Topological Groups

Danny Nygård Hansen

29th October 2021

1 • Definitions and basic properties

Definition 1.1: *Topological groups*

A *topological group* is a triple $(G,\mu,\mathcal{T})$ such that (G,μ) is a group, $(G,\mathcal{T})$ is a topological space, and both the multiplication $\mu\colon G\times G\to G$ and the inversion map $\iota\colon G\to G$ given by $g\mapsto g^{-1}$ are continuous.

In the sequel we will omit mentioning the multiplication and topology of a topological group and simply write G. It is custom to assume that a topological group be T_1 (or Hausdorff, which is actually implied by it being T_1, as we will see), but we omit this assumption since our focus is precisely on the basic topological properties of topological groups.

If G is a topological group and $g\in G$, then we define maps $l_g, r_g\colon G\to G$ by $l_g(h)=gh$ and $r_g(h)=hg$; that is, l_g and r_g are left- and right-multiplication by g, respectively. Both of these are homeomorphisms, with $l_g^{-1}=l_{g^{-1}}$ and $r_g^{-1}=r_{g^{-1}}$. Hence if $a,b\in G$, then there is a homeomorphism on G that takes a to b, namely $l_{ba^{-1}}$. Thus a topological group is *homogeneous* as a topological space.

We begin by collecting some basic properties of topological groups:

Proposition 1.2

Let G be a topological group.

(i) *If $A,B\subseteq G$, then $\overline{A}\,\overline{B}\subseteq\overline{AB}$.*

(ii) *If H is a subgroup of G, then so is $\overline{H}$.*

(iii) *Every open subgroup of G is closed.*

(iv) *If $A,U\subseteq G$ with U open, then AU and UA are open.*

PROOF. We give two proofs of (i). Since the multiplication $\mu\colon G \times G \to G$ is continuous we get

$$\mu(\overline{A}, \overline{B}) = \mu(\overline{A} \times \overline{B}) = \mu(\overline{A \times B}) \subseteq \overline{\mu(A \times B)} = \overline{\mu(A, B)} = \overline{AB}.$$

Alternatively, pick $a \in \overline{A}$ and $b \in \overline{B}$. If U be a neighbourhood of ab, then by continuity of multiplication there are neighbourhoods V_1 and V_2 of a and b respectively such that $V_1 V_2 \subseteq U$. Picking $x \in A \cap V_1$ and $y \in B \cap V_2$ we find that $xy \in AB \cap U$, so $ab \in \overline{AB}$.

Let H be a subgroup of G. By (i) we get

$$\overline{H}\,\overline{H} \subseteq \overline{HH} = \overline{H},$$

so $\overline{H}$ is closed under multiplication. It is also closed under taking inverses, since the inverse map ι is a homeomorphism. This proves (ii). Alternatively, this claim is an easy consequence of basic properties of nets.

Now let H be an open subgroup. Since G is the disjoint union of the cosets of H, we have

$$G \setminus H = \bigcup_{g \in G \setminus H} gH.$$

Since the cosets gH are open, $G \setminus H$ is also open. This proves (iii).

To prove (iv), notice that e.g.

$$AU = \bigcup_{g \in A} gU,$$

so AU is a union of open sets. □

We next remark that the assumption that G be T_1 can be weakened:

PROPOSITION 1.3

Let G be a topological group. If G is T_0, then it is in fact T_1.

PROOF. Assume that G is T_0. By homogeneity it suffices to show that the singleton $\{e\}$ containing the identity $e \in G$ is closed. We show that $G \setminus \{e\}$ is a neighbourhood[1] of all $g \neq e$ in G. Since G is T_0, either $G \setminus \{e\}$ is a neighbourhood of g, or $G \setminus \{g\}$ is a neighbourhood of e. In the first case we are done, so assume the latter. The homeomorphism $l_{g^{-1}}$ maps $G \setminus \{g\}$ to $G \setminus \{e\}$, so the latter set is a neighbourhood of g^{-1}. But the inversion map ι is a homeomorphism that maps $G \setminus \{e\}$ to itself, so this is also a neighbourhood of g. □

[1] If X is a topological space and $x \in X$, then we say that a set $A \subseteq X$ is a neighbourhood if it has an *open subset* U containing x, i.e. $x \in U \subseteq A$.

Recall that a topological space X is called *regular* if it is possible to separate any point from any closed set by disjoint open sets. Our next order of business is to show that any topological group is regular. Before proving this we need some terminology and a lemma:

If G is a topological group and $A \subseteq G$, then we write

$$A^{-1} = \iota(A) = \{a^{-1} \mid a \in A\}.$$

A subset A is called *symmetric* if $A = A^{-1}$. Notice that since ι is a homeomorphism, A is open (closed) if and only if A^{-1} is open (closed).

LEMMA 1.4

Let G be a topological group. If U is a neighbourhood of the identity e, then there is a symmetric neighbourhood V of e such that $VV^{-1} \subseteq U$.

PROOF. Since multiplication is continuous and $ee = e$, there are neighbourhoods V_1 and V_2 of e such that $V_1 V_2 \subseteq U$. Letting

$$V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1},$$

then V has the desired properties. □

PROPOSITION 1.5: *Regularity of topological groups*

Every topological group is regular. In particular, every T_0 topological group is T_3.

PROOF. Let G be a topological group. By homogeneity it is enough to show that if U is a neighbourhood of e, then e has a closed neighbourhood that lies in U.

Let V be a symmetric neighbourhood of e such that $VV^{-1} \subseteq U$. We claim that $\overline{V} \subseteq U$. Let $g \in \overline{V}$. Then every neighbourhood of g intersects V, so in particular $gV \cap V \neq \emptyset$. Choose points $v, w \in V$ such that $gv = w$. It follows that

$$g = wv^{-1} \in VV^{-1} \subseteq U,$$

so $\overline{V} \subseteq U$. □

2 • Coset spaces and quotient groups

2.1. *General properties of coset spaces*

If H is a subgroup of a topological group G, we denote by G/H the set of left cosets of H. Let $q\colon G \to G/H$ be the quotient map and give G/H the quotient topology. We call G/H a *coset space* of G.

Notice that q is in fact open: If $U \subseteq G$ is open, then $q^{-1}(q(U)) = UH$ is also open, so $q(U)$ is open since G/H has the quotient topology coinduced by q.

Proposition 2.1

Let G be a topological group and H a subgroup.

(i) *The coset space G/H is Hausdorff if and only if H is closed.*

(ii) *If G is locally compact*[2]*, then so is G/H.*

Notice that (i) does not assume any separation properties of G. As far as I know, a general coset space G/H with H closed is not necessarily T_3, but of course this is the case of the coset space is in fact a topological quotient group.

Proof. It is easy to show that G/H is T_1 if and only if H is closed.[3] Note that fibres of q are cosets of H, and a coset $gH = l_g(H)$ is closed if and only if H is. But since G/H carries the quotient topology, gH is closed in G if and only if $\{gH\}$ is closed in G/H.

Now assuming that H is closed, we show that G/H is Hausdorff. Let xH and yH be distinct (and hence disjoint) cosets. Then xHy^{-1} is a closed set not containing e, so Lemma 1.4 implies the existence of a symmetric neighbourhood U of e such that $UU \cap xHy^{-1} = \emptyset$. It follows that

$$e \notin UxHy^{-1}U = UxH(Uy)^{-1} = (UxH)(UyH)^{-1},$$

where we use that $U = U^{-1}$ and $H = HH$. That e does not lie in the left-most set is easily proven e.g. by contraposition. It follows that UxH and UyH are disjoint, and since q is open this implies that $q(Ux)$ and $q(Uy)$ are disjoint neighbourhoods of xH and yH in G/H. This proves (i).

To prove (ii), notice that if K is a compact neighbourhood of e in G, then $q(Kx)$ is a compact neighbourhood of xH in G/H. □

This proposition also furnishes a different proof that T_1 implies Hausdorff for topological groups: Simply let $H = \{e\}$, in which case $G \cong G/H$.

2.2. Topological quotient groups

If a subgroup N of a topological group G is normal, we expect that the usual group structure on the (algebraic) quotient group G/N is compatible with the quotient topology. This is indeed the case:

[2] We say that a topological space is locally compact if every point has a compact neighbourhood. There are many non-equivalent definitions of local compactness and this is the least restrictive one. If H is closed then G/H is Hausdorff, so all the usual definitions of local compactness are equivalent for G/H.

[3] If G/H is a topological quotient group, then this is sufficient to show that G/H is Hausdorff, since it is then even T_3 by Proposition 1.5.

THEOREM 2.2: *Topological quotient groups*

If N is a normal subgroup of a topological group G, then G/N is a topological group.

PROOF. If $x,y \in G$ and U is a neighbourhood of $(xN)(yN) = xyN$ in G/N, then continuity of multiplication in G at (x,y) implies the existence of neighbourhoods V and W of x and y respectively, such that $VW \subseteq q^{-1}(U)$. Since q is surjective it follows that $q(V)q(W) \subseteq U$, and because q is also open $q(V)$ and $q(W)$ are neighbourhoods of xN and yH. Hence multiplication is continuous.

Since the inversion map ι on G/N is bijective, it suffices to show that it is open. Let $U \subseteq G/N$ be open and notice that, since q is surjective,

$$\iota(U) = \iota\Big(q(q^{-1}(U))\Big) = q\Big(\iota(q^{-1}(U))\Big).$$

Because $q^{-1}(U)$, and hence $\iota(q^{-1}(U))$, is open in G, it follows that $\iota(U)$ is open since q is open. $\square$

We now explore how a topological group G that is *not* T_0 can be made so by quotienting out by a particular subgroup. First we recall the T_0-identification of a topological space X: Define an equivalence relation $\sim$ on X by letting $x \sim y$ if $\overline{\{x\}} = \overline{\{y\}}$.

LEMMA 2.3: *T_0-identification*

Let $\sim$ be the T_0-identification of a topological space X. Then the quotient space $X/\!\sim$ is T_0.

PROOF. First notice that $q(x) \subseteq \overline{\{x\}}$ for $x \in X$, since if $x \sim x'$ then $x' \in \overline{\{x'\}} = \overline{\{x\}}$. It follows that

$$\overline{\{x\}} = \bigcup_{x' \in \overline{\{x\}}} q(x'),$$

so $\overline{\{x\}}$ is saturated and so is its complement.

Now assume that $q(x) \neq q(y)$. Without loss of generality we may assume that $x \notin \overline{\{y\}}$, from which it follows that $q(x) \in q(\overline{\{y\}}^c)$. We furthermore have $q(y) \notin q(\overline{\{y\}}^c)$, since $\overline{\{y\}}^c$ is a union of equivalence classes not containing y, and $q(\overline{\{y\}}^c)$ is the collection of these equivalence classes. Since $\overline{\{y\}}^c$ is an open saturated set, $q(\overline{\{y\}}^c)$ is a neighbourhood of $q(x)$ not containing $q(y)$. $\square$

Notice that a topological space is T_0 if and only if $\overline{\{x\}} \neq \overline{\{y\}}$ whenever $x \neq y$: This follows since, say, x has a neighbourhood that doesn't contain y, so the closure of $\{y\}$ does not contain x. Hence if X is already T_0, the identification leaves X unchanged. [To do: show that the T_0-identification is the smallest equivalence relation on X that makes it T_0.]

PROPOSITION 2.4

Let G be a topological group. The subgroup $\overline{\{e\}}$ of G is normal. Furthermore, if $\sim$ denotes the T_0-identification of G, then the $\sim$-equivalence classes are the cosets of $\overline{\{e\}}$.

It follows that the quotient group $G/\overline{\{e\}}$ is precisely the T_0-identification $G/\sim$ of G.

PROOF. The subgroup $\overline{\{e\}}$ is the smallest closed subgroup of G, hence it is normal since otherwise intersecting it with one of its conjugates yields a strictly smaller closed subgroup.

It then suffices to show that, for $x, y \in G$, $x \sim y$ if and only if $x\overline{\{e\}} = y\overline{\{e\}}$. But this is clear since e.g. $x\overline{\{e\}} = \overline{\{x\}}$ by continuity of the multiplication on G. □

It is results like the above that lead to the general assumption that topological groups are T_1, since if it is not then we just quotient out by $\overline{\{e\}}$.

References

Aluffi, Paolo (2009). *Algebra: Chapter 0*. American Mathematical Society. 713 pp. ISBN: 978-0-8216-4781-7.

Folland, Gerald B. (2015). *A Course in Abstract Harmonic Analysis*. CRC Press. 305 pp. ISBN: 978-1-4987-2715-0.

McCarty, George (1988). *Topology: An Introduction with Applications to Topological Groups*. Dover. 270 pp. ISBN: 0-486-65633-0.

Rudin, Walter (1991). *Functional Analysis*. 2nd ed. McGraw-Hill, Inc. 424 pp. ISBN: 0-07-054236-8.