

Notes on measure theory and topology

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1 • Introduction

These notes are meant to serve two purposes: Firstly to give an account of (some of) the similarities between topological spaces and measurable spaces. Any student of topology and measure theory have noticed that while σ -algebras generally do not behave as nicely as topologies, we are able to perform many of the same constructions on both structures: Structure-preserving maps (continuous and measurable maps, respectively) are defined the same way, maps induce topologies and σ -algebras in the same way, there are subspaces, products, quotients, and so on.

If we fix a set X , both the set of topologies and the set of σ -algebras on X are complete lattices when ordered by inclusion. I am not aware that such a lattice of structures on a set has a commonly used name, so I have simply called them *structures* in these notes.

Secondly we wish to explore how a topological and measure-theoretical structure on a single set interact.

1.1. Notation

We generally use notation that is standard in topology, measure theory and category theory. The following may or may not be familiar to the reader:

Given a set X we denote its power set by 2^X . If $f: X \rightarrow Y$ is a set function, $\mathcal{E} \subseteq 2^X$ and $\mathcal{F} \subseteq 2^Y$, we write

$$f(\mathcal{E}) = \{f(A) \mid A \in \mathcal{E}\} \quad \text{and} \quad f^{-1}(\mathcal{F}) = \{f^{-1}(B) \mid B \in \mathcal{F}\}.$$

For a set X and a family $\mathcal{D} \subseteq 2^X$ of subsets, we write $\sigma(\mathcal{D})$ for the σ -algebra on X generated by \mathcal{D} , i.e. the smallest σ -algebra containing \mathcal{D} . We do not use any special notation for a topology generated by a family of sets.

If X is a topological space, we denote the Borel σ -algebra on X by $\mathcal{B}(X)$.

2 • Structured sets

2.1. Definitions and basic properties

Let \mathfrak{S} be a map from sets to sets such that $\mathfrak{S}_X := \mathfrak{S}(X)$ is a collection of subsets of 2^X , and such that for all sets X and Y and maps $f: X \rightarrow Y$,

- (1) \mathfrak{S}_X is partially ordered by set inclusion,
- (2) \mathfrak{S}_X is a complete lattice with minimum $\{\emptyset, X\}$ and maximum 2^X ,
- (3) if $\mathcal{F} \in \mathfrak{S}_Y$, then $f^{-1}(\mathcal{F}) \in \mathfrak{S}_X$, and
- (4) if $\mathcal{E} \in \mathfrak{S}_X$, then

$$\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\} \in \mathfrak{S}_Y.$$

We will call such a map \mathfrak{S} a *structure functor*, and it is indeed a functor as we will see in [Subsection 2.4](#).

If X is a set, then a $\mathcal{E} \in \mathfrak{S}_X$ is called a \mathfrak{S} -*structure* on X , and we will call the pair (X, \mathcal{E}) a \mathfrak{S} -*structured set*. We refer to \mathfrak{S}_X as the *lattice of \mathfrak{S} -structures* on X . The minimal structure $\{\emptyset, X\}$ is called the *trivial structure*, and the maximal structure 2^X is called the *discrete structure* on X .

Fix a structure functor \mathfrak{S} . If (X, \mathcal{E}) and (Y, \mathcal{F}) are structured sets, a *homomorphism* from X to Y is a map $f: X \rightarrow Y$ such that $f^{-1}(\mathcal{F}) \subseteq \mathcal{E}$. Clearly the composition of two homomorphisms is again a homomorphism, so the collection of structured sets and homomorphisms form a (locally small) category. Let us denote this category by $\mathbf{Str}_{\mathfrak{S}}$.

The structure $f^{-1}(\mathcal{F})$ in [Item \(3\)](#) is called the *pullback* of \mathcal{F} by f and is denoted $f^*(\mathcal{F})$. Similarly, the structure $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\}$ in [Item \(4\)](#) is called the *pushforward* of \mathcal{E} by f and is denoted $f_*(\mathcal{E})$. The pullback and pushforward by f is defined for all set functions f , not just homomorphisms.

EXAMPLE 2.1. Let \mathfrak{S} denote the map that associates to a set its lattice of topologies. The first two conditions above are obviously satisfied, and the latter two are easily proved. Thus $\mathbf{Str}_{\mathfrak{S}}$ is just the category **Top** of topological spaces. Similarly, if \mathfrak{S} maps a set to its lattice of σ -algebras, then $\mathbf{Str}_{\mathfrak{S}}$ is the category **Mble** of measurable spaces. \lrcorner

In the sequel we fix a structure functor \mathfrak{S} .

LEMMA 2.2

Let X be a set. If $\mathcal{D} \subseteq 2^X$, then there is a smallest element $\langle \mathcal{D} \rangle \in \mathfrak{S}_X$ with $\mathcal{D} \subseteq \langle \mathcal{D} \rangle$.

PROOF. Let $\mathfrak{D} = \{\mathcal{E} \in \mathfrak{S}_X \mid \mathcal{D} \subseteq \mathcal{E}\}$. Since \mathfrak{S}_X is a complete lattice, we can put

$$\langle \mathcal{D} \rangle = \bigwedge_{\mathcal{E} \in \mathfrak{D}} \mathcal{E} \in \mathfrak{S}_X. \quad \square$$

If $\langle \mathcal{D} \rangle = \mathcal{E}$, then we say that \mathcal{D} *generates* or is a *generating set* for \mathcal{E} . It is easy to see that we may characterise joins as a particular generated structure, namely

$$\bigvee_{\alpha \in A} \mathcal{E}_\alpha = \left\langle \bigcup_{\alpha \in A} \mathcal{E}_\alpha \right\rangle. \quad (2.1)$$

PROPOSITION 2.3

Let (X, \mathcal{E}) and (Y, \mathcal{F}) be structured sets, and let $f: X \rightarrow Y$ be any map. For any $\mathcal{D} \subseteq 2^Y$ we have

$$f^{-1}(\langle \mathcal{D} \rangle) = \langle f^{-1}(\mathcal{D}) \rangle.$$

In particular, if $\mathcal{F} = \langle \mathcal{D} \rangle$, then f is a homomorphism if and only if $f^{-1}(\mathcal{D}) \subseteq \mathcal{E}$.

In topology, this proposition is trivial since every element in $\langle \mathcal{D} \rangle$ is a union of finite intersections of elements in \mathcal{D} . The proof below is identical to the one given in measure theory.

PROOF. First notice that $f^{-1}(\mathcal{D}) \subseteq f^{-1}(\langle \mathcal{D} \rangle)$, which implies that

$$\langle f^{-1}(\mathcal{D}) \rangle \subseteq f^{-1}(\langle \mathcal{D} \rangle).$$

For the second inclusion, notice that

$$\mathcal{A} = \{B \subseteq Y \mid f^{-1}(B) \in \langle f^{-1}(\mathcal{D}) \rangle\}$$

is a set structure in Y . Since clearly $\mathcal{D} \subseteq \mathcal{A}$, we also have $\langle \mathcal{D} \rangle \subseteq \mathcal{A}$, which proves the second inclusion. \square

2.2. Initial structures

DEFINITION 2.4: Initial structures

Let $(f_\alpha)_{\alpha \in A}$ be a collection of maps from a set X to structured sets $(X_\alpha, \mathcal{E}_\alpha)$. The *initial structure* \mathcal{E} on X induced by (f_α) is the smallest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigvee_{\alpha \in A} f_\alpha^*(\mathcal{E}_\alpha) = \left\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle.$$

REMARK 2.5. If \mathcal{D}_α is a generating set for \mathcal{E}_α for all $\alpha \in A$, then we may replace \mathcal{E}_α on the right-hand side above with \mathcal{D}_α . This follows immediately from the second part of [Proposition 2.3](#), since the structure $\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha) \rangle$ makes all f_α into homomorphisms.

Note that $\bigvee_{\alpha \in A} f_\alpha^*(\mathcal{D}_\alpha)$ doesn't generally make sense, since \mathcal{D}_α is not necessarily a structure on X_α . \dashv

THEOREM 2.6: Characteristic property of initial structures

Let (X, \mathcal{E}) be a structured set equipped with the initial structure induced by maps $f_\alpha: X \rightarrow X_\alpha$, $\alpha \in A$. If (Y, \mathcal{F}) is a structured set, then $g: Y \rightarrow X$ is a homomorphism if and only if $f_\alpha \circ g$ is a homomorphism for all $\alpha \in A$:

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha \circ g & \uparrow f_\alpha \\ Y & \xrightarrow{g} & X \end{array}$$

Furthermore, the initial structure on X is unique with this property.

PROOF. If g is a homomorphism, then clearly the $f_\alpha \circ g$ are all homomorphisms.

Conversely, assume that all compositions $f_\alpha \circ g$ are homomorphisms. It suffices to show that $g^{-1}(B) \in \mathcal{F}$ for all B from a generating set for \mathcal{E} , so let $B = f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. It follows that

$$g^{-1}(B) = g^{-1}(f_\alpha^{-1}(C)) = (f_\alpha \circ g)^{-1}(C) \in \mathcal{F}$$

as desired.

We now show that the characteristic property uniquely determines a structure on X . First let \mathcal{E} be a structure on X such that a map $g: Y \rightarrow X$ is a homomorphism if and only if each $f_\alpha \circ g$ is a homomorphism. The commutative diagram

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow f_\alpha \\ X & \xrightarrow{\text{id}_X} & X \end{array}$$

then shows that f_α is a homomorphism since id_X trivially is.

Next let \mathcal{E}' be another structure on X with the characteristic property. Consider the commutative diagram

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow f'_\alpha \\ (X, \mathcal{E}) & \xrightarrow{\text{id}_X} & (X, \mathcal{E}') \end{array}$$

where a prime denotes that the domain of a map is (X, \mathcal{E}') but is as a set function the same as its unprimed counterpart. The f_α are homomorphisms, so by the characteristic property applied to \mathcal{E}' we get that id_X is a homomorphism. By symmetry the corresponding primed map $\text{id}'_X: (X, \mathcal{E}') \rightarrow (X, \mathcal{E})$ is also a homomorphism, so (X, \mathcal{E}) and (X, \mathcal{E}') are isomorphic through the identity, and hence $\mathcal{E} = \mathcal{E}'$. \square

EXAMPLE 2.7: Subsets.

Let (X, \mathcal{E}) be a structured set, and let $S \subseteq X$. The inclusion map $\iota_S: S \rightarrow X$ then induces an initial structure on S , namely the pullback $\iota_S^*(\mathcal{E})$. We denote this subset structure by \mathcal{E}_S , and unless otherwise noted subsets of structured sets always carry this structure. By the characteristic property of initial structures, a map $f: Y \rightarrow S$ from a structured set is a homomorphism if and only if $\iota_S \circ f$ is a homomorphism.

On the other hand, if $f: Y \rightarrow X$ is a map with $f(Y) \subseteq S$, then the map $\tilde{f}: Y \rightarrow S$ given by $\tilde{f}(y) = f(y)$ for all $y \in Y$ is a homomorphism if and only if $f = \iota_S \circ \tilde{f}$ is a homomorphism. In other words, whether a map is a homomorphism or not does not depend on the codomain if we agree to equip subsets with the structure induced by their inclusion maps.

If $S = f(Y)$ and $\tilde{f}: Y \rightarrow f(Y)$ is an isomorphism, then we call f an *embedding*. Hence inclusion maps are in particular embeddings. \lrcorner

EXAMPLE 2.8: Products.

Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a collection of structured sets, let $X = \prod_{\alpha \in A} X_\alpha$ be the Cartesian product of the sets X_α , and denote the associated projections by $\pi_\alpha: X \rightarrow X_\alpha$. We define a product structure

$$\mathcal{E} = \bigotimes_{\alpha \in A} \mathcal{E}_\alpha$$

as the initial structure on X induced by the projection maps. Since X is a product of the X_α in the category of sets, the characteristic property of initial structures implies that (X, \mathcal{E}) is a product of the structured sets $(X_\alpha, \mathcal{E}_\alpha)$. \lrcorner

PROPOSITION 2.9: Composition of initial structures

For $\alpha \in A$ and $\lambda \in L_\alpha$, let X_α and $Y_{\alpha\lambda}$ be structured sets such that each X_α carry the initial structure induced by maps $g_{\alpha\lambda}: X_\alpha \rightarrow Y_{\alpha\lambda}$ for $\lambda \in L_\alpha$. Let X be a set, and consider maps $f_\alpha: X \rightarrow X_\alpha$.

Let \mathcal{E}_1 be the initial structure on X induced by the maps f_α , and let \mathcal{E}_2 be the structure induced by the compositions $g_{\alpha\lambda} \circ f_\alpha$. Then $\mathcal{E}_1 = \mathcal{E}_2$.

PROOF. By the characteristic property of initial structures, the f_α are homomorphisms if and only if all the $g_{\alpha\lambda} \circ f_\alpha$ are homomorphisms. If X has the structure \mathcal{E}_1 , then we must have $\mathcal{E}_2 \subseteq \mathcal{E}_1$, since these compositions $g_{\alpha\lambda} \circ f_\alpha$ are homomorphisms. Conversely, if X carries the structure \mathcal{E}_2 , then the f_α are homomorphisms, and so $\mathcal{E}_1 \subseteq \mathcal{E}_2$. \square

EXAMPLE 2.10: Subspace and product structures.

Let $(X_\alpha)_{\alpha \in A}$ be a family of structured sets, and let $S_\alpha \subseteq X_\alpha$ be subsets. Then we may equip the product $S = \prod_{\alpha \in A} S_\alpha$ with a structure by first equipping $X = \prod_{\alpha \in A} X_\alpha$ with the product structure, and then induce the subset structure on S . In the opposite order we may first equip each S_α with the subset structure, and then induce the product structure. These in fact give the same structure since the diagram

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\iota_{S_\alpha}} & X_\alpha \\ \pi_{S_\alpha} \uparrow & & \uparrow \pi_{X_\alpha} \\ S & \xrightarrow{\iota_S} & X \end{array}$$

commutes. \lrcorner

EXAMPLE 2.11: The weak*-topology.

Let X be a topological vector space over the field \mathbb{F} with topological dual X^* , and for $x \in X$ let $\text{ev}_x: X^* \rightarrow \mathbb{F}$ be the evaluation map $\text{ev}_x(\varphi) = \varphi(x)$ for $\varphi \in X^*$. Since X^* is a subset of \mathbb{F}^X , it naturally carries the subspace topology. The product topology on \mathbb{F}^X is induced by the projection maps $\pi_x: \mathbb{F}^X \rightarrow \mathbb{F}$ for $x \in X$. But $\pi_x \circ \iota_{X^*}$ is just the evaluation map ev_x , so the diagram

$$\begin{array}{ccc} & & \mathbb{F} \\ & \nearrow \text{ev}_x & \uparrow \pi_x \\ X^* & \xrightarrow{\iota_{X^*}} & \mathbb{F}^X \end{array}$$

commutes, and the subspace topology on X^* is exactly the weak*-topology. In fact, since the evaluation maps clearly separate points in X^* , the inclusion ι_{X^*} is in fact an embedding, as the following proposition shows: \lrcorner

PROPOSITION 2.12: Embedding into product

Let $f_\alpha: Y \rightarrow X_\alpha$ for $\alpha \in A$ be homomorphisms, let $X = \prod_{\alpha \in A} X_\alpha$, and let $f: Y \rightarrow X$ be the unique homomorphism such that $f_\alpha = \pi_\alpha \circ f$:

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow \pi_\alpha \\ Y & \xrightarrow{f} & X \end{array}$$

Then f is an embedding if and only if Y carries the initial structure induced by the maps f_α and the collection $(f_\alpha)_{\alpha \in A}$ separates points in Y .

Recall that the collection $(f_\alpha)_{\alpha \in A}$ separates points in Y if for each pair $x, y \in Y$ with $x \neq y$ there exists an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$.

PROOF. Notice that f is unique as a set function since X is a product in **Set**, and f is a homomorphism by the characteristic property of initial structures.

First assume that f is an embedding. In particular it is injective, and since the maps π_α separate points in X , the compositions $f_\alpha = \pi_\alpha \circ f$ separate points in Y . Let $\tilde{f}: Y \rightarrow f(Y)$ be the isomorphism such that $f = \iota_{f(Y)} \circ \tilde{f}$. Then since \tilde{f} is an isomorphism, in particular Y carries the initial structure induced by \tilde{f} . But then Y carries the initial structure induced by the maps

$$\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f} = \pi_\alpha \circ f = f_\alpha \quad (2.2)$$

for $\alpha \in A$, as claimed.

Conversely, assume that the f_α separate points in Y and that Y has the initial structure \mathcal{F} induced by the f_α . The f_α are then homomorphisms, and by the characteristic property of initial structures so is f . Furthermore, if $x, y \in Y$ with $x \neq y$, then there is an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$, which implies that $f(x) \neq f(y)$, so f is injective.

Denote the product structure on X by \mathcal{E} . We show that if $B \in \mathcal{F}$, then $f(B) \in \mathcal{E}_{f(Y)}$, which will imply that f is an embedding. It suffices to prove this when B is an element of a generating set for \mathcal{F} , i.e. on the form $f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. By (2.2) we have

$$B = f_\alpha^{-1}(C) = (\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f})^{-1}(C) = \tilde{f}^{-1}((\pi_\alpha \circ \iota_{f(Y)})^{-1}(C)),$$

from which it follows that

$$f(B) = \tilde{f}(B) = (\pi_\alpha \circ \iota_{f(Y)})^{-1}(C) \in \mathcal{E}_{f(Y)}$$

as desired. In the second equality we use that \tilde{f} is surjective. \square

If \mathcal{E} is a structure on a set X , we say that \mathcal{E} is *countably generated* if there is a countable collection of sets $\mathcal{D} \subseteq 2^X$ such that $\mathcal{E} = \langle \mathcal{D} \rangle$.

PROPOSITION 2.13: *Countably generated initial structures*

Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a countable collection of structured sets, and assume that the \mathcal{E}_α are countably generated by collections of sets \mathcal{D}_α . If an initial structure \mathcal{E} is induced on a set X by maps $f_\alpha: X \rightarrow X_\alpha$, then \mathcal{E} is also countably generated.

PROOF. This follows immediately by [Remark 2.5](#) since the generating set $\bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$ is a countable union of countable sets. \square

EXAMPLE 2.14: *Second-countable topological spaces.*

A topology is second-countable if and only if it is countably generated, as we show below. The above proposition then implies that an initial topology induced by a countable family of maps into second-countable spaces is itself second-countable. In particular, subspaces and countable products of second-countable spaces are second-countable.

Now to prove the above claim: Since a basis in particular is a generating set (i.e. a subbasis), second-countable topologies are countably generated. Conversely, let \mathcal{T} be a topology that is generated by a countable set \mathcal{D} . Then a basis \mathcal{B} for \mathcal{T} is obtained by taking finite intersections of elements from \mathcal{D} . The number of these intersections is certainly less than the cardinality of the union

$$\bigcup_{n \in \mathbb{N}} \mathcal{D}^n$$

of all finite products of \mathcal{D} with itself, an element $U_1 \times \cdots \times U_n$ of an n -fold product corresponding to the intersection $\bigcap_{i=1}^n U_i$. But finite products of countable sets are countable, and so are countable unions of countable sets, so the union above is countable. \lrcorner

2.3. Final structures

DEFINITION 2.15: *Final structures*

Let $(f_\alpha)_{\alpha \in A}$ be a collection of maps from structured sets $(X_\alpha, \mathcal{E}_\alpha)$ to a set X . The *final structure* \mathcal{E} on X coinduced by (f_α) is the largest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigwedge_{\alpha \in A} (f_\alpha)_*(\mathcal{E}_\alpha).$$

THEOREM 2.16: *Characteristic property of final structures*

Let (X, \mathcal{E}) be a structured set equipped with the final structure coinduced by

maps $f_\alpha: X_\alpha \rightarrow X$, $\alpha \in A$. If (Y, \mathcal{F}) is a structured set, then $g: X \rightarrow Y$ is a homomorphism if and only if $g \circ f_\alpha$ is a homomorphism for all $\alpha \in A$:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & X \\ & \searrow g \circ f_\alpha & \downarrow g \\ & & Y \end{array}$$

Furthermore, the final structure on X is unique with this property.

PROOF. If g is a homomorphism, then clearly the $g \circ f_\alpha$ are all homomorphisms.

Conversely, assume that the $g \circ f_\alpha$ are homomorphisms. Notice that

$$f_\alpha^{-1}(g^{-1}(\mathcal{F})) = (g \circ f_\alpha)^{-1}(\mathcal{F}) \subseteq \mathcal{E}_\alpha,$$

since the $g \circ f_\alpha$ are homomorphisms. But then $g^{-1}(\mathcal{F}) = g^*(\mathcal{F})$ is a structure on X with respect to which the f_α are homomorphisms, and thus $g^{-1}(\mathcal{F}) \subseteq \mathcal{E}$ since \mathcal{E} is the largest structure with this property. Hence g is a homomorphism.

The proof of uniqueness is analogous to the proof of uniqueness in [Theorem 2.6](#), so we omit it. \square

EXAMPLE 2.17: Quotient structures.

Let X be a structured set, and let Y be any set. If $q: X \rightarrow Y$ is a surjective map, the final structure on Y induced by q is called the *quotient structure*. In particular, if \sim is an equivalence relation on X and $q: X \rightarrow X/\sim$ is the associated natural map, we will always equip X/\sim with the quotient structure induced by q . \lrcorner

EXAMPLE 2.18: Disjoint unions.

If $(X_\alpha)_{\alpha \in A}$ is a collection of structured sets, we equip the disjoint union $X = \coprod_{\alpha \in A} X_\alpha$ with the final structure coinduced by the inclusions $\iota_\alpha: X_\alpha \rightarrow X$.

We will see in [Subsection 2.4](#) that X is a coproduct of the X_α in $\mathbf{Str}_{\mathcal{G}}$, but this fact will be convenient in the proof of [Proposition 2.20](#). \lrcorner

PROPOSITION 2.19: Composition of final structures

For $\alpha \in A$ and $\lambda \in L_\alpha$, let X_α and $Y_{\alpha\lambda}$ be structured sets such that each X_α carry the final structure induced by maps $g_{\alpha\lambda}: Y_{\alpha\lambda} \rightarrow X_\alpha$ for $\lambda \in L_\alpha$. Let X be a set, and consider maps $f_\alpha: X_\alpha \rightarrow X$.

Let \mathcal{E}_1 be the final structure on X induced by the maps f_α , and let \mathcal{E}_2 be the structure induced by the compositions $f_\alpha \circ g_{\alpha\lambda}$. Then $\mathcal{E}_1 = \mathcal{E}_2$.

PROOF. This result is just the dual of [Proposition 2.9](#), so we omit the proof. \square

PROPOSITION 2.20: *Final structures as quotient structures*

Let $f_\alpha: X_\alpha \rightarrow Y$ for $\alpha \in A$ be homomorphisms, let $X = \coprod_{\alpha \in A} X_\alpha$, and let $f: X \rightarrow Y$ be the unique homomorphism such that $f_\alpha = f \circ \iota_\alpha$:

$$\begin{array}{ccc} X_\alpha & & \\ \iota_\alpha \downarrow & \searrow f_\alpha & \\ X & \xrightarrow{f} & Y \end{array}$$

Define an equivalence relation \sim on X by letting $x \sim x'$ if $f(x) = f(x')$ for $x, x' \in X$. Let $q: X \rightarrow X/\sim$ be the associated quotient map, and let $\tilde{f}: X/\sim \rightarrow Y$ be the unique homomorphism such that $\tilde{f} \circ q = f$.

Assume that f is surjective. Then Y has the final structure coinduced by the f_α if and only if \tilde{f} is an isomorphism.

The condition that f is surjective is equivalent to the property that every point in Y is in the image of some f_α . In this case we say that the maps f_α *cover* points in Y .

Notice also that by [Proposition 2.19](#), the final structure on Y coinduced by the f_α is the same as the structure coinduced by f .

PROOF. Notice that f is unique as a set function since X is a coproduct in **Set**, and it is a homomorphism by the characteristic property of final structures.

Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow q & \nearrow \tilde{f} \\ & X/\sim & \end{array}$$

First assume that \tilde{f} is an isomorphism. Then Y in particular carries the final structure coinduced by \tilde{f} . But X/\sim has the final structure coinduced by q , so Y has the final structure coinduced by the map $\tilde{f} \circ q = f$ by [Proposition 2.19](#) as claimed.

Conversely, assume that Y has the final structure \mathcal{F} coinduced by the f_α . Let \mathcal{E} denote the disjoint union structure on X and $\tilde{\mathcal{E}}$ the quotient structure on X/\sim . Notice that \tilde{f} is surjective since f is surjective, and it is injective by definition of \sim . We thus need to prove that \tilde{f}^{-1} is a homomorphism, and it suffices to show that $\tilde{f}(\tilde{\mathcal{E}}) \subseteq \mathcal{F}$. This is the case if and only if

$$q^{-1}(\tilde{\mathcal{E}}) = f^{-1}(\tilde{f}(\tilde{\mathcal{E}})) \subseteq \mathcal{E},$$

and this is true since q is a homomorphism. □

2.4. Categorical properties

The category $\mathbf{Str}_{\mathfrak{S}}$

We first recapitulate some of the above results in categorical terms. The main result is the following:

THEOREM 2.21: Completeness of $\mathbf{Str}_{\mathfrak{S}}$

The category $\mathbf{Str}_{\mathfrak{S}}$ is complete, i.e. it has all small limits.

PROOF. By e.g. Smith (2018, Theorem 60) it is enough to show that $\mathbf{Str}_{\mathfrak{S}}$ has all small products and has equalisers.

Products: We claim that the product (X, \mathcal{E}) considered in Example 2.8 is in fact a product of the objects $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ in $\mathbf{Str}_{\mathfrak{S}}$. If Y is a structured set and $f_\alpha: Y \rightarrow X_\alpha$ are homomorphisms, then since X is a product in \mathbf{Set} there is a unique set function $f: Y \rightarrow X$ such that $f_\alpha = \pi_\alpha \circ f$ for all $\alpha \in A$. But f is also a homomorphism by the characteristic property of the product structure, so (X, \mathcal{E}) is in fact a product in $\mathbf{Str}_{\mathfrak{S}}$.

Equalisers: Let $f, g: X \rightarrow Y$ be any pair of parallel homomorphisms, and let E be the subset of X on which they agree. If $h: Z \rightarrow X$ is any homomorphism such that $f \circ h = g \circ h$, then there is a unique homomorphism $u: Z \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\iota_E} & X \\ \uparrow u & \nearrow h & \downarrow \\ Z & & \end{array} \quad \begin{array}{c} f \\ \rightrightarrows \\ g \end{array} \quad Y$$

We must have $h(Z) \subseteq E$, so we can define u by $u(z) = h(z)$, and u is unique as a set function such that the above diagram commutes. Furthermore, u is a homomorphism by the characteristic property of the subset structure. Thus E along with the inclusion map ι_E is an equaliser of f and g . \square

Functoriality of \mathfrak{S}

As mentioned, \mathfrak{S} is in fact a functor. Its action on a set function $f: X \rightarrow Y$ is defined as the pullback $f^*: \mathfrak{S}_Y \rightarrow \mathfrak{S}_X$.

Let \mathbf{Set} denote the category of sets and \mathbf{CsLat}^\vee the category of complete join-semilattices and join-preserving maps.

PROPOSITION 2.22: Functoriality of \mathfrak{S} , I

The map \mathfrak{S} is a contravariant functor from \mathbf{Set} to \mathbf{CsLat}^\vee .

PROOF. By (2.1) and Proposition 2.3, for any set function f the pullback f^* preserves joins since preimages respect unions, so it is well-defined as a map $\mathbf{Set} \rightarrow \mathbf{CsLat}^\vee$.

The map \mathfrak{S} is also contravariant since if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are set functions, then for $\mathcal{G} \in \mathfrak{S}_Z$ we have

$$(g \circ f)^*(\mathcal{G}) = (g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = (f^* \circ g^*)(\mathcal{G}).$$

Its action on identity functions is clearly trivial, so it is a functor. \square

In the case of topological spaces or measure spaces we can say slightly more: Notice that in both a lattice of topologies or of σ -algebras on a set, intersections of topologies (σ -algebras) are themselves topologies (σ -algebras). A nonempty subset \mathfrak{L} of 2^X , where X is some set, is called an *intersection structure*. If also $X \in \mathfrak{L}$ we call it a *topped intersection structure*. It is easy to show that topped intersection structures are complete lattices ordered by inclusion, and that meets are given by intersections.

If \mathbf{CLat} denotes the category of complete lattices with join- and meet-preserving maps, then we have the following:

PROPOSITION 2.23: *Functoriality of \mathfrak{S} , II*

If \mathfrak{S}_X is an intersection structure for all sets X , then \mathfrak{S} is a contravariant functor from \mathbf{Set} to \mathbf{CLat} .

PROOF. It suffices to show that f^* preserves meets for all set functions $f: X \rightarrow Y$. But this is clear since preimages respect intersections. \square

It is natural to ask whether the pushforward f_* by f gives rise to a covariant functor from \mathbf{Set} into a category of lattices. It is easy to see that f_* is monotone, and a short calculation shows that $(g \circ f)_* = g_* \circ f_*$ if $g: Y \rightarrow Z$ is another set function, so the pushforward does indeed define a covariant functor from \mathbf{Set} to the category \mathbf{Pos} of posets and monotone maps. But f_* does not, as far as I know, generally preserve meets or joins. If all \mathfrak{S}_X are intersection structures, however, it does preserve meets, so it is then a functor into the category \mathbf{CsLat}^\wedge of complete meet-semilattices, though it still does not seem to preserve joins. (I haven't looked too hard for counterexamples.)

The forgetful functor on $\mathbf{Str}_{\mathfrak{S}}$

Just as there is a forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ that sends a topological space to its underlying set, there is a forgetful functor $U: \mathbf{Str}_{\mathfrak{S}} \rightarrow \mathbf{Set}$. As usual, U has a left adjoint D that equips a set with the discrete structure, and it has a right adjoint T that equips the set with the trivial structure, i.e. $D \dashv U \dashv T$.

It follows immediately that U preserves both limits and colimits. Hence if $\mathbf{Str}_{\mathcal{E}}$ e.g. has coproducts – which it has, as we will see later – we already know that they have to be (isomorphic to) disjoint unions of the underlying sets, equipped with an appropriate structure. Contrast this with the situation in the category \mathbf{Grp} of groups: The forgetful functor $U: \mathbf{Grp} \rightarrow \mathbf{Set}$ has a left adjoint, namely the free functor, so U preserves limits. But it does not preserve colimits; for instance, coproducts in \mathbf{Grp} are free products, and their underlying sets are certainly not disjoint unions! Hence U does not have a right adjoint.

Presheaves on structured sets

Let (X, \mathcal{E}) be a structured set, and view \mathcal{E} as a preorder category. Analogous to the case of topological spaces, a presheaf on \mathcal{E} (i.e. a contravariant functor $\mathcal{E} \rightarrow \mathbf{Set}$) is called a *presheaf* on (X, \mathcal{E}) , or simply a presheaf on X if the structure is understood.

As an example, fix a structured set Y and take the presheaf F on X given by $F(B) = \mathbf{Str}_{\mathcal{E}}(B, Y)$, i.e. F sends a set $B \in \mathcal{E}$ to the set of homomorphisms $B \rightarrow Y$. Furthermore, F sends an inclusion $B \subseteq B'$ in \mathcal{E} to the restriction map $\mathbf{Str}_{\mathcal{E}}(B', Y) \rightarrow \mathbf{Str}_{\mathcal{E}}(B, Y)$ given by $f \mapsto f|_B$. A common example of this is the case $\mathbf{Str}_{\mathcal{E}} = \mathbf{Top}$ and $Y = \mathbb{R}$, in which case F sends an open set U to the set of continuous functions $U \rightarrow \mathbb{R}$.

3 • Topology

We remind the reader that a topological space (X, \mathcal{T}) is *second-countable* if there exists a countable basis for \mathcal{T} . Furthermore, (X, \mathcal{T}) is said to be *Lindelöf* if every open cover of X has a countable subcover.

Recall also that a topological property is called *hereditary* if it follows from a space X having this property that any subspace of X also has this property. It is easy to see that second-countability is hereditary, but the Lindelöf property is not:

REMARK 3.1. A space can be Lindelöf without being hereditarily Lindelöf. Let (X, \mathcal{T}) be an uncountable discrete space, and let $y \notin X$. Define a topological space (Y, \mathcal{T}') with underlying space $Y = X \cup \{y\}$ and topology $\mathcal{T}' = \mathcal{T} \cup \{Y\}$. Then (Y, \mathcal{T}') is Lindelöf since any open cover must include Y itself, this being the only open set containing the point y . But the subspace X (whose subspace topology is exactly \mathcal{T}) is clearly not Lindelöf. \lrcorner

If every subspace of X is Lindelöf, then we say that X is *hereditarily Lindelöf*.

PROPOSITION 3.2

If (X, \mathcal{T}) is a second-countable topological space, then it is hereditarily Lindelöf.

PROOF. Every subspace of X is second-countable, so it suffices to show that X is Lindelöf.

Let \mathcal{U} be an open cover of X and let \mathcal{B} be a countable basis for the topology \mathcal{T} . Consider an $x \in X$. Since \mathcal{U} is a cover of X there is some $U_x \in \mathcal{U}$ with $x \in U_x$, and since \mathcal{B} is a basis for \mathcal{T} there is some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U_x$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the subset of open sets obtained in this way. Clearly \mathcal{B}' is a cover of X .

For each $B \in \mathcal{B}'$, the above shows that there exists some $U \in \mathcal{U}$ with $B \subseteq U$. This defines a map $\mathcal{B}' \rightarrow \mathcal{U}$ given by $B \mapsto U$ whose image is a countable cover of X , proving the claim. \square

LEMMA 3.3

Let (X, \mathcal{T}) be a second-countable space. Then every basis for \mathcal{T} contains a countable basis for \mathcal{T} .

PROOF. Let \mathcal{B} be a basis for \mathcal{T} , and let \mathcal{C} be a countable basis. We can write every $C \in \mathcal{C}$ on the form $C = \bigcup_{\alpha \in A} B_\alpha$ for some family $(B_\alpha)_{\alpha \in A} \subseteq \mathcal{B}$. This is in particular an open cover of C , so since X is hereditarily Lindelöf there is a countable subset $A' \subseteq A$ such that $C = \bigcup_{\alpha \in A'} B_\alpha$. For each $C \in \mathcal{C}$ we thus obtain a countable subcollection of sets from \mathcal{B} , and since \mathcal{C} is also countable, the union of all these sets is countable and is clearly a basis for \mathcal{T} . \square

4 • Measure theory

If A is a subset of a measurable space (X, \mathcal{E}) , then recall that we denote by \mathcal{E}_A the initial σ -algebra on A induced by the inclusion $\iota_A: A \rightarrow X$. Similarly, if A is a subspace of a topological space (X, \mathcal{T}) , denote by \mathcal{T}_A the subspace topology on A .

PROPOSITION 4.1

Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$. Then $\mathcal{B}(A) = \mathcal{B}(X)_A$, i.e. $\sigma(\mathcal{T}_A) = \sigma(\mathcal{T})_A$.

PROOF. Notice that

$$\sigma(\mathcal{T}_A) = \sigma(\iota_A^{-1}(\mathcal{T})) = \iota_A^{-1}(\sigma(\mathcal{T})) = \sigma(\mathcal{T})_A$$

by Proposition 2.3. \square

THEOREM 4.2: Products of Borel σ -algebras

Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces, and equip $X = \prod_{\alpha \in A} X_\alpha$ with the product topology \mathcal{T} . Then

$$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) \subseteq \mathcal{B}(X)$$

If A is countable and the spaces X_α are second-countable, then the above inclusion is an equality.

PROOF. Since the projections $\pi_\alpha: X \rightarrow X_\alpha$ are continuous, they are $\mathcal{B}(X)$ - $\mathcal{B}(X_\alpha)$ -measurable. But $\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$ is the smallest σ -algebra on X that makes the projections measurable, which proves the above inclusion.

Now assume that A is countable and that all the X_α are second-countable. From [Example 2.14](#) we know that X is also second-countable. Let

$$\mathcal{D} = \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{T}_\alpha)$$

be a subbasis for the product topology \mathcal{T} , and let \mathcal{B} be the collection of finite intersections of elements in \mathcal{D} . Then \mathcal{B} is a basis for \mathcal{T} , and \mathcal{B} contains a countable basis \mathcal{C} for \mathcal{T} by [Lemma 3.3](#). Since \mathcal{C} is countable, open sets in X are countable unions of finite intersections of elements in \mathcal{D} . Since A is also countable, it suffices to show that

$$\pi_\beta^{-1}(\mathcal{T}_\beta) \subseteq \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$$

for all $\beta \in A$. But this is obvious since the projections are measurable. \square

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