Conceptual foundations of topology

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1.1. Motivation and definitions

Let X be a set. Let $\prec$ be a relation from X to $\wp(X)$. If $x \prec U$, then we say that x is an *inner point* of U, and that U is a *neighbourhood* of x. One intuition for this relation is that $x \prec U$ just when x is 'completely contained' in U, or is 'completely enclosed' in U, with 'room to spare'. For this to be the case we obviously require that

(1) if $x \prec U$, then $x \in U$.

Clearly the empty set cannot be a neighbourhood of any point. On the other hand, since X is the entire universe under consideration we surely want X to be a neighbourhood of x. Hence we have the postulates

(2) $x \nprec \emptyset$, and

(3) $x \prec X$.

In particular, every point has at least one neighbourhood (namely X itself). Furthermore, under the above interpretation, if $V \subseteq X$ is a set with $U \subseteq V$ then V must also be a neighbourhood of x. Hence we have the postulate:

(4) If $x \prec U$ and $U \subseteq V$, then $x \prec V$.

Next assume that both U and V are neighbourhoods of x. That is, x is 'completely contained' in both U and V, so it seems natural enough that x should be 'completely contained' in the intersection $U \cap V$. That is,

(5) If $x \prec U$ and $x \prec V$, then $x \prec U \cap V$.

Now let $\mathcal{N}_x$ denote the collection of neighbourhoods of x. Then the above discussion shows that $\mathcal{N}_x$ is a *filter* on X, i.e. a proper, nonempty subset of $\wp(X)$ that is upward closed and downward directed. Hence we call $\mathcal{N}_x$ the *neighbourhood filter* of x. That is, a neighbourhood filter for a point $x \in X$ is a filter $\mathcal{N}_x$ on X such that all sets in $\mathcal{N}_x$ contain x. A collection $(\mathcal{N}_x)_{x\in X}$ of neighbourhood filters, one for each point in X, is then called a *neighbourhood system* on X.

We summarise the discussion in the following definition:

Definition 1.1: *Pretopological space*

A *pretopology* on a set X is a relation $\prec$ from X to $\wp(X)$ with the following properties:

(i) Centred: For all $x \in X$ and $N \subseteq X$, if $x \prec N$ then $x \in N$.

(ii) Left-total: For all $x \in X$ there exists an $N \subseteq X$ such that $x \prec N$.

(iii) Downward directed: For all $x \in X$ and $N, M \subseteq X$, if $x \prec N$ and $x \prec M$ then $x \prec N \cap M$.

(iv) Upward closed: For all $x \in X$ and $N, M \subseteq X$, if $x \prec N$ and $N \subseteq M$ then $x \prec M$.

The pair $(X, \prec)$ is called a *pretopological space*.

If $x \prec N$ then N is said to be a *neighbourhood* of x, and x an *inner point* of N. The set

$$\mathcal{N}_x = \{N \subseteq X \mid x \prec N\}$$

is called the *neighbourhood filter* of x, and the collection $(\mathcal{N}_x)_{x\in X}$ is called the *neighbourhood system* on X.

1.2. Closures and interiors

Let X be a pretopological space and consider distinct points $x, y \in X$. If there is an $N \in \mathcal{N}_x$ that does not contain y, then in some sense x is 'separated' from y, since it is 'completely enclosed' in something that is disjoint from y. In this case we say that x is *separated from* y. Note however that even if x is separated from y, y might not be separated from x.

Next consider a subset $A \subseteq X$. If $a, a' \in A$ and $x \in X \setminus A$, then it may happen that x is separated from a but not from a'. It might also happen that x is separated from every individual $a \in A$, so there is some $N_a \in \mathcal{N}_x$ with $a \notin N_a$, but no matter which N_a we pick there are still points $a' \in A$ that lie in N_a. However, if there is a $N \in \mathcal{N}_x$ that is completely disjoint from A, then we say that x is *separated from* A. On the other hand, a point that is *not* separated from A is in some sense 'close' to A. Hence we have the following:

DEFINITION 1.2: *Closure and preclosure*

Let X be a pretopological space, and let $A \subseteq X$.

(i) The *preclosure* of A is the set

$$\mathrm{pCl}(A) = \{x \in X \mid \forall N \in \mathcal{N}_x \colon A \cap N \neq \emptyset\},$$

i.e. the set of points that are not separated from A.

(ii) If $A = \mathrm{pCl}(A)$, then A is called *closed*.

(iii) The *closure* $\mathrm{Cl}(A)$ of A is the intersection of all closed sets that contain A.

PROPOSITION 1.3: *Properties of preclosures*

Let X be a pretopological space, and let $A, B \subseteq X$. The preclosure operator $\mathrm{pCl}(\cdot)$ *has the following properties:*

(i) *Increasing: $A \subseteq B$ implies* $\mathrm{pCl}(A) \subseteq \mathrm{pCl}(B)$.

(ii) *Extensive:* $A \subseteq \mathrm{pCl}(A)$.

(iii) *Preserves nullary unions:* $\mathrm{pCl}(\emptyset) = \emptyset$

(iv) *Preserves binary unions:* $\mathrm{pCl}(A \cup B) = \mathrm{pCl}(A) \cup \mathrm{pCl}(B)$.

We also have the following:

(v) *A set $N \subseteq X$ is a neighbourhood of $x \in X$ if and only if $x \notin$* $\mathrm{pCl}(X \setminus N)$.

(vi) *If $A \subseteq F$ and F is closed, then* $\mathrm{pCl}(A) \subseteq F$. *In particular we have the inclusions*

$$A \subseteq \mathrm{pCl}(A) \subseteq \mathrm{pCl}(\mathrm{pCl}(A)) \subseteq \cdots \subseteq \mathrm{Cl}(A).$$

(vii) *If A is closed, then* $\mathrm{pCl}(A) = \mathrm{Cl}(A)$.

PROOF. *Proof of (i)*: This is clear, since if a neighbourhood of $x \in X$ intersects A, then it also intersects B.

Proof of (ii): This is obvious, since if $x \in A$ then any neighbourhood of x intersects A.

Proof of (iii): This is obvious, since every neighbourhood has empty intersection with the empty set.

Proof of (iv): By (i) we have $\mathrm{pCl}(A) \subseteq \mathrm{pCl}(A \cup B)$ and $\mathrm{pCl}(B) \subseteq \mathrm{pCl}(A \cup B)$, so $\mathrm{pCl}(A) \cup \mathrm{pCl}(B) \subseteq \mathrm{pCl}(A \cup B)$. Conversely let $x \notin \mathrm{pCl}(A) \cup \mathrm{pCl}(B)$, so x is both separated from A and B. Hence there exist $N_1, N_2 \in \mathcal{N}_x$ such that $A \cap N_1 = B \cap N_2 = \emptyset$. But then $N_1 \cap N_2$ is a neighbourhood of x separating x from $A \cup B$. Hence $x \notin \mathrm{pCl}(A \cup B)$.

Proof of (v): If N is a neighbourhood of x, then x has a neighbourhood disjoint from $X \setminus N$. Hence $x \notin \mathrm{pCl}(X \setminus N)$. Conversely, if $x \notin \mathrm{pCl}(X \setminus N)$ then x has a neighbourhood M disjoint from $X \setminus N$. But then we must have $M \subseteq N$, so N is also a neighbourhood of x.

Alternatively, this will follow by duality from Proposition 1.10(v).

Proof of (vi): Let $F \subseteq X$ be a closed set containing A. Then since $\mathrm{pCl}(F) = F$ we have $\mathrm{pCl}(A) \subseteq F$ by (i). Repeatedly taking the preclosure on both sides we obtain

$$A \subseteq \mathrm{pCl}(A) \subseteq \mathrm{pCl}(\mathrm{pCl}(A)) \subseteq \cdots \subseteq F.$$

Since $\mathrm{Cl}(A)$ is intersection of all such F, the claim follows.

Proof of (vii): If A is closed, then A is one of the closed sets in the intersection defining $\mathrm{Cl}(A)$, so $\mathrm{Cl}(A) \subseteq A$. The claim then follows from (vi). □

REMARK 1.4. A map $\mathrm{pCl}(\cdot)\colon \wp(X) \to \wp(X)$ satisfying properties (ii), (iii) and (iv) is called a *Čech closure operator* or a *preclosure operator* on X. It is easy to see that these properties imply the other properties by mimicking the proofs above (note that we used (i) in the proof of (iv)). In fact, such an operator can also be used to define a pretopological structure on a set, by declaring that N is a neighbourhood of x if $x \notin \mathrm{pCl}(X \setminus N)$, in accordance with property (v) above. ⌟

Let us try to understand Proposition 1.3 in terms of 'closeness':

Part (i): Closeness is clearly increasing, in the sense that if $A \subseteq B$ and x is close to A, then x is also close to B.

Part (ii): Certainly every point is A is close to A.

Part (iii): Also, certainly nothing is close to the empty set.

Part (iv): If x is close to either A or B, then x is certainly close to $A \cup B$. The converse is less obvious: If x is close to $A \cup B$, is it really the case that x is close to either A or B? We might imagine a notion of closeness where closeness requires a certain point density in a particular area, and perhaps A and B have the required density when put together, but none of them are dense enough on their own.

However, there might be a sense in which x is not properly separated from either A or B. Indeed, for x to be separated from A, x must have a neighbourhood N disjoint from A. But that means that x is 'completely enclosed' in N, and that doesn't seem to be the case if A has some density in the relevant area.

Part (v): This is essentially just a restatement of the definition of preclosure. If x is completely contained in N, then it is not close to $X \setminus N$.

Part (vi): In general, the preclosure of a set is not closed. That is, the preclosure operator is not idempotent. In 'closeness' terms, this means that points that are not close to A might still be close to $\mathrm{pCl}(A)$. For example, say that an integer n is close to a set $A \subseteq \mathbb{Z}$ if the smallest difference between n and any $a \in A$ is at most 1. Then

$$\mathrm{pCl}(A) = \bigcup_{a \in A} \{a-1, a, a+1\},$$

which is clearly not idempotent, though it is a Čech closure operator.

If the sequence of inclusions in (vi) terminates, then we have found all the points that are close to A, and the resulting set is closed. A set is thus closed if it contains all points that are close to it. But for the preclosure on $\mathbb{Z}$ above, the only closed sets are $\emptyset$ and $\mathbb{Z}$ itself.

A second important operation is the following:

Definition 1.5: *Interior and preinterior*

Let $(X, \prec)$ be a pretopological space, and let $A \subseteq X$.

(i) The *preinterior* of A is the set

$$\mathrm{pInt}(A) = \{x \in A \mid x \prec A\}$$

of all inner points of A.

(ii) If $A = \mathrm{pInt}(A)$, then A is called *open*.

(iii) The *interior* $\mathrm{Int}(A)$ of A is the union of all open sets contained in A.

The collection of all open subsets of X is called the *topology* on X.

Remark 1.6. In a topological space, if N is a neighbourhood of a point x, then there exists an open set U such that $x \in U \subseteq N$. We claim that this is not necessarily the case in a pretopological space: Indeed, we show that the interior of a neighbourhood can be empty.

Consider the pretopological space whose underlying set is $\mathbb{R}$, and where $\mathcal{N}_0 = \{N \subseteq \mathbb{R} \mid (-1,1) \subseteq N\}$ and $\mathcal{N}_x = \{\mathbb{R}\}$ for $x \neq 0$. Then $(-1,1)$ is a neighbourhood of 0, but if $A \subseteq (-1,1)$ is to be open, then we must have $\mathrm{pInt}(A) = A$, and this is only possible when $A = \emptyset$.

The trouble is clearly that $(-1,1)$ is a neighbourhood of 0 but not of any of the points that are close to 0, which it would be in a topological space. ⌟

PROPOSITION 1.7

Let $(X,\prec)$ be a pretopological space, and let $\mathcal{T}$ be its topology.

(i) *The empty set and X belong to $\mathcal{T}$.*

(ii) *$\mathcal{T}$ is closed under arbitrary unions.*

(iii) *$\mathcal{T}$ is closed under finite intersections.*

That is, $(X,\mathcal{T})$ is a topological space.

PROOF. *Proof of (i)*: This is obvious.

Proof of (ii): Let $\mathcal{U}$ be a collection of sets from $\mathcal{T}$, and let $\mathbb{U} = \bigcup_{U\in\mathcal{U}} U$. If $x \in \mathbb{U}$, then there is a $U \in \mathcal{U}$ such that $x \in U$. But since U is open we have $x \prec U \subseteq \mathbb{U}$, so $x \prec \mathbb{U}$.

Proof of (iii): By (i) it suffices to consider binary intersections, so let $U, V \in \mathcal{T}$. If $x \in U \cap V$, then $x \prec U$ and $x \prec V$. But then it follows that $x \prec U \cap V$ since the relation $\prec$ is downward directed. □

Closures and interiors are dual to each other:

PROPOSITION 1.8: *Closure–interior duality*

Let $(X,\prec)$ be a pretopological space, and let $A \subseteq X$. Then

$$X \setminus \mathrm{pCl}(A) = \mathrm{pInt}(X \setminus A) \quad \textit{and} \quad \mathrm{pCl}(X \setminus A) = X \setminus \mathrm{pInt}(A).$$

In particular, a set is open if and only if its complement is closed, and

$$X \setminus \mathrm{Cl}(A) = \mathrm{Int}(X \setminus A) \quad \textit{and} \quad \mathrm{Cl}(X \setminus A) = X \setminus \mathrm{Int}(A).$$

PROOF. The first claim follows from the following series of equivalences:

$$\begin{aligned} x \notin \mathrm{pCl}(A) &\Leftrightarrow \exists N \in \mathcal{N}_x \colon A \cap N = \emptyset \\ &\Leftrightarrow \exists N \in \mathcal{N}_x \colon N \subseteq X \setminus A \\ &\Leftrightarrow X \setminus A \in \mathcal{N}_x \\ &\Leftrightarrow x \in \mathrm{pInt}(X \setminus A). \end{aligned}$$

The second part follows by replacing A by $X \setminus A$ and taking complements on both sides.

If $F \subseteq X$ is closed, then

$$X \setminus F = X \setminus \mathrm{pCl}(F) = \mathrm{pInt}(X \setminus F) \subseteq X \setminus F,$$

so $X \setminus F$ is open. Conversely, if $U \subseteq X$ is open then

$$X \setminus U = X \setminus \mathrm{pInt}(U) = \mathrm{pCl}(X \setminus U) \supseteq X \setminus U,$$

which implies that $\mathrm{pCl}(X \setminus U) = X \setminus U$, so $X \setminus U$ is closed.

Finally, let $\mathcal{F}$ be the collection of closed sets containing A, and let $\mathcal{U}$ be the collection of open sets contained in $X \setminus A$. Then the duality between open and closed sets implies that

$$X \setminus \mathrm{Cl}(A) = X \setminus \bigcap_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} X \setminus F = \bigcup_{U \in \mathcal{U}} U = \mathrm{Int}(X \setminus A),$$

as claimed. The second part of the last claim follows by taking complements. □

Corollary 1.9

If A is a subset of a pretopological space, then $\mathrm{Int}(A)$ *is open and* $\mathrm{Cl}(A)$ *is closed.*

Proof. The first claim is obvious from Proposition 1.7, and the second follows by duality. □

We may use this correspondence to obtain properties of preinteriors analogous to those of preclosures in Proposition 1.3:

Proposition 1.10: *Properties of preinteriors*

Let X be a pretopological space, and let $A, B \subseteq X$. The preinterior operator $\mathrm{pInt}(\cdot)$ *has the following properties:*

(i) *Increasing: $A \subseteq B$ implies* $\mathrm{pInt}(A) \subseteq \mathrm{pInt}(B)$.

(ii) *Anti-extensive:* $\mathrm{pInt}(A) \subseteq A$.

(iii) *Preserves nullary intersections:* $\mathrm{pInt}(X) = X$

(iv) *Preserves binary unions:* $\mathrm{pInt}(A \cap B) = \mathrm{pInt}(A) \cap \mathrm{pInt}(B)$.

We also have the following:

(v) *A set $N \subseteq X$ is a neighbourhood of $x \in X$ if and only if $x \in$* $\mathrm{pInt}(N)$.

(vi) *If $U \subseteq A$ and U is open, then $U \subseteq$* $\mathrm{pInt}(A)$. *In particular we have the inclusions*

$$\mathrm{Int}(A) \subseteq \cdots \subseteq \mathrm{pInt}(\mathrm{pInt}(A)) \subseteq \mathrm{pInt}(A) \subseteq A$$

(vii) *If A is open, then* $\mathrm{pInt}(A) = \mathrm{Int}(A)$.

PROOF. Every point follows easily from Proposition 1.3 by duality. However, note that (v) is simply a restatement of the definition of preinterior. □

1.3. Convergence and continuity

We begin by considering continuity. Intuitively, a function $f : X \to Y$ is continuous at a point $x \in X$ if points in X that are close to x are mapped to points in Y that are close to $f(x)$. Usually this is formalised as follows:

DEFINITION 1.11: *Continuity*

Let X and Y be pretopological spaces, and let $x \in X$. A map $f : X \to Y$ is *continuous at* x if, for all $A \subseteq X$, $x \in \mathrm{pCl}(A)$ implies that $f(x) \in \mathrm{pCl}(f(A))$, i.e.

$$\forall A \subseteq X : x \in \mathrm{pCl}(A) \Rightarrow f(x) \in \mathrm{pCl}(f(A)).$$

If f is continuous at x for all $x \in X$, then we say that f is *continuous*.

That is, we require that if A is a set that x is close to, then $f(A)$ is a set that $f(x)$ is close to. But this seems like the opposite of what we were saying before, since there is a difference between points close to x and points that x is close to. This is obvious in our formalism, since the 'close to' relation is heterogeneous: Indeed, points can only be close to *sets*, not to points, nor can sets be close to sets. Hence if x is supposed to be close to something, or something is supposed to be close to x, then that something must be a set.

Indeed, it seems difficult to define a general *homogeneous* closeness relation using closures, since if $x \in A$ and $y \in B$, then x might lie in $\mathrm{pCl}(B)$ while y doesn't lie in $\mathrm{pCl}(A)$. Hence we use the heterogeneous relation above.

As in topological spaces we can characterise global continuity more simply:

PROPOSITION 1.12

Let $f : X \to Y$ be a map between pretopological spaces. Then f is continuous if and only if

$$f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$$

for all $A \subseteq X$.

PROOF. If f is continuous and $A \subseteq X$, then for all $x \in \mathrm{pCl}(A)$ we have $f(x) \in \mathrm{pCl}(f(A))$, which just says that $f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$. Conversely, if f satisfies the above condition and $x \in \mathrm{pCl}(A)$, then $f(x) \in f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$, so f is continuous at x. □

Next we wish to explicitly relate continuity to neighbourhoods, as is standardly done in topology. In a topological space it is well-known that continuity as defined in Definition 1.11 is equivalent to preimages of open sets being open. In pretopological spaces we have something similar, though we must focus on neighbourhoods:

Proposition 1.13

Let $f\colon X \to Y$ be a map between pretopological spaces, and let $x \in X$. Then f is continuous at x if and only if it has the following property: For all $N \in \mathcal{N}_{f(x)}$ there is an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$, i.e.

$$\forall N \in \mathcal{N}_{f(x)} \exists M \in \mathcal{N}_x \colon f(M) \subseteq N.$$

In other words, f is continuous at x if $f^{-1}(N) \in \mathcal{N}_x$ for all $N \in \mathcal{N}_{f(x)}$.

Proof. First assume that f has the above property. Let $A \subseteq X$ be such that $x \in \mathrm{pCl}(A)$, and let $N \in \mathcal{N}_{f(x)}$. Then there exists an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$. Since M is a neighbourhood of x we have $A \cap M \neq \emptyset$. It follows that

$$\emptyset \neq f(A \cap M) \subseteq f(A) \cap f(M) \subseteq f(A) \cap N,$$

so since N was arbitrary we have $f(x) \in \mathrm{pCl}(f(A))$ as desired.

Conversely, assume that f is continuous at x, and let $N \in \mathcal{N}_{f(x)}$. Assume towards a contradiction that $x \in \mathrm{pCl}(f^{-1}(Y \setminus N))$. Since $f(f^{-1}(Y \setminus N)) \subseteq Y \setminus N$ we then have, by continuity,

$$f(x) \in \mathrm{pCl}(Y \setminus N) = Y \setminus \mathrm{pInt}(N),$$

which is a contradiction since $f(x) \in \mathrm{pInt}(N)$. Hence x has a neighbourhood M disjoint from $f^{-1}(Y \setminus N)$, and this has the property that $f(M) \subseteq N$. □

Let us try to understand this result: If $f\colon X \to Y$ is continuous at $x \in X$, then Definition 1.11 is supposed to say that f takes points in X that are close to x and map them to points in Y that are close to $f(x)$. What Proposition 1.13 says is that if (and only if) f is continuous, then no matter which neighbourhood N of $f(x)$ we consider, we can always find a neighbourhood M of x such that f maps M to N. Since it is in some sense ‘harder’ to map a set into N the smaller N is – i.e., we might have make M smaller if N is made smaller – this seems to suggest another interpretation of neighbourhoods: Namely as formalising a sense of closeness. It is sometimes put as follows:

Every neighbourhood N of a point $x \in X$ defines a notion of closeness, let’s call it N-*closeness*. We say that a point $x' \in X$ is N-close to x if $x' \in N$. We may then restate Proposition 1.13 as follows:

> *A map $f\colon X \to Y$ is continuous at $x \in X$ if and only if it has the following property: For every $N \in \mathcal{N}_{f(x)}$ there is an $M \in \mathcal{N}_x$ such that if x' is M-close to x, then $f(x')$ is N-close to $f(x)$.*

This is then entirely analogous to the ε-δ-definition of continuity in metric spaces, where we might have said that x' is δ-close to x if $\rho(x,x') < \delta$, though of course N-closeness is not generally symmetric.

We might wonder whether the above gives us the right notion of closeness. What would happen, for instance, if we for x' to be N-close to x required that x' be an *inner point* of N? In a topological space we might think of this as more natural, especially if we are accustomed to thinking of neighbourhoods as by definition open. This would also not *prima facie* contradict our earlier interpretation of neighbourhoods as 'wholly containing' its inner points. I do not know of any conceptual problems with this approach, but while the two are practically equivalent in topological spaces, in a *pretopological* space neighbourhoods of a point do not behave nicely with respect to points that are 'close' to it, as we saw in Remark 1.6.

We now turn to convergence. Let $\mathcal{F}$ be a system of subsets of a pretopological space X. We think of the sets in $\mathcal{F}$ as describing the state of some system $\mathcal{S}$ in X, where $F \in \mathcal{F}$ means that $\mathcal{S}$ eventually lies in F. If $F \subseteq F'$ then $\mathcal{S}$ of course also eventually lies in F'. And for any two $F_1, F_2 \in \mathcal{F}$, if $\mathcal{S}$ eventually lies in F_1 and eventually lies in F_2, then it seems natural that $\mathcal{S}$ will eventually lie in $F_1 \cap F_2$. Finally, clearly the system cannot lie in the empty set, and on the other hand it trivially lies in X, so in total $\mathcal{F}$ is a filter on X.

Let us consider what it can mean for $\mathcal{S}$ to 'converge' to a point $x \in X$. Can we use preclosures to define such a notion? This seems rather difficult, since $\mathcal{S}$ is not a set so it does not have a preclosure. All we know is that $\mathcal{S}$ somehow 'eventually' lies in F for all $F \in \mathcal{F}$,[1] but the exact nature of $\mathcal{S}$ is unknown to us. Perhaps we can instead use the notion of closeness given to us by the neighbourhood filter $\mathcal{N}_x$ of x: If we trust this notion, then for $\mathcal{S}$ to converge to x we might require that, for all $N \in \mathcal{N}_x$, $\mathcal{S}$ is eventually N-close to x. That is, $\mathcal{S}$ should eventually lie in N. But since $\mathcal{F}$ is precisely the collection of sets in which $\mathcal{S}$ eventually lies, this just means that we should require that $\mathcal{N}_x \subseteq \mathcal{F}$. This naturally leads to the following definition:

Definition 1.14: *Convergence*

Let X be a pretopological space, and let $x \in X$. A filter $\mathcal{F}$ on X *converges* to x if $\mathcal{N}_x \subseteq \mathcal{F}$. In this case we write $\mathcal{F} \to x$.

[1] This kind of situation is often described as $\mathcal{S}$ lying in all $F \in \mathcal{F}$, but this wording makes it easy to conflate the current situation with $\mathcal{S}$ lying in all F *simultaneously*.

Equivalently, a net u in X converges to x if u is eventually in N for all $N \in \mathcal{N}_x$. We similarly write $u \to x$.

That is, convergence of filters and nets is entirely analogous to convergence in topological spaces.

Furthermore, as in topological spaces for a filter $\mathcal{F}$ to converge to x, it suffices that $\mathcal{B} \subseteq \mathcal{F}$ for a filter basis $\mathcal{B}$ for $\mathcal{N}_x$. However, contrary to in topological spaces the collection of open sets containing x does not necessarily constitute a filter basis for $\mathcal{N}_x$: Indeed, we saw in Remark 1.6 that a neighbourhood need not contain any open sets at all. In general pretopological spaces we thus cannot use open sets as a substitute for neighbourhoods in our theory of convergence. Below we will recover the well-known connection between convergence and continuity, showing that open sets also do not suffice to characterise continuity.

We want to understand the relationship between convergence and continuity. Recall that the *pushforward* of a filter $\mathcal{F}$ on a set X by a set function $f \colon X \to Y$ is the filter $f(\mathcal{F})$ on Y generated by the filter basis $\{f(F) \mid F \in \mathcal{F}\}$. Let us try to understand this construction conceptually: If $\mathcal{S}$ is some system in X, then we might imagine mapping that to Y by f to obtain a system $f(\mathcal{S})$. For instance, if $\mathcal{S}$ is a net u, then $f(\mathcal{S})$ is just the net $f \circ u$. If $\mathcal{S}$ eventually lies in some set $F \subseteq X$, then $f(\mathcal{S})$ will simultaneously lie in $f(F)$. If $\mathcal{F}$ is the filter of sets in which $\mathcal{S}$ eventually lie, then $f(\mathcal{S})$ will thus eventually lie in all sets in $\{f(F) \mid F \in \mathcal{F}\}$, and hence in all the sets contained in the filter that this collection (which is indeed a filter basis) generates.

Next suppose that $\mathcal{F}$ converges to some $x \in X$. What about the limit behaviour of the pushforward $f(\mathcal{F})$? Does this converge to $f(x)$? For it to do so we require that $\mathcal{N}_{f(x)} \subseteq f(\mathcal{F})$, i.e. $f(\mathcal{F})$ should eventually lie in N for all neighbourhoods N of $f(x)$. When are we sure of this if we already know that $\mathcal{F}$ eventually lies in M for all neighbourhoods M of x? If f is continuous then this should certainly happen: For then if $\mathcal{S}$ just gets close enough to x, then $f(\mathcal{S})$ can get as close as desired to $f(x)$. More precisely we have the following result:

PROPOSITION 1.15: *Continuity, filters and nets*

Let $f \colon X \to Y$ be a function between pretopological spaces, and let $x \in X$. Then f is continuous at x if and only if $\mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x)$, i.e. if $f(\mathcal{N}_x) \to f(x)$. In particular, the following are equivalent:

(i) *f is continuous at x.*

(ii) *For all filters $\mathcal{F}$ on X, $\mathcal{F} \to x$ implies that $f(\mathcal{F}) \to f(x)$.*

(iii) *For all nets u in X, $u \to x$ implies that $f(u) \to f(x)$.*

PROOF. If f is continuous at x and $N \in \mathcal{N}_{f(x)}$, then there exists an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$. But $f(M)$ lies in $f(\mathcal{N}_x)$, and hence so does N.

Conversely, if $\mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x)$ and $N \in \mathcal{N}_{f(x)}$, then by definition of $f(\mathcal{N}_x)$ there exists an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$, so f is continuous at x.

(i) ⇒ (ii): Recall that $\mathcal{F} \to x$ means that $\mathcal{N}_x \subseteq \mathcal{F}$. Hence $\mathcal{N}_{f(x)} \subseteq f(\mathcal{N}_x) \subseteq f(\mathcal{F})$, so $f(\mathcal{F}) \to f(x)$.

(ii) ⇒ (i): This follows by setting $\mathcal{F} = \mathcal{N}_x$.

(ii) ⇔ (iii): This follows immediately from the correspondence between nets and filters. □

Before we said that if f is continuous at x, then $\mathcal{F} \to x$ implies that $f(\mathcal{F}) \to f(x)$. Proposition 1.15 gives a converse: namely that if this implication holds for all filters $\mathcal{F}$, then f is continuous at x. In fact, the proposition shows that it suffices that the implication holds for the filter $\mathcal{N}_x$. Since $\mathcal{N}_x$ always converges to x, the relevant implication

$$\mathcal{N}_x \to x \quad \Rightarrow \quad f(\mathcal{N}_x) \to f(x)$$

holds just when $f(\mathcal{N}_x) \to f(x)$. How should be understand this convergence? Indeed, what does it even mean that $\mathcal{N}_x \to x$? What is the system in X whose convergence is described by $\mathcal{N}_x$?

2 • Topological spaces

Our discussion of pretopological spaces began with an analysis of *inner points* and *neighbourhoods*. We said that x is an inner point of N when x is 'completely contained' in N. The axioms, however, did not place any restrictions on the relationship between neighbourhoods of different points. However, it seems natural enough that there should be at least some such restrictions.

For example, we might find it problematic that we do not have a good notion of closeness between points. We have only said what it means for a point to be close to a set, not for two points to be close to each other. We might try to say that x is close to y if $x \in \mathrm{pCl}(\{y\})$. But then it seems quite natural that x should be close to y iff y is close to x. Sadly, we are not guaranteed that $y \in \mathrm{pCl}(\{x\})$ whenever $x \in \mathrm{pCl}(\{y\})$. This is one restriction we might make on our spaces.[2]

[2] In the usual topological context, this leads to the theory of R_0-*spaces*. A topological space X is an R_0-space if any two topologically distinguishable points in X are separated. That is, if $x, y \in X$ and e.g. x has a neighbourhood disjoint from y, then y has a neighbourhood disjoint from x. A space is T_1 if and only if it is both T_0 and R_0.

Another is to consider that if x is an inner point of N, then it seems natural that points that are 'close enough to x' should also be inner points of N. Otherwise, why would x be an *inner* point? We considered above the notion of N-closeness: A point y is N-close to x if $y \in N$. Perhaps we should require that, if x is an inner point of N, then there should be some neighbourhood $U \subseteq N$ of x such that points that are U-close to x are also inner points of N. Hence we have the postulate:

(6) For all $x \in X$ and $N \in \mathcal{N}_x$, there is a $U \subseteq N$ with $x \prec U$ such that $y \prec N$ for all $y \in U$.

Accepting this postulate requires accepting its consequences:

PROPOSITION 2.1

Let X be a pretopological space. Then the following are equivalent:

(i) *For all $x \in X$ and $N \in \mathcal{N}_x$, there is a $U \subseteq N$ with $x \prec U$ such that $y \prec N$ for all $y \in U$.*

(ii) *For all $x \in X$, $\mathcal{U}_x$ is a filter basis for $\mathcal{N}_x$.*

(iii) $\mathrm{pCl}(A)$ *is closed for all $A \subseteq X$; equivalently* $\mathrm{pInt}(A)$ *is open for all $A \subseteq X$.*

(iv) $\mathrm{pCl}(A) = \mathrm{Cl}(A)$ *for all $A \subseteq X$; equivalently* $\mathrm{pInt}(A) = \mathrm{Int}(A)$ *for all $A \subseteq X$.*

(v) $\mathrm{pCl}(\cdot)$ *is idempotent; equivalently* $\mathrm{pInt}(\cdot)$ *is idempotent.*

PROOF. The equivalences in parts (iii), (iv) and (v) follow by duality (cf. Proposition 1.8).

(i) ⇒ (ii): Let $x \in X$ and $N \in \mathcal{N}_x$, and let $U = \mathrm{pInt}(N)$. We claim that U is open. For each $y \in U$ we have $y \prec N$, so there is a $U_y \subseteq N$ such that $y \prec U_y$, and such that $z \prec N$ for all $z \in U_y$. But then $U_y \subseteq U$, so $y \prec U$. Hence U is open as desired.

(ii) ⇒ (i): Let $x \in X$ and $N \in \mathcal{N}_x$, and let $U \subseteq N$ be an open set containing x. Since U is open, all its points are inner points, hence inner points of N.

(ii) ⇒ (iii): Let $A \subseteq X$. We show that $\mathrm{pInt}(A)$ is open. For $x \in \mathrm{pInt}(A)$ we have $x \prec A$, so there is an open $U_x \subseteq A$ with $x \in U_x \subseteq A$. But then $U_x \subseteq \mathrm{pInt}(A)$ by Proposition 1.10(vi), so $\mathrm{pInt}(A) = \bigcup_{x \in \mathrm{pInt}(A)} U_x$ is a union of open sets, hence open by Proposition 1.7.

(iii) ⇒ (ii): Let $x \in X$ and $N \in \mathcal{N}_x$ and notice that $x \in \mathrm{pInt}(N) \subseteq N$ with $\mathrm{pInt}(N)$ open.

(iii) ⇒ (iv): Let $A \subseteq X$. Since $\mathrm{pInt}(A)$ is open we have $\mathrm{pInt}(A) \subseteq \mathrm{Int}(A)$ by definition of the interior. The opposite inclusion always holds, so we have equality.

(iv) ⇒ (v): Let $A \subseteq X$. Since $\mathrm{Int}(A)$ is open by Corollary 1.9, we have $\mathrm{Int}(A) = \mathrm{pInt}(\mathrm{Int}(A))$. But then it follows that $\mathrm{pInt}(A) = \mathrm{pInt}(\mathrm{pInt}(A))$.

(v) ⇒ (iii): Let $A \subseteq X$ Since the preinterior operator is idempotent, we have $\mathrm{pInt}(\mathrm{pInt}(A)) = \mathrm{pInt}(A)$. But then $\mathrm{pInt}(A)$ is open by definition. □

Notice that Proposition 2.1(ii) remedies the issue in Remark 1.6 by ensuring that every neighbourhood of a point contains a nonempty open set. This is essentially a restatement of Proposition 2.1(i), relating it to the filter structure that the $\prec$-relation induces on X.

Most notably, postulate [TODO ref] is equivalent to the idempotence of $\mathrm{pCl}(\cdot)$. Recall from Remark 1.4 that $\mathrm{pCl}(\cdot)$ is always a Čech closure operator. If it is also idempotent, then it is called a *Kuratowski closure operator*. The content of Proposition 2.1 is first of all that if $\mathrm{pCl}(\cdot)$ is a Kuratowski closure operator, then $N \subseteq X$ is a neighbourhood of a point $x \in X$ just when x has an open neighbourhood contained in N. Hence the neighbourhoods of x are precisely the neighbourhoods of x in the topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is the topology on X. Therefore, just as a set X can be equipped with a pretopological structure by defining a Čech closure operator on it, X can be equipped with a *topological* structure by defining a Kuratowski closure operator on it. What's more, that structure will be compatible with Kuratowski closure operator in the sense of Proposition 2.1.

Furthermore, we can give a conceptual interpretation of idempotence of $\mathrm{pCl}(\cdot)$ and try to understand exactly what we are assuming when accepting postulate [TODO ref]: Recall that we said that a point x is 'close' to a set A if $x \in \mathrm{pCl}(A)$. Then Proposition 2.1(v) says that x is close to A if and only if it is close to $\mathrm{pCl}(A)$. In other words, taking the (pre)closure of a set in effect creates a sort of boundary, such that all points outside the (pre)closure are separated from it by this boundary.