Notes on measure theory and topology

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1 • Introduction

These notes are meant to serve two purposes: Firstly to give an account of (some of) the similarities between topological spaces and measurable spaces. Any student of topology and measure theory have noticed that while σ-algebras generally do not behave as nicely as topologies, we are able to perform many of the same constructions on both structures: Structure-preserving maps (countinuous and measurable maps, respectively) are defined the same way, maps induce topologies and σ-algebras in the same way, there are subspaces, products, quotients, and so on.

If we fix a set X, both the set of topologies and the set of σ-algebras on X are complete lattices when ordered by inclusion. I am not aware that such a lattice of structures on a set has a commonly used name, so I have simply called them *structures* in these notes.

Secondly we wish to explore how a topological and measure-theoretical structure on a single set interact.

1.1. Notation

We generally use notation that is standard in topology, measure theory and category theory. The following may or may not be familiar to the reader:

Given a set X we denote its power set by 2^X. If $f\colon X \to Y$ is a set function, $\mathcal{E} \subseteq 2^X$ and $\mathcal{F} \subseteq 2^Y$, we write

$$f(\mathcal{E}) = \{f(A) \mid A \in \mathcal{E}\} \quad \text{and} \quad f^{-1}(\mathcal{F}) = \{f^{-1}(B) \mid B \in \mathcal{F}\}.$$

For a set X and a family $\mathcal{D} \subseteq 2^X$ of subsets, we write $\sigma(\mathcal{D})$ for the σ-algebra on X generated by $\mathcal{D}$, i.e. the smallest σ-algebra containing $\mathcal{D}$. We do not use any special notation for a topology generated by a family of sets.

If X is a topological space, we denote the Borel σ-algebra on X by $\mathcal{B}(X)$.

2 • Structured sets

2.1. Definitions and basic properties

Let $\mathfrak{S}$ be a map from sets to sets such that $\mathfrak{S}_X := \mathfrak{S}(X)$ is a collection of subsets of 2^X, and such that for all sets X and Y and maps $f: X \to Y$,

(1) $\mathfrak{S}_X$ is partially ordered by set inclusion,

(2) $\mathfrak{S}_X$ is a complete lattice with minimum $\{\emptyset, X\}$ and maximum 2^X,

(3) if $\mathcal{F} \in \mathfrak{S}_Y$, then $f^{-1}(\mathcal{F}) \in \mathfrak{S}_X$, and

(4) if $\mathcal{E} \in \mathfrak{S}_X$, then
$$\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\} \in \mathfrak{S}_Y.$$

We will call such a map $\mathfrak{S}$ a *structure functor*, and it is indeed a functor as we will see in Subsection 2.4.

If X is a set, then a $\mathcal{E} \in \mathfrak{S}_X$ is called a $\mathfrak{S}$-*structure* on X, and we will call the pair $(X, \mathcal{E})$ a $\mathfrak{S}$-*structured set*. We refer to $\mathfrak{S}_X$ as the *lattice of* $\mathfrak{S}$-*structures* on X. The minimal structure $\{\emptyset, X\}$ is called the *trivial structure*, and the maximal structure 2^X is called the *discrete structure* on X.

Fix a structure functor $\mathfrak{S}$. If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are structured sets, a *homomorphism* from X to Y is a map $f: X \to Y$ such that $f^{-1}(\mathcal{F}) \subseteq \mathcal{E}$. Clearly the composition of two homomorphisms is again a homomorphism, so the collection of structured sets and homomorphisms form a (locally small) category. Let us denote this category by $\mathbf{Str}_{\mathfrak{S}}$.

The structure $f^{-1}(\mathcal{F})$ in Item (3) is called the *pullback* of $\mathcal{F}$ by f and is denoted $f^*(\mathcal{F})$. Similarly, the structure $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\}$ in Item (4) is called the *pushforward* of $\mathcal{E}$ by f and is denoted $f_*(\mathcal{E})$. The pullback and pushforward by f is defined for all set functions f, not just homomorphisms.

EXAMPLE 2.1. Let $\mathfrak{S}$ denote the map that associates to a set its lattice of topologies. The first two conditions above are obviously satisfied, and the latter two are easily proved. Thus $\mathbf{Str}_{\mathfrak{S}}$ is just the category **Top** of topological spaces. Similarly, if $\mathfrak{S}$ maps a set to its lattice of σ-algebras, then $\mathbf{Str}_{\mathfrak{S}}$ is the category **Mble** of measurable spaces. ⌟

In the sequel we fix a structure functor $\mathfrak{S}$.

LEMMA 2.2

Let X be a set. If $\mathcal{D} \subseteq 2^X$, then there is a smallest element $\langle \mathcal{D} \rangle \in \mathfrak{S}_X$ with $\mathcal{D} \subseteq \langle \mathcal{D} \rangle$.

PROOF. Let $\mathfrak{D} = \{\mathcal{E} \in \mathfrak{S}_X \mid \mathcal{D} \subseteq \mathcal{E}\}$. Since $\mathfrak{S}_X$ is a complete lattice, we can put

$$\langle \mathcal{D} \rangle = \bigwedge_{\mathcal{E} \in \mathfrak{D}} \mathcal{E} \in \mathfrak{S}_X. \qquad \square$$

If $\langle \mathcal{D} \rangle = \mathcal{E}$, then we say that $\mathcal{D}$ *generates* or is a *generating set* for $\mathcal{E}$. It is easy to see that we may characterise joins as a particular generated structure, namely

$$\bigvee_{\alpha \in A} \mathcal{E}_\alpha = \Big\langle \bigcup_{\alpha \in A} \mathcal{E}_\alpha \Big\rangle. \tag{2.1}$$

PROPOSITION 2.3

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be structured sets, and let $f \colon X \to Y$ be any map. For any $\mathcal{D} \subseteq 2^Y$ we have

$$f^{-1}\big(\langle \mathcal{D} \rangle\big) = \big\langle f^{-1}(\mathcal{D}) \big\rangle.$$

In particular, if $\mathcal{F} = \langle \mathcal{D} \rangle$, then f is a homomorphism if and only if $f^{-1}(\mathcal{D}) \subseteq \mathcal{E}$.

In topology, this proposition is trivial since every element in $\langle \mathcal{D} \rangle$ is a union of finite intersections of elements in $\mathcal{D}$. The proof below is identical to the one given in measure theory.

PROOF. First notice that $f^{-1}(\mathcal{D}) \subseteq f^{-1}(\langle \mathcal{D} \rangle)$, which implies that

$$\big\langle f^{-1}(\mathcal{D}) \big\rangle \subseteq f^{-1}\big(\langle \mathcal{D} \rangle\big).$$

For the second inclusion, notice that

$$\mathcal{A} = \Big\{ B \subseteq Y \;\Big|\; f^{-1}(B) \in \big\langle f^{-1}(\mathcal{D}) \big\rangle \Big\}$$

is a set structure in Y. Since clearly $\mathcal{D} \subseteq \mathcal{A}$, we also have $\langle \mathcal{D} \rangle \subseteq \mathcal{A}$, which proves the second inclusion. $\square$

2.2. Initial structures

DEFINITION 2.4: *Initial structures*

Let $(f_\alpha)_{\alpha \in A}$ be a collection of maps from a set X to structured sets $(X_\alpha, \mathcal{E}_\alpha)$. The *initial structure* $\mathcal{E}$ on X induced by (f_α) is the smallest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigvee_{\alpha \in A} f_\alpha^*(\mathcal{E}_\alpha) = \Big\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{E}_\alpha) \Big\rangle.$$

REMARK 2.5. If $\mathcal{D}_\alpha$ is a generating set for $\mathcal{E}_\alpha$ for all $\alpha \in A$, then we may replace $\mathcal{E}_\alpha$ on the right-hand side above with $\mathcal{D}_\alpha$. This follows immediately from the second part of Proposition 2.3, since the structure $\left\langle \bigcup_{\alpha\in A} f_\alpha^{-1}(\mathcal{D}_\alpha) \right\rangle$ makes all f_α into homomorphisms.

Note that $\bigvee_{\alpha\in A} f_\alpha^*(\mathcal{D}_\alpha)$ doesn't generally make sense, since $\mathcal{D}_\alpha$ is not necessarily a structure on X_α. ⌟

THEOREM 2.6: *Characteristic property of initial structures*

Let $(X,\mathcal{E})$ be a structured set equipped with the initial structure induced by maps $f_\alpha \colon X \to X_\alpha$, $\alpha \in A$. If $(Y,\mathcal{F})$ is a structured set, then $g \colon Y \to X$ is a homomorphism if and only if $f_\alpha \circ g$ is a homomorphism for all $\alpha \in A$:

$$\begin{tikzcd} & X_\alpha \\ Y \arrow[ur, "f_\alpha\circ g"] \arrow[r, "g"'] & X \arrow[u, "f_\alpha"'] \end{tikzcd}$$

Furthermore, the initial structure on X is unique with this property.

PROOF. If g is a homomorphism, then clearly the $f_\alpha \circ g$ are all homomorphisms.

Conversely, assume that all compositions $f_\alpha \circ g$ are homomorphisms. It suffices to show that $g^{-1}(B) \in \mathcal{F}$ for all B from a generating set for $\mathcal{E}$, so let $B = f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. It follows that

$$g^{-1}(B) = g^{-1}(f_\alpha^{-1}(C)) = (f_\alpha \circ g)^{-1}(C) \in \mathcal{F}$$

as desired.

We now show that the characteristic property uniquely determines a structure on X. First let $\mathcal{E}$ be a structure on X such that a map $g \colon Y \to X$ is a homomorphism if and only if each $f_\alpha \circ g$ is a homomorphism. The commutative diagram

$$\begin{tikzcd} & X_\alpha \\ X \arrow[ur, "f_\alpha"] \arrow[r, "\mathrm{id}_X"'] & X \arrow[u, "f_\alpha"'] \end{tikzcd}$$

then shows that f_α is a homomorphism since id_X trivially is.

Next let $\mathcal{E}'$ be another structure on X with the characteristic property. Consider the commutative diagram

$$\begin{tikzcd} & X_\alpha \\ (X,\mathcal{E}) \arrow[ur, "f_\alpha"] \arrow[r, "\mathrm{id}_X"'] & (X,\mathcal{E}') \arrow[u, "f'_\alpha"'] \end{tikzcd}$$

where a prime denotes that the domain of a map is $(X,\mathcal{E}')$ but is as a set function the same as its unprimed counterpart. The f_α are homomorphisms, so by the characteristic property applied to $\mathcal{E}'$ we get that id_X is a homomorphism. By symmetry the corresponding primed map $\mathrm{id}'_X\colon (X,\mathcal{E}') \to (X,\mathcal{E})$ is also a homomorphism, so $(X,\mathcal{E})$ and $(X,\mathcal{E}')$ are isomorphic through the identity, and hence $\mathcal{E} = \mathcal{E}'$. □

Example 2.7: Subsets.
Let $(X,\mathcal{E})$ be a structured set, and let $S \subseteq X$. The inclusion map $\iota_S\colon S \to X$ then induces an initial structure on S, namely the pullback $\iota_S^*(\mathcal{E})$. We denote this subset structure by $\mathcal{E}_S$, and unless otherwise noted subsets of structured sets always carry this structure. By the characteristic property of initial structures, a map $f\colon Y \to S$ from a structured set is a homomorphism if and only if $\iota_S \circ f$ is a homomorphism.

On the other hand, if $f\colon Y \to X$ is a map with $f(Y) \subseteq S$, then the map $\tilde{f}\colon Y \to S$ given by $\tilde{f}(y) = f(y)$ for all $y \in Y$ is a homomorphism if and only if $f = \iota_S \circ \tilde{f}$ is a homomorphism. In other words, whether a map is a homomorphism or not does not depend on the codomain if we agree to equip subsets with the structure induced by their inclusion maps.

If $S = f(Y)$ and $\tilde{f}\colon Y \to f(Y)$ is an isomorphism, then we call f an *embedding*. Hence inclusion maps are in particular embeddings. ⌟

Example 2.8: Products.
Let $(X_\alpha,\mathcal{E}_\alpha)_{\alpha\in A}$ be a collection of structured sets, let $X = \prod_{\alpha\in A} X_\alpha$ be the Cartesian product of the sets X_α, and denote the associated projections by $\pi_\alpha\colon X \to X_\alpha$. We define a product structure

$$\mathcal{E} = \bigotimes_{\alpha\in A} \mathcal{E}_\alpha$$

as the initial structure on X induced by the projection maps. Since X is a product of the X_α in the category of sets, the characteristic property of initial structures implies that $(X,\mathcal{E})$ is a product of the structured sets $(X_\alpha,\mathcal{E}_\alpha)$. ⌟

PROPOSITION 2.9: *Composition of initial structures*

For $\alpha \in A$ and $\lambda \in L_\alpha$, let X_α and $Y_{\alpha\lambda}$ be structured sets such that each X_α carry the initial structure induced by maps $g_{\alpha\lambda}\colon X_\alpha \to Y_{\alpha\lambda}$ for $\lambda \in L_\alpha$. Let X be a set, and consider maps $f_\alpha\colon X \to X_\alpha$.

Let $\mathcal{E}_1$ be the initial structure on X induced by the maps f_α, and let $\mathcal{E}_2$ be the structure induced by the compositions $g_{\alpha\lambda} \circ f_\alpha$. Then $\mathcal{E}_1 = \mathcal{E}_2$.

PROOF. By the characteristic property of initial structures, the f_α are homomorphisms if and only if all the $g_{\alpha\lambda} \circ f_\alpha$ are homomorphisms. If X has the structure $\mathcal{E}_1$, then we must have $\mathcal{E}_2 \subseteq \mathcal{E}_1$, since these compositions $g_{\alpha\lambda} \circ f_\alpha$ are homomorphisms. Conversely, if X carries the structure $\mathcal{E}_2$, then the f_α are homomorphisms, and so $\mathcal{E}_1 \subseteq \mathcal{E}_2$. □

EXAMPLE 2.10: Subspace and product structures.

Let $(X_\alpha)_{\alpha\in A}$ be a family of structured sets, and let $S_\alpha \subseteq X_\alpha$ be subsets. Then we may equip the product $S = \prod_{\alpha\in A} S_\alpha$ with a structure by first equipping $X = \prod_{\alpha\in A} X_\alpha$ with the product structure, and then induce the subset structure on S. In the opposite order we may first equip each S_α with the subset structure, and then induce the product structure. These in fact give the same structure since the diagram

$$\begin{tikzcd} S_\alpha \arrow[r, "\iota_{S_\alpha}"] & X_\alpha \\ S \arrow[u, "\pi_{S_\alpha}"] \arrow[r, "\iota_S"'] & X \arrow[u, "\pi_{X_\alpha}"'] \end{tikzcd}$$

commutes. ⌟

EXAMPLE 2.11: The weak*-topology.

Let X be a topological vector space over the field $\mathbb{F}$ with topological dual X^*, and for $x \in X$ let $\mathrm{ev}_x\colon X^* \to \mathbb{F}$ be the evaluation map $\mathrm{ev}_x(\varphi) = \varphi(x)$ for $\varphi \in X^*$. Since X^* is a subset of $\mathbb{F}^X$, it naturally carries the subspace topology. The product topology on $\mathbb{F}^X$ is induced by the projection maps $\pi_x\colon \mathbb{F}^X \to \mathbb{F}$ for $x \in X$. But $\pi_x \circ \iota_{X^*}$ is just the evaluation map ev_x, so the diagram

$$\begin{tikzcd} & \mathbb{F} \\ X^* \arrow[ur, "\mathrm{ev}_x"] \arrow[r, "\iota_{X^*}"'] & \mathbb{F}^X \arrow[u, "\pi_x"'] \end{tikzcd}$$

commutes, and the subspace topology on X^* is exactly the weak*-topology. In fact, since the evaluation maps clearly separate points in X^*, the inclusion ι_{X^*} is in fact an embedding, as the following proposition shows: ⌟

Proposition 2.12: *Embedding into product*

Let $f_\alpha \colon Y \to X_\alpha$ for $\alpha \in A$ be homomorphisms, let $X = \prod_{\alpha \in A} X_\alpha$, and let $f \colon Y \to X$ be the unique homomorphism such that $f_\alpha = \pi_\alpha \circ f$:

$$\begin{tikzcd} & X_\alpha \\ Y \arrow[ur, "f_\alpha"] \arrow[r, "f"'] & X \arrow[u, "\pi_\alpha"'] \end{tikzcd}$$

Then f is an embedding if and only if Y carries the initial structure induced by the maps f_α and the collection $(f_\alpha)_{\alpha \in A}$ separates points in Y.

Recall that the collection $(f_\alpha)_{\alpha \in A}$ separates points in Y if for each pair $x, y \in Y$ with $x \neq y$ there exists an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$.

Proof. Notice that f is unique as a set function since X is a product in **Set**, and f is a homomorphism by the characteristic property of initial structures.

First assume that f is an embedding. In particular it is injective, and since the maps π_α separate points in X, the compositions $f_\alpha = \pi_\alpha \circ f$ separate points in Y. Let $\tilde{f} \colon Y \to f(Y)$ be the isomorphism such that $f = \iota_{f(Y)} \circ \tilde{f}$. Then since $\tilde{f}$ is an isomorphism, in particular Y carries the initial structure induced by $\tilde{f}$. But then Y carries the initial structure induced by the maps

$$\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f} = \pi_\alpha \circ f = f_\alpha \tag{2.2}$$

for $\alpha \in A$, as claimed.

Conversely, assume that the f_α separate points in Y and that Y has the initial structure $\mathcal{F}$ induced by the f_α. The f_α are then homomorphisms, and by the characteristic property of initial structures so is f. Furthermore, if $x, y \in Y$ with $x \neq y$, then there is an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$, which implies that $f(x) \neq f(y)$, so f is injective.

Denote the product structure on X by $\mathcal{E}$. We show that if $B \in \mathcal{F}$, then $f(B) \in \mathcal{E}_{f(Y)}$, which will imply that f is an embedding. It suffices to prove this when B is an element of a generating set for $\mathcal{F}$, i.e. on the form $f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. By (2.2) we have

$$B = f_\alpha^{-1}(C) = (\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f})^{-1}(C) = \tilde{f}^{-1}\bigl((\pi_\alpha \circ \iota_{f(Y)})^{-1}(C)\bigr),$$

from which it follows that

$$f(B) = \tilde{f}(B) = (\pi_\alpha \circ \iota_{f(Y)})^{-1}(C) \in \mathcal{E}_{f(Y)}$$

as desired. In the second equality we use that $\tilde{f}$ is surjective. $\square$

If $\mathcal{E}$ is a structure on a set X, we say that $\mathcal{E}$ is *countably generated* if there is a countable collection of sets $\mathcal{D} \subseteq 2^X$ such that $\mathcal{E} = \langle \mathcal{D} \rangle$.

PROPOSITION 2.13: *Countably generated initial structures*

Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a countable collection of structured sets, and assume that the $\mathcal{E}_\alpha$ are countably generated by collections of sets $\mathcal{D}_\alpha$. If an initial structure $\mathcal{E}$ is induced on a set X by maps $f_\alpha : X \to X_\alpha$, then $\mathcal{E}$ is also countably generated.

PROOF. This follows immediately by Remark 2.5 since the generating set $\bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$ is a countable union of countable sets. □

EXAMPLE 2.14: Second-countable topological spaces.
A topology is second-countable if and only if it is countably generated, as we show below. The above proposition then implies that an initial topology induced by a countable family of maps into second-countable spaces is itself second-countable. In particular, subspaces and countable products of second-countable spaces are second-countable.

Now to prove the above claim: Since a basis in particular is a generating set (i.e. a subbasis), second-countable topologies are countably generated. Conversely, let $\mathcal{T}$ be a topology that is generated by a countable set $\mathcal{D}$. Then a basis $\mathcal{B}$ for $\mathcal{T}$ is obtained by taking finite intersections of elements from $\mathcal{D}$. The number of these intersections is certainly less than the cardinality of the union

$$\bigcup_{n \in \mathbb{N}} \mathcal{D}^n$$

of all finite products of $\mathcal{D}$ with itself, an element $U_1 \times \cdots \times U_n$ of an n-fold product corresponding to the intersection $\bigcap_{i=1}^n U_i$. But finite products of countable sets are countable, and so are countable unions of countable sets, so the union above is countable. ⌟

2.3. Final structures

DEFINITION 2.15: *Final structures*

Let $(f_\alpha)_{\alpha \in A}$ be a collection of maps from structured sets $(X_\alpha, \mathcal{E}_\alpha)$ to a set X. The *final structure* $\mathcal{E}$ on X coinduced by (f_α) is the largest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigwedge_{\alpha \in A} (f_\alpha)_*(\mathcal{E}_\alpha).$$

THEOREM 2.16: *Characteristic property of final structures*

Let $(X, \mathcal{E})$ be a structured set equipped with the final structure coinduced by *maps $f_\alpha \colon X_\alpha \to X$, $\alpha \in A$. If $(Y, \mathcal{F})$ is a structured set, then $g \colon X \to Y$ is a homomorphism if and only if $g \circ f_\alpha$ is a homomorphism for all $\alpha \in A$:*

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "g \circ f_\alpha"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

Furthermore, the final structure on X is unique with this property.

PROOF. If g is a homomorphism, then clearly the $g \circ f_\alpha$ are all homomorphisms.

Conversely, assume that the $g \circ f_\alpha$ are homomorphisms. Notice that

$$f_\alpha^{-1}(g^{-1}(\mathcal{F})) = (g \circ f_\alpha)^{-1}(\mathcal{F}) \subseteq \mathcal{E}_\alpha,$$

since the $g \circ f_\alpha$ are homomorphisms. But then $g^{-1}(\mathcal{F}) = g^*(\mathcal{F})$ is a structure on X with respect to which the f_α are homomorphisms, and thus $g^{-1}(\mathcal{F}) \subseteq \mathcal{E}$ since $\mathcal{E}$ is the largest structure with this property. Hence g is a homomorphism.

The proof of uniqueness is analogous to the proof of uniqueness in Theorem 2.6, so we omit it. □

EXAMPLE 2.17: Quotient structures.
Let X be a structured set, and let Y be any set. If $q \colon X \to Y$ is a surjective map, the final structure on Y induced by q is called the *quotient structure*. In particular, if $\sim$ is an equivalence relation on X and $q \colon X \to X/{\sim}$ is the associated natural map, we will always equip $X/{\sim}$ with the quotient structure induced by q. ⌟

EXAMPLE 2.18: Disjoint unions.
If $(X_\alpha)_{\alpha \in A}$ is a collection of structured sets, we equip the disjoint union $X = \coprod_{\alpha \in A} X_\alpha$ with the final structure coinduced by the inclusions $i_\alpha \colon X_\alpha \to X$.

We will see in Subsection 2.4 that X is a coproduct of the X_α in $\mathbf{Str}_{\mathfrak{S}}$, but this fact will be convenient in the proof of Proposition 2.20. ⌟

PROPOSITION 2.19: *Composition of final structures*

For $\alpha \in A$ and $\lambda \in L_\alpha$, let X_α and $Y_{\alpha\lambda}$ be structured sets such that each X_α carry the final structure induced by maps $g_{\alpha\lambda} \colon Y_{\alpha\lambda} \to X_\alpha$ for $\lambda \in L_\alpha$. Let X be a set, and consider maps $f_\alpha \colon X_\alpha \to X$.

Let $\mathcal{E}_1$ be the final structure on X induced by the maps f_α, and let $\mathcal{E}_2$ be the structure induced by the compositions $f_\alpha \circ g_{\alpha\lambda}$. Then $\mathcal{E}_1 = \mathcal{E}_2$.

PROOF. This result is just the dual of Proposition 2.9, so we omit the proof. □

Proposition 2.20: *Final structures as quotient structures*

Let $f_\alpha \colon X_\alpha \to Y$ for $\alpha \in A$ be homomorphisms, let $X = \coprod_{\alpha \in A} X_\alpha$, and let $f \colon X \to Y$ be the unique homomorphism such that $f_\alpha = f \circ i_\alpha$:

$$\begin{tikzcd} X_\alpha \arrow[d, "i_\alpha"'] \arrow[dr, "f_\alpha"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Define an equivalence relation $\sim$ on X by letting $x \sim x'$ if $f(x) = f(x')$ for $x, x' \in X$. Let $q \colon X \to X/{\sim}$ be the associated quotient map, and let $\tilde{f} \colon X/{\sim} \to Y$ be the unique homomorphism such that $\tilde{f} \circ q = f$.

Assume that f is surjective. Then Y has the final structure coinduced by the f_α if and only if $\tilde{f}$ is an isomorphism.

The condition that f is surjective is equivalent to the property that every point in Y is in the image of some f_α. In this case we say that the maps f_α *cover points* in Y.

Notice also that by Proposition 2.19, the final structure on Y coinduced by the f_α is the same as the structure coinduced by f.

Proof. Notice that f is unique as a set function since X is a coproduct in **Set**, and it is a homomorphism by the characteristic property of final structures.

Consider the commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X/{\sim} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

First assume that $\tilde{f}$ is an isomorphism. Then Y in particular carries the final structure coinduced by $\tilde{f}$. But $X/{\sim}$ has the final structure coinduced by q, so Y has the final structure coinduced by the map $\tilde{f} \circ q = f$ by Proposition 2.19 as claimed.

Conversely, assume that Y has the final structure $\mathcal{F}$ coinduced by the f_α. Let $\mathcal{E}$ denote the disjoint union structure on X and $\tilde{\mathcal{E}}$ the quotient structure on $X/{\sim}$. Notice that $\tilde{f}$ is surjective since f is surjective, and it is injective by definition of $\sim$. We thus need to prove that $\tilde{f}^{-1}$ is a homomorphism, and it suffices to show that $\tilde{f}(\tilde{\mathcal{E}}) \subseteq \mathcal{F}$. This is the case if and only if

$$q^{-1}(\tilde{\mathcal{E}}) = f^{-1}\bigl(\tilde{f}(\tilde{\mathcal{E}})\bigr) \subseteq \mathcal{E},$$

and this is true since q is a homomorphism. □

2.4. Categorical properties

The category $\mathbf{Str}_{\mathfrak{S}}$

We first recapitulate some of the above results in categorical terms. The main result is the following:

THEOREM 2.21: *Completeness of* $\mathbf{Str}_{\mathfrak{S}}$

The category $\mathbf{Str}_{\mathfrak{S}}$ *is complete, i.e. it has all small limits.*

PROOF. By e.g. Smith (2018, Theorem 60) it is enough to show that $\mathbf{Str}_{\mathfrak{S}}$ has all small products and has equalisers.

Products: We claim that the product $(X, \mathcal{E})$ considered in Example 2.8 is in fact a product of the objects $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ in $\mathbf{Str}_{\mathfrak{S}}$. If Y is a structured set and $f_\alpha \colon Y \to X_\alpha$ are homomorphisms, then since X is a product in **Set** there is a unique set function $f \colon Y \to X$ such that $f_\alpha = \pi_\alpha \circ f$ for all $\alpha \in A$. But f is also a homomorphism by the characteristic property of the product structure, so $(X, \mathcal{E})$ is in fact a product in $\mathbf{Str}_{\mathfrak{S}}$.

Equalisers: Let $f, g \colon X \to Y$ be any pair of parallel homomorphisms, and let E be the subset of X on which they agree. If $h \colon Z \to X$ is any homomorphism such that $f \circ h = g \circ h$, then there is a unique homomorphism $u \colon Z \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\iota_E"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
Z \arrow[u, dashed, "u"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

We must have $h(Z) \subseteq E$, so we can define u by $u(z) = h(z)$, and u is unique as a set function such that the above diagram commutes. Furthermore, u is a homomorphism by the characteristic property of the subset structure. Thus E along with the inclusion map ι_E is an equaliser of f and g. $\square$

Functoriality of $\mathfrak{S}$

As mentioned, $\mathfrak{S}$ is in fact a functor. Its action on a set function $f \colon X \to Y$ is defined as the pullback $f^* \colon \mathfrak{S}_Y \to \mathfrak{S}_X$.

Let **Set** denote the category of sets and $\mathbf{CsLat}^{\vee}$ the category of complete join-semilattices and join-preserving maps.

PROPOSITION 2.22: *Functoriality of* $\mathfrak{S}$*, I*

The map $\mathfrak{S}$ *is a contravariant functor from* **Set** *to* $\mathbf{CsLat}^{\vee}$.

PROOF. By (2.1) and Proposition 2.3, for any set function f the pullback f^* preserves joins since preimages respect unions, so it is well-defined as a map $\mathbf{Set} \to \mathbf{CsLat}^{\vee}$.

The map $\mathfrak{S}$ is also contravariant since if $f\colon X \to Y$ and $g\colon Y \to Z$ are set functions, then for $\mathcal{G} \in \mathfrak{S}_Z$ we have

$$(g \circ f)^*(\mathcal{G}) = (g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = (f^* \circ g^*)(\mathcal{G}).$$

Its action on identity functions is clearly trivial, so it is a functor. □

In the case of topological spaces or measure spaces we can say slightly more: Notice that in both a lattice of topologies or of σ-algebras on a set, intersections of topologies (σ-algebras) are themselves topologies (σ-algebras). A nonempty subset $\mathfrak{L}$ of 2^X, where X is some set, is called an *intersection structure*. If also $X \in \mathfrak{L}$ we call it a *topped intersection structure*. It is easy to show that topped intersection structures are complete lattices ordered by inclusion, and that meets are given by intersections.

If **CLat** denotes the category of complete lattices with join- and meet-preserving maps, then we have the following:

PROPOSITION 2.23: *Functoriality of $\mathfrak{S}$, II*

If $\mathfrak{S}_X$ is an intersection structure for all sets X, then $\mathfrak{S}$ is a contravariant functor from **Set** *to* **CLat**.

PROOF. It suffices to show that f^* preserves meets for all set functions $f\colon X \to Y$. But this is clear since preimages respect intersections. □

It is natural to ask whether the pushforward f_* by f gives rise to a covariant functor from **Set** into a category of lattices. It is easy to see that f_* is monotone, and a short calculation shows that $(g \circ f)_* = g_* \circ f_*$ if $g\colon Y \to Z$ is another set function, so the pushforward does indeed define a covariant functor from **Set** to the category **Pos** of posets and monotone maps. But f_* does not, as far as I know, generally preserve meets or joins. If all $\mathfrak{S}_X$ are intersection structures, however, it does preserve meets, so it is then a functor into the category $\mathbf{CsLat}^{\wedge}$ of complete meet-semilattices, though it still does not seem to preserve joins. (I haven't looked too hard for counterexamples.)

The forgetful functor on $\mathbf{Str}_{\mathfrak{S}}$

Just as there is a forgetful functor $\mathbf{Top} \to \mathbf{Set}$ that sends a topological space to its underlying set, there is a forgetful functor $U\colon \mathbf{Str}_{\mathfrak{S}} \to \mathbf{Set}$. As usual, U has a left adjoint D that equips a set with the discrete structure, and it has a right adjoint T that equips the set with the trivial structure, i.e. $D \dashv U \dashv T$.

It follows immediately that U preserves both limits and colimits. Hence if $\mathbf{Str}_{\mathfrak{S}}$ e.g. has coproducts – which it has, as we will see later – we already know that they have to be (isomorphic to) disjoint unions of the underlying sets, equipped with an appropriate structure. Contrast this with the situation in the category **Grp** of groups: The forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint, namely the free functor, so U preserves limits. But it does not preserve colimits; for instance, coproducts in **Grp** are free products, and their underlying sets are certainly not disjoint unions! Hence U does not have a right adjoint.

Presheaves on structured sets

Let $(X,\mathcal{E})$ be a structured set, and view $\mathcal{E}$ as a preorder category. Analogous to the case of topological spaces, a presheaf on $\mathcal{E}$ (i.e. a contravariant functor $\mathcal{E} \to \mathbf{Set}$) is called a *presheaf* on $(X,\mathcal{E})$, or simply a presheaf on X if the structure is understood.

As an example, fix a structured set Y and take the presheaf F on X given by $F(B) = \mathbf{Str}_{\mathfrak{S}}(B,Y)$, i.e. F sends a set $B \in \mathcal{E}$ to the set of homomorphisms $B \to Y$. Furthermore, F sends an inclusion $B \subseteq B'$ in $\mathcal{E}$ to the restriction map $\mathbf{Str}_{\mathfrak{S}}(B',Y) \to \mathbf{Str}_{\mathfrak{S}}(B,Y)$ given by $f \mapsto f|_B$. A common example of this is the case $\mathbf{Str}_{\mathfrak{S}} = \mathbf{Top}$ and $Y = \mathbb{R}$, in which case F sends an open set U to the set of continuous functions $U \to \mathbb{R}$.

3 • Topology

We remind the reader that a topological space $(X,\mathcal{T})$ is *second-countable* if there exists a countable basis for $\mathcal{T}$. Furthermore, $(X,\mathcal{T})$ is said to be *Lindelöf* if every open cover of X has a countable subcover.

Recall also that a topological property is called *hereditary* if it follows from a space X having this property that any subspace of X also has this property. It is easy to see that second-countability is hereditary, but the Lindelöf property is not:

Remark 3.1. A space can be Lindelöf without being hereditarily Lindelöf. Let $(X,\mathcal{T})$ be an uncountable discrete space, and let $y \notin X$. Define a topological space $(Y,\mathcal{T}')$ with underlying space $Y = X \cup \{y\}$ and topology $\mathcal{T}' = \mathcal{T} \cup \{Y\}$. Then $(Y,\mathcal{T}')$ is Lindelöf since any open cover must include Y itself, this being the only open set containing the point y. But the subspace X (whose subspace topology is exactly $\mathcal{T}$) is clearly not Lindelöf. ⌟

If every subspace of X is Lindelöf, then we say that X is *hereditarily Lindelöf*.

Proposition 3.2

If $(X,\mathcal{T})$ is a second-countable topological space, then it is hereditarily Lindelöf.

Proof. Every subspace of X is second-countable, so it suffices to show that X is Lindelöf.

Let $\mathcal{U}$ be an open cover of X and let $\mathcal{B}$ be a countable basis for the topology $\mathcal{T}$. Consider an $x \in X$. Since $\mathcal{U}$ is a cover of X there is some $U_x \in \mathcal{U}$ with $x \in \mathcal{U}$, and since $\mathcal{B}$ is a basis for $\mathcal{T}$ there is some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U_x$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the subset of open sets obtained in this way. Clearly $\mathcal{B}'$ is a cover of X.

For each $B \in \mathcal{B}'$, the above shows that there exists some $U \in \mathcal{U}$ with $B \subseteq U$. This defines a map $\mathcal{B}' \to \mathcal{U}$ given by $B \mapsto U$ whose image is a countable cover of X, proving the claim. □

Lemma 3.3

Let $(X,\mathcal{T})$ be a second-countable space. Then every basis for $\mathcal{T}$ contains a countable basis for $\mathcal{T}$.

Proof. Let $\mathcal{B}$ be a basis for $\mathcal{T}$, and let $\mathcal{C}$ be a countable basis. We can write every $C \in \mathcal{C}$ on the form $C = \bigcup_{\alpha \in A} B_\alpha$ for some family $(B_\alpha)_{\alpha \in A} \subseteq \mathcal{B}$. This is in particular an open cover of C, so since X is hereditarily Lindelöf there is a countable subset $A' \subseteq A$ such that $C = \bigcup_{\alpha \in A'} B_\alpha$. For each $C \in \mathcal{C}$ we thus obtain a countable subcollection of sets from $\mathcal{B}$, and since $\mathcal{C}$ is also countable, the union of all these sets is countable and is clearly a basis for $\mathcal{T}$. □

4 • Measure theory

If A is a subset of a measurable space $(X,\mathcal{E})$, then recall that we denote by $\mathcal{E}_A$ the initial σ-algebra on A induced by the inclusion $\iota_A \colon A \to X$. Similarly, if A is a subspace of a topological space $(X,\mathcal{T})$, denote by $\mathcal{T}_A$ the subspace topology on A.

Proposition 4.1

Let $(X,\mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\mathcal{B}(A) = \mathcal{B}(X)_A$, i.e. $\sigma(\mathcal{T}_A) = \sigma(\mathcal{T})_A$.

Proof. Notice that

$$\sigma(\mathcal{T}_A) = \sigma(\iota_A^{-1}(\mathcal{T})) = \iota_A^{-1}(\sigma(\mathcal{T})) = \sigma(\mathcal{T})_A$$

by Proposition 2.3. □

THEOREM 4.2: *Products of Borel σ-algebras*

Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ be a family of topological spaces, and equip $X = \prod_{\alpha\in A} X_\alpha$ with the product topology $\mathcal{T}$. Then

$$\bigotimes_{\alpha\in A} \mathcal{B}(X_\alpha) \subseteq \mathcal{B}(X)$$

If A is countable and the spaces X_α are second-countable, then the above inclusion is an equality.

PROOF. Since the projections $\pi_\alpha\colon X \to X_\alpha$ are continuous, they are $\mathcal{B}(X)$-$\mathcal{B}(X_\alpha)$-measurable. But $\bigotimes_{\alpha\in A} \mathcal{B}(X_\alpha)$ is the smallest σ-algebra on X that makes the projections measurable, which proves the above inclusion.

Now assume that A is countable and that all the X_α are second-countable. From Example 2.14 we know that X is also second-countable. Let

$$\mathcal{D} = \bigcup_{\alpha\in A} \pi_\alpha^{-1}(\mathcal{T}_\alpha)$$

be a subbasis for the product topology $\mathcal{T}$, and let $\mathcal{B}$ be the collection of finite intersections of elements in $\mathcal{D}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$, and $\mathcal{B}$ contains a countable basis $\mathcal{C}$ for $\mathcal{T}$ by Lemma 3.3. Since $\mathcal{C}$ is countable, open sets in X are countable unions of finite intersections of elements in $\mathcal{D}$. Since A is also countable, it suffices to show that

$$\pi_\beta^{-1}(\mathcal{T}_\beta) \subseteq \bigotimes_{\alpha\in A} \mathcal{B}(X_\alpha)$$

for all $\beta \in A$. But this is obvious since the projections are measurable. □

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