Taylor, *Classical Mechanics*

Danny Nygård Hansen

30th October 2022

4 • Energy

4.9. Energy of Interaction of Two Particles

Consider two particles numbered 1 and 2, and let particle i act on particle $j \neq i$ via a force $\boldsymbol{F}_{ij}$. We assume that the force depends only on the position of the two particles, and perhaps time. Focusing on $\boldsymbol{F}_{12}$ we thus have e.g.[1] $\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$. Assuming that the two particles are isolated, we have

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1 + \boldsymbol{h}, \boldsymbol{r}_2 + \boldsymbol{h}, t) = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$$

for all vectors $\boldsymbol{h}$, i.e., the force is translation invariant.

Now assume that $\boldsymbol{r}_2 = \boldsymbol{0}$ at some time[2] t, which we can always accomplish by changing coordinates. Further assume that the force $(\boldsymbol{r}_1, t) \mapsto \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{0}, t)$ is derived from a potential $U_t = U_t(\boldsymbol{r}_1)$, parametrised by t. That is, we require that the line integral of the above force between any two points is independent of path, when we keep t fixed. Next, no longer fix particle 2 at the origin. Since the force is translation invariant, it follows that[3]

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \boldsymbol{F}_{12}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t) = -\nabla U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

Next define a new potential $U_{12} = U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. Denoting by ∇_1 the gradient operator with respect to the first three arguments, i.e. the three coordinates of $\boldsymbol{r}_1$, we thus find that

$$\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\nabla_1 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t).$$

[1] We use the physicist's notation to describe the domain of functions; the codomain is either $\mathbb{R}$ or $\mathbb{R}^3$, and we distinguish these by denoting vector-valued functions with boldface letters, similar to other vector-valued quantities. Thus the notation $\boldsymbol{F}_{12} = \boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$ means that $\boldsymbol{F}_{12}$ is a function $\Omega \to \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ is the set of permitted values of $(\boldsymbol{r}_1, \boldsymbol{r}_2, t)$.

[2] As far as I can tell, the following arguments do not require that we measure the positions of the particles in an inertial frame, so we may set $\boldsymbol{r}_2 = \boldsymbol{0}$ at all t.

[3] By $\nabla U_t(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ below we mean, seemingly contrary to Taylor, the value of the function ∇U_t at the point $\boldsymbol{r}_1 - \boldsymbol{r}_2$.

We similarly find that

$$\boldsymbol{F}_{21}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \nabla_1 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = -\nabla_2 U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2, t),$$

where the operator ∇_2 is defined analogously to ∇_1.