Separation Axioms

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1 • Introduction

In these notes we give an overview of some of the most important separation axioms in point-set topology. For each axiom we consider the following:

(1) What the axiom *means* intuitively. Of course we understand what it means to 'separate' points or sets in a topological space and how the ability to do so is useful. But many of the axioms are equivalent to properties of a space that (seemingly) have nothing to do with separation, and whose importance are perhaps more intuitively clear.

(2) How the axiom behaves with respect to topological constructions like subspaces and products.

(3) Conditions under which a space satisfies a given axiom.

(4) Other properties of the axioms. Each axiom interacts with the surrounding theory in different ways, sometimes to produce spaces with even more structure.

Our attention is primarily focused on both ends of the spectrum: The weakest axiom we consider, and the one we have most to say about, is T_0. We say less about the T_1 and Hausdorff axioms since these are well-known, and there is also rather little to say about regular spaces. More interesting are complete regularity and normality on which we spend a lot of time: normality because of its importance and particularly the Urysohn lemma, and complete regularity because of its connection with rings of continuous functions.

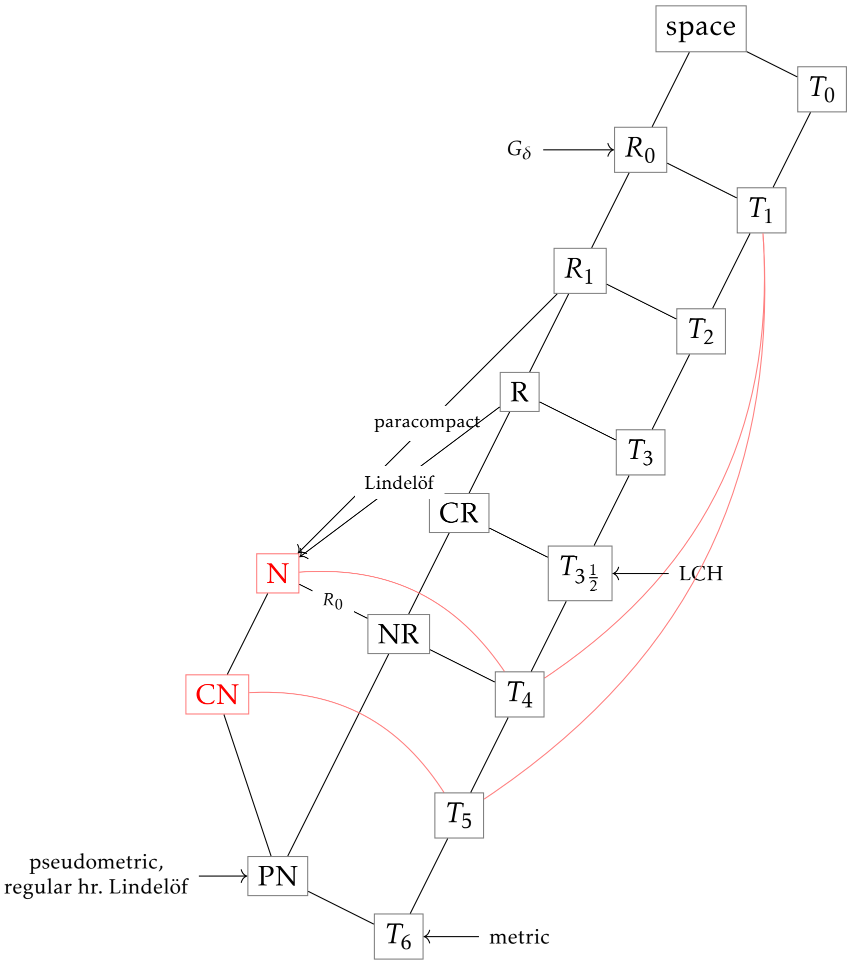

Figure 1: TODO caption

1.1. Notation

If X and Y are topological spaces, we denote the set of continuous maps $X \to Y$ by $C(X, Y)$. If Y is a real/complex topological vector space, then $C(X, Y)$ is a real/complex vector space when addition and scalar multiplication are defined pointwise. In the case $Y = \mathbb{R}$ we simply write $C(X)$, and this is an algebra over $\mathbb{R}$. We also write $C_b(X)$ for the subalgebra of bounded functions.

1.2. Overview of the axioms

It is clear that an R_1-space is automatically R_0. If a space X is regular and points $x, y \in X$ are topologically distinguishable, then x (say) has an open neighbourhood U not containing y. But then U^c is a closed set not containing

x, so these are separated by neighbourhoods by regularity.

Clearly a completely regular space is regular, and to see that a normal R_0-space X is completely regular, hence regular, let $x \in X$ and let $F \subseteq X$ be closed. Then $\overline{\{x\}}$ is disjoint from F by R_0 (cf. TODO ref), and Urysohn's lemma implies that $\overline{\{x\}}$ and F are separated by a continuous function. Hence X is completely regular. Note that we use the term 'normal regular' even though such a space satisfies the (apparently) stronger property of complete regularity.

Finally notice that since perfect normality is hereditary (cf. Corollary 9.2), a perfectly normal space is completely normal. It is also R_0 since it is G_δ, so it is regular.

Each of the T_n-axioms for $n > 0$ are obtained from an axiom by adding either the T_0- or T_1-axiom. It usually suffices to add T_0, but in the case of (complete) normality we must add T_1 to obtain the T_4-axiom. Indeed, the Sierpiński space is completely normal and T_0, but is not T_1 since $\{1\}$ is not closed, so it is not even R_0. In the presence of the T_1-axiom, a normal space is R_0 and hence completely regular, so T_4 is indeed stronger than $T_{3\frac{1}{2}}$. Furthermore, a perfectly normal T_0-space is G_δ and hence R_0, so it is automatically T_1. So T_6 is also stronger than T_5.

We note the following:

- Regular Lindelöf spaces are paracompact (cf. [TODO ref]). Hence Lindelöf LCH-spaces (in particular manifolds) are paracompact.
- Lindelöf LCH-spaces are exhaustible by compact sets (cf. [TODO prove/Lee]), which makes possible a second proof that these spaces are paracompact (cf [TODO prove/Lee]).
- Paracompact R_1-spaces are normal, hence normal regular (cf. [TODO prove]).
- Regular Lindelöf spaces are normal. This follows from the above points, but we give a direct proof in [TODO ref].
- Hereditarily Lindelöf, regular spaces (in particular manifolds) are perfectly normal. This is proved in [TODO ref].

2 • Preliminary definitions and results

If X is a topological space and $A \subseteq X$, then we say that a set $N \subseteq X$ is a *neighbourhood* of A if there is an open set U in X such that $A \subseteq U \subseteq N$. The family of neighbourhoods of a set A is called the *neighbourhood filter of* A and is denoted $\mathcal{N}_A$. A *neighbourhood basis at* A is a filter basis for $\mathcal{N}_A$. If $A = \{x\}$ is a singleton we also write $\mathcal{N}_x$ and call N a neighbourhood of x.

If X is a topological space, then we say that two subsets A and B are *separated* if either A has a neighbourhood disjoint from B, or vice versa. Separated sets are clearly disjoint. We say that A and B are *separated by neighbourhoods* if there exist neighbourhoods of the two sets that are disjoint. If there such neighbourhoods that are also, then A and B are *separated by closed neighbourhoods*.

Furthermore, A and B are said to be *separated by a continuous function* if there is a continuous function $f: X \to \mathbb{R}$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$. We say that A and B are *precisely separated by a continuous function* if $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ for a continuous $f: X \to \mathbb{R}$. In either case we may assume that $f(X) \subseteq [0,1]$ [TODO].

Note that these properties are progressively stronger. This is obvious for most of them, and we note that if A and B are separated by a continuous function f, then the preimages under f of the intervals $[-\frac{1}{3}, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{4}{3}]$ are closed neighbourhood separating A and B.

2.1. Compactness

If X is a topological space, we say that a collection $\mathcal{U}$ of subsets of X is a *cover* of X if $X = \bigcup_{U \in \mathcal{U}} U$. A *subcover* of a cover $\mathcal{U}$ is a subcollection of $\mathcal{U}$ that itself is a cover. An *open cover* is a cover consisting of open sets.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, we say that $\mathcal{U}$ *refines* $\mathcal{V}$ and is a *refinement* of V if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. Notice that a subcover is in particular a refinement.

A collection $\mathcal{A}$ of subsets of X is called *locally finite* if every point of X has a neighbourhood that intersects finitely many elements in $\mathcal{A}$. It is easy to show that if $\mathcal{A}$ is locally finite, then $\overline{\mathcal{A}} = \{\overline{A} \mid A \in \mathcal{A}\}$ is also locally finite: For if $x \in X$ and $U \subseteq X$ is an open neighbourhood of x and $A \in \mathcal{A}$ does not intersect U, then $A \subseteq U^c$, and so $\overline{A} \subseteq U^c$.

Definition 2.1

Let X be a topological space.

(i) X is *compact* if every open cover of X has a finite subcover.

(ii) X is *countably compact* if every countable open cover of X has a finite subcover.

(iii) X is *Lindelöf* if every open cover of X has a countable subcover.

(iv) X is *paracompact* if every open cover of X has an open locally finite refinement.

Clearly a space is compact if and only if it is both countably compact and Lindelöf, and a compact space is also paracompact. A subset $A \subseteq X$ is called *precompact* in X if its closure $\overline{A}$ in X is compact.

Proposition 2.2

Compactness, countable compactness, Lindelöf and paracompactness are weakly hereditary.

This result is of course standard for compact spaces, but we include the proof to illustrate that it is identical to the proof for the other cases.

Proof. Let X be compact/countably compact/Lindelöf/paracompact, and let $A \subseteq X$ be closed. If $\mathcal{U}$ is an open cover (countable if X is countably compact) of A, then there is a (again countable) collection $\mathcal{V}$ of sets open in X such that $\mathcal{U} = \{V \cap A \mid V \in \mathcal{V}\}$. By adjoining A^c to $\mathcal{V}$ we obtain an open cover of X (which is countable), so it has a finite/countable/locally finite subcover $\mathcal{V}'$. Discarding A^c and intersecting every other set in $\mathcal{V}'$ with A yields a finite/countable/locally finite subcover of $\mathcal{U}$, proving the claim. □

Proposition 2.3

The continuous image of a compact/countably compact/Lindelöf space is compact/countably compact/Lindelöf.

Proof. Let $f: X \to Y$ be continuous, and assume that X is compact/countably compact/Lindelöf. Let $\mathcal{U}$ be an open cover of $f(X)$ (countable if X is countably compact). Then $f^{-1}(\mathcal{U})$ is an open cover of X (again countable), so it has a finite/countable subcover $f^{-1}(\mathcal{U}')$. Notice that

$$f(X) = f\Bigl(f^{-1}\Bigl(\bigcup \mathcal{U}'\Bigr)\Bigr) \subseteq \bigcup \mathcal{U}',$$

so $\mathcal{U}'$ is a finite/countable subcover of $\mathcal{U}$ as desired. □

Remark 2.4. Note that the continuous image of a paracompact space need not be paracompact. Let $(X, \mathcal{T})$ be a space that is not paracompact, and let $\mathcal{T}_\mathrm{d}$ be the discrete topology on X. Then $(X, \mathcal{T}_\mathrm{d})$ is paracompact, but the identity map $\mathrm{id}_X : (X, \mathcal{T}_\mathrm{d}) \to (X, \mathcal{T})$ is continuous. ⌟

Lemma 2.5

Let X be a topological space, and let $A \subseteq X$ be a set that can be separated from points that lie in a set $B \subseteq X$. Then A can be separated from compact sets contained in B.

This is an example of the general principle that compact sets often act like points.

PROOF. Let $K \subseteq X$ be a compact set contained in B. Since A can be separated from points in B, then for every $x \in K$ there are disjoint open sets U_x and V_x such that $x \in U_x$ and $A \subseteq V_x$. The collection $(U_x)_{x\in K}$ is an open cover of K, so there is a finite subcover $(U_{x_i})_{i=1}^n$. Let $U = \bigcup_{i=1}^n U_{x_i}$ and $V = \bigcap_{i=1}^n V_{x_i}$. Then U and V are disjoint open sets containing K and A respectively. □

2.2. Local compactness

DEFINITION 2.6: *Local compactness*

A topological space is called

(i) *weakly locally compact* if every point has a compact neighbourhood,

(ii) *weakly locally precompact* if every point has a precompact (open) neighbourhood,

(iii) *(strongly) locally compact* if every point has a neighbourhood basis of compact sets, and

(iv) *(strongly) locally precompact* if every point has a neighbourhood basis of precompact (open) sets.

Clearly a point has a precompact neighbourhood if and only if it has a precompact *open* neighbourhood. A locally compact Hausdorff space is also called an *LCH space*.

PROPOSITION 2.7

Let X be a topological space. If X has any of the properties in Definition 2.6, then X is weakly locally compact. If X is Hausdorff, then all properties in Definition 2.6 are equivalent.

PROOF. The first claim is obvious. For the second claim, assume that X is Hausdorff and weakly locally compact, and let $x \in X$. Then x has a compact neighbourhood K, i.e., there is an open set U with $x \in U \subseteq K$. Since K is closed we have $\overline{U} \subseteq K$, so U is precompact.

Next let U be a precompact open neighbourhood of x. Then ∂U is compact, so x and ∂U are separated by Proposition 5.5, i.e., there exist disjoint open sets V, W such that $x \in V$ and $\partial U \subseteq W$. Replacing V with $V \cap U$ we may assume that $V \subseteq U \setminus W$. Hence $\overline{V} \subseteq \overline{U} \setminus W \subseteq U$, so X is (strongly) locally compact. □

Lemma 2.8

All properties in Definition 2.6 are weakly hereditary. Open subsets of strongly locally compact spaces are also strongly locally compact.

Proof. The first claim is obvious, since the intersection of a compact set and a closed set is compact. The second claim is also obvious. □

Lemma 2.9

If X is weakly locally compact and $K \subseteq X$ is compact, then K has a compact neighbourhood. In particular, if X is strongly locally compact and $K \subseteq U \subseteq X$ with U open, then K has a compact neighbourhood contained in U.

Proof. For the first claim, note that each point of K has a compact neighbourhood, hence is covered by finitely many such neighbourhoods. Their union is a compact neighbourhood of K. The second claim then follows from Lemma 2.8. □

2.3. *Other types of compactness*

If A is a subset and x a point of X, then x is called a *limit point* of A if every neighbourhood of x contains a point of A distinct from x. Furthermore, x is an *ω-limit point* of A if every neighbourhood of x contains infinitely many points from A.

Definition 2.10

Let X be a topological space. Then X is called

(i) *exhaustible by compact sets* if there is a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of X, called an *exhaustion by compact sets*, such that $K_n \subseteq K_{n+1}^\circ$ and $X = \bigcup_{n\in\mathbb{N}} K_n$.

(ii) *hemicompact* if there is a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of X, called an *admissible sequence*, such that for every compact $K \subseteq X$ there is an $n \in \mathbb{N}$ with $K \subseteq K_n$.

(iii) *σ-compact* if there is a sequence $(K_n)_{n\in\mathbb{N}}$ of subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$.

(iv) *limit point compact* if every infinite subset of X has a limit point in X.

(v) *sequentially compact* if every sequence in X has a subsequence that converges to a point in X.

Proposition 2.11

In the notation of Definition 2.10, we have the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ Lindelöf. If X is weakly locally compact, then these properties are equivalent.

Proof. If $(K_n)_{n\in\mathbb{N}}$ is an exhaustion of X and $K \subseteq X$ is compact, then K is covered by finitely many K_n. Choosing the largest of these K_n shows that X is hemicompact.

If X is hemicompact, then an admissible sequence covers X since any singleton is compact. If X is σ-compact, $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact sets that cover X, and $\mathcal{U}$ is a open covering of X, then each K_n is covered by finitely many sets from $\mathcal{U}$, and so X is covered by countably many sets from $\mathcal{U}$.

Conversely assume that X is weakly locally compact. If X is Lindelöf, let K_x be a compact neighbourhood of $x \in X$. Then X is covered by the interiors of the K_x, hence by countably many such sets. But then X is σ-compact.

If X is instead σ-compact, let $(K_n)_{n\in\mathbb{N}}$ be a covering of X by compact sets. Let $C_1 = K_1$ and assume that $C_1, \ldots, C_n$ have been defined with $C_i \subseteq C_{i+1}^\circ$. Then $C_n \cup K_n$ is compact and has a compact neighbourhood C_{n+1} by Lemma 2.9. The sequence $(C_n)_{n\in\mathbb{N}}$ is thus an exhaustion of X by compact sets. □

Proposition 2.12

If X is hemicompact and first countable, then X is weakly locally compact.

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be an admissible sequence in X with $K_n \subseteq K_{n+1}$, and assume that there is a point $x \in X$ with no compact neighbourhood. If $(U_n)_{n\in\mathbb{N}}$ is a neighbourhood basis at x with $U_n \subseteq U_{n+1}$, then since K_n is not a neighbourhood of x there is a point $x_n \in U_n \setminus K_n$. The set $\{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is then compact but is not contained in any K_n. □

Theorem 2.13

(i) *Compactness implies limit point compactness.*

(ii) *For first countable T_1-spaces, limit point compactness implies sequential compactness.*

(iii) *For second countable spaces, sequential compactness implies compactness.*

In particular, the three properties are equivalent for second countable T_1-spaces.

Proof. TODO Lee. □

Lemma 2.14

A totally bounded metric space is second countable.

Proof. TODO Lee. □

Lemma 2.15

Total boundedness is hereditary.

Proof. TODO □

Theorem 2.16

If S is a metric space, then the following are equivalent:

(i) *S is compact.*

(ii) *S is limit point compact.*

(iii) *S is sequentially compact.*

(iv) *S is complete and totally bounded.*

In particular, compact metric spaces are complete and second countable.

Proof. Assume that S is sequentially compact. Then [TODO Lee] shows that S is totally bounded, so it is second countable by [TODO lem ref], and hence compact by [TODO ref]. Furthermore, any Cauchy sequence in S has a convergent subsequence, so is itself convergent. Hence S is complete.

Conversely assume that S is complete and totally bounded, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. By total boundedness, there is a ball B_1 with radius 1 that contains x_n for infinitely many n. But B_1 is also totally bounded by [TODO ref], so there is a ball B_2 in B_1 with radius $\frac{1}{2}$ containing x_n for infinitely many n. We recursively construct a decreasing sequence $B_1 \supseteq B_2 \supseteq \cdots$ such that B_{k+1} is a ball in B_k, such that B_k has radius $\frac{1}{k}$, and such that B_k contains x_n for infinitely many n. We thus extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \in B_k$. This is clearly a Cauchy sequence, and is hence convergent. Thus (x_n) has a convergent subsequence, and S is sequentially compact. □

2.4. Metric spaces

Lemma 2.17

Let (S,ρ) be a pseudometric space, and let $A \subseteq S$ be nonempty. Define a map

$\rho(\cdot, A)\colon S \to [0,\infty)$ by

$$\rho(x,A) = \inf_{a\in A} \rho(x,a).$$

This map has the following properties:

(i) *$\rho(x,A) = 0$ if and only if $x \in \overline{A}$.*

(ii) *If $y \in S$, then $\rho(x,A) \leq \rho(x,y) + \rho(y,A)$.*

(iii) *$\rho(\cdot, A)$ is continuous.*

Proof. First we prove (i). Notice that $\rho(x,A) = 0$ if and only if for any $r > 0$ there is an $a \in A$ such that $\rho(x,a) < r$. But this is true precisely when any ball $B(x,r)$ intersects A, i.e. when $x \in \overline{A}$.

For any $a \in A$ we have

$$\rho(x,A) \leq \rho(x,a) \leq \rho(x,y) + \rho(y,a),$$

and since this is true for any $a \in A$, (ii) follows.

Finally, (iii) follows immediately from (ii), since

$$\rho(x,A) - \rho(y,A) \leq \rho(x,y)$$

for all $x,y \in S$. □

3 • The T_0 axiom

3.1. *Definition and the T_0-identification*

Definition 3.1

A topological space X satisfies the T_0 axiom and is called a T_0*-space* or said to be *Kolmogorov* if, for every pair of distinct points $x,y \in X$, either x has a neighbourhood that does not contain y, or vice versa.

We begin by giving an alternative characterisation of T_0-spaces: Let X be a topological space. We define an ordering on X called the *specialisation preorder* by letting $x \leq y$ if $\mathcal{N}_x \subseteq \mathcal{N}_y$, or equivalently if $x \in \overline{\{y\}}$, for $x,y \in X$. It is clear that $\leq$ is in fact a preorder, and so it determines an equivalence relation $\equiv$; that is, $x \equiv y$ if and only if $x \leq y$ and $y \leq x$.

If $x \equiv y$, then we say that x and y are *topologically indistinguishable* since then x and y have the same neighbourhoods, i.e., $\mathcal{N}_x = \mathcal{N}_y$.

It is clear that X is T_0 if and only if the relation $\equiv$ is trivial, that is if x and y are topologically indistinguishable precisely when $x = y$. The quotient space $X/{\equiv}$ is called the T_0*-identification* or the *Kolmogorov quotient* of X, and it is indeed T_0:

THEOREM 3.2: *The T_0-identification*

Let X be a topological space, and let $q\colon X \to X/{\equiv}$ be the quotient map onto the T_0-identification of X. Then

(i) *open and closed sets are saturated,*

(ii) *q is an open and closed map,*

(iii) *$X/{\equiv}$ is T_0, and*

(iv) *if $\sim$ is an equivalence relation on X such that $X/{\sim}$ is T_0, then $\equiv \subseteq \sim$.*[1]

Part (iv) expresses the fact that $\equiv$ is the most conservative equivalence relation on X that makes the corresponding quotient a T_0-space.

PROOF. We first show that all open sets are saturated. Let U be an open set of X, and let $x \in U$. If $x \equiv x'$, then U is also a neighbourhood of x', so $x' \in U$; in other words, U is a union of fibres. Hence U is saturated with respect to q. Complements of saturated sets are also unions of fibres, hence saturated, so closed sets are also saturated. Since q is a quotient map, it follows that it takes saturated open (closed) subsets of X to open (closed) subsets of $X/{\equiv}$,[2] and hence it is both open and closed.

Now we show that $X/{\equiv}$ is T_0. Assume that $x \not\equiv y$. Without loss of generality we may assume the existence of an element $U \in \mathcal{N}_x \setminus \mathcal{N}_y$, and that U is open. Since q is open, $q(U)$ is a neighbourhood of $q(x)$ in $X/{\equiv}$. And U is saturated so $U = q^{-1}(q(U))$, and because $y \notin U$ it follows that $q(y) \notin q(U)$. Hence $q(U)$ is a neighbourhood of $q(x)$ that is not a neighbourhood of $q(y)$, and thus $X/{\equiv}$ is T_0.

Finally, let $X/{\sim}$ be T_0, and let $p\colon X \to X/{\sim}$ be the quotient map. If $x \nsim y$ then $p(x) \neq p(y)$, so without loss of generality we may choose an open set $U \subseteq X/{\sim}$ with $p(x) \in U$ and $p(y) \notin U$. Then $x \in p^{-1}(U)$ and $y \notin p^{-1}(U)$, i.e. $p^{-1}(U)$ is a neighbourhood of x that is not a neighbourhood of y, so $x \not\equiv y$. □

COROLLARY 3.3

Let $(X, \mathcal{T})$ be a topological space, and let $q\colon X \to X/{\equiv}$ be the quotient map onto the T_0-identification of X. Denote the topology on $X/{\equiv}$ by $\mathcal{T}_{\equiv}$. Then q induces a bijection $q_*\colon \mathcal{T} \to \mathcal{T}_{\equiv}$ given by $q_*(U) = q(U)$ whose inverse q^* is given by $q^*(V) = q^{-1}(V)$.

[1] Both $\equiv$ and $\sim$ are subsets of $X \times X$, so this inclusion means that, for all $x, y \in X$, if $x \equiv y$ then $x \sim y$.

[2] If $U \subseteq X$ is saturated, then $q(U)$ is open in Y if and only if $q^{-1}(q(U)) = U$ is open in X. See also Lee (2011, Proposition 3.60).

Proof. Since q is surjective we have $q(q^{-1}(V)) = V$ for all $V \in \mathcal{T}_\equiv$, and by Theorem 3.2(i) every $U \in \mathcal{T}$ is saturated, so $q^{-1}(q(U)) = U$. □

This corollary implies that most topological properties are preserved in the T_0-identification. Taking the T_0-identification of a space that is already T_0 leaves the space unchanged but, preempting terminology we will introduce later, the T_0-identification of a regular space is regular, and the same is true for completely regular, normal and paracompact spaces. The proofs are trivial.

One might also expect that continuous functions on a space are unchanged in the T_0-identification, and this is in fact the case:

Proposition 3.4

Let X be a topological space, and let $q\colon X \to X/{\equiv}$ be the quotient map onto its T_0-identification. For every T_0-space Y the pullback map

$$\begin{aligned} q^*\colon C(X/{\equiv}, Y) &\to C(X, Y), \\ f &\mapsto f \circ q, \end{aligned}$$

is a bijection. If Y is a T_0 topological vector space[3], then q^* is a linear isomorphism, and it is an algebra isomorphism in the case $Y = \mathbb{R}$.

Proof. Let $q\colon X \to X/{\equiv}$ be the quotient map, and let $g \in C(X, Y)$. We claim that if $x \equiv y$ in X, then $g(x) = g(y)$. For if $g(x) \neq g(y)$, then since Y is T_0 we can, without loss of generality, choose a neighbourhood U of $g(x)$ not containing $g(y)$. Then $g^{-1}(U)$ is a neighbourhood of x not containing y, so $x \not\equiv y$.

Thus every $g \in C(X, Y)$ descends to a map $\tilde{g} \in C(X/{\equiv}, Y)$ with $g = \tilde{g} \circ q$, showing surjectivity, and $\tilde{g}$ is unique, showing injectivity. Hence q^* is a bijection.

Now let Y be a T_0 topological vector space over $\mathbb{R}$. Let $f, g \in C(X/{\equiv}, Y)$ and $\alpha \in \mathbb{R}$. Then

$$q^*(\alpha f + g) = (\alpha f + g) \circ q = \alpha(f \circ q) + (g \circ q) = \alpha q^*(f) + q^*(g).$$

Hence q^* is linear and thus a linear isomorphism.

Finally, if $Y = \mathbb{R}$ then q^* respects multiplication by a similar argument to the above and is thus an algebra isomorphism. □

Proposition 3.5

Let X carry the initial topology induced by maps $f_\alpha\colon X \to X_\alpha$ for $\alpha \in A$. For all $x, y \in X$ we have $x \leq y$ (resp. $x \equiv y$) if and only if $f_\alpha(x) \leq f_\alpha(y)$ (resp. $f_\alpha(x) \equiv f_\alpha(y)$)

[3] One can show that a topological group is T_1 if it is T_0, and furthermore is always regular, so a T_0 topological group is T_3.

for all $\alpha \in A$.

PROOF. Let $x, y \in X$, and assume that $x \le y$. If $U \in \mathcal{N}_{f_\alpha(x)}$, then $f_\alpha^{-1}(U) \in \mathcal{N}_x \subseteq \mathcal{N}_y$ by continuity. That is, $y \in f_\alpha^{-1}(U)$, so $f_\alpha(y) \in U$ implying that $U \in \mathcal{N}_{f_\alpha(y)}$. Hence $f_\alpha(x) \le f_\alpha(y)$ as claimed.

Conversely, assume that $f_\alpha(x) \le f_\alpha(y)$ for all $\alpha \in A$. Notice that it suffices to show that $\mathcal{B}_x \subseteq \mathcal{N}_y$ for some neighbourhood basis $\mathcal{B}_x$ at x. We construct $\mathcal{B}_x$ as follows: If $\mathcal{T}_\alpha$ is the topology on X_α and $\mathcal{T}$ the topology on X, then $\mathcal{S} = \bigcup_{\alpha \in A} f_\alpha^{-1}(\mathcal{T}_\alpha)$ is a subbasis for $\mathcal{T}$, and the set $\mathcal{B}$ of finite intersections of elements in $\mathcal{S}$ is a basis for $\mathcal{T}$. We then let $\mathcal{B}_x = \mathcal{B} \cap \mathcal{N}_x$.

An arbitrary element in $\mathcal{B}_x$ is thus on the form $U = \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_i)$, with $U_i \in \mathcal{T}_{\alpha_i}$. It follows that $f_{\alpha_i}(x) \in U_i$ so that $U_i \in \mathcal{N}_{f_{\alpha_i}(x)} \subseteq \mathcal{N}_{f_{\alpha_i}(y)}$, implying that that $f_{\alpha_i}(y) \in U_i$. Hence $y \in f_{\alpha_i}^{-1}(U_i)$, so $y \in U$ and $U \in \mathcal{N}_y$ as desired. □

We explore the T_0-identification in the context of metric spaces: Let (S, ρ) be a pseudometric space, and define a relation $\sim$ on S by $x \sim y$ if and only if $\rho(x, y) = 0$. This is clearly an equivalence relation. Let $\tilde{S} = S/{\sim}$ and define a map $\tilde{\rho} \colon \tilde{S} \times \tilde{S} \to [0, \infty)$ by $\tilde{\rho}([x], [y]) = \rho(x, y)$. This is well-defined, since if $x \sim x'$ and $y \sim y'$ then

$$\rho(x, y) \le \rho(x, x') + \rho(x', y') + \rho(y', y) = \rho(x', y').$$

It is obvious that $\tilde{\rho}$ is then a metric on $\tilde{S}$, and we call $(\tilde{S}, \tilde{\rho})$ the *metric identification* of (S, ρ).

PROPOSITION 3.6

Let (S, ρ) be a pseudometric space. Then the relation $\sim$ defined above and the topological indistinguishability relation $\equiv$ coincide.

PROOF. It suffices to show that $x \sim y$ if and only if $\mathcal{N}_x = \mathcal{N}_y$ for all $x, y \in S$. Assume that $x \sim y$ and let $U \in \mathcal{N}_x$. Then there is an $r > 0$ such that $B(x, r) \subseteq U$. But we clearly have $y \in B(x, r)$, so $U \in \mathcal{N}_y$. This shows that $\mathcal{N}_x \subseteq \mathcal{N}_y$, and the opposite inclusion follows by symmetry.

Conversely, assume that $x \nsim y$. Then $0 < r < \rho(x, y)$ for some r, and $B(x, r)$ is a neighbourhood of x but not of y. □

3.2. Operations on T_0-spaces

PROPOSITION 3.7

Let X carry the initial topology induced by maps $f_\alpha \colon X \to X_\alpha$ for $\alpha \in A$. Assume that each X_α is T_0, and that the f_α separate points in X. Then X is also T_0.

In particular, any subspace of a T_0-space is T_0, and a product of T_0-spaces is

T_0. Furthermore, if a nonempty product space is T_0, then every factor is T_0.

We give two proofs of this result, one using the characterisation of the specialisation preorder given in Proposition 3.5, and another more direct proof.

Proof 1. Let $x, y \in X$ with $x \neq y$. Since the f_α separate points in X, there is a $\beta \in A$ such that $x_\beta \neq y_\beta$. Since X_β is T_0, the specialisation preorder on X_β is just equality, so $x_\beta \not\equiv y_\beta$. But then $x \not\equiv y$ by Proposition 3.5.

The final claim follows directly from the same proposition, since if the product space is nonempty, then the projections are surjective. □

Proof 2. Let $x, y \in X$ with $x \neq y$. Since the f_α separate points in X, there is a $\beta \in A$ such that $x_\beta \neq y_\beta$. Without loss of generality pick a neighbourhood U of x_β in X_β that does not contain y_β. Then $f_\beta^{-1}(U)$ is a neighbourhood of x that does not contain y.

For the final claim, assume that $X = \prod_{\alpha \in A} X_\alpha$ is a nonempty T_0 product space, and let $\beta \in A$. Pick a point $y \in X$, and let

$$Y = X_\beta \times \prod_{\alpha \neq \beta} \{y_\alpha\} = \{x \in X \mid x_\alpha = y_\alpha \text{ for } \alpha \neq \beta\}.$$

Then Y is T_0 since it is a subspace of X, so $X_\beta \cong Y$ is also T_0. □

Notice that we could also have proved the first result above by proving it separately for subspaces and products, and then using the fact that X could be embedded in the product $\prod_{\alpha \in A} X_\alpha$ since the f_α separate points[4]. In fact, we may use this product embedding to characterise the T_0-spaces as follows:

First recall that the *Sierpiński space* is the space S with the underlying set $\{0, 1\}$ and the topology $\{\emptyset, \{1\}, S\}$, so that the specialisation preorder on S is just the usual ordering on $\{0, 1\} \subseteq \mathbb{N}$. If X is any set, then the set of functions S^X is just the indicator functions on subsets of X, so that $S^X \cong \mathcal{P}(X)$. If $(X, \mathcal{T})$ is a topological space, then an indicator function $\mathbf{1}_U \colon X \to S$ is continuous just when $U = \mathbf{1}_U^{-1}(1)$ is open. It follows that there is a bijection betwen $\mathcal{T}$ and the set $C(X, S)$ of continuous maps $X \to S$.

Furthermore, X has the initial topology induced by $C(X, S)$, since removing an open set U from $\mathcal{T}$ would make the (continuous) indicator function $\mathbf{1}_U$ discontinuous. Also notice that $C(X, S)$ separates points in X just when X is T_0. Hence if X is T_0, then X can be embedded into the product $S^{\mathcal{T}}$ by the map $f \colon X \to S^{\mathcal{T}}$ with the property that $\pi_U \circ f = \mathbf{1}_U$. Since subspaces and products of T_0-spaces are T_0, the converse also holds.

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[4] See e.g. my notes on measure theory and topology, or Willard (1970, Theorem 8.12).

Proposition 3.8

Let $(X_\alpha)_{\alpha\in A}$ be a collection of T_0-spaces. Then the disjoint union $X = \coprod_{\alpha\in A} X_\alpha$ is also T_0.

This proposition in fact holds for all final topologies on a set X with the property that the coinducing maps $(f_\alpha)_{\alpha\in A}$ are injective, and the property that the images $f_\alpha(X_\alpha)$ form a partition of X. Under these assumptions the f_α cover X and are both open and closed (the latter of which we shall not use below). I am not aware of any other interesting final topologies with these properties.

Proof. First recall that the canonical injections ι_α are open, and that their images form a partition of X. For $x, y \in X$ with $x \neq y$, if $x \in X_\alpha$ and $y \in X_\beta$ for $\alpha \neq \beta$ then e.g. X_α is a neighbourhood of x not containing y. If instead $\alpha = \beta$, then since X_α is T_0 the point x has a neighbourhood not containing y. □

4 • The R_0 and T_1 axioms

4.1. *Definition and equivalent properties*

Definition 4.1

A topological space X satisfies the R_0 axiom and is called an R_0-*space* or is said to be *symmetric* if points $x, y \in X$ are separated whenever $x \not\equiv y$.

Furthermore, X satisfies the T_1 axiom and is called a T_1-*space* or is said to be *Fréchet* if X is both R_0 and T_0, i.e., if any two distinct points in X are separated.

In any topological space we have the implications

$$\text{separated} \quad \Rightarrow \quad \text{topologically distinguishable} \quad \Rightarrow \quad \text{distinct.}$$

The first implication can be reversed just when the space is R_0, and the second just when the space is T_0. The composite arrow can thus be reversed when the space is T_1.

We begin by giving some properties of topological spaces that are equivalent to the two axioms:

Proposition 4.2

The following are equivalent for a topological space X:

(i) *X is R_0,*

(ii) *the specialisation preorder $\leq$ is symmetric (i.e., $\leq \, = \, \equiv$),*

(iii) $[x]_{\equiv} = \overline{\{x\}}$ *for all* $x \in X$,

(iv) $[x]_{\equiv}$ *is closed for all* $x \in X$,

(v) *the sets* $\overline{\{x\}}$ *for* $x \in X$ *form a partition of* X,

(vi) $\overline{\{x\}}$ *is the intersection of all neighbourhoods of* x, *for all* $x \in X$,

(vii) *every neighbourhood of* x *contains* $\overline{\{x\}}$, *for all* $x \in X$, *and*

(viii) *every open set is a union of closed sets.*

PROOF. *(i) ⇒ (ii)*: Assume that $x \leq y$. Then x and y are not separated, and so $x \equiv y$.

(ii) ⇒ (iii): The inclusion '$\subseteq$' always holds, so assume that $y \in \overline{\{x\}}$. This means that $y \leq x$, so we also have $x \leq y$, i.e., $x \equiv y$.

(iii) ⇔ (iv): Since $\{x\} \subseteq [x]_{\equiv} \subseteq \overline{\{x\}}$, this is obvious.

(iii) ⇒ (v): The $\equiv$-equivalence classes are a partition of X.

(v) ⇒ (vi): The hypothesis implies that $y \notin \overline{\{x\}}$ if and only if $x \notin \overline{\{y\}}$, so $y \notin \overline{\{x\}}$ if and only if x has a neighbourhood not containing y. It follows that $\overline{\{x\}} = \bigcap_{N \in \mathcal{N}_x} N$.

(vi) ⇒ (vii): This is obvious.

(vii) ⇒ (viii): If U is open and $x \in U$, then U is a neighbourhood of x, and so $\overline{\{x\}} \subseteq U$. Hence $U = \bigcup_{x \in U} \overline{\{x\}}$.

(viii) ⇒ (i): Let $x, y \in X$, and assume that $x \not\equiv y$. Without loss of generality, assume that x has an open neighbourhood U not containing y. Then there is a closed set F with $x \in F \subseteq U$, so F^c is an open neighbourhood of y not containing x. □

PROPOSITION 4.3

The following are equivalent for a topological space X:

(i) X *is* T_1,

(ii) *each singleton of* X *is closed,*

(iii) *each singleton of* X *is the intersection of all its neighbourhoods,*

(iv) *each subset of* X *is the intersection of all its neighbourhoods, and*

(v) *for all subsets* $A \subseteq X$, *a point* $x \in X$ *is a limit point of* A *if and only if it is an* ω*-limit point of* A.

PROOF. If x is a limit point of A and N is a neighbourhood of x, then $A \cap N$ contains a point y_1. Since $\{y_1\}$ is closed, $N \setminus \{y_1\}$ is also a neighbourhood of x, so $A \cap N$ contains a point y_2 distinct from y_1. Continuing in this way yields infinitely many points in $A \cap N$.

Conversely, assume that there is some singleton $\{x\}$ that is not closed. Its closure thus contains a point $y \neq x$, so y is a limit point of $\{x\}$, but it is obviously not an ω-limit point.

The equivalence of the first three properties follow from the equivalence of (i), (iii) and (vi) of Proposition 4.2. The final property (iv) above follows since if $A \subseteq X$, then

$$A = X \setminus \bigcup_{x \notin A} \{x\} = \bigcap_{x \notin A} X \setminus \{x\},$$

so A is an intersection of open sets. □

PROOF. *(i)* $\Rightarrow$ *(ii)*: If X is T_1 and $x \in X$, then every point $y \in X \setminus \{x\}$ has a neighbourhood disjoint from $\{x\}$ so $X \setminus \{x\}$ is open.

(ii) $\Rightarrow$ *??* : If $A \subseteq X$, then

$$A = X \setminus \bigcup_{x \notin A} \{x\} = \bigcap_{x \notin A} X \setminus \{x\},$$

so A is an intersection of open sets.

?? $\Rightarrow$ *(i)*: If $x, y \in X$ with $x \neq y$, then there is an open subset containing x and not y, and vice versa. □

COROLLARY 4.4

A quotient space $X/\!\sim$ is T_1 if and only if every $\sim$-equivalence class is closed in X.

PROOF. Fibres of the quotient map are precisely the equivalence classes, so by the definition of the quotient topology, singletons of $X/\!\sim$ are closed if and only if the corresponding equivalence class is closed as a subset of X. □

4.2. Operations on R_0- and T_1-spaces

PROPOSITION 4.5

Let X carry the initial topology induced by maps $f_\alpha \colon X \to X_\alpha$ for $\alpha \in A$. If each X_α is R_0, then so is X. If X is R_0 and f_α is surjective, then X_α is also R_0.

PROOF. Follows directly from Proposition 3.5. □

Proposition 4.6

Let X carry the initial topology induced by maps $f_\alpha : X \to X_\alpha$ for $\alpha \in A$. Assume that each X_α is T_1, and that the f_α separate points in X. Then X is also T_1.

In particular, any subspace of a T_1-space is T_1, and a product of T_1-spaces is T_1. Furthermore, if a nonempty product space is T_1, then every factor is T_1.

Proof. This follows from Proposition 4.5 and Proposition 3.7. □

Proposition 4.7

Let $(X_\alpha)_{\alpha \in A}$ be a collection of T_1-spaces. Then the disjoint union $X = \coprod_{\alpha \in A} X_\alpha$ is also T_1.

Proof. Similar to the proof of Proposition 3.8. □

4.3. Conditions for the R_0 and T_1 axioms

Proposition 4.8

Every G_δ-space is R_0.

Proof. This follows immediately from Proposition 4.2(viii). □

Proposition 4.9

The closed image[5] of a T_1-space is T_1.

Proof. Let $f : X \to Y$ be a closed map from a T_1-space X to a topological space Y, and let $y \in f(X)$. Then there is some $x \in X$ with $f(x) = y$, and since $\{x\}$ is closed in X and f is closed, $\{y\}$ is closed in Y and hence closed in $f(X)$. □

5 • The R_1 and T_2 axioms: Preregular and Hausdorff spaces

5.1. Definition and equivalent properties

Definition 5.1

A topological space X satisfies the R_1 axiom and is called an *R_1-space* or is said to be *preregular* if points $x, y \in X$ are separated by neighbourhoods whenever $x \not\equiv y$.

Furthermore, X satisfies the T_2 axiom and is called a *T_2-space* or is said

[5] By 'closed image' we mean the image of a closed (not necessarily continuous) map.

to be *Hausdorff* if X is both R_1 and T_0, i.e., if any two distinct points in X are separated by neighbourhoods.

PROPOSITION 5.2

A topological space X is R_1 if and only if $\overline{\{x\}}$ is the intersection of all its closed neighbourhoods, for all $x \in X$.

PROOF. If X is R_1 with $y \notin \overline{\{x\}}$, then x and y are separated by neighbourhoods, say $x \in U$ and $y \in V$. Then V^c is a closed neighbourhood of $\overline{\{x\}}$ not containing y.

For the converse, if $x \not\equiv y$ then assume without loss of generality that $y \notin \overline{\{x\}}$. Then $\overline{\{x\}}$ has a closed neighbourhood F not containing y. Then F^c separates y from a neighbourhood of x. □

PROPOSITION 5.3

The following are equivalent for a topological space X:

(i) *X is Hausdorff,*

(ii) *$\{x\}$ is the intersection of its closed neighbourhoods for $x \in X$,*

(iii) *limits of nets (and hence of filters) in X are unique, and*

(iv) *the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.*

PROOF. *(i) ⇒ (iii)*: Let $(x_\alpha)_{\alpha\in A}$ be a net in X, and assume that $x_\alpha \to x$ and $x_\alpha \to y$. For every pair of neighbourhoods U of x and V of y, (x_α) is eventually in $U \cap V$. Hence x and y have no pair of disjoint neighbourhoods, so $x = y$.

(iii) ⇒ (iv): If Δ were not closed, then there would exist a net $(x_\alpha)_{\alpha\in A}$ in X such that $(x_\alpha, x_\alpha) \to (x,y)$ where $x \neq y$, so the limit of (x_α) would not be unique.

(iv) ⇒ (i): Let $x, y \in X$ be distinct points so that $(x,y) \notin \Delta$. If Δ is closed, then (x,y) has a neighbourhood $U \times V$ in $X \times X$ disjoint from Δ. But then U and V are disjoint neighbourhoods of x and y respectively, so X is Hausdorff. □

5.2. Operations on Hausdorff spaces

PROPOSITION 5.4

Let X carry the initial topology induced by maps $f_\alpha : X \to X_\alpha$ for $\alpha \in A$. Assume that each X_α is Hausdorff, and that the f_α separate points in X. Then X is also Hausdorff.

In particular, any subspace of a T_0-space is T_0, and a product of T_0-spaces is T_0. Furthermore, if a nonempty product space is T_0, then every factor is T_0.

PROOF. TODO □

5.3. Further properties of Hausdorff spaces

PROPOSITION 5.5

In a Hausdorff space, disjoint compact sets can be separated.

PROOF. Let K_1 and K_2 be disjoint compact sets in a Hausdorff space X, and fix a point $x \in K_1$. Since X is Hausdorff, x can be separated from every $y \in K_2$. It follows from Lemma 2.5 that x can be separated from K_2. But then K_2 can be separated from every point in K_1, so another application of Lemma 2.5 yields the desired claim. □

PROPOSITION 5.6

If $f, g\colon X \to Y$ are continuous and Y is Hausdorff, then the set $\{f = g\} = \{x \in X \mid f(x) = g(x)\}$ is closed. In particular, if f and g agree on a dense subset of X, then $f = g$.

PROOF. Let $x \in X$ be such that $f(x) \neq g(x)$. Since Y is Hausdorff, $f(x)$ and $g(x)$ have disjoint neighbourhoods U and V respectively. Then $f^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x on which f and g differ. Thus $\{f \neq g\}$ is open which proves the claim.

Alternatively we may argue using nets[6]: Let $(x_\alpha)_{\alpha \in A}$ be a net in $\{f = g\}$ such that $x_\alpha \to x$. By continuity we have $f(x_\alpha) \to f(x)$ and $g(x_\alpha) \to g(x)$, and since limits are unique in Y by Proposition 5.3(iii) and $f(x_\alpha) = g(x_\alpha)$ we have $f(x) = g(x)$. □

6 • Regular and T_3-spaces

6.1. Definition and equivalent properties

DEFINITION 6.1

A topological space X is *regular* if, for every point $x \in X$ and closed subset $A \subseteq X$ with $x \notin A$, x has a neighbourhood U and A a neighbourhood V with

[6] We refrain from using nets (or filters) as far as possible, or at least also provide proofs that do not depend on them. In this case nets do in fact clarify the necessity of the Hausdorff assumption, so we also include a proof using nets.

$U \cap V = \emptyset$.

If furthermore X is T_1, then X is said to satisfy the T_3 axiom and is called a T_3-*space*.

Notice that a regular space is *not* necessarily Hausdorff since singletons are not closed. Of course a T_3-space is Hausdorff.

PROPOSITION 6.2

A topological space X is regular if and only if every $x \in X$ has a neighbourhood basis of closed sets.

PROOF. Assume that X is regular, and let U be an open neighbourhood of $x \in X$. Then U^c is closed, so there exist disjoint open sets V and W with $x \in V$ and $U^c \subseteq W$. Then $x \in V \subseteq W^c \subseteq U$, so W^c is the desired closed neighbourhood.

Conversely, let $x \in X$ and $A \subseteq X$ closed with $x \notin A$. Then A^c is an open neighbourhood of x, so A^c contains a closed neighbourhood B of X. Then B° and B^c are disjoint open neighbourhoods of x and A respectively. □

6.2. Operations on regular spaces

PROPOSITION 6.3

Let X carry the initial topology induced by maps $f_\alpha \colon X \to X_\alpha$ for $\alpha \in A$. Assume that each X_α is regular. Then X is also regular.

In particular, any subspace of a regular space is regular, and a product of regular spaces is regular. Furthermore, if a nonempty product space is regular, then every factor is regular.

Notice that we do *not* require that the f_α separate points in X since we do not need to distinguish individual points. If the f_α do separate points in X, then by Proposition 4.6 the above also holds with 'regular' replaced with 'T_3'.

PROOF. Assume that each X_α is regular. We prove that each $x \in X$ has a neighbourhood basis of closed sets in accordance with Proposition 6.2, so let U be an open neighbourhood of x. Then x lies in some basic neighbourhood $\bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$, where U_{α_i} is open in X_{α_i}. Hence $f_{\alpha_i}(x) \in U_{\alpha_i}$, and since X_{α_i} is regular $f_{\alpha_i}(x)$ has a closed neighbourhood F_{α_i} contained in U_{α_i}. But then $f_{\alpha_i}^{-1}(F_{\alpha_i}) \subseteq f_{\alpha_i}^{-1}(U_{\alpha_i})$ is a closed neighbourhood of x, and finally $\bigcap_{i=1}^n f_{\alpha_i}^{-1}(F_{\alpha_i})$ is a closed neighbourhood of x contained in U as desired.

The final claim follows as in the proof of Proposition 3.7 since each factor is homeomorphic to a subspace of the product. □

PROPOSITION 6.4

Let $(X_\alpha)_{\alpha\in A}$ be a collection of regular spaces. Then the disjoint union $X = \coprod_{\alpha\in A} X_\alpha$ is also regular.

As with Proposition 3.8 we may immediately generalise this result to a larger class of final topologies: Namely those coinduced by maps $(f_\alpha)_{\alpha\in A}$ that are both open and closed, and that cover X. Note that this assumption is weaker than the previous assumptions, just as the assumptions in Proposition 6.3 were weaker than those of Proposition 3.7. Again I am not aware of any interesting applications of this fact.

PROOF. Let $y \in X$, and let U be a neighbourhood of y. There is then an $\alpha \in A$ and an $x \in X_\alpha$ such that $\iota_\alpha(x) = y$. Then $\iota_\alpha^{-1}(U)$ is an open neighbourhood of x, so x has a closed neighbourhood F contained therein. It follows that $y \in \iota_\alpha(F) \subseteq U$, and $\iota_\alpha(F)$ is a closed neighbourhood of y since ι_α is open and closed. □

6.3. *Further properties of regular spaces*

PROPOSITION 6.5

In a regular space, compact sets can be separated from disjoint closed sets.

PROOF. This is a direct consequence of Lemma 2.5. □

PROPOSITION 6.6

A regular Lindelöf space is paracompact.

PROOF. Let X be a regular Lindelöf space, and let $\mathcal{U}$ be an open cover of X. For every $x \in X$ pick a $U_x \in \mathcal{U}$ with $x \in U_x$. By regularity x has a neighbourhood V_x such that $\overline{V}_x \subseteq U_x$. Then $\{V_x \mid x \in X\}$ is also an open cover of X, so it has countable subcover $\{V_{x_n} \mid n \in \mathbb{N}\}$ since X is Lindelöf.

For $n \in \mathbb{N}$ define sets
$$W_n = U_{x_n} \setminus \bigcup_{k<n} \overline{V}_{x_k}.$$
For $x \in X$ there is a smallest $k \in \mathbb{N}$ such that $x \in \overline{V}_{x_k}$, so $x \in W_k$. Hence $\{W_n \mid n \in \mathbb{N}\}$ is also an open cover of X, and it is clearly a refinement of $\mathcal{U}$. It is also locally finite, since $x \in V_{x_k}$ for some $k \in \mathbb{N}$, but V_{x_k} does not intersect W_n for $n > k$. Hence X is paracompact. □

6.4. *Conditions for regularity*

Corollary 6.7

Pseudometric spaces are regular.

Proof. This will follow from Proposition 7.6 since completely regular spaces are regular. □

Corollary 6.8

Locally compact Hausdorff spaces are regular.

Proof. This will follow from Theorem 7.8 since completely regular spaces are regular.

Alternatively, by Proposition 2.7 every point of an LCH space has a neighbourhood basis of compact (and hence closed) sets, which implies regularity by Proposition 6.2. □

7 • Completely regular and Tychonoff spaces

7.1. *Definition and equivalent properties*

Definition 7.1

A topological space X is *completely regular* if, for every point $x \in X$ and closed subset $A \subseteq X$ with $x \notin A$, there is a continuous function $f: X \to [0,1]$ with $f(x) = 0$ and $f(A) = 1$. Such a function is said to *separate* x and A.

If furthermore X is T_1, then X is said to satisfy the $T_{3\frac{1}{2}}$-axiom and is called *Tychonoff*.

A completely regular space is indeed regular: If f separates x and A, then $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ are disjoint neighbourhoods of x and A respectively.

We now prove that a space is completely regular precisely when the bounded continuous functions on the space induce the topology. Of course, these functions are already continuous, so this says that there are *enough* continuous functions for them to characterise the topology.

To prove this we take a small detour by studying the defining property of completely regular spaces in greater generality. We say that a collection $(f_\alpha)_{\alpha \in A}$ of functions $f_\alpha: X \to X_\alpha$ between topological spaces *separates points from closed sets* if whenever $C \subseteq X$ is closed and $x \notin C$, then $f_\alpha(x) \notin \overline{f_\alpha(C)}$ for some $\alpha \in A$.

Proposition 7.2

A collection $(f_\alpha)_{\alpha\in A}$ of functions $f_\alpha\colon X \to X_\alpha$ between topological spaces separates points from closed sets if and only if the sets $f_\alpha^{-1}(V)$, for $\alpha \in A$ and $V \subseteq X_\alpha$ open, form a basis for the topology on X.

Proof. First assume that (f_α) separates points from closed sets, let $U \subseteq X$ be open and let $x \in U$. Then U^c is closed, so there is some $\alpha \in A$ such that $f_\alpha(x) \notin \overline{f_\alpha(U^c)}$. Then

$$U^c \subseteq f_\alpha^{-1}\big(f_\alpha(U^c)\big) \subseteq f_\alpha^{-1}\big(\overline{f_\alpha(U^c)}\big).$$

So letting $V = \overline{f_\alpha(U^c)}^c$ we find that $x \in f_\alpha^{-1}(V) \subseteq U$ as desired.

Conversely, assume that the sets $f_\alpha^{-1}(V)$ form a basis for the topology on X. Let $x \in X$ and $C \subseteq X$ closed with $x \notin C$. There is an $\alpha \in A$ and an open $V \subseteq X_\alpha$ such that $x \in f_\alpha^{-1}(V) \subseteq C^c$. Then V is a neighbourhood of $f_\alpha(x)$ disjoint from $f_\alpha(C)$, so $f_\alpha(x) \notin \overline{f_\alpha(C)}$. ∎

Corollary 7.3

If $(f_\alpha)_{\alpha\in A}$ is a collection of functions $f_\alpha\colon X \to X_\alpha$ between topological spaces which separates points from closed sets, then X carries the weak topology induced by the maps f_α.

Proof. Proposition 7.2 shows that the collection of preimages $f_\alpha^{-1}(V)$, for $\alpha \in A$ and $V \subseteq X_\alpha$ open, forms a basis for the topology on X, so it in particular generates the topology. ∎

Theorem 7.4

A topological space X is completely regular if and only if it has the weak topology induced by $C_b(X)$.

Proof. If X is completely regular, then $C_b(X)$ separates points from closed sets by definition, so Corollary 7.3 shows that X carries the the weak topology induced by $C_b(X)$.

More directly, let $U \subseteq X$ be open and nonempty, and let $x \in U^c$. By complete regularity there is a continuous $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f \equiv 1$ on U^c. Now let $V_x = f^{-1}([0,1))$, and notice that this is open in the weak topology. But $V_x \subseteq U$, so U is a union of sets on this form, hence weakly open.

Conversely, suppose that X has the weak topology induced by $C_b(X)$. Let $U \subseteq X$ be open, and let $x \in U$. Then there are functions $f_1, \dots, f_n \in C_b(X)$ and subbasic open sets $V_1, \dots, V_n \subseteq \mathbb{R}$ such that

$$x \in \bigcap_{i=1}^n f_i^{-1}(V_i) \subseteq U.$$

By changing the sign on the f_i if necessary, we may assume that each V_i is on the form (a_i, ∞). Define functions $g_i \colon X \to \mathbb{R}$ by $g_i(x) = (f_i(x) - a_i) \vee 0$. Then $g_i^{-1}(0,\infty) = f_i^{-1}(a_i,\infty)$, so

$$x \in \bigcap_{i=1}^{n} g_i^{-1}(0,\infty) \subseteq U.$$

Let $g = g_1 g_2 \cdots g_n$. Then $g(x) > 0$, so $x \in g^{-1}(0,\infty)$. Furthermore, if $g(y) > 0$ then each $g_i(y) > 0$. it follows that

$$x \in g^{-1}(0,\infty) \subseteq U.$$

Then $g(x) \neq 0$, but $g(U^c) = 0$, so X is completely regular. □

Recall that the *Hilbert cube* is the countably infinite power $[0,1]^{\mathbb{N}}$. More generally if A is some (possibly uncountable) set, then a power $[0,1]^A$ is called a *Tychonoff cube*. Notice that if $(I_a)_{a \in A}$ is a family of (nonempty) compact intervals, then since each I_a is homeomorphic to $[0,1]$, the product $\prod_{a\in A} I_a$ is homeomorphic to the Tychonoff cube $[0,1]^A$.

Proposition 7.5

A topological space is Tychonoff if and only if it is homeomorphic to a subspace of a Tychonoff cube.

Proof. Note that any Tychonoff cube is Tychonoff since it is the product of Tychonoff spaces [TODO prove this?]. Furthermore, any subspace of a Tychonoff space is Tychonoff, so this proves the 'if' direction.

For the other implication, let X be a Tychonoff space. The topology on X is the the initial topology induced by the functions in $C = C_b(X)$, each of which can be considered a map into a compact interval, which we may take to be $[0,1]$. Since X is T_1 and completely regular, C separates points in X, so the product map into $[0,1]^C$ is an embedding[7]. □

7.2. Conditions for complete regularity

Proposition 7.6

Pseudometric spaces are completely regular.

In Proposition 8.5 we will see that pseudometric spaces are also normal, but since a pseudometric space is not necessarily T_1, this does not imply that it is (completely) regular. Hence the necessity of the present proposition.

[7] See notes on topology and measure theory.

PROOF. Let (S,ρ) be a pseudometric space, $x \in S$, and let $A \subseteq S$ be closed with $x \notin A$. Since A is closed, the map $y \mapsto \rho(y,A)$ is zero on A and nonzero at y, and it is continuous by Lemma 2.17. □

PROPOSITION 7.7

Normal R_0-spaces are completely regular.

PROOF. Let F be closed in a normal R_0-space X, and let $x \in X \setminus F$. Then $\overline{\{x\}}$ is disjoint from F by Proposition 4.2(vi), so Urysohn's lemma [TODO ref] implies that X is completely regular. □

We now wish to show that locally compact Hausdorff spaces are completely regular. In the presence of the Hausdorff axiom, complete regularity is weaker than normality, and *compact* spaces are normal, so it is perhaps not surprising that *locally* compact spaces are completely regular.

To show this we will prove a version of Urysohn's lemma for locally compact Hausdorff spaces. This relies on the Urysohn lemma for normal spaces covered in the next section, but we place this discussion here since we are interested in it in the context of completely regular spaces.

THEOREM 7.8: *Urysohn's Lemma, locally compact version*

Let X be a locally compact Hausdorff space, and let $K \subseteq U \subseteq X$ with K compact and U open. Then there exists a continuous function $f: X \to [0,1]$ such that $f(K) = 1$ and f vanishes outside a compact subset of U.

PROOF. By Lemma 2.9 there is a precompact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$. Since compact Hausdorff spaces are normal, we can apply Urysohn's lemma for normal spaces to $\overline{V}$: This yields a continuous function $f: \overline{V} \to [0,1]$ with $f(K) = 1$ and $f(\partial V) = 0$. Extend f to X by letting $f(\overline{V}^c) = 1$.

We claim that f is continuous on X. Let $B \subseteq [0,1]$ be closed. If $0 \notin B$, then $f^{-1}(B) = (f|_{\overline{V}})^{-1}(B)$ is closed in $\overline{V}$, hence also in X. On the other hand, if $0 \in B$, then

$$f^{-1}(B) = (f|_{\overline{V}})^{-1}(B) \cup \overline{V}^c = (f|_{\overline{V}})^{-1}(B) \cup V^c,$$

where the last equality follows since $\partial V \subseteq (f|_{\overline{V}})^{-1}(B)$. Again $f^{-1}(B)$ is closed, so f is continuous. □

COROLLARY 7.9

Locally compact Hausdorff spaces are completely regular, hence Tychonoff.

PROOF. Let X be a locally compact Hausdorff space, let $x \in X$ and $A \subseteq X$ be a closed subset. In the notation of Urysohn's lemma, let $K = \{x\}$ and $U = A^c$, which yields a continuous function $f: X \to [0,1]$ with $f(x) = 1$ and $f(A) = 0$. □

7.3. Further properties of completely regular spaces

Proposition 7.10

Let X be a topological space. There exists a Tychonoff space Y such that $C_b(X)$ and $C_b(Y)$ are isomorphic as rings.

Hence, if one is interested in studying rings of bounded functions, then one may as well assume that the domain is Tychonoff.

Proof. Let X' be X equipped with the weak topology induced by $C_b(X)$. Then since replacing the topology with a weaker one does not introduce any new continuous functions, we have $C_b(X) = C_b(X')$. Hence X' is completely regular by Theorem 7.4.

Now consider the T_0-identification $X'/{\equiv}$ of X'. By Proposition 3.4 we have $C(X') \cong C(X'/{\equiv})$, and this isomorphism clearly restricts to an isomorphism $C_b(X') \cong C_b(X'/{\equiv})$, proving the claim. □

7.4. Compactification

If X is a topological space, then a *compactification* of X is a pair (K, i), where K is a compact space and $i\colon X \to K$ is an embedding whose image is dense in K. This is a *Hausdorff compactification* if K is also Hausdorff.

Consider the category $\mathbf{CHaus}_X$ whose objects are arrows $X \to K$ in **Top** from X to a compact Hausdorff space K, and whose arrows are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "j"'] & K \arrow[d, "f"] \\ & H \end{tikzcd}$$

in **Top**.

Compare this to the category that defines the free group on some set A: The category $\mathbf{CHaus}_X$ is precisely the same, except that A is replaced with X, **Set** is replaced with **Top**, and **Grp** is replaced with **CHaus**.

Suppose that $i_X\colon X \to \beta X$ is an initial object in $\mathbf{CHaus}_X$. This this then the analogue of the free group $F(A)$ on A, but there are key differences: For instance, the map $j\colon A \to F(A)$ is always injective, since if $a, b \in A$, then the map $A \to \mathbb{Z}$ with $a \mapsto 0$ and $b \mapsto 1$ must factor through j, and so $j(a) \neq j(b)$. Notice that this works because there are many set functions from A. But unless X is sufficiently nice, there may not be many continuous maps from X to spaces like $[0, 1]$, and so i_X is not necessarily injective.

Theorem 7.11: *Stone–Čech extension*

If X is a topological space, then $\mathbf{CHaus}_X$ has an initial object. That is, there exists a pair $(\beta X, i_X)$, where βX is a compact Hausdorff space and $i_X \colon X \to \beta X$ is continuous, with the following universal property: For any continuous map $f\colon X \to K$ into a compact Hausdorff space K there is a unique continuous map $\beta f \colon \beta X \to K$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & \beta X \arrow[d, "\beta f"] \\ & K \end{tikzcd}$$

commutes. This characterises βX up to unique homeomorphism compatible with i_X, and the image of i_X in βX is dense.

The pair $(\beta X, i_X)$ is called the *Stone–Čech extension* of X. If i_X is an embedding, then we call $(\beta X, i_X)$ the *Stone–Čech compactification* of X.

Proof. Let $C = C(X, [0,1])$ and consider the product $[0,1]^C$. Define the evaluation map $\mathrm{ev}\colon X \to [0,1]^C$ by letting $\mathrm{ev}(x) = \mathrm{ev}_x$, where $\mathrm{ev}_x \colon C \to [0,1]$ is given by $\mathrm{ev}_x(g) = g(x)$. Notice that for $g \in C$ we have $\pi_g \circ \mathrm{ev}(x) = \pi_g(\mathrm{ev}_x) = \mathrm{ev}_x(g) = g(x)$, so ev is continuous.

Let βX be the closure of $\mathrm{ev}(X)$ in $[0,1]^C$, which is compact by Tychonoff's theorem and also Hausdorff. We first verify the universal property in the case $K = [0,1]$. Given $f\colon X \to [0,1]$ we again have $f = \pi_f \circ \mathrm{ev}$, so letting i_X be the corestriction of ev to βX and βf the restriction of π_f to βX makes the diagram commute. Notice also that this fixes the value of βf on $\mathrm{ev}(X)$, and hence on its closure since $[0,1]^C$ is Hausdorff.

Next assume that K is a Tychonoff cube $[0,1]^A$, and let $f\colon X \to [0,1]^A$. For each $a \in A$ there is thus a map $\pi_a \circ f \colon X \to [0,1]$, which yields a map $\beta(\pi_a \circ f)\colon X \to [0,1]$. Let $F = \langle \beta(\pi_a \circ f)\rangle_{a\in A}$ be the A-fold product of all such maps. We claim that $\beta f = F$ works. To see this, consider the diagram

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"] \arrow[ddr, bend right, "\pi_a \circ f"'] & \beta X \arrow[d, "F"] \arrow[dd, bend left, "\beta(\pi_a \circ f)"] \\ & {[0,1]^A} \arrow[d, "\pi_a"] \\ & {[0,1]} \end{tikzcd}$$

The left and right triangles clearly commute, and the middle triangle commutes since

$$\pi_a \circ f = \beta(\pi_a \circ f) \circ i_X = \pi_a \circ F \circ i_X.$$

Furthermore, F is unique with the property that $f = F \circ i_X$, since if G is another such map then, as above,

$$\beta(\pi_a \circ f) \circ i_X = \pi_a \circ f = \pi_a \circ G \circ i_X.$$

Hence $\pi_a \circ G = \beta(\pi_a \circ f)$ by uniqueness for maps into $[0,1]$, and G is uniquely characterised by this equation as a map into a product.

Finally let K be a general compact Hausdorff space. Since it is automatically Tychonoff, by Proposition 7.5 it embeds into a Tychonoff cube $[0,1]^A$. Let $\iota\colon K \hookrightarrow [0,1]^A$ be such an embedding, let $f\colon X \to K$ be continuous and consider the diagram

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] \arrow[ddr, bend right, "\iota\circ f"'] & \beta X \arrow[d, "\beta f"] \arrow[dd, bend left, "\beta(\iota\circ f)"] \\
& K \arrow[d, "\iota"] \\
& {[0,1]^A}
\end{tikzcd}$$

Now let $\beta f(y)$ be the unique $z \in K$ such that $\iota(z) = \beta(\iota \circ f)(y)$, which exists since the images of the maps $\iota \circ f$ and $\beta(\iota \circ f)$ coincide, and the image of the former is contained in the image of ι. This means that $\beta(\iota \circ f) = \iota \circ \beta f$, which implies that βf is continuous since ι is an embedding.

Hence the left and right triangles in the above diagram commute, and to see that the middle triangle also commutes notice that

$$\iota \circ f = \beta(\iota \circ f) \circ i_X = \iota \circ \beta f \circ i_X.$$

Hence $f = \beta f \circ i_X$. For uniqueness of βf with this property, if G is another such map then

$$\iota \circ G \circ i_X = \iota \circ f = \beta(\iota \circ f) \circ i_X,$$

so $\iota \circ G = \beta(\iota \circ f)$, which uniquely determines G as a map into a subspace.

Uniqueness up to unique homemorphism follows since i_X is an initial object in $\mathbf{CHaus}_X$.

Finally we show that $i_X(X)$ is dense in βX. This follows immediately for the concrete construction above, and hence for any pair also satisfying the universal property by uniqueness. We can also give a proof only using the universal property: Assume that $i_X(X)$ is not dense in βX and let $f\colon X \to [0,1]$ be identically 1. Then this factors through βX with $\beta f \equiv 1$. Choose a $z \in \beta X \setminus \overline{i_X(X)}$, and by complete regularity of βX let $\varphi\colon \beta X \to [0,1]$ be such that $\varphi(x) = 0$ and $\varphi \equiv 1$ on $\overline{i_X(X)}$. Then f also factors through βX with $\beta f = \varphi$, and so $\beta f(z) = 0$, which contradicts the uniqueness of βf. ◻

REMARK 7.12. In the proof above, if X is Tychonoff then we claim that i_X is an embedding into the concrete version of βX as a subspace of $[0,1]^C$. For then X has the initial topology induced by the maps in C, and since it is T_1 and completely regular these maps separate points in X. Hence the evaluation map, of which i_X is a corestriction, embeds X into $[0,1]^C$. Hence i_X embeds X into βX. ⌟

COROLLARY 7.13

If $(\beta X, i_X)$ is a Stone–Čech extension of a topological space X, then i_X is an embedding if and only if X is Tychonoff.

PROOF. The 'if' direction follows from the above corollary. The converse follows since any subspace of a Tychonoff space is Tychonoff. ◻

LEMMA 7.14

If X is a Hausdorff space and $A \subseteq X$ is locally compact and dense in X, then A is open in X.

PROOF. Let $x \in A$. By local compactness of A, x has a compact neighbourhood $K \subseteq A$. This implies that there is an open set $U \subseteq X$ containing x such that $U \cap A \subseteq K$. The set $U \setminus K$ is open, so if it is nonempty it contains an element of A, which would imply that $U \cap A$ contains an element not in K. Hence $U \setminus K$ is empty, so $U \subseteq K \subseteq A$, and thus x is an inner point of A. ◻

COROLLARY 7.15

Let X be a Tychonoff space. Then the following are equivalent:

(i) *X is locally compact.*

(ii) *$i(X)$ is open in every Hausdorff compactification $i\colon X \hookrightarrow cX$.*

(iii) *$i(X)$ is open in some Hausdorff compactification $i\colon X \hookrightarrow cX$.*

PROOF. If X is locally compacdt and $i\colon X \to cX$ is a Hausdorff compactification of X, then $i(X)$ is also locally compact and dense in cX, so [TODO ref lemma] implies that $i(X)$ is open in cX. Since X has a Hausdorff compactification, namely the Stone–Čech compactification, (ii) implies (iii). Finally, open subsets of locally compact Hausdorff spaces are locally compact, so (iii) implies (i). ◻

PROPOSITION 7.16

If X is a topological space, then every Hausdorff compactification of X is a quotient of βX, and the quotient map is closed.

PROOF. If $i\colon X \to cX$ is a compactification of X, then the universal property of βX yields a continuous map $\beta i\colon \beta X \to cX$ with the same image as i. Since βX is compact and cX is Hausdorff, βi is automatically closed. It is also surjective since its image is compact closed and dense in cX. □

We also have a similar construction to the Stone–Čech extension that instead constructs Tychonoff spaces. Let $\mathbf{Tych}_X$ be the category whose objects are arrows $X \to T$ in $\mathbf{Top}$ from X to a Tychonoff space T, and arrows are commutative diagrams

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "q"'] & T \arrow[d, "f"] \\ & S \end{tikzcd}$$

in **Top**.

PROPOSITION 7.17: *Tychonoffication*

If X is a topological space, then $\mathbf{Tych}_X$ has an initial object. That is, there exists a pair $(\tau X, p_X)$, where τX is a Tychonoff space and $p_X\colon X \to \tau X$ is continuous, with the following universal property: For any continuous map $f\colon X \to T$ into a Tychonoff space T there is a unique continuous map $\tau f\colon \tau X \to T$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "p_X"] \arrow[dr, "f"'] & \tau X \arrow[d, "\tau f"] \\ & T \end{tikzcd}$$

commutes. This characterises τX up to unique homeomorphism compatible with p_X. Furthermore, p_X is surjective.

PROOF. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "p_X"] \arrow[rr, bend left, "i_X"] \arrow[d, "f"'] & \tau X \arrow[r, hook, "\iota_X"] \arrow[d, "\tilde{\varphi}"] & \beta X \arrow[d, "\varphi=\beta(i_T\circ f)"] \\ T \arrow[r, "p_T"'] \arrow[rr, bend right, "i_T"'] & \tau T \arrow[r, hook, "\iota_T"'] & \beta T \end{tikzcd}$$

Here $\tau X = i_X(X)$, p_X is the corestriction of i_X onto τX, ι_X is the inclusion map, and similarly for the bottom row. In particular p_T is a homeomorphism since T is already Tychonoff. This also clearly makes the outer rectangle commute. Furthermore, notice that

$$\varphi(\tau X) = (\varphi \circ i_X)(X) = (i_T \circ f)(X) \subseteq i_T(T) = \tau T.$$

We thus let $\tilde{\varphi}$ be the restriction of φ to τX, which is also corestricted to τT. Clearly $\tilde{\varphi}$ is continuous and makes the left and right rectangles above commute.

Next let $\tau f = p_T^{-1} \circ \tilde{\varphi}$. For uniqueness assume that $F \circ p_X = f$. Then

$$i_T \circ F \circ p_X = i_T \circ f = \varphi \circ i_X = \varphi \circ \iota_X \circ p_X = \iota_T \circ \tilde{\varphi} \circ p_X = i_T \circ p_T^{-1} \circ \tilde{\varphi} \circ p_X = i_T \circ \tau f \circ p_X.$$

Since i_T is injective and p_X is surjective, it follows that $F = \tau f$. □

If X is a topological space, then a *one-point compactification* of X is a compactification $i\colon X \to X^*$ such that $X^* \setminus i(X)$ is a singleton. We usually denote the sole element of this set by ∞ and think of it as a 'point at infinity'. We also identify X and $i(X)$, so we will always consider X a genuine subset of X^*.

LEMMA 7.18

If X is a topological space and X^* is a one-point compactification, then any open neighbourhood of ∞ in X^* is on the form $(X \setminus C) \cup \{\infty\}$, where C is compact and closed in X.

PROOF. Let U be such an open neighbourhood. Then $X^* \setminus U = X \setminus U$ is closed in X^* and hence compact, and $X \setminus U = (X^* \setminus U) \cap X$ is furthermore closed in X. □

LEMMA 7.19

If X is a non-compact topological space and X^* a one-point of compactification, then X is open in X^*.

PROOF. Assume that X is not open in X^*. Then there is some $x \in X$ whose neighbourhoods in X^* all contain ∞. Since X is dense in X^*, every open neighbourhood of x is also an open neighbourhood of ∞, and hence has compact complement.[8] Any open cover $\mathcal{U}$ of X contains an open neighbourhood U of

[8] Since any open cover of X must contain a neighbourhood of x, hence of ∞, we might attempt to argue using compactness of X^*. But this does not work, since not every neighbourhood of ∞ need be a neighbourhood of x. Hence it might be possible to find an open cover of X^* such that any finite subcover contains a neighbourhood of ∞ (which is not a neighbourhood of x) without which the subcover is not a cover of X.

x, and $X \setminus U$ is compact and hence covered by a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$. Then $\mathcal{U}_0 \cup \{U\}$ is a finite subcover of X. □

If X is compact then it may or may not be open in X^*: For instance, a singleton has two different one-point compactifications, namely the Sierpiński space (in which the original point is open) and the two-point space with the indiscrete topology (in which it is not).

8 • Normal and T_4-spaces

8.1. *Definition and equivalent properties*

DEFINITION 8.1

A topological space X is *normal* if, for every pair of disjoint closed subsets $A, B \subseteq X$, A has a neighbourhood U and B a neighbourhood V with $U \cap V = \emptyset$.

If furthermore X is T_1, then X is said to satisfy the T_4 axiom and is called a *T_4-space*.

PROPOSITION 8.2

Let X be a topological space. Then the following are equivalent:

(i) *X is normal.*

(ii) *If $F \subseteq U$ with F closed and U open, then there is an open set V such that $F \subseteq V \subseteq \overline{V} \subseteq F$.*

(iii) *If $F \subseteq U$ with F closed and U open, then there is a sequence $(V_n)_{n\in\mathbb{N}}$ of open sets such that $F \subseteq \bigcup_{n\in\mathbb{N}} V_n$ and $\overline{V}_n \subseteq U$.*

PROOF. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, so assume that (iii) holds. If A and B are disjoint closed sets, then let (V_n) and (W_n) be sequences corresponding to the inclusions $A \subseteq B^c$ and $B \subseteq A^c$, respectively. Now let

$$V_n' = V_n \setminus \bigcup_{i\leq n} \overline{W}_i \quad \text{and} \quad W_n' = W_n \setminus \bigcup_{i\leq n} \overline{V}_i,$$

and put $V = \bigcup_{n\in\mathbb{N}} V_n'$ and $W = \bigcup_{n\in\mathbb{N}} W_n'$. We easily check that V_n' and W_m' are disjoint for all $m, n \in \mathbb{N}$, so V and W are disjoint open sets separating A and B, as desired. □

If X is a topological space and $A, B \subseteq X$ are closed sets, then a continuous function $f : X \to [0,1]$ with $f(A) = 0$ and $f(B) = 1$ is called a *Urysohn function* for A and B.

Theorem 8.3: *Urysohn's Lemma*

A topological space X is normal if and only if there is a Urysohn function for every pair of closed subsets of X.

Proof. See Lee [TODO ref]. □

Theorem 8.4: *The Tietze extension theorem*

A topological space X is normal if and only if any continuous function $f\colon A \to \mathbb{R}$ on a closed set $A \subseteq X$ can be extended to a continuous function on all of X, i.e. there exists a continuous $F\colon X \to \mathbb{R}$ such that $f = F|_A$.

Furthermore, if $a, b \in \mathbb{R}$ and $f(A) \subseteq [a, b]$ then F can be chosen such that $F(X) \subseteq [a, b]$.

Proof. TODO □

8.2. *Conditions for normality*

Proposition 8.5

Pseudometric spaces are perfectly normal.

Proof. Let (S, ρ) be a pseudometric space, and let $A, B \subseteq S$ be disjoint closed subsets. For $a \in A$ let $r_a = \rho(a, B)/2 > 0$, and for $b \in B$ let $r_b = \rho(b, A)/2 > 0$. Let

$$U = \bigcup_{a \in A} B(a, r_a) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, r_b).$$

We claim that U and V are disjoint. Let $x \in U$ and $y \in V$. Then $x \in B(a, r_a)$ and $y \in B(b, r_b)$ for some $a \in A$ and $b \in B$. Then

$$\rho(a, b) \leq \rho(a, x) + \rho(x, y) + \rho(y, b) < \rho(x, y) + r_a + r_b,$$

which implies that

$$0 \leq \rho(a, b) - r_a - r_b < \rho(x, y),$$

where the first inequality follows since

$$\rho(a, b) = \frac{\rho(a, b) + \rho(a, b)}{2} \geq \frac{\rho(a, B)}{2} + \frac{\rho(b, A)}{2} = r_a + r_b.$$

Hence S is normal.

We next show that S is G_δ, so let $F \subseteq S$ be closed and for $n \in \mathbb{N}$ let $F_n = \{x \in S \mid \rho(x, F) < \frac{1}{n}\}$. Then each F_n is open and $F = \bigcap_{n \in \mathbb{N}} F_n$ as desired. □

PROPOSITION 8.6

Every paracompact R_1-space is normal regular.

The proof below will show directly that such a space is regular, but this also follows since a normal R_0-space is normal regular (in particular completely regular).

PROOF. Let X be a paracompact R_1-space, A a closed subset and $q \in X \setminus A$. For every $p \in A$ the points p and q are topologically distinguishable (since A^c is a neighbourhood of q), so there exist disjoint open neighbourhoods U_p and V_p of p and q respectively. Each $p \in A$ thus has a neighbourhood U_p such that $q \notin \overline{U}_p$.

The sets U_p is an open cover of A, so by paracompactness of A (cf. Proposition 2.2) we obtain a locally finite subcover $\mathcal{U}$. Letting $\mathbb{U} = \bigcup_{U \in \mathcal{U}} U$ and $\mathbb{V} = X \setminus \overline{\mathbb{U}}$ we then have two disjoint open sets, and by local finiteness of $\mathcal{U}$ we have $\overline{\mathbb{U}} = \bigcup_{U \in \mathcal{U}} \overline{U}$, so $\mathbb{V}$ contains q.

This shows that X is regular. The same argument then shows that X is normal, by using regularity instead of the R_1-axiom. □

PROPOSITION 8.7

A regular Lindelöf space is normal. If it is also hereditarily Lindelöf, then it is G_δ and hence perfectly normal.

PROOF. Let X be a regular Lindelöf space, and let $F \subseteq U$ with F closed and U open. By regularity, each point $x \in F$ has an open neighbourhood V_x with $\overline{V}_x \subseteq U$. Since F is also Lindelöf by [TODO ref], it is covered by countably many of the sets V_x. Hence X satisfies the property in [TODO ref] and is hence normal.

Now assume that X is also hereditarily Lindelöf, and let $U \subseteq X$ be open. For each $x \in U$ there is a closed neighbourhood N_x of x with $N_x \subseteq U$ (cf. Proposition 6.2). The sets N_x thus cover U, but then so does a countable subcollection, so U is F_σ. □

We thus find that compact Hausdorff implies normal, and that regular Lindelöf implies normal. That is, normality follows from a compactness property along with a separability property. We cannot weaken compactness to Lindelöf and still have normality, but also 'strengthening' Hausdorff to regularity retains normality. (Of course, it is not regularity but T_3 that is stronger than Hausdorff.)

9 • Perfectly normal spaces

THEOREM 9.1: *The Vedenisoff theorem*

Let X be a topological space. The following are equivalent:

(i) *X is perfectly normal.*

(ii) *Every closed subset of X is a zero set of a function $X \to [0,1]$.*

(iii) *Every open subset of X is a cozero set of a function $X \to [0,1]$.*

(iv) *If $A, B \subseteq X$ are disjoint closed sets, then A and B are precisely separated by a continuous function $X \to [0,1]$.*

PROOF. *(i) $\Rightarrow$ (ii)*: Let $F \subseteq X$ be closed, and let $F = \bigcap_{n\in\mathbb{N}} U_n$, where each U_n is open. By Urysohn [TODO ref], there are continuous functions $f_n\colon X \to [0,1]$ with $f_n(x) = 0$ for all $x \in F$, and $f_n(y) = 1$ for all $y \in U_n^c$. Now define $f\colon X \to [0,1]$ by

$$f = \sum_{n=1}^{\infty} 2^{-n} f_n.$$

Clearly $f^{-1}(\{0\}) = F$, and f is continuous (for instance by the Weierstrass M-test).

(ii) $\Leftrightarrow$ (iii): This is obvious.

(ii) $\Rightarrow$ (iv): Let A and B be zero sets of continuous functions $f, g\colon X \to [0,1]$ respectively, and define $h = f/(f+g)$. Then A and B are precisely separated by h.

(iv) $\Rightarrow$ (i): Clearly X is normal. If F is a closed subset of X, then F is disjoint from $\emptyset$, and the hypothesis yields a function $f\colon X \to [0,1]$ with $F = f^{-1}(\{0\})$. We then have $F = \bigcap_{n\in\mathbb{N}} f^{-1}([0, \frac{1}{n}))$ as desired. □

COROLLARY 9.2

Perfect normality is hereditary.

PROOF. Let X be perfectly normal, let $A \subseteq X$, and let $F \subseteq A$ be closed in A. Then there is a closed set $\hat{F}$ in X such that $F = \hat{F} \cap A$. By [TODO Vedenisoff] $\hat{F}$ is the zero set of some continuous function $f\colon X \to [0,1]$. But then F is the zero set of the function $f|_A$, so A is also perfectly normal. □

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