

# Notes on measure theory and topology

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## 1 • Introduction

These notes are meant to serve two purposes: Firstly to give an account of (some of) the similarities between topological spaces and measurable spaces. Any student of topology and measure theory have noticed that while  $\sigma$ -algebras generally do not behave as nicely as topologies, we are able to perform many of the same constructions on both structures: Structure-preserving maps (continuous and measurable maps, respectively) are defined the same way, maps induce topologies and  $\sigma$ -algebras in the same way, there are subspaces, products, quotients, and so on.

If we fix a set  $X$ , both the set of topologies and the set of  $\sigma$ -algebras on  $X$  are complete lattices when ordered by inclusion. I am not aware that such a lattice of structures on a set has a commonly used name, so I have simply called them *structures* in these notes.

Secondly we wish to explore how a topological and measure-theoretical structure on a single set interact.

### 1.1. Notation

We generally use notation that is standard in topology, measure theory and category theory. The following may or may not be familiar to the reader:

Given a set  $X$  we denote its power set by  $2^X$ . If  $f: X \rightarrow Y$  is a set function,  $\mathcal{E} \subseteq 2^X$  and  $\mathcal{F} \subseteq 2^Y$ , we write

$$f(\mathcal{E}) = \{f(A) \mid A \in \mathcal{E}\} \quad \text{and} \quad f^{-1}(\mathcal{F}) = \{f^{-1}(B) \mid B \in \mathcal{F}\}.$$

For a set  $X$  and a family  $\mathcal{D} \subseteq 2^X$  of subsets, we write  $\sigma(\mathcal{D})$  for the  $\sigma$ -algebra on  $X$  generated by  $\mathcal{D}$ , i.e. the smallest  $\sigma$ -algebra containing  $\mathcal{D}$ . We do not use any special notation for a topology generated by a family of sets.

If  $X$  is a topological space, we denote the Borel  $\sigma$ -algebra on  $X$  by  $\mathcal{B}(X)$ .

## 2 • Structured sets

### 2.1. Definitions and basic properties

Let  $\mathfrak{S}$  be a map from sets to sets such that  $\mathfrak{S}_X := \mathfrak{S}(X)$  is a collection of subsets of  $2^X$ , and such that for all sets  $X$  and  $Y$  and maps  $f: X \rightarrow Y$ ,

- (1)  $\mathfrak{S}_X$  is partially ordered by set inclusion,
- (2)  $\mathfrak{S}_X$  is a complete lattice with minimum  $\{\emptyset, X\}$  and maximum  $2^X$ ,
- (3) if  $\mathcal{F} \in \mathfrak{S}_Y$ , then  $f^{-1}(\mathcal{F}) \in \mathfrak{S}_X$ , and
- (4) if  $\mathcal{E} \in \mathfrak{S}_X$ , then

$$\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\} \in \mathfrak{S}_Y.$$

We will call such a map  $\mathfrak{S}$  a *structure functor*, and it is indeed a functor as we will see in [Subsection 2.4](#).

If  $X$  is a set, then a  $\mathcal{E} \in \mathfrak{S}_X$  is called a  $\mathfrak{S}$ -*structure* on  $X$ , and we will call the pair  $(X, \mathcal{E})$  a  $\mathfrak{S}$ -*structured set*. We refer to  $\mathfrak{S}_X$  as the *lattice of  $\mathfrak{S}$ -structures* on  $X$ . The minimal structure  $\{\emptyset, X\}$  is called the *trivial structure*, and the maximal structure  $2^X$  is called the *discrete structure* on  $X$ .

Fix a structure functor  $\mathfrak{S}$ . If  $(X, \mathcal{E})$  and  $(Y, \mathcal{F})$  are structured sets, a *homomorphism* from  $X$  to  $Y$  is a map  $f: X \rightarrow Y$  such that  $f^{-1}(\mathcal{F}) \subseteq \mathcal{E}$ . Clearly the composition of two homomorphisms is again a homomorphism, so the collection of structured sets and homomorphisms form a (locally small) category. Let us denote this category by  $\mathbf{Str}_{\mathfrak{S}}$ .

The structure  $f^{-1}(\mathcal{F})$  in [Item \(3\)](#) is called the *pullback* of  $\mathcal{F}$  by  $f$  and is denoted  $f^*(\mathcal{F})$ . Similarly, the structure  $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\}$  in [Item \(4\)](#) is called the *pushforward* of  $\mathcal{E}$  by  $f$  and is denoted  $f_*(\mathcal{E})$ . The pullback and pushforward by  $f$  is defined for all set functions  $f$ , not just homomorphisms.

**EXAMPLE 2.1.** Let  $\mathfrak{S}$  denote the map that associates to a set its lattice of topologies. The first two conditions above are obviously satisfied, and the latter two are easily proved. Thus  $\mathbf{Str}_{\mathfrak{S}}$  is just the category **Top** of topological spaces. Similarly, if  $\mathfrak{S}$  maps a set to its lattice of  $\sigma$ -algebras, then  $\mathbf{Str}_{\mathfrak{S}}$  is the category **Mble** of measurable spaces. ┘

In the sequel we fix a structure functor  $\mathfrak{S}$ .

#### LEMMA 2.2

Let  $X$  be a set. If  $\mathcal{D} \subseteq 2^X$ , then there is a smallest element  $\langle \mathcal{D} \rangle \in \mathfrak{S}_X$  with  $\mathcal{D} \subseteq \langle \mathcal{D} \rangle$ .

**PROOF.** Let  $\mathfrak{D} = \{\mathcal{E} \in \mathfrak{S}_X \mid \mathcal{D} \subseteq \mathcal{E}\}$ . Since  $\mathfrak{S}_X$  is a complete lattice, we can put

$$\langle \mathcal{D} \rangle = \bigwedge_{\mathcal{E} \in \mathfrak{D}} \mathcal{E} \in \mathfrak{S}_X. \quad \square$$

If  $\langle \mathcal{D} \rangle = \mathcal{E}$ , then we say that  $\mathcal{D}$  *generates* or is a *generating set* for  $\mathcal{E}$ . It is easy to see that we may characterise joins as a particular generated structure, namely

$$\bigvee_{\alpha \in A} \mathcal{E}_\alpha = \left\langle \bigcup_{\alpha \in A} \mathcal{E}_\alpha \right\rangle. \quad (2.1)$$

### PROPOSITION 2.3

Let  $(X, \mathcal{E})$  and  $(Y, \mathcal{F})$  be structured sets, and let  $f: X \rightarrow Y$  be any map. For any  $\mathcal{D} \subseteq 2^Y$  we have

$$f^{-1}(\langle \mathcal{D} \rangle) = \langle f^{-1}(\mathcal{D}) \rangle.$$

In particular, if  $\mathcal{F} = \langle \mathcal{D} \rangle$ , then  $f$  is a homomorphism if and only if  $f^{-1}(\mathcal{D}) \subseteq \mathcal{E}$ .

In topology, this proposition is trivial since every element in  $\langle \mathcal{D} \rangle$  is a union of finite intersections of elements in  $\mathcal{D}$ . The proof below is identical to the one given in measure theory.

**PROOF.** First notice that  $f^{-1}(\mathcal{D}) \subseteq f^{-1}(\langle \mathcal{D} \rangle)$ , which implies that

$$\langle f^{-1}(\mathcal{D}) \rangle \subseteq f^{-1}(\langle \mathcal{D} \rangle).$$

For the second inclusion, notice that

$$\mathcal{A} = \{B \subseteq Y \mid f^{-1}(B) \in \langle f^{-1}(\mathcal{D}) \rangle\}$$

is a set structure in  $Y$ . Since clearly  $\mathcal{D} \subseteq \mathcal{A}$ , we also have  $\langle \mathcal{D} \rangle \subseteq \mathcal{A}$ , which proves the second inclusion.  $\square$

## 2.2. Initial structures

### DEFINITION 2.4: Initial structures

Let  $(f_\alpha)_{\alpha \in A}$  be a collection of maps from a set  $X$  to structured sets  $(X_\alpha, \mathcal{E}_\alpha)$ . The *initial structure*  $\mathcal{E}$  on  $X$  induced by  $(f_\alpha)$  is the smallest structure on  $X$  that makes all  $f_\alpha$  homomorphisms. That is,

$$\mathcal{E} = \bigvee_{\alpha \in A} f_\alpha^*(\mathcal{E}_\alpha) = \left\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle.$$

**REMARK 2.5.** If  $\mathcal{D}_\alpha$  is a generating set for  $\mathcal{E}_\alpha$  for all  $\alpha \in A$ , then we may replace  $\mathcal{E}_\alpha$  on the right-hand side above with  $\mathcal{D}_\alpha$ . This follows immediately from the second part of [Proposition 2.3](#), since the structure  $\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha) \rangle$  makes all  $f_\alpha$  into homomorphisms.

Note that  $\bigvee_{\alpha \in A} f_\alpha^*(\mathcal{D}_\alpha)$  doesn't generally make sense, since  $\mathcal{D}_\alpha$  is not necessarily a structure on  $X_\alpha$ .  $\lrcorner$

**THEOREM 2.6: Characteristic property of initial structures**

Let  $(X, \mathcal{E})$  be a structured set equipped with the initial structure induced by maps  $f_\alpha: X \rightarrow X_\alpha$ ,  $\alpha \in A$ . If  $(Y, \mathcal{F})$  is a structured set, then  $f: Y \rightarrow X$  is a homomorphism if and only if  $f_\alpha \circ f$  is a homomorphism for all  $\alpha \in A$ :

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha \circ f & \uparrow f_\alpha \\ Y & \xrightarrow{f} & X \end{array}$$

Furthermore, the initial structure on  $X$  is unique with this property.

**PROOF.** If  $f$  is a homomorphism, then clearly the  $f_\alpha \circ f$  are all homomorphisms.

Conversely, assume that all compositions  $f_\alpha \circ f$  are homomorphisms. It suffices to show that  $f^{-1}(B) \in \mathcal{F}$  for all  $B$  from a generating set for  $\mathcal{E}$ , so let  $B = f_\alpha^{-1}(C)$  for some  $\alpha \in A$  and  $C \in \mathcal{E}_\alpha$ . It follows that

$$f^{-1}(B) = f^{-1}(f_\alpha^{-1}(C)) = (f_\alpha \circ f)^{-1}(C) \in \mathcal{F}$$

as desired.

We now show that the characteristic property uniquely determines a structure on  $X$ . First notice that the commutative diagram

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow f_\alpha \\ X & \xrightarrow{\text{id}_X} & X \end{array}$$

shows that  $f_\alpha$  is a homomorphism, and that this only depends on  $\mathcal{E}$  having the characteristic property above, and not on the concrete definition of  $\mathcal{E}$ .

Now assume that  $\mathcal{E}'$  is a structure on  $X$  with the characteristic property. Consider the commutative diagram

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow f'_\alpha \\ (X, \mathcal{E}) & \xrightarrow{\text{id}_X} & (X, \mathcal{E}') \end{array}$$

where a prime denotes that the domain of a map is  $(X, \mathcal{E}')$  but is as a set function the same as its unprimed counterpart. The  $f_\alpha$  are homomorphisms, so by the characteristic property applied to  $\mathcal{E}'$  we get that  $\text{id}_X$  is a homomorphism.

Finally consider the analogous diagram with primes interchanged:

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f'_\alpha & \uparrow f_\alpha \\ (X, \mathcal{E}') & \xrightarrow{\text{id}'_X} & (X, \mathcal{E}) \end{array}$$

The  $f'_\alpha$  are homomorphisms as we showed above, since  $\mathcal{E}'$  by assumption satisfies the characteristic property. Applying the characteristic property to  $\mathcal{E}$  then shows that  $\text{id}'_X$  is a homomorphism. Thus  $(X, \mathcal{E})$  and  $(X, \mathcal{E}')$  are isomorphic through the identity, hence  $\mathcal{E} = \mathcal{E}'$ .  $\square$

#### EXAMPLE 2.7: Subsets.

Let  $(X, \mathcal{E})$  be a structured set, and let  $S \subseteq X$ . The inclusion map  $\iota_S: S \rightarrow X$  then induces an initial structure on  $S$ , namely the pullback  $\iota_S^*(\mathcal{E})$ . We denote this subset structure by  $\mathcal{E}_S$ , and unless otherwise noted subsets of structured sets always carry this structure. By the characteristic property of initial structures, a map  $f: Y \rightarrow S$  from a structured set is a homomorphism if and only if  $\iota_S \circ f$  is a homomorphism.

On the other hand, if  $f: Y \rightarrow X$  is a map with  $f(Y) \subseteq S$ , then the map  $\tilde{f}: Y \rightarrow S$  given by  $\tilde{f}(y) = f(y)$  for all  $y \in Y$  is a homomorphism if and only if  $f = \iota_S \circ \tilde{f}$  is a homomorphism. In other words, whether a map is a homomorphism or not does not depend on the codomain if we agree to equip subsets with the structure induced by their inclusion maps.

If  $S = f(Y)$  and  $\tilde{f}: Y \rightarrow f(Y)$  is an isomorphism, then we call  $f$  an *embedding*.  $\lrcorner$

#### EXAMPLE 2.8: Products.

Let  $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$  be a collection of structured sets, let  $X = \prod_{\alpha \in A} X_\alpha$  be the Cartesian product of the sets  $X_\alpha$ , and denote the associated projections by  $\pi_\alpha: X \rightarrow X_\alpha$ . We define a product structure

$$\mathcal{E} = \bigotimes_{\alpha \in A} \mathcal{E}_\alpha$$

as the initial structure on  $X$  induced by the projection maps. Since  $X$  is a product of the  $X_\alpha$  in the category of sets, the characteristic property of initial structures implies that  $(X, \mathcal{E})$  is a product of the structured sets  $(X_\alpha, \mathcal{E}_\alpha)$ .  $\lrcorner$

**PROPOSITION 2.9: Composition of initial structures**

For  $\alpha \in A$  and  $\lambda \in \Lambda_\alpha$ , let  $X_\alpha$  and  $Y_{\alpha\lambda}$  be structured sets such that each  $X_\alpha$  carry the initial structure induced by maps  $g_{\alpha\lambda}: X_\alpha \rightarrow Y_{\alpha\lambda}$  for  $\lambda \in \Lambda_\alpha$ . Let  $X$  be a set, and consider maps  $f_\alpha: X \rightarrow X_\alpha$ .

Let  $\mathcal{E}_1$  be the initial structure on  $X$  induced by the maps  $f_\alpha$ , and let  $\mathcal{E}_2$  be the structure induced by the compositions  $g_{\alpha\lambda} \circ f_\alpha$ . Then  $\mathcal{E}_1 = \mathcal{E}_2$ .

**PROOF.** By the characteristic property of initial structures, the  $f_\alpha$  are homomorphisms if and only if all the  $g_{\alpha\lambda} \circ f_\alpha$  are homomorphisms. If  $X$  has the structure  $\mathcal{E}_1$ , then we must have  $\mathcal{E}_2 \subseteq \mathcal{E}_1$ , since these compositions  $g_{\alpha\lambda} \circ f_\alpha$  are homomorphisms. Conversely, if  $X$  carries the structure  $\mathcal{E}_2$ , then the  $f_\alpha$  are homomorphisms, and so  $\mathcal{E}_1 \subseteq \mathcal{E}_2$ .  $\square$

**EXAMPLE 2.10: Subspace and product structures.**

Let  $(X_\alpha)_{\alpha \in A}$  be a family of structured sets, and let  $S_\alpha \subseteq X_\alpha$  be subsets. Then we may equip the product  $S = \prod_{\alpha \in A} S_\alpha$  by first equipping  $X = \prod_{\alpha \in A} X_\alpha$  with the product structure, and then induce the subset structure on  $S$ . In the opposite order we may first equip each  $S_\alpha$  with the subset structure, and then induce the product structure. These in fact give the same structure since the diagram

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\iota_{S_\alpha}} & X_\alpha \\ \pi_{S_\alpha} \uparrow & & \uparrow \pi_{X_\alpha} \\ S & \xrightarrow{\iota_S} & X \end{array}$$

commutes.  $\lrcorner$

**EXAMPLE 2.11: The weak\*-topology.**

Let  $X$  be a topological vector space over the field  $\mathbb{F}$  with topological dual  $X^*$ , and for  $x \in X$  let  $\text{ev}_x: X^* \rightarrow \mathbb{F}$  be the evaluation map  $\text{ev}_x(\varphi) = \varphi(x)$  for  $\varphi \in X^*$ . Since  $X^*$  is a subset of  $\mathbb{F}^X$ , it naturally carries the subspace topology. The product topology on  $\mathbb{F}^X$  is induced by the projection maps  $\pi_x: \mathbb{F}^X \rightarrow \mathbb{F}$  for  $x \in X$ . But  $\pi_x \circ \iota_{X^*}$  is just the evaluation map  $\text{ev}_x$ , so the subspace topology on  $X^*$  is exactly the weak\*-topology.  $\lrcorner$

**PROPOSITION 2.12: Embedding into product**

Let  $f_\alpha: Y \rightarrow X_\alpha$  for  $\alpha \in A$  be homomorphisms, let  $X = \prod_{\alpha \in A} X_\alpha$ , and let  $f: Y \rightarrow X$  be the unique homomorphism such that  $f_\alpha = \pi_\alpha \circ f$ :

$$\begin{array}{ccc} & & X_\alpha \\ & \nearrow f_\alpha & \uparrow \pi_\alpha \\ Y & \xrightarrow{f} & X \end{array}$$

Then  $f$  is an embedding if and only if  $Y$  carries the initial structure induced by the maps  $f_\alpha$  and the collection  $(f_\alpha)_{\alpha \in A}$  separates points in  $Y$ .

**PROOF.** Notice that  $f$  is unique as a set function since  $X$  is a product in **Set**, and  $f$  is a homomorphism by the characteristic property of initial structures.

First assume that  $f$  is an embedding. In particular it is injective, and since the maps  $\pi_\alpha$  separate points in  $X$ , the compositions  $f_\alpha = \pi_\alpha \circ f$  separate points in  $Y$ . Let  $\tilde{f}: Y \rightarrow f(Y)$  be the isomorphism such that  $f = \iota_{f(Y)} \circ \tilde{f}$ . Then since  $\tilde{f}$  is an isomorphism, in particular  $Y$  carries the initial structure induced by  $\tilde{f}$ . But then  $Y$  carries the initial structure induced by the maps

$$\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f} = \pi_\alpha \circ f = f_\alpha \quad (2.2)$$

for  $\alpha \in A$ , as claimed.

Conversely, assume that the  $f_\alpha$  separate points in  $Y$  and that  $Y$  has the initial structure  $\mathcal{F}$  induced by the  $f_\alpha$ . The  $f_\alpha$  are then homomorphisms, and by the characteristic property of initial structures so is  $f$ . Furthermore, if  $x, y \in Y$  with  $x \neq y$ , then there is an  $\alpha \in A$  such that  $f_\alpha(x) \neq f_\alpha(y)$ , which implies that  $f(x) \neq f(y)$ , so  $f$  is injective.

Denote the product structure on  $X$  by  $\mathcal{E}$ . We show that if  $B \in \mathcal{F}$ , then  $f(B) \in \mathcal{E}_{f(Y)}$ , which will imply that  $f$  is an embedding. It suffices to prove this when  $B$  is an element of a generating set for  $\mathcal{F}$ , i.e. on the form  $f_\alpha^{-1}(C)$  for some  $\alpha \in A$  and  $C \in \mathcal{E}_\alpha$ . By (2.2) we have

$$B = f_\alpha^{-1}(C) = (\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f})^{-1}(C) = \tilde{f}^{-1}\left((\pi_\alpha \circ \iota_{f(Y)})^{-1}(C)\right),$$

from which it follows that

$$f(B) = \tilde{f}(B) = (\pi_\alpha \circ \iota_{f(Y)})^{-1}(C) \in \mathcal{E}_{f(Y)}.$$

as desired.  $\square$

If  $\mathcal{E}$  is a structure on a set  $X$ , we say that  $\mathcal{E}$  is *countably generated* if there is a countable collection of sets  $\mathcal{D} \subseteq 2^X$  such that  $\mathcal{E} = \langle \mathcal{D} \rangle$ .

**PROPOSITION 2.13:** *Countably generated initial structures*

Let  $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$  be a countable collection of structured sets, and assume that the  $\mathcal{E}_\alpha$  are countably generated by collections of sets  $\mathcal{D}_\alpha$ . If an initial structure  $\mathcal{E}$  is induced on a set  $X$  by maps  $f_\alpha: X \rightarrow X_\alpha$ , then  $\mathcal{E}$  is also countably generated.

**PROOF.** This follows immediately by Remark 2.5 since the generating set  $\bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$  is a countable union of countable sets.  $\square$

**EXAMPLE 2.14:** Second-countable topological spaces.

A topology is second-countable if and only if it is countably generated, as we show below. The above proposition then implies that an initial topology induced by a countable family of maps into second-countable spaces is itself second-countable. In particular, subspaces and countable products of second-countable spaces are second-countable.

Now to prove the above claim: Since a basis in particular is a generating set (i.e. a subbasis), second-countable topologies are countably generated. Conversely, let  $\mathcal{T}$  be a topology that is generated by a countable set  $\mathcal{D}$ . Then a basis  $\mathcal{B}$  for  $\mathcal{T}$  is obtained by taking finite intersections of elements from  $\mathcal{D}$ . The number of these intersections is certainly less than the cardinality of the union

$$\bigcup_{n \in \mathbb{N}} \prod_{i=1}^n \mathcal{D}$$

of all finite products of  $\mathcal{D}$  with itself, an element  $U_1 \times \cdots \times U_n$  of an  $n$ -fold product corresponding to the intersection  $\bigcap_{i=1}^n U_i$ . But finite products of countable sets are countable, and so are countable unions of countable sets, so the union above is countable.  $\lrcorner$

## 2.3. Final structures

**DEFINITION 2.15:** Final structures

Let  $(f_\alpha)_{\alpha \in A}$  be a collection of maps from structured sets  $(X_\alpha, \mathcal{E}_\alpha)$  to a set  $X$ . The *final structure*  $\mathcal{E}$  on  $X$  coinduced by  $(f_\alpha)$  is the largest structure on  $X$  that makes all  $f_\alpha$  homomorphisms. That is,

$$\mathcal{E} = \bigwedge_{\alpha \in A} (f_\alpha)_*(\mathcal{E}_\alpha).$$

**THEOREM 2.16:** Characteristic property of final structures

Let  $(X, \mathcal{E})$  be a structured set equipped with the final structure coinduced by maps  $f_\alpha: X_\alpha \rightarrow X$ ,  $\alpha \in A$ . If  $(Y, \mathcal{F})$  is a structured set, then  $f: X \rightarrow Y$  is a homomorphism if and only if  $f \circ f_\alpha$  is a homomorphism for all  $\alpha \in A$ :

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & X \\ & \searrow f \circ f_\alpha & \downarrow f \\ & & Y \end{array}$$

Furthermore, the final structure on  $X$  is unique with this property.



**PROOF.** If  $f$  is a homomorphism, then clearly the  $f \circ f_\alpha$  are all homomorphisms.

Conversely, assume that the  $f \circ f_\alpha$  are homomorphisms. Notice that

$$f_\alpha^{-1}(f^{-1}(\mathcal{F})) = (f \circ f_\alpha)^{-1}(\mathcal{F}) \subseteq \mathcal{E}_\alpha,$$

since the  $f \circ f_\alpha$  are homomorphisms. But then  $f^{-1}(\mathcal{F}) = f^*(\mathcal{F})$  is a structure on  $X$  with respect to which the  $f_\alpha$  are homomorphisms, and thus  $f^{-1}(\mathcal{F}) \subseteq \mathcal{E}$  since  $\mathcal{E}$  is the largest structure with this property. Hence  $f$  is a homomorphism.

The proof of uniqueness is analogous to the proof of uniqueness in [Theorem 2.6](#), so we omit it.  $\square$

**EXAMPLE 2.17: Quotient structures.**

Let  $X$  be a structured set, and let  $Y$  be any set. If  $q: X \rightarrow Y$  is a surjective map, the final structure on  $Y$  induced by  $q$  is called the *quotient structure*. In particular, if  $\sim$  is an equivalence relation on  $X$  and  $q: X \rightarrow X/\sim$  is the associated natural map, we will always equip  $X/\sim$  with the quotient structure induced by  $q$ .  $\lrcorner$

**EXAMPLE 2.18: Disjoint unions.**

If  $(X_\alpha)_{\alpha \in A}$  is a collection of structured sets, we equip the disjoint union  $X = \coprod_{\alpha \in A} X_\alpha$  with the final structure coinduced by the inclusions  $i_\alpha: X_\alpha \rightarrow X$ .

We will see in [Subsection 2.4](#) that  $X$  is a coproduct of the  $X_\alpha$  in  $\mathbf{Str}_\mathcal{G}$ , but this fact will be convenient in the proof of [Proposition 2.20](#).  $\lrcorner$

**PROPOSITION 2.19: Composition of final structures**

For  $\alpha \in A$  and  $\lambda \in \Lambda_\alpha$ , let  $X_\alpha$  and  $Y_{\alpha\lambda}$  be structured sets such that each  $X_\alpha$  carry the final structure induced by maps  $g_{\alpha\lambda}: Y_{\alpha\lambda} \rightarrow X_\alpha$  for  $\lambda \in \Lambda_\alpha$ . Let  $X$  be a set, and consider maps  $f_\alpha: X_\alpha \rightarrow X$ .

Let  $\mathcal{E}_1$  be the final structure on  $X$  induced by the maps  $f_\alpha$ , and let  $\mathcal{E}_2$  be the structure induced by the compositions  $g_{\alpha\lambda} \circ f_\alpha$ . Then  $\mathcal{E}_1 = \mathcal{E}_2$ .

**PROOF.** This result is just the dual of [Proposition 2.9](#), so we omit the proof.  $\square$

**PROPOSITION 2.20: Final structures as quotient structures**

Let  $f_\alpha: X_\alpha \rightarrow Y$  for  $\alpha \in A$  be homomorphisms, let  $X = \coprod_{\alpha \in A} X_\alpha$ , and let  $f: X \rightarrow Y$  be the unique homomorphism such that  $f_\alpha = f \circ i_\alpha$ :

$$\begin{array}{ccc} X_\alpha & & \\ i_\alpha \downarrow & \searrow f_\alpha & \\ X & \xrightarrow{f} & Y \end{array}$$

Define an equivalence relation  $\sim$  on  $X$  by letting  $x \sim x'$  if  $f(x) = f(x')$  for  $x, x' \in X$ . Let  $q: X \rightarrow X/\sim$  be the associated quotient map, and let  $\tilde{f}: X/\sim \rightarrow Y$  be the unique homomorphism such that  $\tilde{f} \circ q = f$ .

Assume that  $f$  is surjective. Then  $Y$  has the final structure coinduced by the  $f_\alpha$  if and only if  $\tilde{f}$  is an isomorphism.

The condition that  $f$  is surjective is equivalent to the property that every point in  $Y$  is in the image of some  $f_\alpha$ . In this case we say that the maps  $f_\alpha$  cover points in  $Y$ .

Notice also that by [Proposition 2.19](#), the final structure on  $Y$  coinduced by the  $f_\alpha$  is the same as the structure coinduced by  $f$ .

**PROOF.** Notice that  $f$  is unique as a set function since  $X$  is a coproduct in **Set**, and it is a homomorphism by the characteristic property of final structures.

Consider the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow q \quad \nearrow \tilde{f} & \\ & X/\sim & \end{array}$$

First assume that  $\tilde{f}$  is an isomorphism. Then  $Y$  in particular carries the final structure coinduced by  $\tilde{f}$ . But  $X/\sim$  has the final structure coinduced by  $q$ , so  $Y$  has the final structure coinduced by the map  $\tilde{f} \circ q = f$  by [Proposition 2.19](#) as claimed.

Conversely, assume that  $Y$  has the final structure  $\mathcal{F}$  coinduced by the  $f_\alpha$ . Let  $\mathcal{E}$  denote the disjoint union structure on  $X$  and  $\tilde{\mathcal{E}}$  the quotient structure on  $X/\sim$ . Notice that  $\tilde{f}$  is surjective since  $f$  is surjective, and it is injective by definition of  $\sim$ . We thus need to prove that  $\tilde{f}^{-1}$  is a homomorphism, and it suffices to show that  $\tilde{f}(\tilde{\mathcal{E}}) \subseteq \mathcal{F}$ . This is the case if and only if

$$q^{-1}(\tilde{\mathcal{E}}) = f^{-1}(\tilde{f}(\tilde{\mathcal{E}})) \subseteq \mathcal{E},$$

and this is true since  $q$  is a homomorphism. □

## 2.4. Categorical properties

### The category $\mathbf{Str}_\mathbb{S}$

We first recapitulate some of the above results in categorical terms. The main result is the following:

**THEOREM 2.21:** *Completeness of  $\mathbf{Str}_\mathbb{S}$*

The category  $\mathbf{Str}_{\mathfrak{S}}$  is complete, i.e. it has all small limits.

**PROOF.** By e.g. Smith (2018, Theorem 60) it is enough to show that  $\mathbf{Str}_{\mathfrak{S}}$  has all small products and has equalisers.

*Products:* We claim that the product  $(X, \mathcal{E})$  considered in Example 2.8 is in fact a product of the objects  $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$  in  $\mathbf{Str}_{\mathfrak{S}}$ . If  $Y$  is a structured set and  $f_\alpha: Y \rightarrow X_\alpha$  are homomorphisms, then since  $X$  is a product in  $\mathbf{Set}$  there is a unique set function  $f: Y \rightarrow X$  such that  $f_\alpha = \pi_\alpha \circ f$  for all  $\alpha \in A$ . But  $f$  is also a homomorphism by the characteristic property of the product structure, so  $(X, \mathcal{E})$  is in fact a product in  $\mathbf{Str}_{\mathfrak{S}}$ .

*Equalisers:* Let  $f, g: X \rightarrow Y$  be any pair of parallel homomorphisms, and let  $E$  be the subset of  $X$  on which they agree. If  $h: Z \rightarrow X$  is any homomorphism such that  $f \circ h = g \circ h$ , then there is a unique homomorphism  $u: Z \rightarrow E$  such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\iota_E} & X \\ \uparrow u & \nearrow h & \downarrow \\ Z & & \end{array} \quad \begin{array}{c} f \\ \rightrightarrows \\ g \end{array} Y$$

We must have  $h(Z) \subseteq E$ , so we can define  $u$  by  $u(z) = h(z)$ , and  $u$  is unique as a set function such that the above diagram commutes. Furthermore,  $u$  is a homomorphism by the characteristic property of the subset structure. Thus  $E$  along with the inclusion map  $\iota_E$  is an equaliser of  $f$  and  $g$ .  $\square$

### Functoriality of $\mathfrak{S}$

As mentioned,  $\mathfrak{S}$  is in fact a functor. Its action on a set function  $f: X \rightarrow Y$  is defined as the pullback  $f^*: \mathfrak{S}_Y \rightarrow \mathfrak{S}_X$ .

Let  $\mathbf{Set}$  denote the category of sets and  $\mathbf{CsLat}^\vee$  the category of complete join-semilattices and join-preserving maps.

**PROPOSITION 2.22:** *Functoriality of  $\mathfrak{S}$ , I*

The map  $\mathfrak{S}$  is a contravariant functor from  $\mathbf{Set}$  to  $\mathbf{CsLat}^\vee$ .

**PROOF.** By (2.1) and Proposition 2.3, for any set function  $f$  the pullback  $f^*$  preserves joins since preimages respect unions, so it is well-defined as a map  $\mathbf{Set} \rightarrow \mathbf{CsLat}^\vee$ .

The map  $\mathfrak{S}$  is also contravariant since if  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are set functions, then for  $\mathcal{G} \in \mathfrak{S}_Z$  we have

$$(g \circ f)^*(\mathcal{G}) = (g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = (f^* \circ g^*)(\mathcal{G}).$$

Its action on identity functions is clearly trivial, so it is a functor.  $\square$

In the case of topological spaces or measure spaces we can say slightly more: Notice that in both a lattice of topologies or of  $\sigma$ -algebras on a set, intersections of topologies ( $\sigma$ -algebras) are themselves topologies ( $\sigma$ -algebras). A nonempty subset  $\mathfrak{L}$  of  $2^X$ , where  $X$  is some set, is called an *intersection structure*. If also  $X \in \mathfrak{L}$  we call it a *topped intersection structure*. It is easy to show that topped intersection structures are complete lattices ordered by inclusion, and that meets are given by intersections.

If **CLat** denotes the category of complete lattices with join- and meet-preserving maps, then we have the following:

**PROPOSITION 2.23: Functoriality of  $\mathfrak{S}$ , II**

*If  $\mathfrak{S}_X$  is an intersection structure for all sets  $X$ , then  $\mathfrak{S}$  is a contravariant functor from **Set** to **CLat**.*

**PROOF.** It suffices to show that  $f^*$  preserves meets for all set functions  $f: X \rightarrow Y$ . But this is clear since preimages respect intersections.  $\square$

It is natural to ask whether the pushforward  $f_*$  by  $f$  gives rise to a covariant functor from **Set** into a category of lattices. It is easy to see that  $f_*$  is monotone, and a short calculation shows that  $(g \circ f)_* = g_* \circ f_*$  if  $g: Y \rightarrow Z$  is another set function, so the pushforward does indeed define a covariant functor from **Set** to the category **Pos** of posets and monotone maps. But  $f_*$  does not, as far as I know, generally preserve meets or joins. If all  $\mathfrak{S}_X$  are intersection structures, however, it does preserve meets, so it is then a functor into the category **CsLat**<sup>^</sup> of complete meet-semilattices, though it still does not seem to preserve joins. (I haven't looked too hard for counterexamples.)

*The forgetful functor on **Str** <sub>$\mathfrak{S}$</sub>*

Just as there is a forgetful functor **Top**  $\rightarrow$  **Set** that sends a topological space to its underlying set, there is a forgetful functor  $U: \mathbf{Str}_{\mathfrak{S}} \rightarrow \mathbf{Set}$ . As usual,  $U$  has a left adjoint  $D$  that equips a set with the discrete structure, and it has a right adjoint  $T$  that equips the set with the trivial structure, i.e.  $D \dashv U \dashv T$ .

It follows immediately that  $U$  preserves both limits and colimits. Hence if **Str** <sub>$\mathfrak{S}$</sub>  e.g. has coproducts – which it has, as we will see later – we already know that they have to be (isomorphic to) disjoint unions of the underlying sets, equipped with an appropriate structure. Contrast this with the situation in the category **Grp** of groups: The forgetful functor  $U: \mathbf{Grp} \rightarrow \mathbf{Set}$  has a left adjoint, namely the free functor, so  $U$  preserves limits. But it does not preserve colimits; for instance, coproducts in **Grp** are free products, and their underlying sets are certainly not disjoint unions! Hence  $U$  does not have a right adjoint.

### Presheaves on structured sets

Let  $(X, \mathcal{E})$  be a structured set, and view  $\mathcal{E}$  as a preorder category. Analogous to the case of topological spaces, a presheaf on  $\mathcal{E}$  (i.e. a contravariant functor  $\mathcal{E} \rightarrow \mathbf{Set}$ ) is called a *presheaf* on  $(X, \mathcal{E})$ , or simply a presheaf on  $X$  if the structure is understood.

As an example, fix a structured set  $Y$  and take the presheaf  $F$  on  $X$  given by  $F(B) = \mathbf{Str}_{\mathcal{E}}(B, Y)$ , i.e.  $F$  sends a set  $B \in \mathcal{E}$  to the set of homomorphisms  $B \rightarrow Y$ . Furthermore,  $F$  sends an inclusion  $B \subseteq B'$  in  $\mathcal{E}$  to the restriction map  $\mathbf{Str}_{\mathcal{E}}(B', Y) \rightarrow \mathbf{Str}_{\mathcal{E}}(B, Y)$  given by  $f \mapsto f|_B$ . A common example of this is the case  $\mathbf{Str}_{\mathcal{E}} = \mathbf{Top}$  and  $Y = \mathbb{R}$ , in which case  $F$  sends an open set  $U$  to the set of continuous functions  $U \rightarrow \mathbb{R}$ .

## 3 • Topology

We remind the reader that a topological space  $(X, \mathcal{T})$  is *second-countable* if there exists a countable basis for  $\mathcal{T}$ . Furthermore,  $(X, \mathcal{T})$  is said to be *Lindelöf* if every open cover of  $X$  has a countable subcover.

Recall also that a topological property is called *hereditary* if it follows from a space  $X$  having this property that any subspace of  $X$  also has this property. It is easy to see that second-countability is hereditary, but the Lindelöf property is not:

**REMARK 3.1.** A space can be Lindelöf without being hereditarily Lindelöf. Let  $(X, \mathcal{T})$  be an uncountable discrete space, and let  $y \notin X$ . Define a topological space  $(Y, \mathcal{T}')$  with underlying space  $Y = X \cup \{y\}$  and topology  $\mathcal{T}' = \mathcal{T} \cup \{Y\}$ . Then  $(Y, \mathcal{T}')$  is Lindelöf since any open cover must include  $Y$  itself, this being the only open set containing the point  $y$ . But the subspace  $X$  (whose subspace topology is exactly  $\mathcal{T}$ ) is clearly not Lindelöf.  $\lrcorner$

If every subspace of  $X$  is Lindelöf, then we say that  $X$  is *hereditarily Lindelöf*.

### PROPOSITION 3.2

*If  $(X, \mathcal{T})$  is a second-countable topological space, then it is hereditarily Lindelöf.*

**PROOF.** Every subspace of  $X$  is second-countable, so it suffices to show that  $X$  is Lindelöf.

Let  $\mathcal{U}$  be an open cover of  $X$  and let  $\mathcal{B}$  be a countable basis for the topology  $\mathcal{T}$ . Consider an  $x \in X$ . Since  $\mathcal{U}$  is a cover of  $X$  there is some  $U_x \in \mathcal{U}$  with  $x \in U_x$ , and since  $\mathcal{B}$  is a basis for  $\mathcal{T}$  there is some  $B_x \in \mathcal{B}$  with  $x \in B_x \subseteq U_x$ . Let  $\mathcal{B}' \subseteq \mathcal{B}$  be the subset of open sets obtained in this way. Clearly  $\mathcal{B}'$  is a cover of  $X$ .

For each  $B \in \mathcal{B}'$ , the above shows that there exists some  $U \in \mathcal{U}$  with  $B \subseteq U$ . This defines a map  $\mathcal{B}' \rightarrow \mathcal{U}$  given by  $B \mapsto U$  whose image is a countable cover of  $X$ , proving the claim.  $\square$

### LEMMA 3.3

*Let  $(X, \mathcal{T})$  be a second-countable space. Then every basis for  $\mathcal{T}$  contains a countable basis for  $\mathcal{T}$ .*

**PROOF.** Let  $\mathcal{B}$  be a basis for  $\mathcal{T}$ , and let  $\mathcal{C}$  be a countable basis. We can write every  $C \in \mathcal{C}$  on the form  $C = \bigcup_{\alpha \in A} B_\alpha$  for some family  $(B_\alpha)_{\alpha \in A} \subseteq \mathcal{B}$ . This is in particular an open cover of  $C$ , so since  $X$  is hereditarily Lindelöf there is a countable subset  $A' \subseteq A$  such that  $C = \bigcup_{\alpha \in A'} B_\alpha$ . For each  $C \in \mathcal{C}$  we thus obtain a countable subcollection of sets from  $\mathcal{B}$ , and since  $\mathcal{C}$  is also countable, the union of all these sets is countable and is clearly a basis for  $\mathcal{T}$ .  $\square$

## 4 • Measure theory

If  $A$  is a subset of a measurable space  $(X, \mathcal{E})$ , then recall that we denote by  $\mathcal{E}_A$  the initial  $\sigma$ -algebra on  $A$  induced by the inclusion  $\iota_A: A \rightarrow X$ . Similarly, if  $A$  is a subspace of a topological space  $(X, \mathcal{T})$ , denote by  $\mathcal{T}_A$  the subspace topology on  $A$ .

### PROPOSITION 4.1

*Let  $(X, \mathcal{T})$  be a topological space, and let  $A \subseteq X$ . Then  $\mathcal{B}(A) = \mathcal{B}(X)_A$ , i.e.  $\sigma(\mathcal{T}_A) = \sigma(\mathcal{T})_A$ .*

**PROOF.** Notice that

$$\sigma(\mathcal{T}_A) = \sigma(\iota_A^{-1}(\mathcal{T})) = \iota_A^{-1}(\sigma(\mathcal{T})) = \sigma(\mathcal{T})_A$$

by Proposition 2.3.  $\square$

### THEOREM 4.2: Products of Borel $\sigma$ -algebras

*Let  $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$  be a family of topological spaces, and equip  $X = \prod_{\alpha \in A} X_\alpha$  with the product topology  $\mathcal{T}$ . Then*

$$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) \subseteq \mathcal{B}(X)$$

*If  $A$  is countable and the spaces  $X_\alpha$  are second-countable, then the above inclusion is an equality.*

**PROOF.** Since the projections  $\pi_\alpha: X \rightarrow X_\alpha$  are continuous, they are  $\mathcal{B}(X)$ - $\mathcal{B}(X_\alpha)$ -measurable. But  $\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$  is the smallest  $\sigma$ -algebra on  $X$  that makes the projections measurable, which proves the above inclusion.

Now assume that  $A$  is countable and that all the  $X_\alpha$  are second-countable. From [Example 2.14](#) we know that  $X$  is also second-countable. Let

$$\mathcal{D} = \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{T}_\alpha)$$

be a subbasis for the product topology  $\mathcal{T}$ , and let  $\mathcal{B}$  be the collection of finite intersections of elements in  $\mathcal{D}$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{T}$ , and  $\mathcal{B}$  contains a countable basis  $\mathcal{C}$  for  $\mathcal{T}$  by [Lemma 3.3](#). Since  $\mathcal{C}$  is countable, open sets in  $X$  are countable unions of finite intersections of elements in  $\mathcal{D}$ . Since  $A$  is also countable, it suffices to show that

$$\pi_\beta^{-1}(\mathcal{T}_\beta) \subseteq \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$$

for all  $\beta \in A$ . But this is obvious since the projections are measurable.  $\square$

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