Notes on measure theory and topology

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1 • Introduction

These notes are meant to serve two purposes: Firstly to give an account of (some of) the similarities between topological spaces and measurable spaces. Any student of topology and measure theory have noticed that while σ-algebras generally do not behave as nicely as topologies, we are able to perform many of the same constructions on both structures: Structure-preserving maps (countinuous and measurable maps, respectively) are defined the same way, maps induce topologies and σ-algebras in the same way, there are subspaces, products, quotients, and so on.

If we fix a set X, both the set of topologies and the set of σ-algebras on X are complete lattices when ordered by inclusion. I am not aware that such a lattice of structures on a set has a commonly used name, so I have simply called them *structures* in these notes.

Secondly we wish to explore how a topological and measure-theoretical structure on a single set interact.

2 • Structured sets

2.1. Definitions and basic properties

Let $\mathfrak{S}$ be a map from sets to sets such that $\mathfrak{S}_X := \mathfrak{S}(X)$ is a collection of subsets of 2^X, and such that for all sets X and Y and maps $f \colon X \to Y$,

(1) $\mathfrak{S}_X$ is partially ordered by set inclusion,

(2) $\mathfrak{S}_X$ is a complete lattice with minimum $\{\emptyset, X\}$ and maximum 2^X,

(3) if $\mathcal{F} \in \mathfrak{S}_Y$, then $f^{-1}(\mathcal{F}) \in \mathfrak{S}_X$, and

(4) if $\mathcal{E} \in \mathfrak{S}_X$, then
$$\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\} \in \mathfrak{S}_Y.$$

We will call such a map $\mathfrak{S}$ a *structure functor*, and it is indeed a functor as we will see in Subsection 2.3.

If X is a set, then a $\mathcal{E} \in \mathfrak{S}_X$ is called a $\mathfrak{S}$-*structure* on X, and we will call the pair $(X, \mathcal{E})$ a $\mathfrak{S}$-*structured set*. We refer to $\mathfrak{S}_X$ as the *lattice of* $\mathfrak{S}$-*structures* on X. The minimal structure $\{\emptyset, X\}$ is called the *trivial structure*, and the maximal structure 2^X is called the *discrete structure* on X.

Fix a structure functor $\mathfrak{S}$. If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are structured sets, a *homomorphism* from X to Y is a map $f : X \to Y$ such that $f^{-1}(B) \in \mathcal{E}$ for all $B \in \mathcal{F}$. Clearly the composition of two homomorphisms is again a homomorphism, so the collection of structured sets and homomorphisms form a (locally small) category. Let us denote this category by $\mathbf{Str}_{\mathfrak{S}}$.

The structure $f^{-1}(\mathcal{F})$ in Item (3) is called the *pullback* of $\mathcal{F}$ by f and is denoted $f^*(\mathcal{F})$. Similarly, the structure $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{E}\}$ in Item (4) is called the *pushforward* of $\mathcal{E}$ by f and is denoted $f_*(\mathcal{E})$. The pullback and pushforward by f is defined for all set functions f, not just homomorphisms.

Example 2.1. Let $\mathfrak{S}$ denote the map that associates to a set its lattice of topologies. The first two conditions above are obviously satisfied, and the latter two are easily proved. Thus $\mathbf{Str}_{\mathfrak{S}}$ is just the category **Top** of topological spaces. Similarly, if $\mathfrak{S}$ maps a set to its lattice of σ-algebras, then $\mathbf{Str}_{\mathfrak{S}}$ is the category **Mble** of measurable spaces. ⌟

In the sequel we fix a structure functor $\mathfrak{S}$.

Lemma 2.2

Let X be a set. If $\mathcal{D} \subseteq 2^X$, then there is a smallest element $\langle \mathcal{D} \rangle \in \mathfrak{S}_X$ with $\mathcal{D} \subseteq \langle \mathcal{D} \rangle$.

Proof. Let $\Sigma(\mathcal{D}) = \{\mathcal{E} \in \mathfrak{S}_X \mid \mathcal{D} \subseteq \mathcal{E}\}$. Since $\mathfrak{S}_X$ is a complete lattice, we can put
$$\langle \mathcal{D} \rangle = \bigwedge_{\mathcal{E} \in \Sigma(\mathcal{D})} \mathcal{E} \in \mathfrak{S}_X. \qquad \square$$

If $\langle \mathcal{D} \rangle = \mathcal{E}$, then we say that $\mathcal{D}$ *generates* or is a *generating set* for $\mathcal{E}$. It is easy to see that we may characterise joins as a particular generated structure, namely
$$\bigvee_{\alpha \in A} \mathcal{E}_\alpha = \Big\langle \bigcup_{\alpha \in A} \mathcal{E}_\alpha \Big\rangle. \tag{2.1}$$

Proposition 2.3

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be structured sets, and let $f : X \to Y$ be any map. For any

$\mathcal{D} \subseteq 2^Y$ we have

$$f^{-1}\big(\langle \mathcal{D} \rangle\big) = \big\langle f^{-1}(\mathcal{D}) \big\rangle.$$

In particular, if $\mathcal{F} = \langle \mathcal{D} \rangle$, then f is a homomorphism if and only if $f^{-1}(D) \in \mathcal{E}$ for all $D \in \mathcal{D}$.

In topology, this proposition is trivial since every element in $\langle \mathcal{D} \rangle$ is a union of finite intersections of elements in $\mathcal{D}$. The proof below is identical to the one given in measure theory.

PROOF. First notice that $f^{-1}(\mathcal{D}) \subseteq f^{-1}(\langle \mathcal{D} \rangle)$, which implies that

$$\big\langle f^{-1}(\mathcal{D}) \big\rangle \subseteq f^{-1}\big(\langle \mathcal{D} \rangle\big).$$

For the second inclusion, notice that

$$\mathcal{A} = \Big\{ B \subseteq Y \,\Big|\, f^{-1}(B) \in \big\langle f^{-1}(\mathcal{D}) \big\rangle \Big\}$$

is a set structure in Y. Since clearly $\mathcal{D} \subseteq \mathcal{A}$, we also have $\langle \mathcal{D} \rangle \subseteq \mathcal{A}$, which proves the second inclusion. $\square$

2.2. *Initial structures*

DEFINITION 2.4: *Initial structures*

Let $(f_\alpha)_{\alpha \in A}$ be a collection of maps from a set X to structured sets $(X_\alpha, \mathcal{E}_\alpha)$. The *initial structure* $\mathcal{E}$ on X induced by (f_α) is the smallest structure on X that makes all f_α homomorphisms. That is,

$$\mathcal{E} = \bigvee_{\alpha \in A} f_\alpha^{-1}(\mathcal{E}_\alpha) = \Big\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{E}_\alpha) \Big\rangle.$$

REMARK 2.5. If $\mathcal{D}_\alpha$ is a generating set for $\mathcal{E}_\alpha$ for all $\alpha \in A$, then we may replace $\mathcal{E}_\alpha$ on the right-hand side above with $\mathcal{D}_\alpha$. This follows immediately from the second part of Proposition 2.3, since the structure $\big\langle \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha) \big\rangle$ makes all f_α into homomorphisms.

Note that $\bigvee_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$ doesn't generally make sense, since $\mathcal{D}_\alpha$ is not necessarily a structure on X_α. ⌟

THEOREM 2.6: *Characteristic property of initial structures*

Let $(X, \mathcal{E})$ be a structured set equipped with the initial structure induced by maps $f_\alpha \colon X \to X_\alpha$, $\alpha \in A$. If $(Y, \mathcal{F})$ is a structured set, then $f \colon Y \to X$ is a homomorph-

ism if and only if $f_\alpha \circ f$ is a homomorphism for all $\alpha \in A$:

$$\begin{tikzcd} X \arrow[r, "f_\alpha"] & X_\alpha \\ Y \arrow[u, "f"] \arrow[ur, "f_\alpha \circ f"'] & \end{tikzcd}$$

In particular, the maps f_α are homomorphisms. Furthermore, the initial structure on X is unique with this property.

PROOF. If f is a homomorphism, then clearly the $f_\alpha \circ f$ are all homomorphisms.

Conversely, assume that all compositions $f_\alpha \circ f$ are homomorphisms. It suffices to show that $f^{-1}(B) \in \mathcal{F}$ for all B from a generating set for $\mathcal{E}$, so let $B = f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. It follows that

$$f^{-1}(B) = f^{-1}(f_\alpha^{-1}(C)) = (f_\alpha \circ f)^{-1}(C) \in \mathcal{F}$$

as desired. It now follows that f_α is a homomorphism because the diagram

$$\begin{tikzcd} X \arrow[r, "f_\alpha"] & X_\alpha \\ X \arrow[u, "\mathrm{id}_X"] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

commutes, and since id_X is a homomorphism. Notice that this only depends on $\mathcal{E}$ having the characteristic property above, and not on the concrete definition of $\mathcal{E}$.

Now assume that $\mathcal{E}'$ is a structure on X with the characteristic property of the initial structure. Consider the commutative diagram

$$\begin{tikzcd} (X,\mathcal{E}') \arrow[r, "f_\alpha'"] & X_\alpha \\ (X,\mathcal{E}) \arrow[u, "\mathrm{id}_X"] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

where a prime denotes that the domain of a map is $(X,\mathcal{E}')$ but is as a set function the same as its unprimed counterpart. The f_α are homomorphisms, so by the characteristic property applied to $\mathcal{E}'$ we get that id_X is a homomorphism.

Finally consider the analogous diagram with primes interchanged:

$$\begin{tikzcd} (X,\mathcal{E}) \arrow[r, "f_\alpha"] & X_\alpha \\ (X,\mathcal{E}') \arrow[u, "\mathrm{id}_X'"] \arrow[ur, "f_\alpha'"'] & \end{tikzcd}$$

The f'_α are homomorphisms, since this fact only depends on $\mathcal{E}'$ satisfying the characteristic property of initial structures. Applying the characteristic property to $\mathcal{E}$ then shows that id'_X is a homomorphism. Thus $(X,\mathcal{E})$ and $(X,\mathcal{E}')$ are isomorphic through the identity, hence $\mathcal{E} = \mathcal{E}'$. □

EXAMPLE 2.7: Subsets.
Let $(X,\mathcal{E})$ be a structured set, and let $S \subseteq X$. The inclusion map $\iota_S \colon S \to X$ then induces an initial structure on S, namely the pullback $\iota_S^*(\mathcal{E})$. We denote this subset structure by $\mathcal{E}_S$, and unless otherwise noted subsets of structured sets always carry this structure. By the characteristic property of initial structures, a map $f \colon Y \to S$ from a structured set is a homomorphism if and only if $\iota_S \circ f$ is a homomorphism.

On the other hand, if $f \colon Y \to X$ is a map with $f(Y) \subseteq S$, then the map $\tilde{f} \colon Y \to S$ given by $\tilde{f}(y) = f(y)$ for all $y \in Y$ is a homomorphism if and only if $f = \iota_S \circ \tilde{f}$ is a homomorphism. In other words, whether a map is a homomorphism or not does not depend on the codomain if we agree to equip subsets with the structure induced by their inclusion maps.

If $S = f(Y)$ and $\tilde{f} \colon Y \to f(Y)$ is an isomorphism, then we call f an *embedding*. ⌟

EXAMPLE 2.8: Products.
Let $(X_\alpha,\mathcal{E}_\alpha)_{\alpha\in A}$ be a collection of structured sets, let $X = \prod_{\alpha\in A} X_\alpha$ be the Cartesian product of the sets X_α, and denote the associated projections by $\pi_\alpha \colon X \to X_\alpha$. We define a product structure

$$\mathcal{E} = \bigotimes_{\alpha\in A} \mathcal{E}_\alpha$$

as the initial structure on X induced by the projection maps. Since X is a product of the X_α in the category of sets, the characteristic property of initial structures implies that $(X,\mathcal{E})$ is a product of the structured sets $(X_\alpha,\mathcal{E}_\alpha)$. ⌟

PROPOSITION 2.9: *Composition of initial structures*

Assume that X has the initial structure induced by a family of maps $f_\alpha \colon X \to X_\alpha$ for $\alpha \in A$, and that each set X_α has the initial structure induced by maps $g_{\alpha\lambda} \colon X_\alpha \to Y_{\alpha\lambda}$ for $\lambda \in \Lambda_\alpha$. Then X carries the initial structure induced by the maps $g_{\alpha\lambda} \circ f_\alpha \colon X \to Y_{\alpha\lambda}$ for $\alpha \in A$ and $\lambda \in \Lambda_\alpha$.

PROOF. Let $\mathcal{F}_{\alpha\lambda}$ be the set structure on $Y_{\alpha\lambda}$. By definition we have

$$\mathcal{E}_\alpha = \bigvee_{\lambda\in\Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda}) = \Big\langle \bigcup_{\lambda\in\Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda}) \Big\rangle.$$

Since the union on the right-hand side is a generating set for $\mathcal{E}_\alpha$, Remark 2.5 implies that

$$\mathcal{E} = \Big\langle \bigcup_{\alpha\in A} f_\alpha^{-1}\Big(\bigcup_{\lambda\in\Lambda_\alpha} g_{\alpha\lambda}^{-1}(\mathcal{F}_{\alpha\lambda})\Big)\Big\rangle = \Big\langle \bigcup_{\alpha\in A}\bigcup_{\lambda\in\Lambda_\alpha} (g_{\alpha\lambda}\circ f_\alpha)^{-1}(\mathcal{F}_{\alpha\lambda})\Big\rangle,$$

proving the claim. □

Example 2.10: Subspace and product structures.

Let $(X_\alpha)_{\alpha\in A}$ be a family of structured sets, and let $S_\alpha \subseteq X_\alpha$ be subsets. Then we may equip the product $S = \prod_{\alpha\in A} S_\alpha$ by first equipping $X = \prod_{\alpha\in A} X_\alpha$ with the product structure, and then induce the subset structure on S. In the opposite order we may first equip each S_α with the subset structure, and then induce the product structure. These in fact give the same structure since the diagram

$$\begin{tikzcd} & S_\alpha \arrow[r, "\iota_{S_\alpha}"] & X_\alpha \\ S \arrow[ur, "\pi_{S_\alpha}"] \arrow[r, "\iota_S"'] & X \arrow[ur, "\pi_{X_\alpha}"'] & \end{tikzcd}$$

commutes. ⌟

Example 2.11: The weak*-topology.

Let X be a topological vector space over the field $\mathbb{F}$ with topological dual X^*, and for $x\in X$ let $\mathrm{ev}_x\colon X^*\to\mathbb{F}$ be the evaluation map $\mathrm{ev}_x(\varphi)=\varphi(x)$ for $\varphi\in X^*$. Since X^* is a subset of $\mathbb{F}^X$, it naturally carries the subspace topology. The product topology on $\mathbb{F}^X$ is induced by the projection maps $\pi_x\colon \mathbb{F}^X\to\mathbb{F}$ for $x\in X$. But $\pi_x\circ\iota_{X^*}$ is just the evaluation map ev_x, so the subspace topology on X^* is exactly the weak*-topology. ⌟

Proposition 2.12: *Embedding into product*

Let $f_\alpha\colon Y\to X_\alpha$ for $\alpha\in A$, let $X=\prod_{\alpha\in A} X_\alpha$, and let $f\colon Y\to X$ be the unique map such that $f_\alpha = \pi_\alpha\circ f$:

$$\begin{tikzcd} & & X_\alpha \\ Y \arrow[urr, bend left, "f_\alpha"] \arrow[r, "f"'] & X \arrow[ur, "\pi_\alpha"'] & \end{tikzcd}$$

Then f is an embedding if and only if Y carries the initial structure induced by the maps f_α and the collection $(f_\alpha)_{\alpha\in A}$ separates points in Y.

Proof. First assume that f is an embedding. In particular it is injective, and since the maps π_α separate points in X, the compositions $f_\alpha=\pi_\alpha\circ f$ separate points in Y. Let $\tilde{f}\colon Y\to f(Y)$ be the isomorphism such that $f=\iota_{f(Y)}\circ\tilde{f}$. Then since $\tilde{f}$ is an isomorphism, in particular Y carries the initial structure induced by $\tilde{f}$. But then Y carries the initial structure induced by the maps

$$\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f} = \pi_\alpha \circ f = f_\alpha \tag{2.2}$$

for $\alpha \in A$, as claimed.

Conversely, assume that the f_α separate points in Y and that Y has the initial structure $\mathcal{F}$ induced by the f_α. The f_α are then homomorphisms, and by the characteristic property of initial structures so is f. Furthermore, if $x, y \in Y$ with $x \neq y$, then there is an $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$, which implies that $f(x) \neq f(y)$, so f is injective.

Denote the product structure on X by $\mathcal{E}$. We show that if $B \in \mathcal{F}$, then $f(B) \in \mathcal{E}_{f(Y)}$, which will imply that f is an embedding. It suffices to prove this when B is an element of a generating set for $\mathcal{F}$, i.e. on the form $f_\alpha^{-1}(C)$ for some $\alpha \in A$ and $C \in \mathcal{E}_\alpha$. By (2.2) we have

$$B = f_\alpha^{-1}(C) = (\pi_\alpha \circ \iota_{f(Y)} \circ \tilde{f})^{-1}(C) = \tilde{f}^{-1}\big((\pi_\alpha \circ \iota_{f(Y)})^{-1}(C)\big),$$

from which it follows that

$$f(B) = \tilde{f}(B) = (\pi_\alpha \circ \iota_{f(Y)})^{-1}(C) \in \mathcal{E}_{f(Y)}.$$

as desired. □

If $\mathcal{E}$ is a structure on a set X, we say that $\mathcal{E}$ is *countably generated* if there is a countable collection of sets $\mathcal{D} \subseteq 2^X$ such that $\mathcal{E} = \langle \mathcal{D} \rangle$.

PROPOSITION 2.13: *Countably generated initial structures*

Let $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ be a countable collection of structured sets, and assume that the $\mathcal{E}_\alpha$ are countably generated by collections of sets $\mathcal{D}_\alpha$. If an initial structure $\mathcal{E}$ is induced on a set X by maps $f_\alpha \colon X \to X_\alpha$, then $\mathcal{E}$ is also countably generated.

PROOF. This follows immediately by Remark 2.5 since the generating set $\bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{D}_\alpha)$ is a countable union of countable sets. □

EXAMPLE 2.14: Second-countable topological spaces.

We claim that a topology is second-countable if and only if it is countably generated. Since a basis in particular is a generating set (i.e. a subbasis), second-countable topologies are countably generated. Let $\mathcal{T}$ be a topology that is generated by a countable set $\mathcal{D}$. Then a basis $\mathcal{B}$ for $\mathcal{T}$ is obtained by taking finite intersections of elements from $\mathcal{D}$. The number of these intersections is certainly less than the cardinality of the union

$$\bigcup_{n \in \mathbb{N}} \prod_{i=1}^{n} \mathcal{D}$$

of all finite products of $\mathcal{D}$ with itself, an element $U_1 \times \cdots \times U_n$ of an n-fold product corresponding to the intersection $\bigcap_{i=1}^{n} U_i$. But finite products of countable sets are countable, and so are countable unions of countable sets, so the union above is countable.

The above proposition then implies that an initial topology induced by a countable family of maps into second-countable spaces is itself second-countable. In particular, subspaces and countable products of second-countable spaces are second-countable. ⌟

2.3. Categorical properties

The category $\mathbf{Str}_{\mathfrak{S}}$

We first recapitulate some of the above results in categorical terms. The main result is the following:

THEOREM 2.15: *Completeness of* $\mathbf{Str}_{\mathfrak{S}}$

The category $\mathbf{Str}_{\mathfrak{S}}$ *is complete, i.e. it has all small limits.*

PROOF. By e.g. Smith 2018, Theorem 60 it is enough to show that $\mathbf{Str}_{\mathfrak{S}}$ has all small products and has equalisers.

Products: We claim that the product $(X, \mathcal{E})$ considered in Example 2.8 is in fact a product of the objects $(X_\alpha, \mathcal{E}_\alpha)_{\alpha \in A}$ in $\mathbf{Str}_{\mathfrak{S}}$. If Y is a structured set and $f_\alpha Y \to X_\alpha$ are homomorphisms, then since X is a product in $\mathbf{Set}$ there is a unique set function $f : Y \to X$ such that $f_\alpha = \pi_\alpha \circ f$ for all $\alpha \in A$. But f is also a homomorphism by the characteristic property of the product structure, so $(X, \mathcal{E})$ is in fact a product in $\mathbf{Str}_{\mathfrak{S}}$.

Equalisers: Let $f, g : X \to Y$ be any pair of parallel homomorphisms, and let E be the subset of X on which they agree. If $h: Z \to X$ is any homomorphism such that $f \circ h = g \circ h$, then there is a unique homomorphism $u : Z \to E$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\iota_E"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[u, dashed, "u"] \arrow[ur, "h"'] & & \end{tikzcd}$$

We must have $h(Z) \subseteq E$, so we can define u by $u(z) = h(z)$, and u is unique as a set function such that the above diagram commutes. Furthermore, u is a homomorphism by the characteristic property of the subset structure. Thus E along with the inclusion map ι_E is an equaliser of f and g. □

Functoriality of $\mathfrak{S}$

As mentioned, $\mathfrak{S}$ is in fact a functor. Its action on a set function $f: X \to Y$ is defined as the pullback $f^*: \mathfrak{S}_Y \to \mathfrak{S}_X$.

Let **Set** denote the category of sets and $\mathbf{CsLat}^\vee$ the category of complete join-semilattices and join-preserving maps.

PROPOSITION 2.16: *Functoriality of $\mathfrak{S}$, I*

The map $\mathfrak{S}$ is a contravariant functor from **Set** *to* $\mathbf{CsLat}^\vee$.

PROOF. By (2.1) and Proposition 2.3, for any set function f the pullback f^* preserves joins since preimages respect unions, so it is well-defined as a map $\mathbf{Set} \to \mathbf{CsLat}^\vee$.

The map $\mathfrak{S}$ is also contravariant since if $f: X \to Y$ and $g: Y \to Z$ are set functions, then for $\mathcal{G} \in \mathfrak{S}_Z$ we have

$$(g \circ f)^*(\mathcal{G}) = (g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = (f^* \circ g^*)(\mathcal{G}).$$

Its action on identity functions is clearly trivial, so it is a functor. □

In the case of topological spaces or measure spaces we can say slightly more: Notice that in both a lattice of topologies or of σ-algebras on a set, intersections of topologies (σ-algebras) are themselves topologies (σ-algebras). A nonempty subset $\mathfrak{L}$ of 2^X, where X is some set, is called an *intersection structure*. If also $X \in \mathfrak{L}$ we call it a *topped intersection structure*. It is easy to show that topped intersection structures are complete lattices ordered by inclusion, and that meets are given by intersections.

If **CLat** denotes the category of complete lattices with join- and meet-preserving maps, then we have the following:

PROPOSITION 2.17: *Functoriality of $\mathfrak{S}$, II*

If $\mathfrak{S}_X$ is an intersection structure for all sets X, then $\mathfrak{S}$ is a contravariant functor from **Set** *to* **CLat**.

PROOF. It suffices to show that f^* preserves meets for all set functions $f: X \to Y$. But this is clear since preimages respect intersections. □

It is natural to ask whether the pushforward f_* by f gives rise to a covariant functor from **Set** into a category of lattices. It is easy to see that f_* is monotone, and a short calculation shows that $(g \circ f)_* = g_* \circ f_*$ if $g: Y \to Z$ is another set function, so the pushforward does indeed define a covariant functor from **Set** to the category **Pos** of posets and monotone maps. But f_* does not, as far as I know, generally preserve meets or joins. If all $\mathfrak{S}_X$ are intersection structures, however, it does preserve meets, so it is then a functor into the category $\mathbf{CsLat}^{\wedge}$ of complete meet-semilattices, though it still does not seem to preserve joins. (I haven't looked too hard for counterexamples.)

The forgetful functor on $\mathbf{Str}_{\mathfrak{S}}$

Just as there is a forgetful functor $\mathbf{Top} \to \mathbf{Set}$ that sends a topological space to its underlying set, there is a forgetful functor $U\colon \mathbf{Str}_{\mathfrak{S}} \to \mathbf{Set}$. As usual, U has a left adjoint D that equips a set with the discrete structure, and it has a right adjoint T that equips the set with the trivial structure, i.e. $D \dashv U \dashv T$.

It follows immediately that U preserves both limits and colimits. Hence if $\mathbf{Str}_{\mathfrak{S}}$ e.g. has coproducts – which it has, as we will see later – we already know that they have to be (isomorphic to) disjoint unions of the underlying sets, equipped with an appropriate structure. Contrast this with the situation in the category $\mathbf{Grp}$ of groups: The forgetful functor $U\colon \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint, namely the free functor, so U preserves limits. But it does not preserve colimits; for instance, coproducts in $\mathbf{Grp}$ are free products, and their underlying sets are certainly not disjoint unions! Hence U does not have a right adjoint.

Presheaves on structured sets

Let $(X,\mathcal{E})$ be a structured set, and view $\mathcal{E}$ as a preorder category. Analogous to the case of topological spaces, a presheaf on $\mathcal{E}$ (i.e. a contravariant functor $\mathcal{E} \to \mathbf{Set}$) is called a *presheaf* on $(X,\mathcal{E})$, or simply a presheaf on X if the structure is understood.

As an example, fix a structured set Y and take the presheaf F on X given by $F(B) = \mathbf{Str}_{\mathfrak{S}}(B,Y)$, i.e. F sends a set $B \in \mathcal{E}$ to the set of homomorphisms $B \to Y$. Furthermore, F sends an inclusion $B \subseteq B'$ in $\mathcal{E}$ to the restriction map $\mathbf{Str}_{\mathfrak{S}}(B',Y) \to \mathbf{Str}_{\mathfrak{S}}(B,Y)$ given by $f \mapsto f|_B$. A common example of this is the case $\mathbf{Str}_{\mathfrak{S}} = \mathbf{Top}$ and $Y = \mathbb{R}$, in which case F sends an open set U to the set of continuous functions $U \to \mathbb{R}$.

3 • Topology

3.1. Countability axioms

We remind the reader that a topological space $(X,\mathcal{T})$ is *second-countable* if there exists a countable basis for $\mathcal{T}$. Furthermore, $(X,\mathcal{T})$ is said to be *Lindelöf* if every open cover of X has a countable subcover.

Recall also that a topological property is called *hereditary* if it follows from a space X having this property that any subspace of X also has this property. It is easy to see that second-countability is hereditary, but the Lindelöf property is not:

REMARK 3.1. A space can be Lindelöf without being hereditarily Lindelöf. Let $(X,\mathcal{T})$ be an uncountable discrete space, and let $y \notin X$. Define a topological space $(Y,\mathcal{T}')$ with underlying space $Y = X \cup \{y\}$ and topology $\mathcal{T}' = \mathcal{T} \cup \{Y\}$. Then $(Y,\mathcal{T}')$ is Lindelöf since any open cover must include Y itself, this being the only open set containing the point y. But the subspace X (whose subspace topology is exactly $\mathcal{T}$) is clearly not Lindelöf. ⌟

If every subspace of X is Lindelöf, then we say that X is *hereditarily Lindelöf*.

PROPOSITION 3.2

If $(X,\mathcal{T})$ is a second-countable topological space, then it is hereditarily Lindelöf.

PROOF. Every subspace of X is second-countable, so it suffices to show that X is Lindelöf.

Let $\mathcal{U}$ be an open cover of X and let $\mathcal{B}$ be a countable basis for the topology $\mathcal{T}$. Consider an $x \in X$. Since $\mathcal{U}$ is a cover of X there is some $U_x \in \mathcal{U}$ with $x \in \mathcal{U}$, and since $\mathcal{B}$ is a basis for $\mathcal{T}$ there is some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U_x$. Let $\mathcal{B}' \subseteq \mathcal{B}$ be the subset of open sets obtained in this way. Clearly $\mathcal{B}'$ is a cover of X.

For each $B \in \mathcal{B}'$, the above shows that there exists some $U \in \mathcal{U}$ with $B \subseteq U$. This defines a map $\mathcal{B}' \to \mathcal{U}$ given by $B \mapsto U$ whose image is a countable cover of X, proving the claim. □

LEMMA 3.3

Let $(X,\mathcal{T})$ be a second-countable space. Then every basis for $\mathcal{T}$ contains a countable basis for $\mathcal{T}$.

PROOF. Let $\mathcal{B}$ be a basis for $\mathcal{T}$, and let $\mathcal{C}$ be a countable basis. We can write every $C \in \mathcal{C}$ on the form $C = \bigcup_{\alpha\in A} B_\alpha$ for some family $\{B_\alpha \mid \alpha \in A\} \subseteq \mathcal{B}$. This is in particular an open cover of C, so since X is hereditarily Lindelöf there is a countable subset $A' \subseteq A$ such that $C = \bigcup_{\alpha\in A'} B_\alpha$. For each $C \in \mathcal{C}$ we thus obtain a countable subcollection of sets from $\mathcal{B}$, and since $\mathcal{C}$ is also countable, the union of all these sets is countable and is clearly a basis for $\mathcal{T}$. □

4 • Measure theory

If A is a subset of a measurable space $(X,\mathcal{E})$, then recall that we denote by $\mathcal{E}_A$ the initial σ-algebra on A induced by the inclusion $\iota_A \colon A \to X$. Similarly, if A is a subspace of a topological space $(X,\mathcal{T})$, denote by $\mathcal{T}_A$ the subspace topology on A.

Proposition 4.1

Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\mathcal{B}(A) = \mathcal{B}(X)_A$, i.e. $\sigma(\mathcal{T}_A) = \sigma(\mathcal{T})_A$.

Proof. Notice that

$$\sigma(\mathcal{T}_A) = \sigma(\iota_A^{-1}(\mathcal{T})) = \iota_A^{-1}(\sigma(\mathcal{T})) = \sigma(\mathcal{T})_A$$

by Proposition 2.3. □

Proposition 4.2

Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a family of topological spaces, and equip $X = \prod_{\alpha \in A} X_\alpha$ with the product topology $\mathcal{T}$. Then

$$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) \subseteq \mathcal{B}(X)$$

If A is countable and the spaces X_α are second-countable, then the above inclusion is an equality.

Proof. Since the projections $\pi_\alpha \colon X \to X_\alpha$ are continuous, they are $\mathcal{B}(X)$-$\mathcal{B}(X_\alpha)$-measurable. But $\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$ is the smallest σ-algebra on X that makes the projections measurable, which proves the above inclusion.

Now assume that A is countable and that all the X_α are second-countable. From Example 2.14 we know that X is also second-countable. Let

$$\mathcal{D} = \bigcup_{\alpha \in A} \pi_\alpha^{-1}(\mathcal{T}_\alpha)$$

be a subbasis for the product topology $\mathcal{T}$, and let $\mathcal{B}$ be the collection of finite intersections of elements in $\mathcal{D}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$, and $\mathcal{B}$ contains a countable basis $\mathcal{C}$ for $\mathcal{T}$ by Lemma 3.3. Since $\mathcal{C}$ is countable, open sets in X are countable unions of finite intersections of elements in $\mathcal{D}$. Since A is also countable, it suffices to show that

$$\pi_\beta^{-1}(\mathcal{T}_\beta) \subseteq \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$$

for all $\beta \in A$. But this is obvious since the projections are measurable. □

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