Notes on measure theory and topology

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1 • Introduction

These notes are meant to serve two purposes: Firstly to give an account of (some of) the similarities between topological spaces and measurable spaces. Any student of topology and measure theory have noticed that while σ-algebras generally do not behave as nicely as topologies, we are able to perform many of the same constructions on both structures: Structure-preserving maps (countinuous and measurable maps, respectively) are defined the same way, maps induce topologies and σ-algebras in the same way, there are subspaces, products, quotients, and so on.

If we fix a set X, both the set of topologies and the set of σ-algebras on X are complete lattices when ordered by inclusion. I am not aware that such a lattice of structures on a set has a commonly used name, so I have simply called them *structures* in these notes.

Secondly we wish to explore how a topological and measure-theoretical structure on a single set interact.

2 • Structured sets

2.1. Definitions and basic properties

Let $\mathfrak{S}$ be a map from sets to sets such that $\mathfrak{S}_X := \mathfrak{S}(X)$ is a collection of subsets of 2^X, and such that for all sets X and Y and maps $f\colon X \to Y$,

(1) $\mathfrak{S}_X$ is partially ordered by set inclusion,

(2) $\mathfrak{S}_X$ is a complete lattice with minimum $\{\emptyset, X\}$ and maximum 2^X,

(3) if $\mathcal{F} \in \mathfrak{S}_Y$, then $f^{-1}(\mathcal{F}) \in \mathfrak{S}_X$, and

(4) if $\mathcal{E} \in \mathfrak{S}_X$, then
$$\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\} \in \mathfrak{S}_Y.$$

We will call such a map $\mathfrak{S}$ a *structure functor*, and it is indeed a functor as we will see in Subsection 2.2.

If X is a set, then a $\mathcal{E} \in \mathfrak{S}_X$ is called a $\mathfrak{S}$-*structure* on X, and we will call the pair $(X, \mathcal{E})$ a $\mathfrak{S}$-*structured set*. We refer to $\mathfrak{S}_X$ as the *lattice of* $\mathfrak{S}$-*structures* on X. The minimal structure $\{\emptyset, X\}$ is called the *trivial structure*, and the maximal structure 2^X is called the *discrete structure* on X.

Fix a structure functor $\mathfrak{S}$. If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are structured sets, a *homomorphism* from X to Y is a map $f: X \to Y$ such that $f^{-1}(B) \in \mathcal{E}$ for all $B \in \mathcal{F}$. Clearly the composition of two homomorphisms is again a homomorphism, so the collection of structured sets and homomorphisms form a (locally small) category. Let us denote this category by $\mathbf{Str}_{\mathfrak{S}}$.

The structure $f^{-1}(\mathcal{F})$ in Item (3) is called the *pullback* of $\mathcal{F}$ by f and is denoted $f^*(\mathcal{F})$. Similarly, the structure $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\}$ in Item (4) is called the *pushforward* of $\mathcal{E}$ by f and is denoted $f_*(\mathcal{E})$.

Example 2.1. Let $\mathfrak{S}$ denote the map that associates to a set its lattice of topologies. The first two conditions above are obviously satisfied, and the latter two are easily proved. Thus $\mathbf{Str}_{\mathfrak{S}}$ is just the category **Top** of topological spaces. Similarly, if $\mathfrak{S}$ maps a set to its lattice of σ-algebras, then $\mathbf{Str}_{\mathfrak{S}}$ is the category **Mble** of measurable spaces. ⌟

In the sequel we fix a structure functor $\mathfrak{S}$.

Lemma 2.2

Let X be a set. If $\mathcal{D} \subseteq 2^X$, then there is a smallest element $\langle \mathcal{D} \rangle \in \mathfrak{S}_X$ with $\mathcal{D} \subseteq \langle \mathcal{D} \rangle$.

Proof. Let $\Sigma(\mathcal{D}) = \{\mathcal{E} \in \mathfrak{S}_X \mid \mathcal{D} \subseteq \mathcal{E}\}$. Since $\mathfrak{S}_X$ is a complete lattice, we can put
$$\langle \mathcal{D} \rangle = \bigwedge_{\mathcal{E} \in \Sigma(\mathcal{D})} \mathcal{E} \in \mathfrak{S}_X. \qquad \square$$

If $\langle \mathcal{D} \rangle = \mathcal{E}$, then we say that $\mathcal{D}$ *generates* or is a *generating set* for $\mathcal{E}$. It is easy to see that we may characterise joins as a particular generated structure, namely
$$\bigvee_{\alpha \in A} \mathcal{E}_\alpha = \left\langle \bigcup_{\alpha \in A} \mathcal{E}_\alpha \right\rangle. \tag{2.1}$$

Proposition 2.3

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be structured sets, and let $f: X \to Y$ be any map. For any $\mathcal{D} \in 2^Y$ we have
$$f^{-1}\big(\langle \mathcal{D} \rangle\big) = \big\langle f^{-1}(\mathcal{D}) \big\rangle.$$

In particular, if $\mathcal{F} = \langle\mathcal{D}\rangle$, then f is a homomorphism if and only if $f^{-1}(D) \in \mathcal{E}$ for all $D \in \mathcal{D}$.

In topology, this proposition is trivial since every element in $\langle\mathcal{D}\rangle$ is a union of finite intersections of elements in $\mathcal{D}$. The proof below is identical to the one given in measure theory.

Proof. First notice that $f^{-1}(\mathcal{D}) \subseteq f^{-1}(\langle\mathcal{D}\rangle)$, which implies that

$$\left\langle f^{-1}(\mathcal{D})\right\rangle \subseteq f^{-1}\big(\langle\mathcal{D}\rangle\big).$$

For the second inclusion, notice that

$$\mathcal{A} = \left\{B \subseteq Y \;\middle|\; f^{-1}(B) \in \left\langle f^{-1}(\mathcal{D})\right\rangle\right\}$$

is a set structure in Y. Since clearly $\mathcal{D} \subseteq \mathcal{A}$, we also have $\langle\mathcal{D}\rangle \subseteq \mathcal{A}$, which proves the second inclusion. □

2.2. Categorical properties

Functoriality of $\mathfrak{S}$

As mentioned, $\mathfrak{S}$ is in fact a functor. Its action on a set function $f: X \to Y$ is defined as the pullback $f^*: \mathfrak{S}_Y \to \mathfrak{S}_X$.

Let **Set** denote the category of sets and $\mathbf{CsLat}^{\vee}$ the category of complete join-semilattices and join-preserving maps.

Proposition 2.4: *Functoriality of $\mathfrak{S}$, I*

The map $\mathfrak{S}$ is a contravariant functor from **Set** *to* $\mathbf{CsLat}^{\vee}$.

Proof. By (2.1) and Proposition 2.3, for any set function f the pullback f^* preserves joins since preimages respect unions, so it is well-defined as a map $\mathbf{Set} \to \mathbf{CsLat}^{\vee}$.

The map $\mathfrak{S}$ is also contravariant since if $f: X \to Y$ and $g: Y \to Z$ are set functions, then for $\mathcal{G} \in \mathfrak{S}_Z$ we have

$$(g \circ f)^*(\mathcal{G}) = (g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = (f^* \circ g^*)(\mathcal{G}).$$

Its action on identity functions is clearly trivial, so it is a functor. □

In the case of topological spaces or measure spaces we can say slightly more: Notice that in both a lattice of topologies or of σ-algebras on a set, intersections of topologies (σ-algebras) are themselves topologies (σ-algebras). A nonempty subset $\mathfrak{L}$ of 2^X, where X is some set, is called an *intersection structure*. If also $X \in \mathfrak{L}$ we call it a *topped intersection structure*. It is easy to show that topped intersection structures are complete lattices ordered by inclusion, and that meets are given by intersections.

If **CLat** denotes the category of complete lattices with join- and meet-preserving maps, then we have the following:

PROPOSITION 2.5: *Functoriality of $\mathfrak{S}$, II*

If $\mathfrak{S}_X$ is an intersection structure for all sets X, then $\mathfrak{S}$ is a contravariant functor from **Set** *to* **CLat**.

PROOF. It suffices to show that f^* preserves meets for all set functions $f\colon X \to Y$. But this is clear since preimages respect intersections. □

It is natural to ask whether the pushforward f_* by f gives rise to a covariant functor from **Set** into a category of lattices. Since f_* is monotone, and we easily see that $(g \circ f)_* = g_* \circ f_*$ if $g\colon Y \to Z$ is a set function, so this does indeed define a covariant functor from **Set** to the category **Pos** of posets and monotone maps. But f_* does not, as far as I know, generally preserve meets or joins. If all $\mathfrak{S}_X$ are intersection structures, however, it does preserve meets, so it is then a functor into the category $\mathbf{CsLat}^{\wedge}$ of complete meet-semilattices, though it still does not seem to preserve joins. (I haven't looked too hard for counterexamples.)

The forgetful functor on $\mathbf{Str}_{\mathfrak{S}}$

Just as there is a forgetful functor **Top** → **Set** that sends a topological space to its underlying set, there is a forgetful functor $U\colon \mathbf{Str}_{\mathfrak{S}} \to \mathbf{Set}$. As usual, U has a left adjoint D that equips a set with the discrete structure, and it has a right adjoint T that equips the set with the trivial structure, i.e. $D \dashv U \dashv T$.

It follows immediately that U preserves both limits and colimits. Hence if $\mathbf{Str}_{\mathfrak{S}}$ e.g. has coproducts – which it has, as we will see later – we already know that they have to be (isomorphic to) disjoint unions of the underlying sets, equipped with an appropriate structure.

Presheaves on structured sets

Let $(X, \mathcal{E})$ be a structured set, and view $\mathcal{E}$ as a preorder category. Analogous to the case of topological spaces, a presheaf on $\mathcal{E}$ (i.e. a contravariant functor $\mathcal{E} \to$ **Set**) is called a *presheaf* on $(X, \mathcal{E})$, or simply a presheaf on X if the structure is understood.

As an example, fix a structured set Y and take the presheaf F on X given by $F(B) = \mathbf{Str}_{\mathfrak{S}}(B, Y)$, i.e. F sends a set $B \in \mathcal{E}$ to the set of homomorphisms $B \to Y$. Furthermore, F sends an inclusion $B \subseteq B'$ in $\mathcal{E}$ to the restriction map $\mathbf{Str}_{\mathfrak{S}}(B', Y) \to \mathbf{Str}_{\mathfrak{S}}(B, Y)$ given by $f \mapsto f|_B$. (This of course requires that we have a notion of subsets inheriting structures. Indeed we do, see Example 2.9.)

A common example of this is the case $\mathbf{Str}_{\mathfrak{S}} = \mathbf{Top}$ and $Y = \mathbb{R}$, in which case F sends an open set U to the set of continuous functions $U \to \mathbb{R}$.

2.3. *Initial structures*

DEFINITION 2.6: *Initial structures*

Let $(f_\alpha)_{\alpha\in A}$ be a collection of maps from a set X to structured sets $(X_\alpha, \mathcal{E}_\alpha)$. The *initial structure* $\mathcal{E}$ on X induced by (f_α) is the smallest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigvee_{\alpha\in A} f_\alpha^{-1}(\mathcal{E}_\alpha) = \Big\langle \bigcup_{\alpha\in A} f_\alpha^{-1}(\mathcal{E}_\alpha) \Big\rangle.$$

REMARK 2.7. If $\mathcal{D}_\alpha$ is a generating set for $\mathcal{E}_\alpha$ for all $\alpha \in A$, then we may replace $\mathcal{E}_\alpha$ above with $\mathcal{D}_\alpha$. This follows immediately from the second part of Proposition 2.3, since the structure $\big\langle \bigcup_{\alpha\in A} f_\alpha^{-1}(\mathcal{D}_\alpha) \big\rangle$ makes all f_α into homomorphisms. ⌟

THEOREM 2.8: *Characteristic property of initial structures*

Let $(X, \mathcal{E})$ be a structured set equipped with the initial structure induced by maps $f_\alpha : X \to X_\alpha$, $\alpha \in A$. If $(Y, \mathcal{F})$ is a structured set, then $f : Y \to X$ is a homomorphism if and only if $f_\alpha \circ f$ is a homomorphism for all $\alpha \in A$:

$$\begin{tikzcd} X \arrow[r, "f_\alpha"] & X_\alpha \\ Y \arrow[u, "f"] \arrow[ur, "f_\alpha\circ f"'] & \end{tikzcd}$$

In particular, the maps f_α are homomorphisms.

Furthermore, the initial structure on X is unique with this property.

PROOF. If f is a homomorphism, then clearly all $f_\alpha \circ f$ are all homomorphisms.

Conversely, assume that all compositions $f_\alpha \circ f$ are homomorphisms. It suffices to show that $f^{-1}(B) \in \mathcal{F}$ for all B from a generating set for $\mathcal{E}$, so let $B = f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. It follows that

$$f^{-1}(B) = f^{-1}(f_\alpha^{-1}(C)) = (f_\alpha \circ f)^{-1}(C) \in \mathcal{F}$$

as desired. It now follows that f_α is a homomorphism since the diagram

$$\begin{tikzcd} X \arrow[r, "f_\alpha"] & X_\alpha \\ X \arrow[u, "\mathrm{id}_X"] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

commutes, and id_X is always a homomorphism.

Now assume that $\mathcal{E}'$ is a structure on X with the characteristic property of the initial structure. Consider the commutative diagram

$$\begin{tikzcd} (X,\mathcal{E}') \arrow[r, "f'_\alpha"] & X_\alpha \\ (X,\mathcal{E}) \arrow[u, "\mathrm{id}_X"] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

where a prime denotes that the domain of a map is $(X,\mathcal{E}')$ but is otherwise the same as its unprimed counterpart. The f'_α are homomorphisms, since this fact only depends on $(X,\mathcal{E}')$ satisfying the characteristic property of initial structures. It then follows from this property that id_X is a homomorphism.

In the same way the diagram

$$\begin{tikzcd} (X,\mathcal{E}) \arrow[r, "f_\alpha"] & X_\alpha \\ (X,\mathcal{E}') \arrow[u, "\mathrm{id}'_X"] \arrow[ur, "f'_\alpha"'] & \end{tikzcd}$$

shows that id'_X is a homomorphism, and so $(X,\mathcal{E})$ and $(X,\mathcal{E}')$ are isomorphic through the identity, hence $\mathcal{E} = \mathcal{E}'$. $\square$

EXAMPLE 2.9: Subsets.

Let $(X,\mathcal{E})$ be a structured set, and let $S \subseteq X$. The inclusion map $\iota_S \colon S \to X$ then induces an initial structure on S, namely the pullback $\mathcal{E}_S = \iota_S^*(\mathcal{E})$. By the characteristic property of initial structures, a map $f \colon Y \to S$ from a structured set is a homomorphism if and only if $\iota_S \circ f$ is a homomorphism.

On the other hand, if $f \colon Y \to X$ is a map with $f(Y) \subseteq S$, then the map $\tilde{f} \colon Y \to S$ given by $\tilde{f}(y) = f(y)$ for all $y \in Y$ is a homomorphism if and only if $f = \iota_S \circ \tilde{f}$ is a homomorphism. In other words, whether a map is a homomorphism or not does not depend on the codomain if we agree to equip subsets with the structure induced by their inclusion maps.

If $S = f(Y)$ and $\tilde{f} \colon Y \to f(Y)$ is an isomorphism, then we call f an *embedding*. ⌟

EXAMPLE 2.10: Products.
Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a collection of structured sets, let $X = \prod_{\alpha \in A} X_\alpha$ be the Cartesian product of the sets X_α, and denote the associated projections by $\pi_\alpha \colon X \to X_\alpha$. We define a product structure

$$\mathcal{E} = \bigotimes_{\alpha \in A} \mathcal{E}_\alpha$$

as the initial structure on X induced by the projection maps. Since X is a product of the X_α in the category of sets, the characteristic property of initial structures implies that $(X, \mathcal{E})$ is a product of the structured sets $(X_\alpha, \mathcal{E}_\alpha)$. This shows that the category $\mathbf{Str}_{\mathfrak{S}}$ has all small products. ⌟

PROPOSITION 2.11: *Composition of initial structures*

Assume that X has the initial structure induced by a family of maps $f_\alpha \colon X \to X_\alpha$ for $\alpha \in A$, and that each set X_α has the initial structure induced by maps $g_{\alpha\lambda} \colon X_\alpha \to Y_{\alpha\lambda}$ for $\lambda \in \Lambda_\alpha$. Then X carries the initial structure induced by the maps $g_{\alpha\lambda} \circ f_\alpha \colon X \to Y_{\alpha\lambda}$ for $\alpha \in A$ and $\lambda \in \Lambda_\alpha$.

PROOF. Let $\mathcal{F}_{\alpha\lambda}$ be the set structure on $Y_{\alpha\lambda}$. By definition we have

$$\mathcal{E}_\alpha = \bigvee_{\lambda \in \Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda}) = \Bigl\langle \bigcup_{\lambda \in \Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda}) \Bigr\rangle.$$

Since the union on the right-hand side is a generating set for $\mathcal{E}_\alpha$, Remark 2.7 implies that

$$\mathcal{E} = \Bigl\langle \bigcup_{\alpha \in A} f_\alpha^{-1}\Bigl(\bigcup_{\lambda \in \Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda}) \Bigr) \Bigr\rangle = \Bigl\langle \bigcup_{\alpha \in A} \bigcup_{\lambda \in \Lambda_\alpha} (g_{\alpha\lambda} \circ f_\alpha)^{-1}(\mathcal{F}_{\alpha\lambda}) \Bigr\rangle,$$

proving the claim. □

EXAMPLE 2.12: Subspace and product structures.
Let $(X_\alpha)_{\alpha \in A}$ be a family of structured sets, and let $S_\alpha \subseteq X_\alpha$ be subsets. Then we may equip the product $S = \prod_{\alpha \in A} S_\alpha$ by first equipping $X = \prod_{\alpha \in A} X_\alpha$ with the product structure, and then induce the subset structure on S. In the opposite order we may first equip each S_α with the subset structure, and then induce the product structure. These in fact give the same structure since the diagram

$$\begin{tikzcd}
 & S_\alpha \arrow[r, "\iota_{S_\alpha}"] & X_\alpha \\
S \arrow[ur, "\pi_{S_\alpha}"] \arrow[r, "\iota_S"'] & X \arrow[ur, "\pi_{X_\alpha}"'] &
\end{tikzcd}$$

commutes. ⌟

EXAMPLE 2.13: The weak*-topology.
Let X be a topological vector space over the field $\mathbb{F}$ with topological dual X^*, and for $x \in X$ let $\mathrm{ev}_x \colon X^* \to \mathbb{F}$ be the evaluation map $\mathrm{ev}_x(\varphi) = \varphi(x)$ for $\varphi \in X^*$. Since X^* is a subset of $\mathbb{F}^X$, it naturally carries the subspace topology. The product topology on $\mathbb{F}^X$ is induced by the projection maps $\pi_x \colon \mathbb{F}^X \to \mathbb{F}$ for $x \in X$. But $\pi_x \circ \iota_{X^*}$ is just the evaluation map ev_x, so the subspace topology on X^* is exactly the weak*-topology. ⌟

PROPOSITION 2.14: *Embedding into product*

Let $f_\alpha \colon Y \to X_\alpha$ for $\alpha \in A$, let $X = \prod_{\alpha \in A} X_\alpha$, and let $f \colon Y \to X$ be the unique map such that $f_\alpha = \pi_\alpha \circ f$:

$$\begin{tikzcd} & & X_\alpha \\ Y \arrow[rr, bend left, "f_\alpha"] \arrow[r, "f"'] & X \arrow[ur, "\pi_\alpha"'] & \end{tikzcd}$$

Then f is an embedding if and only if Y carries the initial structure induced by the maps f_α and the collection $(f_\alpha)_{\alpha \in A}$ separates points in Y.

PROOF. First assume that f is an embedding. In particular it is injective, and since the maps π_α separate points in X, the compositions $f_\alpha = \pi_\alpha \circ f$ separate points in Y. Let $\tilde{f} \colon Y \to f(Y)$ be the isomorphism such that $f = \iota_{f(Y)} \circ \tilde{f}$. Then since $\tilde{f}$ is an isomorphism, in particular Y carries the initial structure induced by $\tilde{f}$. But then Y carries the initial structure induced by the maps

$$\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f} = \pi_\alpha \circ f = f_\alpha \tag{2.2}$$

for $\alpha \in A$, as claimed.

Conversely, assume that the f_α separate points in Y and that Y has the initial structure $\mathcal{F}$ induced by the f_α. The f_α are then homomorphisms, and by the characteristic property of initial structures so is f. Furthermore, if $x, y \in Y$ with $x \neq y$, then there is an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$, which implies that $f(x) \neq f(y)$, so f is injective.

Denote the product structure on X by $\mathcal{E}$. We show that if $B \in \mathcal{F}$, then $f(B) \in \mathcal{E}_{f(Y)}$, which will imply that f is an embedding. It suffices to prove this when B is an element of a generating set for $\mathcal{F}$, i.e. on the form $f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. By (2.2) we have

$$B = f_\alpha^{-1}(C) = (\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f})^{-1}(C) = \tilde{f}^{-1}\big((\pi_\alpha \circ \iota_{f(Y)})^{-1}(C)\big),$$

from which it follows that

$$f(B) = \tilde{f}(B) = (\pi_\alpha \circ \iota_{f(Y)})^{-1}(C) \in \mathcal{E}_{f(Y)}.$$

as desired. □

If $\mathcal{E}$ is a structure on a set X, we say that $\mathcal{E}$ is *countably generated* if there is a countable collection of sets $\mathcal{D} \subseteq 2^X$ such that $\mathcal{E} = \langle \mathcal{D} \rangle$.

PROPOSITION 2.15: *Countably generated initial structures*

Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a countable collection of structured sets, and assume that the $\mathcal{E}_\alpha$ are countably generated by collections of sets $\mathcal{D}_\alpha$. If an initial structure $\mathcal{E}$ is induced on a set X by maps $f_\alpha \colon X \to X_\alpha$, then $\mathcal{E}$ is also countably generated.

PROOF. This follows immediately by Remark 2.7 since the generating set $\bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$ is a countable union of countable sets. □

EXAMPLE 2.16: Second-countable topological spaces.
The above proposition implies that an initial topology induced by a countable family of maps into second-countable spaces is itself second-countable. In particular, subspaces and countable products of second-countable spaces are second-countable. Since this holds for subspaces, we say that second-countability is a *hereditary* property.[1] ⌟

3 • Topology

3.1. Countability axioms

We remind the reader that a topological space $(X, \mathcal{T})$ is *second-countable* if there exists a countable basis for $\mathcal{T}$. Furthermore, $(X, \mathcal{T})$ is said to be *Lindelöf* if every open cover of X has a countable subcover.

Recall also that a topological property is called *hereditary* if it follows from a space X having this property that any subspace of X also has this property. It is easy to see that second-countability is hereditary, but the Lindelöf property is not:

REMARK 3.1. A space can be Lindelöf without being hereditarily Lindelöf. Let $(X, \mathcal{T})$ be an uncountable discrete space, and let $y \notin X$. Define a topological space $(Y, \mathcal{T}')$ with underlying space $Y = X \cup \{y\}$ and topology $\mathcal{T}' = \mathcal{T} \cup \{Y\}$. Then $(Y, \mathcal{T}')$ is Lindelöf since any open cover must include Y itself, this being the only open set containing the point y. But the subspace X (whose subspace topology is exactly $\mathcal{T}$) is clearly not Lindelöf. ⌟

If every subspace of X is Lindelöf, then we say that X is *hereditarily Lindelöf*.

[1] Contrast this e.g. with the Lindelöf property: If X is an uncountable discrete space, then X is not Lindelöf. However, if we take an element $\infty \notin X$, the space $\tilde{X} = X \cup \{\infty\}$ whose open sets are the open sets in X and $\tilde{X}$ itself is Lindelöf. Properties like compactness and (path-)connectedness are also not hereditary.

Proposition 3.2

If $(X, \mathcal{T})$ is a second-countable topological space, then it is hereditarily Lindelöf.

Proof. Every subspace of X is second-countable, so it suffices to show that X is Lindelöf.

Let $\mathcal{U}$ be an open cover of X and let $\mathcal{B}$ be a countable basis for the topology $\mathcal{T}$. Consider an $x \in X$. Since $\mathcal{U}$ is a cover of X there is some $U_x \in \mathcal{U}$ with $x \in \mathcal{U}$, and since $\mathcal{B}$ is a basis for $\mathcal{T}$ there is some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U_x$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the subset of open sets obtained in this way. Clearly $\mathcal{B}'$ is a cover of X.

For each $B \in \mathcal{B}'$, the above shows that there exists some $U \in \mathcal{U}$ with $B \subseteq U$. This defines a map $\mathcal{B}' \to \mathcal{U}$ given by $B \mapsto U$ whose image is a countable cover of X, proving the claim. □

Lemma 3.3

Let $(X, \mathcal{T})$ be a second-countable space. Then every basis for $\mathcal{T}$ contains a countable basis for $\mathcal{T}$.

Proof. Let $\mathcal{B}$ be a basis for $\mathcal{T}$, and let $\mathcal{C}$ be a countable basis. We can write every $C \in \mathcal{C}$ on the form $C = \bigcup_{\alpha \in A} B_\alpha$ for some family $\{B_\alpha \mid \alpha \in A\} \subseteq \mathcal{B}$. This is in particular an open cover of C, so since X is hereditarily Lindelöf there is a countable subset $A' \subseteq A$ such that $C = \bigcup_{\alpha \in A'} B_\alpha$. For each $C \in \mathcal{C}$ we thus obtain a countable subcollection of sets from $\mathcal{B}$, and since $\mathcal{C}$ is also countable, the union of all these sets is countable and is clearly a basis for $\mathcal{T}$. □

Proposition 3.4

Let $(X_n)_{n \in \mathbb{N}}$ be a countable family of second-countable topological spaces, and let $X = \prod_{n \in \mathbb{N}} X_n$ be the product space equipped with the product topology. Then X is second-countable.

Proof. Let $\mathcal{B}_n$ be a countable basis for the topology on X_n. For a finite subset $A \subseteq \mathbb{N}$ let

$$\mathcal{B}_A = \left\{ \prod_{n \in \mathbb{N}} U_n \;\middle|\; U_n \in \mathcal{B}_n \text{ if } n \in A \text{ and } U_n = X_n \text{ otherwise} \right\}.$$

Now let $\mathcal{B}$ be the union of all such $\mathcal{B}_A$, and notice that $\mathcal{B}$ is countable since each $\mathcal{B}_A$ is countable and there are countably many finite subsets of $\mathbb{N}$. Clearly all sets in $\mathcal{B}$ are open in X. Conversely, if $U \subseteq X$ is open and $x = (x_n)_{n \in \mathbb{N}} \in U$, then there exist open sets $V_n \subseteq X_n$ such that

$$x \in \prod_{n \in \mathbb{N}} V_n \subseteq U,$$

where for all but finitely many n we have $V_n = X_n$. For the remaining n there exist $U_n \in \mathcal{B}_n$ such that $x_n \in U_n \subseteq V_n$, since $\mathcal{B}_n$ is a basis for the topology on X_n. Letting $U_n = X_n$ if $V_n = X_n$ we get

$$x \in \prod_{n\in\mathbb{N}} U_n \subseteq \prod_{n\in\mathbb{N}} V_n \subseteq U.$$

And $\prod_{n\in\mathbb{N}} U_n \in \mathcal{B}$, so $\mathcal{B}$ is a basis for the topology on X. □

4 • Measure theory

If A is a subset of a measurable space $(X, \mathcal{E})$, then denote by $\mathcal{E}_A$ the σ-algebra on A induced by the inclusion $i\colon A \to X$. That is, let

$$\mathcal{E}_A = i^{-1}(\mathcal{E}) = \{i^{-1}(B) \mid B \in \mathcal{E}\} = \{A \cap B \mid B \in \mathcal{E}\}.$$

Similarly, if A is a subspace of a topological space $(X, \mathcal{T})$, denote by $\mathcal{T}_A$ the subspace topology on A.

Proposition 4.1

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\mathcal{B}(A) = \mathcal{B}(X)_A$, i.e. $\sigma(\mathcal{T}_A) = \sigma(\mathcal{T})_A$.

Proof. Notice that

$$\sigma(\mathcal{T}_A) = \sigma(i^{-1}(\mathcal{T})) = i^{-1}(\sigma(\mathcal{T})) = \sigma(\mathcal{T})_A,$$

where we use [reference to pulling i^{-1} out of σ]. □

Proposition 4.2

Let $\{X_\alpha \mid \alpha \in A\}$ be a family of topological spaces, and let $X = \prod_{\alpha\in A} X_\alpha$. Then

$$\bigotimes_{\alpha\in A} \mathcal{B}(X_\alpha) \subseteq \mathcal{B}(X)$$

If A is countable and the spaces X_α are second-countable, then the above inclusion is an equality.

Proof. Let $\mathcal{T}$ be the product topology on X, and let $\mathcal{D}$ be the generating set of the product σ-algebra on X. Since $\mathcal{D}$ also generates the product topology (but doesn't it generate the box topology??) $\mathcal{T}$ we have $\mathcal{D} \subseteq \mathcal{T}$, proving the first claim. (No, I think we need to look at generators of each σ-algebra in the product. Or use continuity of projections! Yes, better.)

Now assume that A is countable and that all the X_α are second-countable. To prove the second claim it suffices to show that $\mathcal{T} \subseteq \sigma(\mathcal{D})$. First recall that X is also second-countable by Proposition 3.4. Consider the basis for the product topology on X that consists of finite intersections of elements in $\mathcal{D}$, and let $\mathcal{D}'$ be a countable basis contained therein (cf. Lemma 3.3). Then every set in $\mathcal{T}$ is a countable union of sets in $\mathcal{D}'$, and so $\mathcal{T} \subseteq \sigma(\mathcal{D}') = \sigma(\mathcal{D})$ as desired. ☐