Topological Groups

Danny Nygård Hansen

20th October 2022

1 • Introduction

The purpose of these notes is to clarify some of the basic properties of topological groups. We in particular hope to shed light on the reasonability of the common assumption that topological groups be Hausdorff.

Most of the results below can be found in various forms in a variety of books or articles (see the references), but I have not come across a resource that develops this theory in precisely this way. One example is the T_0-identification of a topological group, as we will see below.

2 • Topological groups

2.1. Definitions and basic properties

Definition 2.1: *Topological groups*

A *topological group* is a triple $(G, \mu, \mathcal{T})$ such that (G, μ) is a group, $(G, \mathcal{T})$ is a topological space, and both the multiplication $\mu\colon G \times G \to G$ and the inversion map $\iota\colon G \to G$ given by $g \mapsto g^{-1}$ are continuous.

In the sequel we will simply write G, and it will be clear from context whether G is to be thought of as a set, group, topological space or topological group. The identity on G will be denoted e_G, or simply e. It is custom to assume that a topological group be T_1 (or Hausdorff, which is actually implied by it being T_1, as will follow from Proposition 2.4), but we omit this assumption since our focus is precisely on the basic topological properties of topological groups.

Let G be a topological group and let $g \in G$. We define maps $\lambda_g, \rho_g\colon G \to G$ by $\lambda_g(h) = gh$ and $\rho_g(h) = hg$; that is, λ_g and ρ_g are given by left- and right-multiplication by g, respectively. Both of these are homeomorphisms, with $\lambda_g^{-1} = \lambda_{g^{-1}}$ and $\rho_g^{-1} = \rho_{g^{-1}}$. Hence if $a, b \in G$, then there is a homeomorphism on

G that takes a to b, namely $\lambda_{ba^{-1}}$. Thus a topological group is *homogeneous* as a topological space. In particular,[1] $\mathcal{N}_g = \lambda_g(\mathcal{N}_e) = g\mathcal{N}_e$ for all $g \in H$, so the topology on G is determined by any neighbourhood basis at e. A neighbourhood basis at e is called a *local basis*.

If G is a topological group and $A \subseteq G$, then we write

$$A^{-1} = \iota(A) = \{a^{-1} \mid a \in A\}.$$

A subset A is called *symmetric* if $A = A^{-1}$. Notice that since ι is a homeomorphism, A is open (closed) if and only if A^{-1} is open (closed).

We begin by collecting some basic properties of topological groups:

Proposition 2.2: *Properties of topological groups*

Let G be a topological group, and let $A, B \subseteq G$.

(i) $\overline{A}\,\overline{B} \subseteq \overline{AB}$.

(ii) *If $U \subseteq G$ is open, then AU and UA are open.*

(iii) $\overline{A} = \bigcap_{U \in \mathcal{N}_0} AU$.

(iv) *Assume that G is Hausdorff. If $ab = ba$ for all $a \in A$, $b \in B$, then $ab = ba$ for all $a \in \overline{A}$, $b \in \overline{B}$.*

As we will see in Proposition 2.4, the Hausdorff assumption in (iv) can be weakened to just T_0. However, this assumption cannot be dropped: Any nontrivial, nonabelian group is a topological group in the trivial topology, but then the closure of the trivial subgroup is the whole group and hence not abelian.

Proof. We give two proofs of (i). Since the multiplication $\mu\colon G \times G \to G$ is continuous we get

$$\mu(\overline{A}, \overline{B}) = \mu(\overline{A} \times \overline{B}) = \mu(\overline{A \times B}) \subseteq \overline{\mu(A \times B)} = \overline{\mu(A, B)} = \overline{AB}.$$

Alternatively, consider $a \in \overline{A}$ and $b \in \overline{B}$. If U is a neighbourhood of ab, then by continuity of multiplication there are neighbourhoods V_1 and V_2 of a and b respectively such that $V_1 V_2 \subseteq U$. Picking $x \in A \cap V_1$ and $y \in B \cap V_2$ we find that $xy \in AB \cap U$, so $ab \in \overline{AB}$.

To prove (ii), notice that e.g.

$$AU = \bigcup_{g \in A} gU,$$

[1] If X is a topological space and $A \subseteq X$, then we say that a set $N \subseteq X$ is a *neighbourhood* of A if there is an open set U in X such that $A \subseteq U \subseteq N$. The family of neighbourhoods of a set A is called the *neighbourhood filter of* A and is denoted $\mathcal{N}_A$. If $A = \{x\}$ is a singleton we also write $\mathcal{N}_x$ and call N a neighbourhood of x.

so AU is a union of open sets.

To prove (iii), notice that $g \in \overline{A}$ if and only if $A \cap V \neq \emptyset$ for all $V \in \mathcal{N}_g$, if and only if $A \cap gU \neq \emptyset$ for all $U \in \mathcal{N}_0$. Since $\mathcal{N}_0$ is symmetric, this is the case just when $g \in AU$ for all $U \in \mathcal{N}_0$.

Let $\gamma_a \colon G \to G$ be conjugation by a, i.e. $\gamma_a(g) = aga^{-1}$. Then (iv) says that if γ_a is the identity map on B, then it is the identity map on $\overline{B}$. But since G is Hausdorff, this follows. □

2.2. Separability in topological groups

Recall that a topological space X is *regular* if it is possible to separate any point from any closed set by disjoint open sets.[2] Our next order of business is to show that any topological group is regular.[3] Before proving this we need a lemma:

LEMMA 2.3

Let G be a topological group. If U is an open neighbourhood of the identity e, then there is a symmetric open neighbourhood V of e such that $VV \subseteq U$. In particular, G has a local basis of symmetric open sets.

PROOF. Since multiplication is continuous and $ee = e$, there are open neighbourhoods V_1 and V_2 of e such that $V_1V_2 \subseteq U$. If we let

$$V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1},$$

then V has the desired properties. The final claim follows from the fact that $V \subseteq VV$ since $e \in V$. □

PROPOSITION 2.4: *Regularity of topological groups*

If G is a topological group, $g \in G$, $A \subseteq G$ is closed and $g \notin A$, then there exists a symmetric neighbourhood V of e such that Vg and VA are disjoint. In particular every topological group is regular, and every T_0 topological group is T_3.

PROOF. Since $g \notin A$ we also have $e \notin Ag^{-1}$. But Ag^{-1} is closed, so by Lemma 2.3 there is a symmetric neighbourhood V of e such that $VV \cap Ag^{-1} = \emptyset$. It follows that $Vg \cap VA = \emptyset$ as desired. This shows that G is regular, and the final claim follows since a regular T_0-space is Hausdorff. □

---

[2] In our terminology, a regular Hausdorff space would be called a T_3-*space*.

[3] In fact, topological groups are *completely regular*. The proofs I know of this fact uses the theory of uniform spaces, so we do not cover it in these notes.

2.3. Continuous group homomorphisms

We next explore the relationship between the topological and algebraic structure for maps. Let $A \subseteq G$. A map $f: A \to H$ is said to be *uniformly continuous* if there for each neighbourhood V of e_H exists a neighbourhood U of e_G such that $x^{-1}y \in U$ implies $f(x)^{-1}f(y) \in V$, for all $x, y \in A$. Uniform continuity clearly implies continuity, since the above says that $f(yU) \subseteq f(y)V$, and every neighbourhood of y and $f(y)$ are on the form yU respectively $f(y)V$ by homogeneity.

If f is injective and both $f: A \to f(A)$ and $f^{-1}: f(A) \to A$ are uniformly continuous, then f is called a *unimorphism*. Notice that f need not be surjective for it to be a unimorphism. Notice also that any restriction of a unimorphism is also a unimorphism, since the restriction of a uniformly continuous map is uniformly continuous.

As expected from homogeneity, all topological homomorphisms are automatically uniformly continuous:

PROPOSITION 2.5

Let $\varphi: G \to H$ be a group homomorphism between topological groups that is continuous at some point $g \in G$. Then φ is uniformly continuous.

PROOF. We may assume that φ is continuous at e_G, since

$$\varphi = \varphi \circ \lambda_{g^{-1}} \circ \lambda_g = \lambda_{\varphi(g)^{-1}} \circ \varphi \circ \lambda_g,$$

and the map on the right-hand side is continuous at e_G iff φ is continuous at g.[4] Thus let V be a neighbourhood of e_H. By continuity of φ at e_G, there exists a neighbourhood U of e_G such that $f(U) \subseteq V$. For $x, y \in G$ such that $x^{-1}y \in U$ we thus have

$$f(x)^{-1}f(y) = f(x^{-1}y) \in V,$$

as required for uniform continuity. ☐

2.4. Subgroups

If G is a topological group and H is a subgroup of G, then H is equipped with the subspace topology from G. Clearly the inherited group operations are still continuous, so H is a topological group.

PROPOSITION 2.6: *Properties of subgroups*

Let G be a topological group.

(i) *If H is a subgroup of G, then so is $\overline{H}$. If H is normal in G, then so is $\overline{H}$. If G*

[4] This already shows that φ is continuous.

is T_0 and H is abelian, then so is $\overline{H}$.

(ii) *Every open subgroup of G is closed.*

Proof. Let H be a subgroup of G. By Proposition 2.2(i) we get

$$\overline{H}\,\overline{H} \subseteq \overline{HH} = \overline{H},$$

so $\overline{H}$ is closed under multiplication. It is also closed under taking inverses, since the inverse map ι is a homeomorphism. Hence it is a subgroup. Alternatively, this claim is an easy consequence of basic properties of nets.

Clearly $\overline{H}$ is the smallest closed subgroup that contains H. Hence if H is normal and $\overline{H}$ is not, then intersecting $\overline{H}$ with one of its conjugates yields a strictly smaller closed subgroup containing H.

If G is T_0 and H is abelian, then it follows from Proposition 2.2(iv) with $A = B = H$ that $\overline{H}$ is abelian. This proves (i).

Finally let H be an open subgroup. Since G is the disjoint union of the left cosets of H, we have

$$G \setminus H = \bigcup_{g \in G \setminus H} gH.$$

Since the cosets gH are open, $G \setminus H$ is also open. This proves (ii). □

2.5. Convergence and completeness

If P and Q are preordered sets, recall that the product order on $P \times Q$ is given by $(p,q) \leq (p',q')$ iff $p \leq p'$ and $q \leq q'$. If P and Q are directed, then this induces a direction on $P \times Q$.

Let G be a topological group, and let $(g_i)_{i \in I}$ be a net in G. Recall that g_i converges to $g \in G$ if for every neighbourhood U of e, g_i eventually lies in U. Furthermore, (g_i) is called a *Cauchy net* if the net $(g_i^{-1} g_j)_{(i,j) \in I \times I}$ converges to e. That is, if for every neighbourhood U of 0 there is an $i_0 \in I$ such that $i, j \geq i_0$ implies that $g_i^{-1} g_j \in U$. It is easy to see that a convergent net is Cauchy since the group operations are continuous. Conversely, we say that a subset $A \subseteq G$ is *complete* if every Cauchy net in A converges (in G) to a point in A.

Finally, we say that a metric ρ on G is *left-invariant* if $\rho(ag, ah) = \rho(g, h)$ for all $a, g, h \in G$. Right-invariance is defined analogously.

Remark 2.7. Assume that the topology on $(G, \mathcal{T})$ is generated by a pseudometric ρ. Then $\mathcal{T}$-Cauchy nets do not necessarily correspond to ρ-Cauchy nets. For instance, consider the metric $\rho(x,y) = |\arctan x - \arctan y|$ on $\mathbb{R}$, i.e. the metric induced by the homeomorphism $\arctan\colon \mathbb{R} \to (-\pi/2, \pi/2)$, where the latter interval is considered as a metric subspace of $\mathbb{R}$ with the usual metric d. Then the net $(i)_{i \in \mathbb{N}}$ is Cauchy in $(\mathbb{R}, \rho)$, but not Cauchy in $(\mathbb{R}, d)$.

However, if ρ is either left- or right-invariant then the two notions coincide: This follows easily from the observation that in this case $g^{-1}h \in B_\rho(e,\varepsilon)$ if and only if $\rho(g,h) < \varepsilon$, for all $g,h \in G$ and $\varepsilon > 0$. ⌟

Proposition 2.8: *Complete implies closed*

Let G be a Hausdorff topological group. Then every complete subset of G is closed in G.

Proof. Let A be a complete subset of G, and let $g \in \overline{A}$. Then there is a net $(g_i)_{i\in I}$ in A that converges to g. This is then a Cauchy net in A, hence converges to some $g' \in A$. Since G is Hausdorff limits are unique, so $g = g' \in A$. □

Lemma 2.9

Let G and H be topological groups, let $A \subseteq G$, and let $f\colon A \to H$ be uniformly continuous. If $(g_i)_{i\in I}$ is a Cauchy net in A, then $(f(g_i))_{i\in I}$ is a Cauchy net.

Proof. Let V and U be neighbourhoods as in the discussion above. Let $i_0 \in I$ be such that $i,j \geq i_0$ implies $g_i^{-1}g_j \in U$. Uniform continuity then implies that $f(g_i)^{-1}f(g_j) \in V$, and since V was arbitrary the claim follows. □

Theorem 2.10

Let G and H be topological groups, let $A \subseteq G$, and let $f\colon A \to H$ be a unimorphism. If A is complete, then $B = f(A)$ is also complete.

In particular, completeness is preserved by topological group isomorphisms.

We in fact only need f to be continuous and to have a uniformly continuous right-inverse.

Proof. Let $(h_i)_{i\in I}$ be a Cauchy net in B. Then since $f^{-1}\colon B \to A$ is uniformly continuous, $(f^{-1}(h_i))_{i\in I}$ is a Cauchy net in A, hence convergent to some $g \in A$ by completeness. Since f is continuous, we have $h_i = f(f^{-1}(h_i)) \to f(g) \in B$, so (h_i) converges to a point in B. Thus B is complete. □

3 • Coset spaces and quotient groups

3.1. *General properties of coset spaces*

If H is a subgroup of a topological group G, we denote by G/H the set of left cosets of H. Let $q\colon G \to G/H$ be the quotient map and equip G/H with the quotient topology. We call G/H a *(left) coset space* of G by H.

Notice that q is in fact open: If $U \subseteq G$ is open, then $q^{-1}(q(U)) = UH$ is also open by Proposition 2.2(ii), so $q(U)$ is open since G/H has the quotient topology coinduced by q.

Proposition 3.1: *Properties of coset spaces*

Let G be a topological group and H a subgroup.

(i) *The coset space G/H is regular.*

(ii) *The topology on G/H is homogeneous.*

(iii) *G/H is T_1 (and hence T_3) if and only if H is closed.*

(iv) *G/H is discrete if and only if H is open.*

Notice that (iii) does not assume any separation properties of G.

Proof. We first show that G/H is regular. Let $q(x) \in G/H$, and let $B \subseteq G/H$ be a closed set. Then $A = q^{-1}(B)$ is closed, so by Proposition 2.4 there is a symmetric neighbourhood V of e in G such that $Vx \cap VA = \emptyset$. Since A is a union of left cosets of H we have $A = AH$, so $Vx \cap VAHH = \emptyset$. Since H is symmetric, it follows that

$$q(Vx) \cap q(VA) = VxH \cap VAH = \emptyset.$$

Since q is open the sets $q(Vx)$ and $q(VA)$ are neighbourhoods of $q(x)$ and $q(A) = B$ respectively.

Next we show that G/H is homogeneous. For $g \in G$ define a map $\theta_g \colon G/H \to G/H$ by $\theta(q(x)) = q(gx)$. This is well-defined, since if $xH = yH$, then $gxH = gyH$. Furthermore, the diagram

$$\begin{array}{ccc} G & \xrightarrow{\lambda_g} & G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ G/H & \xrightarrow[\theta_g]{} & G/H \end{array}$$

commutes, so $\theta_g \circ q$ is continuous. But by the characteristic property of the quotient topology on G/H, θ_g is also continuous. Since $\theta_g^{-1} = \theta_{g^{-1}}$ it is in fact a homeomorphism, and $\theta_{yx^{-1}}$ takes $q(x)$ to $q(y)$, which shows homogeneity.

Next we show that G/H is T_1 if and only if H is closed. Note that fibres of q are cosets of H, and a coset $gH = \lambda_g(H)$ is closed if and only if H is. But since G/H carries the quotient topology, gH is closed in G if and only if $\{gH\}$ is closed in G/H. Similarly, H (and hence gH) is *open* in G if and only if $\{gH\}$ is open in G/H, i.e. if and only if G/H is discrete. $\square$

3.2. Topological quotient groups

If a subgroup N of a topological group G is normal, we expect that the usual group structure on the (algebraic) quotient group G/N is compatible with the quotient topology. This is indeed the case:

THEOREM 3.2: *Topological quotient groups*

If N is a normal subgroup of a topological group G, then G/N is a topological group.

PROOF. Let $\mu\colon G\times G\to G$ and $M\colon G/N\times G/N\to G/N$ denote multiplication on G and G/N respectively, let $q\colon G\to G/N$ be the quotient map and define a map $Q\colon G\times G\to G/N\times G/N$ by $Q=q\times q$. Then Q is surjective and open since q is. Notice that the diagram

$$\begin{tikzcd} G\times G \arrow[r, "\mu"] \arrow[d, "Q"'] & G \arrow[d, "q"] \\ G/N\times G/N \arrow[r, "M"'] & G/N \end{tikzcd}$$

commutes. If $V\subseteq G/N$ is open, then so is $Q^{-1}(M^{-1}(V))=\mu^{-1}(q^{-1}(V))$, so applying Q to both sides we find that $M^{-1}(V)$ is open. Hence M is continuous.

Let $I\colon G/N\to G/N$ be the inversion map. Continuity of I is proved similarly by noticing that the diagram

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[d, "q"'] & G \arrow[d, "q"] \\ G/N \arrow[r, "I"'] & G/N \end{tikzcd}$$

commutes, where $\iota\colon G\to G$ is inversion in G. $\square$

PROPOSITION 3.3: *Factorisation through quotient group*

Let $\varphi\colon G\to H$ be a continuous group homomorphism between topological groups, and let N be a normal subgroup of G. If $N\subseteq\ker\varphi$, then there exists a unique set function $\tilde{\varphi}\colon G/N\to H$ such that the diagram

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & H \\ & G/N \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes. Furthermore, $\tilde{\varphi}$ is a continuous group homomorphism.

PROOF. Existence and uniqueness of $\tilde{\varphi}$ follows from the universal property of quotients in the category of sets. Continuity follows from the same property in the category of topological spaces, and $\tilde{\varphi}$ is a group homomorphism by the same property in the category of groups. □

We now explore how a topological group G that is *not* T_0 can be made so by quotienting out by a particular subgroup. To do this justice we first recall the T_0*-identification* of a topological space X: The ordering $x \leq y$ defined by $x \in \overline{\{y\}}$ is called the *specialisation preorder*, and it is easy to show that $x \leq y$ is equivalent to $\mathcal{N}_x \subseteq \mathcal{N}_y$. This order gives rise to an equivalence relation $\equiv$, and we say that two points $x, y \in X$ are *topologically indistinguishable* if $x \equiv y$.

It is clear that X is T_0 if and only if the relation $\equiv$ is trivial, and it is not difficult to show that the quotient space $X/\equiv$, called the T_0*-identification* or the *Kolmogorov quotient* of X, is indeed T_0. In fact, $\equiv$ is the most conservative equivalence relation $\sim$ such that $X/\sim$ is T_0, though we shall not need this fact. [TODO: reference to separation axiom notes]

When we apply this construction to the theory of topological groups, we see that the T_0-identification can be understood in terms of quotient groups:

PROPOSITION 3.4: *T_0-identification of groups*

Let G be a topological group. The subgroup $\overline{\{e\}}$ of G is normal. Furthermore, the $\equiv$-equivalence classes are precisely the left cosets of $\overline{\{e\}}$. It follows that the quotient group $G/\overline{\{e\}}$ is just the T_0-identification $G/\equiv$ of G.

PROOF. The subgroup $\overline{\{e\}}$ is the smallest closed subgroup of G, hence it is normal since otherwise intersecting it with one of its conjugates yields a strictly smaller closed subgroup.

It then suffices to show that, for $x, y \in G$, $x \equiv y$ if and only if $x\overline{\{e\}} = y\overline{\{e\}}$. But this is clear since e.g. $x\overline{\{e\}} = \overline{\{x\}}$ by continuity of multiplication on G. □

It is results like the above that lead to the common assumption that topological groups are T_1, since if it is not then we just quotient out by $\overline{\{e\}}$. One might justify this by arguing as follows: If a topology on a set X is to respect the set structure, then there must be a difference between two distinct points that is captured by the topology. But this difference must lie in which neighbourhoods each points has; the topology simply carries no further information than this. So if $x \neq y$, then it must be the case that $x \not\equiv y$, i.e. that X is T_0.

On the other hand, a *group* structure on a set G certainly respects the set structure, in that different elements may give rise to different actions: just let G act on itself by multiplication.

Thus if the topology and group structure on a topological group are to be truly compatible, then the topology must respect the underlying set structure, i.e. be T_0. In the sequel we shall, however, resist assuming that topological groups are T_0 as far as possible.

There is more to say about the T_0-identification that is not pertinent to this discussion. We refer to Willard (1970, Exercise 13C) and Pirttimäki (2019). [TODO: And my own notes]

4 • Metrisation

If X is a topological space, then we denote by $C(X)$ the space of continuous real-valued functions on X. Further denote by $C_b(X)$ the subspace of bounded functions. We equip $C_b(X)$ with the supremum norm. The category of normed spaces over the field $\mathbb{K}$ and linear maps with norm at most 1 is denoted $\mathbb{K}\text{-}\mathbf{Nor}_1$. The isomorphisms in $\mathbb{K}\text{-}\mathbf{Nor}_1$ are precisely the surjective isometries.

Lemma 4.1

Let G be a Hausdorff topological group that is first countable. Then there exists a bounded continuous function $f\colon G \to [0,1]$ with the following properties:

(i) $f(e) = 1$, *and* $f(x) < 1$ *for all* $x \neq e$.

(ii) *The sets* $V_n = \{x \in G \mid f(x) > 1 - 1/n\}$ *for* $n \in \mathbb{N}$ *form a local basis for* G.

(iii) *For every* $\varepsilon > 0$ *there exists an open neighbourhood* U *of* e *such that* $|f(gx) - f(x)| \leq \varepsilon$ *for all* $g \in U$ *and* $x \in G$.

Proof. .[TODO] See here. □

Theorem 4.2: *The Birkhoff–Kakutani Theorem*

Let G be a topological group. Then G is metrisable if and only if it is Hausdorff and first countable. In this case G is metrisable by a left-invariant metric.

Proof. The ‘only if’ direction is obvious, so we prove the ‘if’ direction.

Consider the map

$$\begin{aligned} \tau\colon G &\to \mathrm{Aut}_{\mathbb{R}\text{-}\mathbf{Nor}_1}(C_b(G)), \\ g &\mapsto \tau_g, \end{aligned}$$

where $\tau_g f(x) = f(g^{-1}x)$ for $f \in C_b(G)$ and $x \in G$. The map is well-defined: If $f, f' \in C_b(G)$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \tau_g(\alpha f + f')(x) &= (\alpha f + f')(g^{-1}x) = \alpha f(g^{-1}x) + f'(g^{-1}x) \\ &= \alpha\tau_g f(x) + \tau_g f'(x) = (\alpha\tau_g f + \tau_g f')(x) \end{aligned}$$

for all $x \in G$, i.e. τ_g is linear as claimed. Furthermore, notice that $\|\tau_g f\|_{\sup} = \|f\|_{\sup}$, since acting on f with τ_g simply permutes the domain G of f, so f and $\tau_g f$ have the same image. Hence τ_g is indeed an automorphism in $\mathbb{R}$-$\mathbf{Nor}_1$. Finally, it is easy to show that τ is a group homomorphism, i.e. a representation of G on $C_b(G)$.

Fixing a function $f \in C_b(G)$ yields a map $G \to C_b(G)$ given by $g \mapsto \tau_g f$, and this induces a pseudometric ρ_f on G. This is more concretely given by

$$\rho_f(g,h) = \|\tau_g f - \tau_h f\|_{\sup} = \sup_{x \in G} \big| f(g^{-1}x) - f(h^{-1}x) \big|.$$

Notice that ρ_f is indeed left-invariant, since

$$\begin{aligned}\rho_f(xg,xh) &= \|\tau_{xg} f - \tau_{xh} f\|_{\sup} = \|\tau_x(\tau_g f - \tau_h f)\|_{\sup} \\ &= \|\tau_g f - \tau_h f\|_{\sup} = \rho_f(g,h),\end{aligned}$$

for all $x \in G$.

Now assume that G is Hausdorff and first countable, and let f be the function from Lemma 4.1. Since f has a unique maximum, the map $g \mapsto \tau_g f$ is injective, so ρ_f is a genuine metric. Furthermore notice that this map is continuous by Lemma 4.1(iii).

It remains to be shown that ρ_f generates the topology on G. First notice that every ρ_f-ball is open in G since it is the preimage of a ball under $g \mapsto \tau_g f$. Conversely, let U be open in G, and let $g \in U$. There is an $n \in \mathbb{N}$ such that $gV_n \subseteq U$ by Lemma 4.1(ii). Now let $h \in G$ with $\rho_f(g,h) < 1/n$. Then

$$\begin{aligned}\frac{1}{n} > \rho_f(g,h) &= \sup_{x \in G} |f(g^{-1}x) - f(h^{-1}x)| \\ &\geq |f(g^{-1}h) - f(e)| \geq 1 - f(g^{-1}h).\end{aligned}$$

Hence $h \in gV_n$, so U is ρ_f-open. This proves the claim. □

5 • Topological vector spaces

5.1. *Convexity*

Below we let $\mathbb{K}$ denote either field of real numbers or the field of complex numbers.

Let X be an $\mathbb{K}$-vector space. For $x, y \in X$ we denote by $[x,y]$ the line segment between x and y, i.e. the set $\{tx + (1-t)y \mid t \in [0,1]\}$. A subset $C \subseteq X$ is *convex* if $[x,y] \subseteq C$ for all $x, y \in X$, or equivalently if $tC + (1-t)C \subseteq C$ for all $t \in [0,1]$. The intersection of all convex sets in X containing a subset $A \subseteq X$ is called the *convex hull* of A and is denoted $\operatorname{Conv}(A)$. This is clearly the smallest convex set containing A. Notice that $[x,y] = \operatorname{Conv}(\{x,y\})$.

Furthermore, a subset S of X is called *star-shaped at* $x \in S$ if $[x,y] \subseteq S$ for all $y \in S$. If there exists an $x \in S$ such that S is star-shaped at x, then S is simply called *star-shaped*. Clearly every non-empty convex set is star-shaped.

Lemma 5.1

Let X and Y be $\mathbb{K}$-vector spaces, let $A, C \subseteq X$ with C convex, and let $T\colon X \to Y$ be a linear map.

(i) *The image $T(C)$ is convex.*

(ii) *If T is an isomorphism, then*

$$T(\operatorname{Conv}(A)) = \operatorname{Conv}(T(A)).$$

Proof. *(i)*: This follows easily since

$$tT(C) + (1-t)T(C) = T(tC + (1-t)C) \subseteq T(C)$$

by linearity.

(ii): Clearly $T(A) \subseteq T(\operatorname{Conv}(A))$, so by (i) and the minimality of the convex hull we have $\operatorname{Conv}(T(A)) \subseteq T(\operatorname{Conv}(A))$. Replacing A by $T(A)$ and T by T^{-1} we get $\operatorname{Conv}(A) \subseteq T^{-1}(\operatorname{Conv}(T(A)))$, and applying T to both sides yields the opposite inclusion. ☐

5.2. *Definitions and basic properties*

Definition 5.2: *Topological vector spaces*

A *topological vector space* over $\mathbb{K}$ is a tuple $(X, +, \cdot, \mathcal{T})$ such that $(X, +, \cdot)$ is a $\mathbb{K}$-vector space, $(X, \mathcal{T})$ is a topological space, and both the addition $+\colon X \times X \to X$ and the scalar multiplication $\cdot\colon \mathbb{K} \times X \to X$ are continuous.

Notice that the inversion map $\iota\colon X \to X$ given by $\iota(x) = -x$ can be written $\iota(x) = (-1)x$, hence is continuous. Thus if $(X, +, \cdot, \mathcal{T})$ is a topological vector space, then $(X, +, \mathcal{T})$ is a topological group.

Definition 5.3

Let X be a $\mathbb{K}$-vector space, and let $A \subseteq X$. Then A is said to be

(i) *balanced* if $\alpha A \subseteq A$ for all $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, and

(ii) *absorbing* if for every $x \in X$ there exists a $t > 0$ such that $x \in tA$.

Assume that X is a topological vector space. Then A is called

(iii) *bounded* if for every neighbourhood U of 0 there exists a $t > 0$ such that $A \subseteq tU$.

Furthermore, X is said to be

(iv) *locally star-shaped* if it has a basis of star-shaped sets,

(v) *locally convex* if it has a basis of convex sets,

(vi) *locally bounded* if each point has a bounded neighbourhood,

(vii) an *F-space* if its topology is induced by a complete invariant metric, and

(viii) a *Fréchet space* if it is a locally convex F-space.

Remark 5.4. We collect a series of elementary results concerning the definitions above.

(i) If $T: X \to Y$ is a linear map between (not necessarily topological) $\mathbb{K}$-vector spaces and $A \subseteq X$ is balanced, then $T(A)$ is also balanced.

(ii) The convex hull of a balanced set is balanced: If $\alpha \in \mathbb{K} \setminus \{0\}$, then multiplication by α is a linear isomorphism. If A is balanced, it follows from Lemma 5.1(ii) that

$$\alpha \operatorname{Conv}(A) = \operatorname{Conv}(\alpha A) \subseteq \operatorname{Conv}(A),$$

so $\operatorname{Conv}(A)$ is also balanced.

(iii) If A is balanced and $\alpha \in \mathbb{K}$ with $|\alpha| = 1$, then both $\alpha A \subseteq A$ and $\alpha^{-1} A \subseteq A$. The latter implies that $A \subseteq \alpha A$, so $\alpha A = A$. Letting $\alpha = -1$ we get that balanced sets are symmetric. If $\mathbb{K} = \mathbb{R}$ and A is convex, then A is symmetric iff it is balanced.

(iv) Absorbing sets automatically contain 0. So do nonempty balanced sets.

(v) There is another common definition of boundedness that is slightly more complicated than the above but useful in some applications. In Corollary 5.7 we show that these definitions are equivalent.

Boundedness in the above sense, call it '$\mathcal{T}$-boundedness', does not generally agree with boundedness with respect to a metric ρ, call it 'ρ-boundedness': Assume that $X \neq 0$ is metrisable by an invariant metric ρ (e.g. assume that X is normable). Then the metric $\rho' = \rho/(1+\rho)$ is also invariant and topologically equivalent to ρ, and every subset of X is ρ'-bounded. However, in Corollary 5.8 we will see that no nontrivial Hausdorff TVS is $\mathcal{T}$-bounded.

However, if $\|\cdot\|$ is a norm on X, then $\mathcal{T}$-boundedness *does* coincide with $\|\cdot\|$-boundedness.

(vi) X is locally star-shaped iff it has a local basis of star-shaped open sets. Similarly for local convexity, local path-connectedness and local connectedness. In Proposition 5.6(i) we prove that all topological vector spaces are locally star-shaped, hence locally (path-)connected.

(vii) X is locally bounded iff 0 has a bounded neighbourhood. Notice that local boundedness is atypical since it does not assume a basis of bounded sets. ⌟

LEMMA 5.5

Let X be a topological vector space, and let $A \subseteq X$.

(i) *If A is a subspace, then so is $\overline{A}$.*

(ii) *If A is convex, then so are $\overline{A}$ and A°. In particular, the convex hull of an open set is open.*

(iii) *If A is balanced, then so is $\overline{A}$. If $0 \in A$, then A° is also balanced.*

(iv) *If A is balanced, then it is also star-shaped at 0, and hence path-connected.*

(v) *If A is bounded, then so are $\overline{A}$ and A°.*

PROOF. *(i)*: Assume that A is a subspace, and let $\alpha \in \mathbb{K}$. Then

$$\alpha\overline{A} + \overline{A} = \overline{\alpha A} + \overline{A} \subseteq \overline{\alpha A + A} \subseteq \overline{A}.$$

(ii): Assume that A is convex, and let $t \in [0,1]$. Then

$$t\overline{A} + (1-t)\overline{A} = \overline{tA} + \overline{(1-t)A} \subseteq \overline{tA + (1-t)A} \subseteq \overline{A},$$

as desired. Since $A^\circ \subseteq A$, for $t \in (0,1)$ we have

$$tA^\circ + (1-t)A^\circ \subseteq A,$$

and the set on the left-hand side is open by Proposition 2.2(ii) (since both t and $1-t$ are nonzero), hence contained in A°. The final claim follows since if A is open, then $\mathrm{Conv}(A)^\circ$ is an open convex set containing A.

(iii): Assume that A is balanced, and let $\alpha \in \mathbb{K}$ with $|A| \leq 1$. Then

$$\alpha\overline{A} = \overline{\alpha A} \subseteq \overline{A}$$

as desired. If $\alpha \neq 0$, then

$$\alpha A^\circ = (\alpha A)^\circ \subseteq \alpha A \subseteq A.$$

The set of the left-hand side is open, so it is contained in A°. If A contains 0, then this also holds when $\alpha = 0$.

(iv): This is obvious if A is empty, so let $x \in A$. For $t \in [0,1]$ we have $tx \in tA \subseteq A$ since A is balanced, so $[0,x] \subseteq A$. Hence A is star-shaped at 0.

(v): Assume that A is bounded. Every subset of A is also bounded, so in particular A° is bounded. Let U be a neighbourhood of 0. By regularity of X (cf. Proposition 2.4) there is a neighbourhood V of 0 such that $\overline{V} \subseteq U$. Since A is bounded there exists a $t > 0$ such that $A \subseteq tV$. We thus have

$$\overline{A} \subseteq \overline{tV} = t\overline{V} \subseteq tU,$$

as desired. □

Proposition 5.6: *Balanced local basis*

Let X be a topological vector space.

(i) *Every open neighbourhood of 0 contains an open balanced neighbourhood of 0. In particular, X has a local basis of balanced open sets. Thus X is locally star-shaped, hence locally path-connected.*

(ii) *Every convex open neighbourhood of 0 contains a convex open balanced neighbourhood of 0. In particular, if X is locally convex, it has a local basis of convex balanced open sets.*

Proof. *(i)*: Let U be an open neighbourhood of 0. Since scalar multiplication is continuous at $(0,0) \in \mathbb{K} \times X$, there exists a $\delta > 0$ and an open neighbourhood $V \subseteq X$ of 0 such that $B(0,\delta)V \subseteq U$. The set $B(0,\delta)V$ is clearly balanced and contains 0, and since

$$B(0,\delta)V = \bigcup_{\alpha \in B(0,\delta)} \alpha V = \bigcup_{\alpha \in B'(0,\delta)} \alpha V,$$

it is a union of open sets and hence itself open. The last claim follows from Lemma 5.5(iv).

(ii): Let U be an open convex neighbourhood of 0. Part (i) furnishes a balanced open neighbourhood V of 0 contained in U. Its convex hull $\operatorname{Conv}(V)$ is then balanced by Remark 5.4(ii) and open by Lemma 5.5(ii). By minimality it is also contained in U, as desired. □

Corollary 5.7: *Alternative characterisation of boundedness*

Let X be a topological vector space, and let $A \subseteq X$. Then the following are equivalent:

(i) *A is bounded, i.e. for every neighbourhood U of 0 there exists a $t > 0$ such that $A \subseteq tU$.*

(ii) *For every neighbourhood U of 0 there exists a $t > 0$ such that $A \subseteq \alpha U$ for all $\alpha \in \mathbb{K}$ with $|\alpha| \geq t$.*

PROOF. *(i) $\Rightarrow$ (ii)*: By Proposition 5.6(i) U contains a balanced neighbourhood V of 0. If A is bounded, then there is a $t > 0$ such that $A \subseteq tV$. Let $\alpha \in \mathbb{K}$ with $|\alpha| \geq t$. Then αV is also balanced by Remark 5.4(i), so since $|t/\alpha| \leq 1$ we have

$$tV = \frac{t}{\alpha}\alpha V \subseteq \alpha V,$$

implying that $A \subseteq \alpha V \subseteq \alpha U$ as desired.

(ii) $\Rightarrow$ (i): This is obvious. □

COROLLARY 5.8: *Hausdorff spaces are unbounded*

Let X be a nontrivial Hausdorff topological vector space. Then X is unbounded.

If X carries the trivial topology, then X is bounded. Hence the Hausdorff assumption cannot simply be dropped.

PROOF. Let $x \in X$ be nonzero, and let U be a neighbourhood of 0 not containing x. Consider the set $A = \{nx \mid n \in \mathbb{N}\}$. Assume towards a contradiction that A is bounded, and choose $t > 0$ in accordance with Corollary 5.7(ii). If $n \in \mathbb{N}$ with $n \geq t$, then $A \subseteq nU$. But since $x \notin U$ we have $nx \notin nU$, a contradiction. Hence A is unbounded, so X itself is unbounded. □

PROPOSITION 5.9

Let X be a topological vector space, and let U be a neighbourhood of 0.

(i) *Let $(r_i)_{i\in I}$ be an unbounded net in $\mathbb{K}$. Then*

$$X = \bigcup_{i\in I} r_i U.$$

In particular, U is absorbing.

(ii) *Every compact set K in X is bounded.*

(iii) *Assume that U is bounded, and let $(\delta_i)_{i\in I}$ be a net in $\mathbb{K} \setminus \{0\}$ with 0 as a cluster point. Then the family $\{\delta_i U \mid i \in I\}$ is a local basis for X.*

PROOF. Let $x \in X$. Since scalar multiplication is continuous, the set $V = \{\alpha \in \mathbb{K} \mid \alpha x \in U\}$ is a neighbourhood of $0 \in \mathbb{K}$, so there exists an $i \in I$ such that $1/r_i \in V$. That is, $(1/r_i)x \in U$, or $x \in r_i U$. To show that U is absorbing, let $I = \mathbb{N}$ and $r_i = i$.

To prove that K is bounded, let $W \subseteq U$ be a balanced neighbourhood of 0. Then (i) implies that

$$K \subseteq \bigcup_{i \in \mathbb{N}} iW,$$

and by compactness we have $K \subseteq \bigcup_{i=1}^{t} iW$ for some $t \in \mathbb{N}$. But then $K \subseteq tW$ since W (and hence tW by Remark 5.4(i)) is balanced. It follows that $K \subseteq tU$, so K is bounded.

Finally let V be a neighbourhood of 0. Since U is bounded we may choose $t > 0$ as in Corollary 5.7(ii) (with U in place of A and V in place of U). There exists an $i \in I$ such that $\delta_i \in B(0, 1/t)$, i.e. $|1/\delta_i| > t$. Hence $U \subseteq (1/\delta_i)V$, i.e. $\delta_i U \subseteq V$ as desired. □

COROLLARY 5.10: *Cauchy sequences are bounded*

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in a topological vector space X. Then the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded. In particular, convergent sequences are bounded.

More generally, we say that a net $(x_i)_{i \in I}$ is bounded if the set $\{x_i \mid i \in I\}$ is bounded. This result, however, does not hold for general nets.

PROOF. Let U be a neighbourhood of 0. By Proposition 5.6(i) we may assume that U is balanced. There exists a neighbourhood V of 0 such that $V + V \subseteq U$, by Lemma 2.3. Since (x_n) is Cauchy, there is an $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $x_m - x_n \in V$. In particular, $x_n \in x_N + V \subseteq V + V \subseteq U$.

Now let $n < N$. Then since U is absorbing by Proposition 5.9(i), there is a $t_n > 0$ such that $x_n \in t_n U$. Letting $t = \max\{1, t_1, \ldots, t_{N-1}\}$ we thus have $x_n \in tU$ for all $n \in \mathbb{N}$ since U is balanced. □

5.3. *Locally convex spaces*

Let X be a topological vector space. To each $A \subseteq X$ we associate the *Minkowski functional* μ_A of A, given by

$$\mu_A(x) = \inf\{t > 0 \mid t^{-1}x \in A\}$$

for $x \in X$. We shall only be interested in μ_A in the case where A is absorbing (and in fact also convex). In this case A is nonempty (since it contains 0) and we have $\mu_A(x) < \infty$ for all $x \in X$.

LEMMA 5.11

Let X be a topological vector space, and let $A \subseteq X$ be a convex, absorbing set. For all $x, y \in X$ we then have

(i) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$, *and*

(ii) $\mu_A(\alpha x) = \mu_{\alpha^{-1}A}(x)$ *for* $\alpha \in \mathbb{K}$, *where* $0^{-1}A := X$. *If* $t \geq 0$, *then* $\mu_A(tx) = t\mu_A(x)$.

Assume furthermore that A *is balanced.*

(iii) μ_A *is a seminorm.*

(iv) *If* A *is open,* $x \in X$ *and* $r > 0$, *then*

$$B_{\mu_A}(x,r) = \{y \in X \mid \mu_A(y-x) < r\} = x + rA,$$

Proof. *(i)*: Let $\varepsilon > 0$, and put $s = \mu_A(x)+\varepsilon$ and $t = \mu_A(y)+\varepsilon$. Since A is convex and contains 0, we have $s^{-1}x, t^{-1}y \in A$. But then

$$\frac{x+y}{s+t} = \frac{t}{s+t}\frac{x}{s} + \frac{s}{s+t}\frac{y}{t} \in A$$

by convexity, which implies that

$$\mu_A(x+y) \leq s+t = \mu_A(x)+\mu_A(y)+2\varepsilon.$$

Since ε was arbitrary, the claim follows.

(ii): For $\alpha \neq 0$ we have

$$\mu_A(\alpha x) = \inf\{s > 0 \mid s^{-1}\alpha x \in A\} = \inf\{s > 0 \mid s^{-1}x \in \alpha^{-1}A\},$$

and if $\alpha = 0$ then since A contains 0 we have

$$\mu_A(0x) = 0 = \mu_{0^{-1}A}(x),$$

with the convention that $0^{-1}A = X$. For $t > 0$ we have

$$\mu_A(tx) = \inf\{s > 0 \mid s^{-1}tx \in A\} = \inf\{tr \mid r > 0, r^{-1}x \in A\} = t\mu_A(x),$$

where we have used the substitution $r = s/t$.

(iii): Let $\alpha \in \mathbb{K}$ and write $\alpha = |\alpha|u$ with $|u| = 1$. Then

$$\mu_A(\alpha x) = \mu_A(|\alpha|ux) = |\alpha|\mu_{u^{-1}A}(x) = |\alpha|\mu_A(x),$$

where we use that $u^{-1}A = A$ since A is balanced.

(iv): First assume that $r = 1$ and $x = 0$. If $y \in A$, then there exists a $t \in (0,1)$ such that $t^{-1}y \in A$ since A is open. Hence $\mu_A(y) < 1$. If instead $y \notin A$, then if $t^{-1}y \in A$ for some $t > 0$, then we must have $t \geq 1$ since A is balanced.

For general $r > 0$ notice that

$$rA = \{y \in X \mid \mu_{rA}(y) < 1\} = \{y \in X \mid r^{-1}\mu_A(y) < 1\} = \{y \in X \mid \mu_A(y) < r\}$$

by (ii). The claim for general $x \in X$ is obvious. □

THEOREM 5.12

Let X be a $\mathbb{K}$-vector space, and let $\mathcal{P}$ be a family of seminorms on X. Let $\mathcal{T}$ denote the topology induced by $\mathcal{P}$, i.e. induced by all open balls $B_p(x,\varepsilon)$ for $p \in \mathcal{P}$, $x \in X$, and $\varepsilon > 0$.

(i) *For each $x \in X$, the finite intersections of the sets $B_p(x,\varepsilon)$, for $p \in \mathcal{P}$ and $\varepsilon > 0$, form a neighbourhood basis at x.*

(ii) *If $(x_i)_{i\in I}$ is a net in X and $x \in X$, then $x_i \to x$ iff $p(x_i - x) \to 0$ for all $p \in \mathcal{P}$.*

(iii) *$(X,\mathcal{T})$ is a locally convex topological vector space.*

PROOF. *(i)*: Let U be a neighbourhood of x. By definition of $\mathcal{T}$,

$$x \in \bigcap_{k=1}^{n} B_{p_k}(x_k,\varepsilon_k) \subseteq U$$

for appropriate $x_k \in X$, $p_k \in \mathcal{P}$ and $\varepsilon_k > 0$. Letting $\delta_k = \varepsilon_k - p_k(x - x_k)$ the triangle inequality implies that

$$x \in \bigcap_{k=1}^{n} B_{p_k}(x,\delta_k) \subseteq \bigcap_{k=1}^{n} B_{p_k}(x_k,\varepsilon_k),$$

proving the claim.

(ii): This is obvious from (i).

(iii): Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be nets in X, and let $(\alpha_i)_{i\in I}$ be a net in $\mathbb{K}$, and assume that $x_i \to x$, $y_i \to y$ and $\alpha_i \to \alpha$. Then eventually $|\alpha_i| \leq |\alpha| + 1$, so eventually

$$\begin{aligned} p\big((\alpha_i x_i + y_i) - (\alpha x + y)\big) &= p\big((\alpha_i x_i - \alpha_i x) + (\alpha_i x - \alpha x) + (y_i - y)\big) \\ &\leq p(\alpha_i x_i - \alpha_i x) + p(\alpha_i x - \alpha x) + p(y_i - y) \\ &\leq (|\alpha| + 1)p(x_i - x) + |\alpha_i - \alpha| p(x) + p(y_i - y), \end{aligned}$$

which goes to zero. Hence the vector operations are continuous. Furthermore, the usual proof that balls are convex also works for balls defined by seminorms, so the balls $B_p(x,\varepsilon)$ are convex. Hence (finite) intersections of balls are convex, so $\mathcal{T}$ has a basis of convex sets. ☐

REMARK 5.13. Let $(X,\mathcal{T})$ be a locally convex topological vector space, and let $\mathcal{B}$ be a local basis of convex balanced open sets, in accordance with Proposition 5.6(ii). By Theorem 5.12, the corresponding family $\mathcal{M} = \{\mu_B \mid B \in \mathcal{B}\}$ of Minkowski functionals generates a vector space topology $\mathcal{T}'$ on X. We claim that $\mathcal{T} = \mathcal{T}'$.

The inclusion $\mathcal{T}' \subseteq \mathcal{T}$ follows since each Minkowski functional is $\mathcal{T}$-continuous (since seminorms are continuous), so the sets $B_\mu(0,\varepsilon) = \mu^{-1}(B(0,\varepsilon))$ are $\mathcal{T}$-open for all $\mu \in \mathcal{M}$. Hence the sets $B_\mu(x,\varepsilon) = x + B_\mu(0,\varepsilon)$ for general $x \in X$ are also $\mathcal{T}$-open by homogeneity.

Conversely, for $B \in \mathcal{B}$ we have $B = B_{\mu_B}(0,1)$ by Lemma 5.11(iv), so $B \in \mathcal{T}'$. Hence $\mathcal{T} \subseteq \mathcal{T}'$ since the latter is a vector space topology, and the former is generated by translates of sets in $\mathcal{B}$. ⌟

This remark in particular implies the following result:

Corollary 5.14

A topological vector space X is locally convex if and only if its topology is generated by a family of seminorms.

Theorem 5.15

A topological vector space X is seminormable if and only if 0 has a convex bounded neighbourhood.

Proof. The ‘only if’ part if obvious, so we prove the converse.

Let U be a convex bounded neighbourhood of 0. By Proposition 5.6(ii) we may assume that U is also balanced, so Lemma 5.11(iii) implies that the Minkowski functional μ_U is a seminorm. It suffices to show that μ_U generates the topology on X.

By Proposition 5.9(iii), the collection $\{rU \mid r > 0\}$ is a local basis. But Lemma 5.11(iv) says that $rU = B_{\mu_U}(0,r)$, so the topology on X coincides with the μ_U-topology. □

5.4. Quotient spaces

Theorem 5.16: *Topological quotient vector spaces*

If M is a subspace of a topological $\mathbb{K}$-vector space X, then X/M is a topological vector space.

Proof. Theorem 3.2 already shows that X/M is a topological group, so it suffices to show that the scalar multiplication on X/M is continuous. To this end, let $\rho\colon \mathbb{K} \times X \to X$ be the scalar multiplication on X, $R\colon \mathbb{K} \times X/M \to X/M$ the scalar multiplication on X/M, $q\colon X \to X/M$ the quotient map, and define $Q\colon \mathbb{K} \times X \to \mathbb{K} \times X/M$ by $Q = \mathrm{id}_{\mathbb{K}} \times q$. Continuity of R then follows by noticing that Q is open and surjective, and that the diagram

$$\begin{tikzcd} \mathbb{K} \times X \arrow[r, "\rho"] \arrow[d, "Q"'] & X \arrow[d, "q"] \\ \mathbb{K} \times X/M \arrow[r, "R"'] & X/M \end{tikzcd}$$

commutes. □

Proposition 5.17: *Factorisation through quotient spaces*

Let $T\colon X \to Y$ be a continuous linear map between topological vector spaces, and let M be a subspace of X. If $M \subseteq \ker T$, then there exists a unique set function $\tilde{T}\colon X/M \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "q"'] & & Y \\ & X/M \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

commutes. Furthermore, $\tilde{T}$ is a continuous linear map.

Proof. Existence and uniqueness of $\tilde{T}$ follows from the universal property of quotients in the category of sets. Continuity follows from the same property in the category of topological spaces, linearity by the same property in the category of $\mathbb{K}$-vector spaces. □

5.5. Continuous linear maps

A linear map $T\colon X \to Y$ between topological vector spaces is *bounded* if $T(A)$ is bounded for every bounded set $A \subseteq X$. If X and Y are normed spaces, this agrees with the usual definition of boundedness: If there is a $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$, then T clearly sends bounded sets to bounded sets. Conversely, the closed ball $\overline{B}(0,1)$ is bounded so $T(\overline{B}(0,1)) \subseteq \overline{B}(0,C)$ for some $C \geq 0$, i.e. $\|Tx\| \leq C$ whenever $\|x\| \leq 1$. Boundedness with respect to $\|\cdot\|$ then follows by linearity.

Continuity and boundedness are equivalent for operators between normed spaces. We begin by exploring the relationship between continuity and boundedness for maps between general topological vector spaces:

Proposition 5.18: *Continuity and boundedness*

Let $T\colon X \to Y$ be a linear map between topological vector spaces. If T is continuous then it is bounded. The converse also holds if X is first countable.

PROOF. Assume that T is continuous, let $A \subseteq X$ be bounded, and let $V \subseteq Y$ be a neighbourhood of 0. Letting $U = T^{-1}(V)$ there exists a $t > 0$ such that $A \subseteq tU$. It follows that $T(A) \subseteq tT(U) = tV$ as desired.

Conversely, assume that X is first countable and that T is bounded but *not* continuous. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing sequence of sets in X such that $\{U_n \mid n \in \mathbb{N}\}$ is a local basis. Since T is not continuous, there is a balanced neighbourhood V of 0 in Y such that $T^{-1}(V)$ is not a neighbourhood of 0 in X. Hence there exists for every $n \in \mathbb{N}$ an $x_n \in \frac{1}{n}U_n$ such that $Tx_n \notin V$. Then $nx_n \to 0$ as $n \to \infty$, so $(nx_n)_{n\in\mathbb{N}}$ is bounded by Corollary 5.10. Hence the sequence $(nTx_n)_{n\in\mathbb{N}}$ is also bounded, so there is a $t > 0$ such that $(nTx_n) \subseteq tV$. Since V is balanced, for $n > t$ we have

$$Tx_n \in \frac{t}{n}V \subseteq V,$$

contradicting the definition of (x_n). Hence T is in fact continuous. □

It turns out that linearity and continuity are closely related. In fact, it will turn out that a linear map is continuous if either its domain or codomain is finite-dimensional, at least in the Hausdorff case. Before proving this we note the following result:

LEMMA 5.19

Let $T\colon X \to Y$ be a linear map between topological vector spaces. If there is a neighbourhood U of 0 in X such that $T(U)$ is bounded, then T is continuous.

PROOF. For any neighbourhood V of 0 in Y there is an $r > 0$ such that $T(rU) = rT(U) \subseteq V$. Since rU is a neighbourhood of 0, T is continuous at 0 and thus continuous by Proposition 2.5. □

THEOREM 5.20: *Finite-dimensional domain*

Let $T\colon X \to Y$ be a linear map between topological $\mathbb{K}$-vector spaces. If X is Hausdorff and finite-dimensional, then T is continuous. If Y is also Hausdorff, then T is a homeomorphism onto its image.

This implies that there exists exactly one Hausdorff topology on a finite-dimensional vector space X that makes X into a topological vector space. Since any finite-dimensional vector space X can be equipped with a norm, all Hausdorff topologies on X are norm topologies. Furthermore, all norms on X are topologically equivalent; in fact all norms are Lipschitz equivalent, though this does not readily follow from the above.

The Hausdorff assumption cannot easily be removed. For instance, if $(X, \|\cdot\|)$ is a normed space and $\mathcal{T}$ is the trivial topology on X, then the identity map $\mathrm{id}_X\colon (X,\mathcal{T}) \to (X,\|\cdot\|)$ is usually not continuous.

PROOF. *$X = \mathbb{K}^d \Rightarrow T$ continuous:* Let $(e_1, \dots, e_d)$ be the standard basis for $\mathbb{K}^d$. Then

$$Tx = \sum_{i=1}^{d} \pi_i(x) T e_i$$

for $x \in X$, where $\pi_i \colon \mathbb{K}^d \to \mathbb{K}$ is the ith projection. Since each π_i is continuous, and since addition and scalar multiplication are continuous in Y, it follows that T is continuous.

$X = \mathbb{K}^d$, Y Hausdorff $\Rightarrow T$ homeomorphism: We may assume that T is surjective. Consider the unit sphere S^1 in $\mathbb{K}^d$. Then $T(S^1)$ is compact (hence closed since Y is Hausdorff) and $0 \notin T(S^1)$, so 0 has a balanced neighbourhood U disjoint from $T(S^1)$. Then $T^{-1}(U)$ is balanced, hence (path-)connected by Lemma 5.5(iv), and disjoint from S^1. Since $T^{-1}(U)$ contains 0 it must lie in $B(0,1)$ and is thus bounded. Hence T^{-1} is continuous by Lemma 5.19, so T is a homeomorphism.

X Hausdorff, $\dim X < \infty \Rightarrow T$ continuous: Let $S \colon \mathbb{K}^d \to X$ be a linear isomorphism, where $d = \dim X$. The above then shows S is a homeomorphism and that TS is continuous. But then $T = (TS)S^{-1}$ is continuous.

X, Y Hausdorff, $\dim X < \infty \Rightarrow T$ homeomorphism: Under these assumptions the map $TS \colon \mathbb{K}^d \to Y$ is a homeomorphism, so T is as well. □

THEOREM 5.21: *Finite-dimensional codomain*

Let $T \colon X \to Y$ be a linear map between topological vector spaces. Assume that Y is finite-dimensional. If $\ker T$ is closed, then T is continuous. The converse also holds if Y is Hausdorff.

PROOF. Let $\pi \colon X \to X/\ker T$ be the quotient map. Then there is (cf. Proposition 5.17) a map $\tilde{T} \colon X/\ker T \to Y$ such that $T = \tilde{T} \circ \pi$, and $\tilde{T}$ is a linear isomorphism onto a subspace of Y, so $X/\ker T$ is a finite-dimensional subspace of X. If $\ker T$ is closed, then by Proposition 3.1(iii) $X/\ker T$ is a Hausdorff topological vector space, so Theorem 5.20 implies that $\tilde{T}$ is continuous. Thus T is also continuous.

Conversely assume that Y is Hausdorff and T is continuous. Then $\{0\}$ is closed in Y, so $\ker T = T^{-1}(\{0\})$ is also closed. □

5.6. Finite-dimensional spaces

COROLLARY 5.22

If M is a finite-dimensional subspace of a Hausdorff topological $\mathbb{K}$-vector space X, then M is closed in X.

PROOF. Let $d = \dim M$, and let $T : \mathbb{K}^d \to M$ be a linear isomorphism, hence a homeomorphism. Then since $\mathbb{K}^d$ is complete as a normed space, hence as a topological vector space, M is also complete by Theorem 2.10. But then it is closed by Proposition 2.8. □

THEOREM 5.23: *F. Riesz's Theorem*

Let X be a Hausdorff topological $\mathbb{K}$-vector space. Then X is locally compact if and only if it is finite-dimensional.

PROOF. If X is finite-dimensional of dimension d, then X isomorphic to $\mathbb{K}^d$ as a vector space, hence as a topological space by Theorem 5.20, and so it is locally compact.

Conversely, if X is locally compact then 0 has a precompact open neighbourhood U. By Proposition 5.9(ii), $\overline{U}$ (and hence U) is also bounded, so the collection $\mathcal{U} = \{2^{-n}U \mid n \in \mathbb{N}\}$ is a local basis by Proposition 5.9(iii).

By compactness of $\overline{U}$ there exists a finite set $\mathcal{B} \subseteq X$ such that $U \subseteq \overline{U} \subseteq \mathcal{B} + \frac{1}{2}U$. Hence if $S = \operatorname{span}\mathcal{B}$, then $U \subseteq S + \frac{1}{2}U$. Since S is a subspace, it follows that $\frac{1}{2}U \subseteq S + \frac{1}{4}U$. Hence

$$U \subseteq S + \tfrac{1}{2}U \subseteq S + (S + \tfrac{1}{4}U) = S + \tfrac{1}{4}U.$$

By induction we thus find that

$$U \subseteq \bigcap_{n\in\mathbb{N}} (S + 2^{-n}U) = \overline{S} = S,$$

where the first equality follows from Proposition 2.2(iii) since $\mathcal{U}$ is a local basis, and the second follows by Corollary 5.22. Since U is absorbing so is S, but S is a subspace so $X = S$. Hence X is finite-dimensional. □

References

Aluffi, Paolo (2009). *Algebra: Chapter 0*. American Mathematical Society. 713 pp. ISBN: 978-0-8216-4781-7.

Folland, Gerald B. (2007). *Real Analysis: Modern Techniques and Their Applications*. 2nd ed. Wiley. 386 pp. ISBN: 0-471-31716-0.

— (2015). *A Course in Abstract Harmonic Analysis*. CRC Press. 305 pp. ISBN: 978-1-4987-2715-0.

Lee, John M. (2011). *Introduction to Topological Manifolds*. 2nd ed. Springer. 433 pp. ISBN: 978-1-4419-7939-1. DOI: 10.1007/978-1-4419-7940-7.

McCarty, George (1988). *Topology: An Introduction with Applications to Topological Groups*. Dover. 270 pp. ISBN: 0-486-65633-0.

Munkres, James R. (2000). *Topology*. 2nd ed. Prentice Hall, Inc. 537 pp. ISBN: 0-13-181629-2.

Narici, Lawrence and Edward Beckenstein (2011). *Topological Vector Spaces*. 2nd ed. CRC Press. 610 pp. ISBN: 978-1-58488-866-6.

Pirttimäki, Teemu (2019). *A survey of Kolmogorov quotients*. arXiv: 1905.01157.

Roman, Steven (2008). *Advanced Linear Algebra*. 3rd ed. Springer. 522 pp. ISBN: 978-0-387-72828-5.

Rudin, Walter (1991). *Functional Analysis*. 2nd ed. McGraw-Hill, Inc. 424 pp. ISBN: 0-07-054236-8.

Willard, Stephen (1970). *General Topology*. Addison-Wesley Publishing Company, Inc. 369 pp.