Taylor, *Classical Mechanics*

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3 • Momentum and Angular Momentum

3.3. The Centre of Mass

REMARK 3.1: The centre of mass and convexity.
Consider collections of particles at positions $\boldsymbol{r}_{11}, \dots, \boldsymbol{r}_{1k}$ and $\boldsymbol{r}_{21}, \dots, \boldsymbol{r}_{2l}$, with masses $m_{11}, \dots, m_{1k}$ and $m_{21}, \dots, m_{2l}$, respectively. Let $M_1 = \sum_{i=1}^k m_{1i}$ and $M_2 = \sum_{j=1}^l m_{2j}$ be the total masses of the two collections of particles, and let

$$\boldsymbol{R}_1 = \frac{1}{M_1} \sum_{i=1}^k m_{1i} \boldsymbol{r}_{1i} \quad \text{and} \quad \boldsymbol{R}_2 = \frac{1}{M_2} \sum_{j=1}^l m_{2j} \boldsymbol{r}_{2j}$$

be their centres of mass. The centre of mass of the whole system is then

$$\begin{aligned}
\boldsymbol{R} &= \frac{1}{M_1 + M_2} \left(\sum_{i=1}^k m_{1i} \boldsymbol{r}_{1i} + \sum_{j=1}^l m_{2j} \boldsymbol{r}_{2j} \right) \\
&= \frac{1}{M_1 + M_2} \left(\frac{M_1}{M_1} \sum_{i=1}^k m_{1i} \boldsymbol{r}_{1i} + \frac{M_2}{M_2} \sum_{j=1}^l m_{2j} \boldsymbol{r}_{2j} \right) \\
&= \frac{M_1 \boldsymbol{R}_1 + M_2 \boldsymbol{R}_2}{M_1 + M_2}.
\end{aligned}$$

This obviously extends to any finite number of collections of particles.

In particular, it follows that the centre of mass of a finite collection of particles lies in the convex hull of their position vectors. For the above shows that $\boldsymbol{R}$ is a convex combination of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, so this follows by induction on the number of particles. In fact, this holds for any weighted average (with non-negative weights) of points in any dimension. ⌟

3.4. Angular Momentum for a Single Particle

REMARK 3.2: Angular momentum in noninertial frames.
Choose an origin O that is not necessarily at rest in any inertial frame, and consider a particle with mass m and position vector $\boldsymbol{r}$ relative to O. The angular momentum of the particle relative to O is then, by definition, $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$. Its time derivative is

$$\dot{\boldsymbol{l}} = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{r} \times \dot{\boldsymbol{p}}.$$

If O is at rest in an inertial frame, then $\dot{\boldsymbol{p}}$ equals the total force $\boldsymbol{F}$ on the particle, so the above equals the torque $\boldsymbol{\Gamma}$. However, if we include in $\boldsymbol{F}$ any inertial forces resulting from measuring $\boldsymbol{r}$ in a noninertial frame and define the torque from this force, then the above still equals the torque. ⌟

3.5. Angular Momentum for Several Particles

REMARK 3.3: König's theorem for angular momenta.
For $i = 1, \ldots, n$ consider a particle with mass m_i and position vector $\boldsymbol{r}_i$ with respect to an origin O at rest in an inertial frame. Furthermore, let $\boldsymbol{p}_i$ be its momentum. If $\boldsymbol{R}$ is the centre of mass of all n particles, then we let $\boldsymbol{r}_i' = \boldsymbol{r} - \boldsymbol{R}$ be the relative position of the ith particle, and let $\boldsymbol{p}_i' = m_i \dot{\boldsymbol{r}}_i'$. Denote by M the total mass of the particles, and let $\boldsymbol{P}$ be the total momentum (equivalently $\boldsymbol{P} = M\dot{\boldsymbol{R}}$). The angular momentum of the ith particle with respect to the centre of mass is then

$$\begin{aligned} \boldsymbol{l}_i' = \boldsymbol{r}_i' \times \boldsymbol{p}_i' &= (\boldsymbol{r}_i - \boldsymbol{R}) \times (\boldsymbol{p}_i - m_i \dot{\boldsymbol{R}}) \\ &= \boldsymbol{l}_i - m_i \boldsymbol{r}_i \times \dot{\boldsymbol{R}} - \boldsymbol{R} \times \boldsymbol{p}_i + m_i \boldsymbol{R} \times \dot{\boldsymbol{R}}. \end{aligned}$$

Therefore, the total angular momentum with respect to the centre of mass is

$$\begin{aligned} \boldsymbol{L}' = \sum_{i=1}^{n} \boldsymbol{l}_i' &= \boldsymbol{L} - M\boldsymbol{R} \times \dot{\boldsymbol{R}} - \boldsymbol{R} \times \boldsymbol{P} + M\boldsymbol{R} \times \dot{\boldsymbol{R}} \\ &= \boldsymbol{L} - \boldsymbol{R} \times \boldsymbol{P} = \boldsymbol{L} - \boldsymbol{L}_{\mathrm{CM}}, \end{aligned}$$

where $\boldsymbol{L}_{\mathrm{CM}} = \boldsymbol{R} \times \boldsymbol{P}$ is the angular momentum of the centre of mass. Hence

$$\boldsymbol{L} = \boldsymbol{L}' + \boldsymbol{L}_{\mathrm{CM}},$$

which is *König's theorem* for angular momenta. ⌟

REMARK 3.4: Decomposition of torque.
Next let each particle i be acted upon by an external force $\boldsymbol{F}_i$, and let $\boldsymbol{F}$ be the total force on the system. If the total external torque relative to the origin O is

$\mathbf{\Gamma}$, then the external torque relative to the centre of mass is

$$\begin{aligned}\mathbf{\Gamma}' &= \sum_{i=1}^{n} \boldsymbol{r}_i' \times \boldsymbol{F}_i = \sum_{i=1}^{n} (\boldsymbol{r}_i - \boldsymbol{R}) \times \boldsymbol{F}_i \\ &= \sum_{i=1}^{n} \mathbf{\Gamma}_i - \boldsymbol{R} \times \boldsymbol{F} = \mathbf{\Gamma} - \mathbf{\Gamma}_{\mathrm{CM}},\end{aligned}$$

where $\mathbf{\Gamma}_{\mathrm{CM}} = \boldsymbol{R} \times \boldsymbol{F}$ is the torque on the centre of mass. That is,

$$\mathbf{\Gamma} = \mathbf{\Gamma}' + \mathbf{\Gamma}_{\mathrm{CM}}.$$

Now notice that

$$\dot{\boldsymbol{L}}_{\mathrm{CM}} = (\dot{\boldsymbol{R}} \times \boldsymbol{P}) + (\boldsymbol{R} \times \dot{\boldsymbol{P}}) = \boldsymbol{R} \times \boldsymbol{F} = \mathbf{\Gamma}_{\mathrm{CM}},$$

since $\dot{\boldsymbol{R}}$ and $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ are parallel. König's theorem thus implies that

$$\dot{\boldsymbol{L}}' = \dot{\boldsymbol{L}} - \dot{\boldsymbol{L}}_{\mathrm{CM}} = \mathbf{\Gamma} - \mathbf{\Gamma}_{\mathrm{CM}} = \mathbf{\Gamma}'.$$

That is, relative to the centre of mass, the external torque is the time derivative of the angular momentum, even if the centre of mass frame is not inertial. ⌟

4 • Energy

4.1. *Kinetic Energy and Work*

Remark 4.1: König's theorem for kinetic energy.
Notice that

$$\begin{aligned}T &= \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \|\boldsymbol{v}_i' + \dot{\boldsymbol{R}}\|^2 \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i (v_i')^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{R}^2 + \dot{\boldsymbol{R}} \cdot \sum_{i=1}^{n} m_i \boldsymbol{v}_i'.\end{aligned}$$

But the last term vanishes since $\sum_{i=1}^{n} m_i \boldsymbol{v}_i'$ is the time derivative of $\sum_{i=1}^{n} m_i \boldsymbol{r}_i'$, which is zero. Hence

$$T = T' + T_{\mathrm{CM}},$$

which is *König's theorem* for kinetic energy. ⌟

4.9. Energy of Interaction of Two Particles

Consider two particles numbered 1 and 2, and let particle i act on particle $j \neq i$ via a force $\boldsymbol{F}_{ij}$. We assume that the force depends only on the position of the two particles, and perhaps time. Focusing on $\boldsymbol{F}_{12}$ we thus have e.g.[1] $\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$. Assuming that the two particles are isolated, we have

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1 + \boldsymbol{h}, \boldsymbol{r}_2 + \boldsymbol{h}, t) = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$$

for all vectors $\boldsymbol{h}$, i.e., the force is translation invariant.

Now assume that $\boldsymbol{r}_2 = \boldsymbol{0}$ at some time[2] t, which we can always accomplish by changing coordinates. Further assume that the force $(\boldsymbol{r}_1, t) \mapsto \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{0}, t)$ is derived from a potential $U_t = U_t(\boldsymbol{r}_1)$, parametrised by t. That is, we require that the line integral of the above force between any two points is independent of path, when we keep t fixed. Next, no longer fix particle 2 at the origin. Since the force is translation invariant, it follows that[3]

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t) = -\nabla U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

Next define a new potential $U_{12} = U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. Denoting by ∇_1 the gradient operator with respect to the first three arguments, i.e. the three coordinates of $\boldsymbol{r}_1$, we thus find that

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\nabla_1 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t).$$

We similarly find that

$$\boldsymbol{F}_{21}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \nabla_1 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\nabla_2 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t),$$

where the operator ∇_2 is defined analogously to ∇_1.

[1] We use the physicist's notation to describe the domain of functions; the codomain is either $\mathbb{R}$ or $\mathbb{R}^3$, and we distinguish these by denoting vector-valued functions with boldface letters, similar to other vector-valued quantities. Thus the notation $\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$ means that $\boldsymbol{F}_{12}$ is a function $\Omega \to \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ is the set of permitted values of $(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$.

[2] As far as I can tell, the following arguments do not require that we measure the positions of the particles in an inertial frame, so we may set $\boldsymbol{r}_2 = \boldsymbol{0}$ at all t.

[3] By $\nabla U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ below we mean, seemingly contrary to Taylor, the value of the function ∇U_t at the point $\boldsymbol{r}_1 - \boldsymbol{r}_2$.