Topological Groups

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1 · Introduction

The purpose of these notes is to clarify some of the basic properties of topological groups. We in particular hope to shed light on the reasonability of the common assumption that topological groups be Hausdorff.

Most of the results below can be found in various forms in a variety of books or articles (see the references), but I have not come across a resource that develops this theory in precisely this way. One example is the T_0-identification of a topological group, as we will see below.

2 · Definitions and basic properties

DEFINITION 2.1: *Topological groups*

A *topological group* is a triple $(G, \mu, \mathcal{T})$ such that (G, μ) is a group, $(G, \mathcal{T})$ is a topological space, and both the multiplication $\mu \colon G \times G \to G$ and the inversion map $\iota \colon G \to G$ given by $g \mapsto g^{-1}$ are continuous.

In the sequel we will simply write G, and it will be clear from context whether G is to be thought of as a set, group, topological space or topological group. It is custom to assume that a topological group be T_1 (or Hausdorff, which is actually implied by it being T_1, as will follow from Proposition 2.5), but we omit this assumption since our focus is precisely on the basic topological properties of topological groups.

Let G be a topological group and let $g \in G$. We define maps $\lambda_g, \rho_g \colon G \to G$ by $\lambda_g(h) = gh$ and $\rho_g(h) = hg$; that is, λ_g and ρ_g are given by left- and right-multiplication by g, respectively. Both of these are homeomorphisms, with $\lambda_g^{-1} = \lambda_{g^{-1}}$ and $\rho_g^{-1} = \rho_{g^{-1}}$. Hence if $a, b \in G$, then there is a homeomorphism on G that takes a to b, namely $\lambda_{ba^{-1}}$. Thus a topological group is *homogeneous* as a topological space.

We begin by collecting some basic properties of topological groups:

PROPOSITION 2.2: *Properties of topological groups*

Let G be a topological group.

(i) *If $A, B \subseteq G$, then $\overline{A}\,\overline{B} \subseteq \overline{AB}$.*

(ii) *If H is a subgroup of G, then so is $\overline{H}$.*

(iii) *Every open subgroup of G is closed.*

(iv) *If $A, U \subseteq G$ with U open, then AU and UA are open.*

PROOF. We give two proofs of (i). Since the multiplication $\mu \colon G \times G \to G$ is continuous we get

$$\mu(\overline{A}, \overline{B}) = \mu(\overline{A} \times \overline{B}) = \mu(\overline{A \times B}) \subseteq \overline{\mu(A \times B)} = \overline{\mu(A, B)} = \overline{AB}.$$

Alternatively, consider $a \in \overline{A}$ and $b \in \overline{B}$. If U is a neighbourhood of ab, then by continuity of multiplication there are neighbourhoods V_1 and V_2 of a and b respectively such that $V_1 V_2 \subseteq U$. Picking $x \in A \cap V_1$ and $y \in B \cap V_2$ we find that $xy \in AB \cap U$, so $ab \in \overline{AB}$.

Let H be a subgroup of G. By (i) we get

$$\overline{H}\,\overline{H} \subseteq \overline{HH} = \overline{H},$$

so $\overline{H}$ is closed under multiplication. It is also closed under taking inverses, since the inverse map ι is a homeomorphism. This proves (ii). Alternatively, this claim is an easy consequence of basic properties of nets.

Now let H be an open subgroup. Since G is the disjoint union of the left cosets of H, we have

$$G \setminus H = \bigcup_{g \in G \setminus H} gH.$$

Since the cosets gH are open, $G \setminus H$ is also open. This proves (iii).

To prove (iv), notice that e.g.

$$AU = \bigcup_{g \in A} gU,$$

so AU is a union of open sets. $\qquad\square$

We next remark that the assumption that G be T_1 can be weakened:

PROPOSITION 2.3: T_0 *implies* T_1

Let G be a topological group. If G is T_0, then it is in fact T_1.

PROOF. Assume that G is T_0. By homogeneity it suffices to show that the singleton $\{e\}$ containing the identity $e \in G$ is closed. We show that $G \setminus \{e\}$ is a neighbourhood[1] of all its points. Let $g \neq e$. Since G is T_0, either $G \setminus \{e\}$ is a neighbourhood of g or $G \setminus \{g\}$ is a neighbourhood of e. In the first case we are done, so assume the latter. The homeomorphism $\lambda_{g^{-1}}$ maps $G \setminus \{g\}$ to $G \setminus \{e\}$, so the latter set is a neighbourhood of g^{-1}. But the inversion map ι is a homeomorphism that maps $G \setminus \{e\}$ to itself, so this is also a neighbourhood of g. This proves the claim. □

Recall that a topological space X is called *regular* if it is possible to separate any point from any closed set by disjoint open sets.[2] Our next order of business is to show that any topological group is regular. Before proving this we need some terminology and a lemma:

If G is a topological group and $A \subseteq G$, then we write

$$A^{-1} = \iota(A) = \{a^{-1} \mid a \in A\}.$$

A subset A is called *symmetric* if $A = A^{-1}$. Notice that since ι is a homeomorphism, A is open (closed) if and only if A^{-1} is open (closed).

LEMMA 2.4

Let G be a topological group. If U is a neighbourhood of the identity e, then there is a symmetric neighbourhood V of e such that $VV \subseteq U$.

PROOF. Since multiplication is continuous and $ee = e$, there are neighbourhoods V_1 and V_2 of e such that $V_1 V_2 \subseteq U$. If we let

$$V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1},$$

then V has the desired properties. □

PROPOSITION 2.5: *Regularity of topological groups*

If G is a topological group, $g \in G$, $A \subseteq G$ is closed and $g \notin A$, then there exists a symmetric neighbourhood V of e such that Vg and VA are disjoint. In particular every topological group is regular, and every T_0 topological group is T_3.

PROOF. Since $g \notin A$ we also have $e \notin Ag^{-1}$. But Ag^{-1} is closed, so by Lemma 2.4 there is a symmetric neighbourhood V of e such that $VV \cap Ag^{-1} = \emptyset$. It follows that $Vg \cap VA = \emptyset$ as desired. This shows that G is regular, and the final claim follows by Proposition 2.3. □

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[1] If X is a topological space and $A \subseteq X$, then we say that a set $N \subseteq X$ is a *neighbourhood* of A if there is an open set U in X such that $A \subseteq U \subseteq N$. The family of neighbourhoods of a set A is called the *neighbourhood filter of A* and is denoted $\mathcal{N}_A$. If $A = \{x\}$ is a singleton we also write $\mathcal{N}_x$ and call N a neighbourhood of x.

[2] In our terminology, a regular T_1-space would be called a T_3-*space*.

3 · Coset spaces and quotient groups

3.1. General properties of coset spaces

If H is a subgroup of a topological group G, we denote by G/H the set of left cosets of H. Let $q \colon G \to G/H$ be the quotient map and equip G/H with the quotient topology. We call G/H a *coset space* of G.

Notice that q is in fact open: If $U \subseteq G$ is open, then $q^{-1}(q(U)) = UH$ is also open, so $q(U)$ is open since G/H has the quotient topology coinduced by q.

PROPOSITION 3.1: *Properties of coset spaces*

Let G be a topological group and H a subgroup.

  (i) *The coset space G/H is regular.*

 (ii) *G/H is T_1 (and hence T_3) if and only if H is closed.*

(iii) *If G is locally compact[3], then so is G/H.*

Notice that (ii) does not assume any separation properties of G.

(The claim (i) is the content of supplementary exercise 7(d) of Munkres (2000, Chapter 2), except that in the exercise H is assumed to be closed. I do not see where this assumption is used.)

PROOF. We first show that G/H is regular. Let $q(x) \in G/H$, and let $B \subseteq G/H$ be a closed set. Then $A = q^{-1}(B)$ is closed, so by Proposition 2.5 there is a symmetric neighbourhood V of e in G such that $Vx \cap VA = \emptyset$. Since A is a union of left cosets of H we have $A = AH$, and a quick calculations shows that $VxH \cap VA = \emptyset$. It follows by a similar calculation that $q(Vx)$ and $q(VA)$ are disjoint, and since q is open these are neighbourhoods of $q(x)$ and $q(A) = B$ respectively.

Next we show that G/H is T_1 if and only if H is closed. Note that fibres of q are cosets of H, and a coset $gH = \lambda_g(H)$ is closed if and only if H is. But since G/H carries the quotient topology, gH is closed in G if and only if $\{gH\}$ is closed in G/H.

To prove (iii), notice that if K is a compact neighbourhood of e in G, then $q(Kx)$ is a compact neighbourhood of xH in G/H. □

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[3] We say that a topological space is locally compact if every point has a compact neighbourhood. There are many non-equivalent definitions of local compactness and this is the least restrictive one. If H is closed then G/H is Hausdorff, so all the usual definitions of local compactness are equivalent for G/H.

3.2. Topological quotient groups

If a subgroup N of a topological group G is normal, we expect that the usual group structure on the (algebraic) quotient group G/N is compatible with the quotient topology. This is indeed the case:

THEOREM 3.2: *Topological quotient groups*

If N is a normal subgroup of a topological group G, then G/N is a topological group.

PROOF. If $x, y \in G$ and U is a neighbourhood of $(xN)(yN) = xyN$ in G/N, then continuity of multiplication in G at (x, y) implies the existence of neighbourhoods V and W of x and y respectively, such that $VW \subseteq q^{-1}(U)$. Since q is surjective it follows that $q(V)q(W) \subseteq U$, and because q is also open $q(V)$ and $q(W)$ are neighbourhoods of xN and yN. Hence multiplication is continuous.

Since the inversion map ι on G/N is bijective, it suffices to show that it is open. Let $U \subseteq G/N$ be open and notice that, since q is surjective,

$$\iota(U) = \iota\big(q(q^{-1}(U))\big) = q\big(\iota(q^{-1}(U))\big).$$

Because $q^{-1}(U)$, and hence $\iota(q^{-1}(U))$, is open in G, it follows that $\iota(U)$ is open since q is open. $\qquad\square$

We now explore how a topological group G that is *not* T_0 can be made so by quotienting out by a particular subgroup. To do this justice we first recall the T_0-*identification* of a topological space X: The ordering $x \le y$ defined by $x \in \overline{\{y\}}$ is called the *specialisation preorder*, and it is easy to show that $x \le y$ is equivalent to $\mathcal{N}_x \subseteq \mathcal{N}_y$. This order gives rise to an equivalence relation $\equiv$, and we say that two points $x, y \in X$ are *topologically indistinguishable* if $x \equiv y$.

It is clear that X is T_0 if and only if the relation $\equiv$ is trivial, and it is not difficult to show that the quotient space $X/\equiv$, called the T_0-*identification* or the *Kolmogorov quotient* of X, is indeed T_0. In fact, $\equiv$ is the most conservative equivalence relation $\sim$ such that $X/\sim$ is T_0, though we shall not need this fact. [reference to separation axiom notes]

When we apply this construction to the theory of topological groups, we see that the T_0-identification can be understood in terms of quotient groups:

PROPOSITION 3.3: T_0-*identification of groups*

Let G be a topological group. The subgroup $\overline{\{e\}}$ of G is normal. Furthermore, the $\equiv$-equivalence classes are precisely the left cosets of $\overline{\{e\}}$. It follows that the quotient group $G/\overline{\{e\}}$ is just the T_0-identification $G/\equiv$ of G.

PROOF. The subgroup $\overline{\{e\}}$ is the smallest closed subgroup of G, hence it is normal since otherwise intersecting it with one of its conjugates yields a strictly smaller closed subgroup.

It then suffices to show that, for $x, y \in G$, $x \equiv y$ if and only if $x\overline{\{e\}} = y\overline{\{e\}}$. But this is clear since e.g. $x\overline{\{e\}} = \overline{\{x\}}$ by continuity of multiplication on G. $\quad\square$

It is results like the above that lead to the common assumption that topological groups are T_1, since if it is not then we just quotient out by $\overline{\{e\}}$. One might justify this by arguing as follows: If a topology on a set X is to respect the set structure, then there must be a difference between two distinct points that is captured by the topology. But this difference must lie in which neighbourhoods each points has; the topology simply carries no further information than this. So if $x \neq y$, then it must be the case that $x \not\equiv y$, i.e. that X is T_0.

On the other hand, a *group* structure on a set G certainly respects the set structure, in that different elements may give rise to different actions: just let G act on itself by multiplication.

Thus if the topology and group structure on a topological group are to be truly compatible, then the topology must respect the underlying set structure, i.e. be T_0. In the sequel we shall, however, resist assuming that topological groups are T_0 as far as possible.

There is more to say about the T_0-identification that is not pertinent to this discussion. We refer to Willard (1970, Exercise 13C) and Pirttimäki (2019). [And my own notes]

4 • Topological vector spaces

The purpose of these notes is not to give a complete discussion of either topological groups or vector spaces. In this section we will thus focus on the very basic aspect of topological vector spaces and refer the reader to the literature for further developments.

[Definition of TVS.]

4.1. Quotient spaces

We begin by reviewing the basic construction and properties of quotients of general *algebraic* vector spaces. So let X be a vector space over a field k, and let S be a subspace of X. Then X is in particular an abelian group under addition, and S is a subgroup. We may then construct the quotient *group X/S* whose elements are the cosets $x + S$ for $x \in X$.

The equivalence relation defining the group is also compatible with the scalar multiplication on X: For if $x, y \in X$ and $x - y \in S$, then for $r \in k$ we have

$$rx - ry = r(x - y) \in S,$$

since S is a subspace. Thus $rx + S = ry + S$, and we may define multiplication by $r \in k$ in X/S by $r(x + S) = rx + S$. It is routine to check that this makes X/S into a k-vector space, called the *quotient space* of X by S. It is easy to see that the quotient map $\pi \colon X \to X/S$ is linear, and that $\ker \pi = S$.

We now specialise to topological vector spaces. Let X be a topological vector space over the field $\mathbb{F}$. If S is a subspace of X, then recall that we give X/S the quotient topology coinduced by the quotient map π, and that π is open.

THEOREM 4.1: *Topological quotient spaces*

Let X be a topological vector space, and let S be a subspace. Then X/S is a topological vector space.

PROOF. By Theorem 3.2 it suffices to show that scalar multiplication is continuous. Let $x \in X$ and $\alpha \in \mathbb{F}$, and let U be a neighbourhood of $\alpha x + S$ in X/S. Continuity of scalar multiplication in S at (α, x) implies the existence of neighbourhoods $V \subseteq \mathbb{F}$ and $W \subseteq X$ of α and x respectively, such that $VW \subseteq \pi^{-1}(U)$. Since π is surjective and linear, it follows that $V\pi(W) \subseteq U$, and since π is also open $\pi(W)$ is a neighbourhood of $x + S$. Hence scalar multiplication in X/S is continuous. $\qquad\square$

Notice that the second part of the proof of Theorem 3.2 is redundant for topological vector spaces. One may thus omit this part if one is only interested in vector spaces and not in groups in their own right.

Next we consider metrics: A pseudo-metric ρ on a vector space X is called *(translation) invariant* if $\rho(x + z, y + z) = \rho(x, y)$ for all $x, y, z \in X$. It is obvious that the pseudo-metric induced by a semi-norm is invariant. It is a theorem that for any metrisable (indeed any first-countable) topological vector space there exists an invariant metric compatible with the topology, see e.g. Rudin (1991, Theorem 1.24).

Assume that X is metrisable by an invariant metric ρ. Define $\tilde{\rho} \colon X/S \times X/S \to [0, \infty)$ by

$$\tilde{\rho}(x + S, y + S) = \inf_{s \in S} \rho(x - y, s).$$

PROPOSITION 4.2: *Metric on quotient space*

The map $\tilde{\rho}$ is an invariant pseudo-metric on X/S compatible with the quotient topology. Furthermore, $\tilde{\rho}$ is a metric if and only if S is closed.

PROOF. It is clear that $\tilde{\rho}$ is positive and symmetric. We prove the triangle inequality: Let $x, y, z \in X$. For any $s \in S$ we find that

$$\rho(x - z, 2s) \le \rho(x - z, s + y - z) + \rho(s + y - z, 2s)$$
$$= \rho(x - y, s) + \rho(x - z, s).$$

Notice that as s runs through S, so does $2s$ and vice-versa. By taking infima over $s \in S$ on both sides we thus get

$$\tilde{\rho}(x, z) \le \tilde{\rho}(x, y) + \tilde{\rho}(y, z)$$

as desired. It is clear from the definition that $\tilde{\rho}$ is invariant.

[Quotient topology – Rudin uses local bases, perhaps add something about those for groups?]

Finally, recall that a pseudo-metric is a metric if and only if the metric topology is T_0. By Proposition 2.3 this is equivalent to it being T_1, and Proposition 3.1(ii) implies that this is the case if and only if S is closed. $\quad\square$

We next consider norms: Let $\|\cdot\|$ be a semi-norm on X. We define a map $\|\cdot\|_S : X/S \to [0, \infty)$ by

$$\|x + S\|_S = \inf_{s \in S}\|x - s\|.$$

PROPOSITION 4.3: *Quotient norm*

The map $\|\cdot\|_S$ is a semi-norm on X/S. If ρ is the pseudo-metric induced by the semi-norm on X and $\tilde{\rho}$ is the corresponding pseudo-metric on X/S, then

$$\|x + S\|_S = \tilde{\rho}(x + S, 0).$$

In particular, $\tilde{\rho}$ is the pseudo-metric induced by $\|\cdot\|_S$, and $\|\cdot\|_S$ is compatible with the quotient topology. Furthermore, it is a norm if and only if S is closed.

PROOF. It is clear that $\|\cdot\|_S$ is positive and absolutely homogeneous. We next prove the triangle inequality:

Let ρ be the metric induced by the semi-norm on X, i.e. $\rho(y, z) = \|y - z\|$ for $y, z \in X$. Notice that since ρ is invariant, for $x \in X$,

$$\|x + S\|_S = \inf_{s \in S}\|x - s\| = \inf_{s \in S} \rho(x - s, 0)$$
$$= \inf_{s \in S} \rho(x - 0, s) = \tilde{\rho}(x + S, 0).$$

If also $y \in X$, then it follows that

$$\|(x + S) + (y + S)\|_S = \tilde{\rho}((x + S) + (y + S), 0) = \tilde{\rho}(x + S, -y + S)$$
$$\leq \tilde{\rho}(x + S, 0) + \tilde{\rho}(0, -y + S) = \|x + S\|_S + \|y + S\|_S,$$

so $\|\cdot\|_S$ is a semi-norm. The rest of the claims follow by Proposition 4.2. $\square$

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