

Taylor, *Classical Mechanics*

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3 • Momentum and Angular Momentum

3.3. *The Centre of Mass*

REMARK 3.1: The centre of mass and convexity.

Consider collections of particles at positions $\mathbf{r}_{11}, \dots, \mathbf{r}_{1k}$ and $\mathbf{r}_{21}, \dots, \mathbf{r}_{2l}$, with masses m_{11}, \dots, m_{1k} and m_{21}, \dots, m_{2l} , respectively. Let $M_1 = \sum_{i=1}^k m_{1i}$ and $M_2 = \sum_{j=1}^l m_{2j}$ be the total masses of the two collections of particles, and let

$$\mathbf{R}_1 = \frac{1}{M_1} \sum_{i=1}^k m_{1i} \mathbf{r}_{1i} \quad \text{and} \quad \mathbf{R}_2 = \frac{1}{M_2} \sum_{j=1}^l m_{2j} \mathbf{r}_{2j}$$

be their centres of mass. The centre of mass of the whole system is then

$$\begin{aligned} \mathbf{R} &= \frac{1}{M_1 + M_2} \left(\sum_{i=1}^k m_{1i} \mathbf{r}_{1i} + \sum_{j=1}^l m_{2j} \mathbf{r}_{2j} \right) \\ &= \frac{1}{M_1 + M_2} \left(\frac{M_1}{M_1} \sum_{i=1}^k m_{1i} \mathbf{r}_{1i} + \frac{M_2}{M_2} \sum_{j=1}^l m_{2j} \mathbf{r}_{2j} \right) \\ &= \frac{M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2}{M_1 + M_2}. \end{aligned}$$

This obviously extends to any finite number of collections of particles.

In particular, it follows that the centre of mass of a finite collection of particles lies in the convex hull of their position vectors. For the above shows that \mathbf{R} is a convex combination of \mathbf{R}_1 and \mathbf{R}_2 , so this follows by induction on the number of particles. In fact, this holds for any weighted average (with non-negative weights) of points in any dimension. \lrcorner

4 • Energy

4.9. Energy of Interaction of Two Particles

Consider two particles numbered 1 and 2, and let particle i act on particle $j \neq i$ via a force \mathbf{F}_{ij} . We assume that the force depends only on the position of the two particles, and perhaps time. Focusing on \mathbf{F}_{12} we thus have e.g.¹ $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$. Assuming that the two particles are isolated, we have

$$\mathbf{F}_{12}(\mathbf{r}_1 + \mathbf{h}, \mathbf{r}_2 + \mathbf{h}, t) = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$$

for all vectors \mathbf{h} , i.e., the force is translation invariant.

Now assume that $\mathbf{r}_2 = \mathbf{0}$ at some time² t , which we can always accomplish by changing coordinates. Further assume that the force $(\mathbf{r}_1, t) \mapsto \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{0}, t)$ is derived from a potential $U_t = U_t(\mathbf{r}_1)$, parametrised by t . That is, we require that the line integral of the above force between any two points is independent of path, when we keep t fixed. Next, no longer fix particle 2 at the origin. Since the force is translation invariant, it follows that³

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbf{F}_{12}(\mathbf{r}_1 - \mathbf{r}_2, t) = -\nabla U_t(\mathbf{r}_1 - \mathbf{r}_2).$$

Next define a new potential $U_{12} = U_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = U_t(\mathbf{r}_1 - \mathbf{r}_2)$. Denoting by ∇_1 the gradient operator with respect to the first three arguments, i.e. the three coordinates of \mathbf{r}_1 , we thus find that

$$\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = -\nabla_1 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t).$$

We similarly find that

$$\mathbf{F}_{21}(\mathbf{r}_1, \mathbf{r}_2, t) = -\mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = \nabla_1 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t) = -\nabla_2 U_{12}(\mathbf{r}_1, \mathbf{r}_2, t),$$

where the operator ∇_2 is defined analogously to ∇_1 .

¹ We use the physicist's notation to describe the domain of functions; the codomain is either \mathbb{R} or \mathbb{R}^3 , and we distinguish these by denoting vector-valued functions with boldface letters, similar to other vector-valued quantities. Thus the notation $\mathbf{F}_{12} = \mathbf{F}_{12}(\mathbf{r}_1, \mathbf{r}_2, t)$ means that \mathbf{F}_{12} is a function $\Omega \rightarrow \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ is the set of permitted values of $(\mathbf{r}_1, \mathbf{r}_2, t)$.

² As far as I can tell, the following arguments do not require that we measure the positions of the particles in an inertial frame, so we may set $\mathbf{r}_2 = \mathbf{0}$ at all t .

³ By $\nabla U_t(\mathbf{r}_1 - \mathbf{r}_2)$ below we mean, seemingly contrary to Taylor, the value of the function ∇U_t at the point $\mathbf{r}_1 - \mathbf{r}_2$.