

Taylor, *Classical Mechanics*

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3 • Momentum and Angular Momentum

3.3. *The Centre of Mass*

REMARK 3.1: The centre of mass and convexity.

Consider collections of particles at positions $\mathbf{r}_{11}, \dots, \mathbf{r}_{1k}$ and $\mathbf{r}_{21}, \dots, \mathbf{r}_{2l}$, with masses m_{11}, \dots, m_{1k} and m_{21}, \dots, m_{2l} , respectively. Let $M_1 = \sum_{i=1}^k m_{1i}$ and $M_2 = \sum_{j=1}^l m_{2j}$ be the total masses of the two collections of particles, and let

$$\mathbf{R}_1 = \frac{1}{M_1} \sum_{i=1}^k m_{1i} \mathbf{r}_{1i} \quad \text{and} \quad \mathbf{R}_2 = \frac{1}{M_2} \sum_{j=1}^l m_{2j} \mathbf{r}_{2j}$$

be their centres of mass. The centre of mass of the whole system is then

$$\begin{aligned} \mathbf{R} &= \frac{1}{M_1 + M_2} \left(\sum_{i=1}^k m_{1i} \mathbf{r}_{1i} + \sum_{j=1}^l m_{2j} \mathbf{r}_{2j} \right) \\ &= \frac{1}{M_1 + M_2} \left(\frac{M_1}{M_1} \sum_{i=1}^k m_{1i} \mathbf{r}_{1i} + \frac{M_2}{M_2} \sum_{j=1}^l m_{2j} \mathbf{r}_{2j} \right) \\ &= \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2}. \end{aligned}$$

This obviously extends to any finite number of collections of particles.

In particular, it follows that the centre of mass of a finite collection of particles lies in the convex hull of their position vectors. For the above shows that \mathbf{R} is a convex combination of \mathbf{R}_1 and \mathbf{R}_2 , so this follows by induction on the number of particles. In fact, this holds for any weighted average (with non-negative weights) of points in any dimension. \lrcorner

3.4. Angular Momentum for a Single Particle

REMARK 3.2: Angular momentum in noninertial frames.

Choose an origin O that is not necessarily at rest in any inertial frame, and consider a particle with mass m and position vector \mathbf{r} relative to O . The angular momentum of the particle relative to O is then, by definition, $\mathbf{l} = \mathbf{r} \times \mathbf{p}$. Its time derivative is

$$\dot{\mathbf{l}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times \dot{\mathbf{p}}.$$

If O is at rest in an inertial frame, then $\dot{\mathbf{p}}$ equals the total force \mathbf{F} on the particle, so the above equals the torque $\mathbf{\Gamma}$. However, if we include in \mathbf{F} any inertial forces resulting from measuring \mathbf{r} in a noninertial frame and define the torque from this force, then the above still equals the torque. \lrcorner

3.5. Angular Momentum for Several Particles

REMARK 3.3: König's theorem for angular momenta.

For $i = 1, \dots, n$ consider a particle with mass m_i and position vector \mathbf{r}_i with respect to an origin O at rest in an inertial frame. Furthermore, let \mathbf{p}_i be its momentum. If \mathbf{R} is the centre of mass of all n particles, then we let $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$ be the relative position of the i th particle, and let $\mathbf{p}'_i = m_i \dot{\mathbf{r}}'_i$. Denote by M the total mass of the particles, and let \mathbf{P} be the total momentum (equivalently $\mathbf{P} = M\dot{\mathbf{R}}$). The angular momentum of the i th particle with respect to the centre of mass is then

$$\begin{aligned} \mathbf{l}'_i &= \mathbf{r}'_i \times \mathbf{p}'_i = (\mathbf{r}_i - \mathbf{R}) \times (\mathbf{p}_i - m_i \dot{\mathbf{R}}) \\ &= \mathbf{l}_i - m_i \mathbf{r}_i \times \dot{\mathbf{R}} - \mathbf{R} \times \mathbf{p}_i + m_i \mathbf{R} \times \dot{\mathbf{R}}. \end{aligned}$$

Therefore, the total angular momentum with respect to the centre of mass is

$$\begin{aligned} \mathbf{L}' &= \sum_{i=1}^n \mathbf{l}'_i = \mathbf{L} - M\mathbf{R} \times \dot{\mathbf{R}} - \mathbf{R} \times \mathbf{P} + M\mathbf{R} \times \dot{\mathbf{R}} \\ &= \mathbf{L} - \mathbf{R} \times \mathbf{P} = \mathbf{L} - \mathbf{L}_{\text{CM}}, \end{aligned}$$

where $\mathbf{L}_{\text{CM}} = \mathbf{R} \times \mathbf{P}$ is the angular momentum of the centre of mass. Hence

$$\mathbf{L} = \mathbf{L}' + \mathbf{L}_{\text{CM}},$$

which is *König's theorem* for angular momenta. \lrcorner

REMARK 3.4: Decomposition of torque.

Next let each particle i be acted upon by an external force \mathbf{F}_i , and let \mathbf{F} be the total force on the system. If the total external torque relative to the origin O is

Γ , then the external torque relative to the centre of mass is

$$\begin{aligned}\Gamma' &= \sum_{i=1}^n \mathbf{r}'_i \times \mathbf{F}_i = \sum_{i=1}^n (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i \\ &= \sum_{i=1}^n \Gamma_i - \mathbf{R} \times \mathbf{F} = \Gamma - \Gamma_{\text{CM}},\end{aligned}$$

where $\Gamma_{\text{CM}} = \mathbf{R} \times \mathbf{F}$ is the torque on the centre of mass. That is,

$$\Gamma = \Gamma' + \Gamma_{\text{CM}}.$$

Now notice that

$$\dot{\mathbf{L}}_{\text{CM}} = (\dot{\mathbf{R}} \times \mathbf{P}) + (\mathbf{R} \times \dot{\mathbf{P}}) = \mathbf{R} \times \mathbf{F} = \Gamma_{\text{CM}},$$

since $\dot{\mathbf{R}}$ and $\mathbf{P} = M\dot{\mathbf{R}}$ are parallel. König's theorem thus implies that

$$\dot{\mathbf{L}}' = \dot{\mathbf{L}} - \dot{\mathbf{L}}_{\text{CM}} = \Gamma - \Gamma_{\text{CM}} = \Gamma'.$$

That is, relative to the centre of mass, the external torque is the time derivative of the angular momentum, even if the centre of mass frame is not inertial. \lrcorner

4 • Energy

4.1. Kinetic Energy and Work

REMARK 4.1: König's theorem for kinetic energy.

Notice that

$$\begin{aligned}T &= \frac{1}{2} \sum_{i=1}^n m_i v_i^2 = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{v}'_i + \dot{\mathbf{R}}\|^2 \\ &= \frac{1}{2} \sum_{i=1}^n m_i (v'_i)^2 + \frac{1}{2} \sum_{i=1}^n m_i \dot{\mathbf{R}}^2 + \dot{\mathbf{R}} \cdot \sum_{i=1}^n m_i \mathbf{v}'_i.\end{aligned}$$

But the last term vanishes since $\sum_{i=1}^n m_i \mathbf{v}'_i$ is the time derivative of $\sum_{i=1}^n m_i \mathbf{r}'_i$, which is zero. Hence

$$T = T' + T_{\text{CM}},$$

which is König's theorem for kinetic energy. \lrcorner

4.9. Energy of Interaction of Two Particles

Consider two particles numbered 1 and 2, and let particle j act on particle $i \neq j$ via a force \mathbf{F}_{ij} . We assume that the force depends only on the position of the two particles, and perhaps time. Focusing on \mathbf{F}_{12} we thus have e.g.¹ $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$. Assuming that the two particles are isolated, we have

$$\mathbf{F}_{12}(\mathbf{r}_1 + \mathbf{h}, \mathbf{r}_2 + \mathbf{h}, t) = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$$

for all vectors \mathbf{h} , i.e., the force is translation invariant.

Now assume that $\mathbf{r}_2 = \mathbf{0}$ at some time² t , which we can always accomplish by changing coordinates. Further assume that the force $\mathbf{r}_1 \mapsto \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{0}, t)$ is derived from a potential $U_t = U_t(\mathbf{r}_1)$, depending on t . That is, we require that the line integral of the above force between any two points is independent of path, when we keep t fixed. Next, no longer fix particle 2 at the origin. Since the force is translation invariant, it follows that³

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{0}, t) = -\nabla U_t(\mathbf{r}_1 - \mathbf{r}_2).$$

Next define a new potential $U_{12} = U_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = U_t(\mathbf{r}_1 - \mathbf{r}_2)$. Denoting by ∇_1 the gradient operator with respect to the first three arguments, i.e. the three coordinates of \mathbf{r}_1 , we thus find that

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = -\nabla_1 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t).$$

It now follows that

$$\mathbf{F}_{21}(\mathbf{r}_1, \mathbf{r}_2, t) = -\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = \nabla_1 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = -\nabla_2 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t),$$

where the operator ∇_2 is defined analogously to ∇_1 . Hence we indeed have the identities $\mathbf{F}_{12} = -\nabla_1 U_{12}$ and $\mathbf{F}_{21} = -\nabla_2 U_{12}$, so both forces are derived from a single potential energy function.⁴

¹ We use the physicist's notation to describe the domain of functions; the codomain is either \mathbb{R} or \mathbb{R}^3 , and we distinguish these by denoting vector-valued functions with boldface letters, similar to other vector-valued quantities. Thus the notation $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$ means that \mathbf{F}_{12} is a function $\Omega \rightarrow \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ is the set of permitted values of $(\mathbf{r}_1, \mathbf{r}_2, t)$.

² As far as I can tell, the following arguments do not require that we measure the positions of the particles in an inertial frame, so we may set $\mathbf{r}_2 = \mathbf{0}$ at all t . In this case we may let the force and the potential be functions of t instead of only being parametrised by t .

³ By $\nabla U_t(\mathbf{r}_1 - \mathbf{r}_2)$ below we mean, seemingly contrary to Taylor, the value of the function ∇U_t at the point $\mathbf{r}_1 - \mathbf{r}_2$.

⁴ Compare Taylor's equation (4.83):

$$(\text{Force on particle 1}) = -\nabla_1 U$$

$$(\text{Force on particle 2}) = -\nabla_2 U$$

This is simply wrong as stated, as Taylor's choice of U requires him to apply e.g. $-\nabla_1$ to the function $(\mathbf{r}_1, \mathbf{r}_2) \mapsto U(\mathbf{r}_1 - \mathbf{r}_2)$, and not to U itself.