldsymbol{z})\rangle,
\end{aligned}$$

implying that $\varphi_0(\alpha \boldsymbol{x}) = \alpha\varphi_0(\boldsymbol{x})$ Thus φ_0 is linear.

To show uniqueness, assume that $A\boldsymbol{x} + \boldsymbol{b} = \varphi(\boldsymbol{x}) = A'\boldsymbol{x} + \boldsymbol{b}'$ for all $\boldsymbol{x} \in \mathbb{R}^d$. Then $\boldsymbol{b} = \varphi(\boldsymbol{0}) = \boldsymbol{b}'$, so $A\boldsymbol{x} = A'\boldsymbol{x}$. Since this holds for all $\boldsymbol{x}$, it follows that $A = A'$ as desired. □

Consider points $\boldsymbol{b}_1, \dots, \boldsymbol{b}_K \in \mathbb{R}^3$ and let $\boldsymbol{b} = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_K)$. Let further $\boldsymbol{c} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_K)$ be a rigid motion of $\boldsymbol{b}$. That is, there is a one-parameter family of isometries $\varphi_t \colon \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_K\} \to \mathbb{R}^3$ such that $\varphi_t(\boldsymbol{b}_i) = \boldsymbol{c}_i(t)$. However, to apply Proposition 5.2 to φ_t it must be defined on $\mathbb{R}^3$. A natural question is then whether it is possible to extend an isometry from a subset of $\mathbb{R}^3$ to all $\mathbb{R}^3$. The answer is affirmative, as we shall see below.

Another natural question is whether such an extension is unique. If the points $\boldsymbol{b}_1, \dots, \boldsymbol{b}_K$ do not lie in a plane, then by renumbering we may assume that $\boldsymbol{b}_2 - \boldsymbol{b}_1$, $\boldsymbol{b}_3 - \boldsymbol{b}_1$, and $\boldsymbol{b}_4 - \boldsymbol{b}_1$ are linearly independent and hence span $\mathbb{R}^3$. The proof below will show that this implies that an extension of φ_t is unique.

Lemma 5.3

Let $A \subseteq \mathbb{R}^d$ and let $\varphi \colon A \to \mathbb{R}^d$ be an isometry. Then φ extends to an isometry $\tilde{\varphi} \colon \mathbb{R}^d \to \mathbb{R}^d$.

Furthermore, we have the following uniqueness results:

(i) *If A contains $d+1$ affinely independent elements, then φ has a unique isometric extension to $\mathbb{R}^d$.*

(ii) *If A contains d affinely independent elements, then φ has a unique isometric extension to $\mathbb{R}^d$ that is also orientation preserving.*[1]

Proof. By translation we may assume that $\boldsymbol{0} \in A$ (if A is empty then the claim is obvious) and that $\varphi(\boldsymbol{0}) = \boldsymbol{0}$. First extend φ by linearity to $\operatorname{span} A$. This is well-defined, since if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in A$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, then by Lemma 5.1 we have

$$\left\| \sum_{i=1}^n \alpha_i \varphi(\boldsymbol{x}_i) \right\|^2 = \sum_{i,j=1}^n \alpha_i \alpha_j \langle \varphi(\boldsymbol{x}_i), \varphi(\boldsymbol{x}_j) \rangle = \sum_{i,j=1}^n \alpha_i \alpha_j \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle = \left\| \sum_{i=1}^n \alpha_i \boldsymbol{x}_i \right\|^2.$$

This also shows that φ extends to an isometry on $\operatorname{span} A$, and in particular φ is linear by Proposition 5.2.

Since φ is linear it maps $\operatorname{span} A$ to a subspace $\varphi(\operatorname{span} A)$ of $\mathbb{R}^d$ with the same dimension. Let U and V be the orthogonal complements of $\operatorname{span} A$ and

[1] By this we mean that if $\tilde{\varphi}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ in accordance with Proposition 5.2, then $A \in \mathrm{SO}(d)$.

$\varphi(\operatorname{span} A)$ respectively, and let $\psi\colon U \to V$ be an isometry. Letting $\tilde{\varphi} = \varphi \oplus \psi$, it follows easily by Pythagoras' theorem that $\tilde{\varphi}$ is also an isometry as desired.

For the first uniqueness claim, notice that if A contains a collection of $d+1$ affinely independent elements and $\boldsymbol{a}$ is one element in such a collection, then $A - \boldsymbol{a}$ contains d linearly independent vectors. Hence its span, which is contained in $\operatorname{span} A$, is d-dimensional.

To prove the second uniqueness claim, notice that an argument similar to the above shows that $\operatorname{span} A$ has dimension at least $d-1$. Then notice that if $\dim U = 1$, then there are exactly two isometries $U \to V$, only one of which makes $\tilde{\varphi}$ orientation preserving. ☐

Motion relative to the centre of mass

In the section 'The inertial tensor', Spivak considers a rigid motion $\boldsymbol{c} = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_K)$ of a collection of points $\boldsymbol{b}_1, \dots, \boldsymbol{b}_K$. These are given by $\boldsymbol{c}_i(t) = B(t)\boldsymbol{b}_i + \boldsymbol{w}(t)$ for some $B\colon \mathbb{R} \to \mathrm{SO}(3)$ and $\boldsymbol{w}\colon \mathbb{R} \to \mathbb{R}^3$.

Spivak then claims that we may choose $\boldsymbol{w}$ to be the centre of mass $\boldsymbol{C}$ given by

$$\boldsymbol{C} = \frac{1}{M}\sum_{i=1}^{K} m_i \boldsymbol{c}_i, \quad \text{where} \quad M = \sum_{i=1}^{K} m_i.$$

Given the uniqueness part of Proposition 5.2, it is not immediately clear that this is possible, even if it is intuitively reasonable. Inserting the rigid motion in the expression for $\boldsymbol{C}$, we get

$$\begin{aligned}\boldsymbol{C}(t) &= \frac{1}{M}\sum_{i=1}^{K} m_i\big(B(t)\boldsymbol{b}_i + \boldsymbol{w}(t)\big) = \frac{1}{M}\sum_{i=1}^{K} m_i B(t)\boldsymbol{b}_i + \boldsymbol{w}(t) \\ &= B(t)\frac{\sum_{i=1}^{K} m_i \boldsymbol{c}_i(0)}{M} + \boldsymbol{w}(t) = B(t)\boldsymbol{C}(0) + \boldsymbol{w}(t).\end{aligned}$$

That is, the centre of mass has the exact same time evolution as each of the individual particles. If we choose our coordinate system such that $\boldsymbol{C}(0) = \boldsymbol{0}$, we thus get $\boldsymbol{C}(t) = \boldsymbol{w}(t)$.

Notice that this does not mean that we consider the motion as seen from the centre of mass sytem. We simply choose an inertial system in which $\boldsymbol{C}(0) = \boldsymbol{0}$.