Conceptual foundations of topology

Danny Nygård Hansen

24th October 2022

1 • Pretopological spaces

1.1. Motivation and definitions

Let X be a set. Let $\prec$ be a relation from X to $\wp(X)$. If $x \prec U$, then we say that x is an *inner point* of U, and that U is a *neighbourhood* of x. One intuition for this relation is that $x \prec U$ just when x is 'completely contained' in U, or is 'completely enclosed' in U, with 'room to spare'. For this to be the case we obviously require that

(1) if $x \prec U$, then $x \in U$.

Clearly the empty set cannot be a neighbourhood of any point. On the other hand, since X is the entire universe under consideration we surely want X to be a neighbourhood of x. Hence we have the postulates

(2) $x \nprec \emptyset$, and

(3) $x \prec X$.

In particular, every point has at least one neighbourhood (namely X itself). Furthermore, under the above interpretation, if $V \subseteq X$ is a set with $U \subseteq V$ then V must also be a neighbourhood of x. Hence we have the postulate:

(4) If $x \prec U$ and $U \subseteq V$, then $x \prec V$.

Next assume that both U and V are neighbourhoods of x. That is, x is 'completely contained' in both U and V, so it seems natural enough that x should be 'completely contained' in the intersection $U \cap V$. That is,

(5) If $x \prec U$ and $x \prec V$, then $x \prec U \cap V$.

Now let $\mathcal{N}_x$ denote the collection of neighbourhoods of x. Then the above shows that $\mathcal{N}_x$ is a *filter* on X, i.e. a proper, nonempty subset of $\wp(X)$ that is upward closed and downward directed. Hence we call $\mathcal{N}_x$ the *neighbourhood filter* of x. That is, a neighbourhood filter for a point $x \in X$ is a filter $\mathcal{N}_x$ on X such that all sets in $\mathcal{N}_x$ contain x. A collection $(\mathcal{N}_x)_{x \in X}$ of neighbourhood filters, one for each point in X, is then called a *neighbourhood system* on X.

We summarise the discussion in the following definition:

DEFINITION 1.1: *Pretopological space*

A *pretopology* on a set X is a relation $\prec$ from X to $\wp(X)$ with the following properties:

(i) Centred: For all $x \in X$ and $N \subseteq X$, if $x \prec N$ then $x \in N$.

(ii) Left-total: For all $x \in X$ there exists an $N \subseteq X$ such that $x \prec N$.

(iii) Downward directed: For all $x \in X$ and $N, M \subseteq X$, if $x \prec N$ and $x \prec M$ then $x \prec N \cap M$.

(iv) Upward closed: For all $x \in X$ and $N, M \subseteq X$, if $x \prec N$ and $N \subseteq M$ then $x \prec M$.

The pair $(X, \prec)$ is called a *pretopological space*.

If $x \prec N$ then N is said to be a *neighbourhood* of x, and x an *inner point* of N. The set

$$\mathcal{N}_x = \{N \subseteq X \mid x \prec N\}$$

is called the *neighbourhood filter* of x, and the collection $(\mathcal{N}_x)_{x \in X}$ is called the *neighbourhood system* on X.

1.2. Closures and interiors

Let X be a pretopological space and consider distinct points $x, y \in X$. If there is an $N \in \mathcal{N}_x$ that does not contain y, then in some sense x is 'separated' from y, since it is 'completely enclosed' in something that is disjoint from y. In this case we say that x is *separated from* y. Note however that even if x is separated from y, y might not be separated from x.

Next consider a subset $A \subseteq X$. If $a, a' \in A$ and $x \in X \setminus A$, then it may happen that x is separated from a but not from a'. It might also happen that x is separated from every individual $a \in A$, so there is some $N_a \in \mathcal{N}_x$ with $a \notin N_a$, but no matter which N_a we pick there are still points $a' \in A$ that lie in N_a. However, if there is a $N \in \mathcal{N}_x$ that is completely disjoint from A, then we say that x is *separated from* A. On the other hand, a point that is *not* separated from A is in some sense 'close' to A. Hence we have the following

Definition 1.2: *Closure and preclosure*

Let X be a pretopological space, and let $A \subseteq X$.

(i) The *preclosure* of A is the set

$$\mathrm{pCl}(A) = \{x \in X \mid \forall N \in \mathcal{N}_x \colon A \cap N \neq \emptyset\},$$

i.e. the set of points that are not separated from A.

(ii) If $A = \mathrm{pCl}(A)$, then A is called *closed*.

(iii) The *closure* $\mathrm{Cl}(A)$ of A is the intersection of all closed sets that contain A.

Proposition 1.3: *Properties of preclosures*

Let X be a pretopological space, and let $A, B \subseteq X$. The preclosure operator $\mathrm{pCl}(\cdot)$ has the following properties:

(i) *Increasing: $A \subseteq B$ implies $\mathrm{pCl}(A) \subseteq \mathrm{pCl}(B)$.*

(ii) *Extensive: $A \subseteq \mathrm{pCl}(A)$.*

(iii) *Preserves nullary unions: $\mathrm{pCl}(\emptyset) = \emptyset$*

(iv) *Preserves binary unions: $\mathrm{pCl}(A \cup B) = \mathrm{pCl}(A) \cup \mathrm{pCl}(B)$.*

We also have the following:

(v) *A set $N \subseteq X$ is a neighbourhood of $x \in X$ if and only if $x \notin \mathrm{pCl}(X \setminus N)$.*

(vi) *There is a potentially infinite series of inclusions*

$$A \subseteq \mathrm{pCl}(A) \subseteq \mathrm{pCl}(\mathrm{pCl}(A)) \subseteq \cdots \subseteq \mathrm{Cl}(A).$$

(vii) *If A is closed, then $\mathrm{pCl}(A) = \mathrm{Cl}(A)$.*

Proof. *Proof of (i)*: This is clear, since if a neighbourhood of $x \in X$ intersects A, then it also intersects B.

Proof of (ii): This is obvious, since if $x \in A$ then any neighbourhood of x intersects A.

Proof of (iii): This is obvious, since every neighbourhood has empty intersection with the empty set.

Proof of (iv): By (i) we have $\mathrm{pCl}(A) \subseteq \mathrm{pCl}(A \cup B)$ and $\mathrm{pCl}(B) \subseteq \mathrm{pCl}(A \cup B)$, so $\mathrm{pCl}(A) \cup \mathrm{pCl}(B) \subseteq \mathrm{pCl}(A \cup B)$. Conversely let $x \notin \mathrm{pCl}(A) \cup \mathrm{pCl}(B)$, so x is both separated from A and B. Hence there exist $N_1, N_2 \in \mathcal{N}_x$ such that

$A \cap N_1 = B \cap N_2 = \emptyset$. But then $N_1 \cap N_2$ is a neighbourhood of x separating x from $A \cup B$. Hence $x \notin \mathrm{pCl}(A \cup B)$.

Proof of (v): If N is a neighbourhood of x, then x has a neighbourhood disjoint from $X \setminus N$. Hence $x \notin \mathrm{pCl}(X \setminus N)$. Conversely, if $x \notin \mathrm{pCl}(X \setminus N)$ then x has a neighbourhood M disjoint from $X \setminus N$. But then we must have $M \subseteq N$, so N is also a neighbourhood of x.

Proof of (vi): Let $F \subseteq X$ be a closed set containing A. Then since $\mathrm{pCl}(F) = F$ we have $\mathrm{pCl}(A) \subseteq F$ by (i). Repeatedly taking the preclosure on both sides we obtain

$$A \subseteq \mathrm{pCl}(A) \subseteq \mathrm{pCl}(\mathrm{pCl}(A)) \subseteq \cdots \subseteq F.$$

Since $\mathrm{Cl}(A)$ is intersection of all such F, the claim follows.

Proof of (vii): If A is closed, then A is one of the closed sets in the intersection defining $\mathrm{Cl}(A)$, so $\mathrm{Cl}(A) \subseteq A$. The claim then follows from (vi). □

REMARK 1.4. A map $\mathrm{pCl}(\cdot)\colon \wp(X) \to \wp(X)$ satisfying properties (ii), (iii) and (iv) is called a *Čech closure operator* or a *preclosure operator* on X. It is easy to see that these properties imply the other properties by mimicking the proofs above (note that we used (i) in the proof of (iv) above). In fact, such an operator can also be used to define a pretopological structure on a set, by declaring that N is a neighbourhood of x if $x \notin \mathrm{pCl}(X \setminus N)$, in accordance with property (v) above. ⌟

Let us try to understand the above results in terms of ‘closeness’:

Part (i): Closeness is clearly increasing, in the sense that if $A \subseteq B$ and x is close to A, then x is also close to B.

Part (ii): Certainly every point is A is close to A.

Part (iii): Also, certainly nothing is close to the empty set.

Part (iv): If x is close to either A or B, then x is certainly close to $A \cup B$. The converse is less obvious: If x is close to $A \cup B$, is it really the case that x is close to either A or B? We might imagine a notion of closeness where closeness requires a certain point density in a particular area, and perhaps A and B have the required density when put together, but none of them are dense enough on their own.

However, there might be a sense in which x is not properly separated from either A or B. Indeed, for x to be separated from A, x must have a neighbourhood N disjoint from A. But that means that x is ‘completely enclosed’ in N, and that doesn't seem to be the case if A has some density in the relevant area.

Part (v): This is essentially just a restatement of the definition of preclosure. If x is completely contained in N, then it is not close to $X \setminus N$.

Part (vi): In general, the preclosure of a set is not closed. That is, the preclosure operator is not idempotent. In 'closeness' terms, this means that points that are not close to A might still be close to $\mathrm{pCl}(A)$. For example, say that an integer n is close to a set $A \subseteq \mathbb{Z}$ if the smallest difference between n and any $a \in A$ is at most 1. Then

$$\mathrm{pCl}(A) = \bigcup_{a \in A} \{a-1, a, a+1\},$$

which is clearly not idempotent, though it is a Čech closure operator.

If the sequence of inclusions in (vi) terminates, then we have found all the points that are close to A, and the resulting set is closed. A set is thus closed if it contains all points that are close to it. But for the preclosure on $\mathbb{Z}$ above, the only closed sets are $\emptyset$ and $\mathbb{Z}$ itself.

A second important operation is the following:

Definition 1.5: *Interior and preinterior*

Let $(X, \prec)$ be a pretopological space, and let $A \subseteq X$.

(i) The *preinterior* of A is the set

$$\mathrm{pInt}(A) = \{x \in A \mid x \prec A\}$$

of all inner points of A.

(ii) If $A = \mathrm{pInt}(A)$, then A is called *open*.

(iii) The *interior* $\mathrm{Int}(A)$ of A is the union of all open sets contained in A.

Remark 1.6. In a topological space, if N is a neighbourhood of a point x, then there exists an open set U such that $x \in U \subseteq N$. We claim that this is not necessarily the case in a pretopological space: Indeed, we show that the interior of a neighbourhood can be empty.

Consider the pretopological space whose underlying set is $\mathbb{R}$, and where $\mathcal{N}_0 = \{N \subseteq \mathbb{R} \mid (-1,1) \subseteq N\}$ and $\mathcal{N}_x = \{\mathbb{R}\}$ for $x \neq 0$. Then $(-1,1)$ is a neighbourhood of 0, but if $A \subseteq (-1,1)$ is to be open, then we must have $\mathrm{pInt}(A) = A$, and this is only possible when $A = \emptyset$.

The trouble is clearly that $(-1,1)$ is a neighbourhood of 0 but not of any of the points that are close to 0, which it would be in a topological space. ⌟

Closures and interiors are dual to each other:

PROPOSITION 1.7

Let $(X, \prec)$ *be a pretopological space, and let* $A \subseteq X$*. Then*

$$X \setminus \mathrm{pCl}(A) = \mathrm{pInt}(X \setminus A) \quad \text{and} \quad \mathrm{pCl}(X \setminus A) = X \setminus \mathrm{pInt}(A).$$

In particular, a set is open if and only if its complement is closed, and

$$X \setminus \mathrm{Cl}(A) = \mathrm{Int}(X \setminus A) \quad \text{and} \quad \mathrm{Cl}(X \setminus A) = X \setminus \mathrm{Int}(A).$$

PROOF. The first claim follows from the following series of equivalences:

$$\begin{aligned} x \notin \mathrm{pCl}(A) &\Leftrightarrow \exists N \in \mathcal{N}_x : A \cap N = \emptyset \\ &\Leftrightarrow \exists N \in \mathcal{N}_x : N \subseteq X \setminus A \\ &\Leftrightarrow X \setminus A \in \mathcal{N}_x \\ &\Leftrightarrow x \in \mathrm{pInt}(X \setminus A). \end{aligned}$$

The second part follows by replacing A by $X \setminus A$ and taking complements on both sides.

If $F \subseteq X$ is closed, then

$$X \setminus F = X \setminus \mathrm{pCl}(F) = \mathrm{pInt}(X \setminus F) \subseteq X \setminus F,$$

so $X \setminus F$ is open. Conversely, if $U \subseteq X$ is open then

$$X \setminus U = X \setminus \mathrm{pInt}(U) = \mathrm{pCl}(X \setminus U) \supseteq X \setminus U,$$

which implies that $\mathrm{pCl}(X \setminus U) = X \setminus U$, so $X \setminus U$ is closed.

Finally, let $\mathcal{F}$ be the collection of closed sets containing A, and let $\mathcal{U}$ be the collection of open sets contained in $X \setminus A$. Then the duality between open and closed sets implies that

$$X \setminus \mathrm{Cl}(A) = X \setminus \bigcap_{F \in \mathcal{F}} F = \bigcup_{F \in \mathcal{F}} X \setminus F = \bigcup_{U \in \mathcal{U}} U = \mathrm{Int}(X \setminus A),$$

as claimed. The second part of the last claim follows by taking complements. □

We may use this correspondence to obtain properties of preinteriors analogous to those of preclosures in [TODO ref], but we will not use these in the sequel.

1.3. *Convergence and continuity*

Let $\mathcal{F}$ be a system of subsets of a pretopological space X. We think of the sets in $\mathcal{F}$ as describing the state of some system $\mathcal{S}$ in X, where $F \in \mathcal{F}$ means that $\mathcal{S}$

eventually lies in F. If $F \subseteq F'$ then $\mathcal{S}$ of course also eventually lies in F'. And for any two $F_1, F_2 \in \mathcal{F}$, if $\mathcal{S}$ eventually lies in F_1 and eventually lies in F_2, then it seems natural that $\mathcal{S}$ will eventually lie in $F_1 \cap F_2$. Finally, clearly the system cannot lie in the empty set, and on the other hand it trivially lies in X, so in total $\mathcal{F}$ is a filter on X.

If $\mathcal{S}$ is somehow supposed to 'converge' to a point $x \in X$, and if N is a neighbourhood of x, then it also seems natural to require that $\mathcal{S}$ eventually lies in N. For if N being a neighbourhood of x means that x is somehow 'completely enclosed' by N 'with room to spare', and if $\mathcal{S}$ is supposed to converge to x, then surely $\mathcal{S}$ must eventually also be enclosed by N, i.e. we require that $N \in \mathcal{F}$. This naturally leads to the following definition:

Definition 1.8

Let X be a pretopological space, and let $x \in X$. A filter $\mathcal{F}$ on X *converges* to x if $\mathcal{N}_x \subseteq \mathcal{F}$.

Equivalently, a net u in X converges to x if u is eventually in N for all $N \in \mathcal{N}_x$.

That is, convergence of filters and nets are entirely analogous to convergence in topological spaces.

Next we turn to continuity. Intuitively, a function $f : X \to Y$ is continuous at a point $x \in X$ if points in X that are close to x are mapped to points in Y that are close to $f(x)$. Usually this is formalised as follows:

Definition 1.9: *Continuity*

Let X and Y be pretopological spaces, and let $x \in X$. A map $f : X \to Y$ is *continuous at* x if, for all $A \subseteq X$, $x \in \mathrm{pCl}(A)$ implies that $f(x) \in \mathrm{pCl}(f(A))$, i.e.

$$\forall A \subseteq X : x \in \mathrm{pCl}(A) \Rightarrow f(x) \in \mathrm{pCl}(f(A)).$$

If f is continuous at x for all $x \in X$, then we say that f is *continuous*.

That is, we require that if A is a set that x is close to, then $f(A)$ is a set that $f(x)$ is close to. But this seems like the opposite of what we were saying before, since there is a difference between points close to x and points that x is close to. This is obvious in our formalism, since the 'close to' relation is heterogeneous: Indeed, points can only be close to *sets*, not to points, nor can sets be close to sets. Hence if x is supposed to be close to something, or something is supposed to be close to x, then that something must be a set.

Indeed, it seems difficult to define a general *homogeneous* closeness relation using closures, since if $x \in A$ and $y \in B$, then x might lie in $\mathrm{pCl}(B)$ while y doesn't lie in $\mathrm{pCl}(A)$. Hence we use the heterogeneous relation above.

As in topological spaces we can characterise global continuity more simply:

PROPOSITION 1.10

Let $f : X \to Y$ be a map between pretopological spaces. Then f is continuous if and only if

$$f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$$

for all $A \subseteq X$.

PROOF. If f is continuous and $A \subseteq X$, then for all $x \in \mathrm{pCl}(A)$ we have $f(x) \in \mathrm{pCl}(f(A))$, which just says that $f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$. Conversely, if f satisfies the above condition and $x \in \mathrm{pCl}(A)$, then $f(x) \in f(\mathrm{pCl}(A)) \subseteq \mathrm{pCl}(f(A))$, so f is continuous at x. □

Next we wish to explicitly relate continuity to neighbourhoods, as is standardly done in topology. In a topological space it is well-known that continuity as defined above is equivalent to preimages of open sets being open. In pretopological spaces we have something similar, though we must focus on neighbourhoods:

PROPOSITION 1.11

Let $f : X \to Y$ be a map between pretopological spaces, and let $x \in X$. Then f is continuous at x if and only if it has the following property: For all $N \in \mathcal{N}_{f(x)}$ there is an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$, i.e.

$$\forall N \in \mathcal{N}_{f(x)} \exists M \in \mathcal{N}_x : f(M) \subseteq N.$$

In other words, f is continuous at x if $f^{-1}(N) \in \mathcal{N}_x$ for all $N \in \mathcal{N}_{f(x)}$. In particular, f is continuous if and only if the preimage of any neighbourhood is a neighbourhood.

PROOF. First assume that f has the above property. Let $A \subseteq X$ be such that $x \in \mathrm{pCl}(A)$, and let $N \in \mathcal{N}_{f(x)}$. Then there exists an $M \in \mathcal{N}_x$ such that $f(M) \subseteq N$. Since M is a neighbourhood of x we have $A \cap M \neq \emptyset$. It follows that

$$\emptyset \neq f(A \cap M) \subseteq f(A) \cap f(M) \subseteq f(A) \cap N,$$

so since N was arbitrary we have $f(x) \in \mathrm{pCl}(f(A))$ as desired.

Conversely, assume that f is continuous at x, and let $N \in \mathcal{N}_{f(x)}$. Assume towards a contradiction that $x \in \mathrm{pCl}(f^{-1}(Y \setminus N))$. For since $f(f^{-1}(Y \setminus N)) \subseteq= Y \setminus N$ we would then have, by continuity,

$$f(x) \in \mathrm{pCl}(Y \setminus N) = Y \setminus \mathrm{pInt}(N),$$

which is a contradiction since $f(x) \in \mathrm{pInt}(N)$. Hence x has a neighbourhood M disjoint from $f^{-1}(Y \setminus N)$, and this has the property that $f(M) \subseteq N$.

For the final claim, this is clear since every neighbourhood is a neighbourhood of some point. □

Let us try to understand this result: If $f: X \to Y$ is continuous at $x \in X$, then our initial definition is supposed to say that f takes points in X that are close to x and map them to points in Y that are close to $f(x)$. What the above says is that if (and only if) f is continuous, then no matter which neighbourhood N of $f(x)$ we consider, we can always find a neighbourhood M of x such that f maps M to N. Since it is in some sense 'harder' to map a set into N the smaller N is – i.e., we require M to be smaller if N is made smaller – this seems to suggest another interpretation of neighbourhoods: Namely as formalising a sense of closeness. It is sometimes put as follows:

Every neighbourhood N of a point $x \in X$ defines a notion of closeness, let's call it *N-closeness*. We say that a point $x' \in X$ is N-close to x if $x' \in N$. We may then restate [TODO ref] as follows:

> *A map $f: X \to Y$ is continuous at $x \in X$ if and only if it has the following property: For every $N \in \mathcal{N}_{f(x)}$ there is an $M \in \mathcal{N}_x$ such that if x' is M-close to x, then $f(x')$ is N-close to $f(x)$.*

This is then entirely analogous to the ε-δ-definition of continuity in metric spaces, where we might have said that x' is δ-close to x if $\rho(x, x') < \delta$.

We might wonder whether the above gives us the right notion of closeness. What would happen, for instance, if we for x' to be N-close to x required that x' be an *inner point* of N? In a topological space we might think of this as more natural, especially if we are accustomed to thinking of neighbourhoods as by definition open. This would also not *prima facie* contradict our earlier interpretation of neighbourhoods as 'wholly containing' its inner points. I do not know of any conceptual problems with this approach, but while the two are practically equivalent in topological spaces, in a *pretopological* space neighbourhoods of a point do not behave nicely with respect to points that are 'close' to it, as we saw in [TODO ref rem].

Assume that for all points $x \in X$, a neighbourhood filter $\mathcal{N}_x$ has been specified. If $U \subseteq X$ has the property that all its points are in fact inner points, i.e. that $x \in U$ implies $x \prec U$, then U is called *open*. The collection of all open sets on X is called its *topology*. Denote the topology on X by $\mathcal{T}$. It is easy to show that $\mathcal{T}$ has the following properties:

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(2) If $\mathcal{U} \subseteq \mathcal{T}$, then $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$.

(3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

PROOF. *Proof of (1)*: Since $\emptyset$ contains no points, it is trivially a neighbourhood of all its points. By postulate [TODO ref], X is a neighbourhood of all $x \in X$.

Proof of (2): Since $V \subseteq \bigcup_{U \in \mathcal{U}} U$ for all $V \in \mathcal{U}$ and V is a neighbourhood of all its points, postulate [TODO ref] implies that the union above is also a neighbourhood of all points in V. All points in the union is contained in some V, so the claim follows.

Proof of (3): This is a direct consequence of postulate [TODO ref]. □

Now define the *interior* of $A \subseteq X$ as the set $\mathrm{Int}(A)$ of all inner points of A, i.e.

$$\mathrm{Int}(A) = \{x \in A \mid x \prec A\}.$$

We claim that $\mathrm{Int}(A)$ is open. For if $x \in \mathrm{Int}(A)$ then $x \prec A$, so postulate [TODO ref] yields a $U \in \mathcal{N}_x$ with $U \subseteq A$ such that $y \prec A$ for all $y \in U$. Hence $U \subseteq \mathrm{Int}(A)$, so since $\mathcal{N}_x$ is upward closed, we have $x \prec \mathrm{Int}(A)$.

This yields the following result:

LEMMA 1.12

Let $x \in X$ and $N \subseteq X$. Then $N \in \mathcal{N}_x$ if and only if there is a $U \in \mathcal{T}$ such that $x \in U \subseteq N$.

PROOF. Let $U = \mathrm{Int}(N)$. Since $x \prec N$ we have $x \in \mathrm{Int}(N)$ as desired. □

Consider distinct points $x, y \in X$. If there is a $U \in \mathcal{N}_x$ that does not contain y, then in some sense x is 'separated' from y, since it is 'completely enclosed' in something that is disjoint from y. In this case we say that x is *separated from* y. Note however that even if x is separated from y, y might not be separated from x (unless X is T_1).

Next consider a subset $A \subseteq X$. If $a, a' \in A$ and $x \in X \setminus A$, then it may happen that x is separated from a but not from a'. It might also happen that x is separated from every individual $a \in A$, so there is some $U_a \in \mathcal{N}_x$ with $a \notin U_a$, but no matter which U_a we pick there are still points $a' \in A$ that lie in U_a. However, if there is a $U \in \mathcal{N}_x$ that is completely disjoint from A, then we say that x is *separated from* A.

If x is not separated from A, i.e. if every $U \in \mathcal{N}_x$ intersects (some point in) A, then x is called an *accumulation point* of A. The set of accumulation points of A is called the *derived set* of A and is denoted A'. Note that A and A' are disjoint by definition. The *closure* of A is the set $\mathrm{Cl}(A) = A \cup A'$. Hence $\mathrm{Cl}(A)$ is the set of points that cannot be separated from A, so these points are in some sense 'close' to A.

Clearly $A \subseteq B$ implies $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$, since if $x \in X$ is separated from B then it is separated from A. That is, the map $A \mapsto \mathrm{Cl}(A)$ is increasing. We furthermore have the following:

PROPOSITION 1.13

The operation $A \mapsto \mathrm{Cl}(A)$ on $\wp(X)$ has the following properties:

(i) $\mathrm{Cl}(\emptyset) = \emptyset$.

(ii) $A \subseteq \mathrm{Cl}(A)$.

(iii) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

(iv) $\mathrm{Cl}((\mathrm{Cl}(A))) = \mathrm{Cl}(A)$.

PROOF. *Proof of (i)*: This is obvious since every point in X is trivially separated from $\emptyset$.

Proof of (ii): This is obvious since $\mathrm{Cl}(A) = A \cup A'$ by definition.

Proof of (iii): Since $A \mapsto \mathrm{Cl}(A)$ is increasing, we have $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(A \cup B)$ and $\mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B)$, so $\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B)$. Conversely let $x \notin \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$, so x is both separated from A and B. Hence there exist $U_1, U_2 \in \mathcal{N}_x$ such that $U_1 \cap A = U_2 \cap B = \emptyset$. But then $U_1 \cap U_2$ is a neighbourhood of x separating x from $A \cup B$. Hence $x \notin \mathrm{Cl}(A \cup B)$.

Proof of (iv): Since $A \subseteq \mathrm{Cl}(A)$ it suffices to show that $\mathrm{Cl}((\mathrm{Cl}(A))) \subseteq \mathrm{Cl}(A)$. Let N be a neighbourhood of $x \in \mathrm{Cl}((\mathrm{Cl}(A)))$, and let U be open with $x \in U \subseteq N$. Then U is also a neighbourhood of x, so $\mathrm{Cl}(A) \cap U \neq \emptyset$. Pick some $y \in \mathrm{Cl}(A) \cap U$. Then U is also a neighbourhood of y, and since $y \in \mathrm{Cl}(A)$ we must have $A \cap U \neq \emptyset$. But then $A \cap N \neq \emptyset$, so $x \in \mathrm{Cl}(A)$. □

It is easy to understand the first three properties above in terms of 'closeness': Certainly nothing is close to the empty set, and every point in A is close to A. Furthermore, something is close to $A \cup B$ if and only if it is close to either A or B. But the last property is difficult to understand intuitively: There are intuitive notions of 'closeness' that satisfy the first three properties but not the fourth. For instance, we might say that two integers are close if their difference is at most 1. That is, for $A \subseteq \mathbb{Z}$ we would define

$$\mathrm{Cl}(A) = \bigcup_{a \in A} \{a-1, a, a+1\}.$$

This clearly satisfies the first three properties, but it is not idempotent. (More properly we would say that an integer n is close to the)

A map $A \mapsto \mathrm{Cl}(A)$ that satisfies the first three properties is called a *Čech closure operator*, and if it also satisfies the fourth it is called a *Kuratowski closure operator*. It is well-known that the latter can be used to define a topological structure on a set. However, we find idempotence to be too unintuitive to take this as a starting point of our discussion.

1.4. Closed sets

We say that a set $F \subseteq X$ is *closed* if $\mathrm{Cl}(F) = F$. In particular, the closure of any set is closed.

1.5. Continuity

1.6. Results