Separation Axioms

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1 • Introduction

In these notes we give an overview of some of the most important separation axioms in point-set topology. For each axiom we consider the following:

(1) What the axiom *means* intuitively. Of course we understand what it means to 'separate' points or sets in a topological space and how the ability to do so is useful. But many of the axioms are equivalent to properties of a space that (seemingly) have nothing to do with separation, and whose importance are perhaps more intuitively clear.

(2) How the axiom behaves with respect to topological constructions like subspaces and products.

(3) Conditions under which a space satisfies a given axiom.

(4) Other properties of the axioms. Each axiom interacts with the surrounding theory in different ways, sometimes to produce spaces with even more structure.

Our attention is primarily focused on both ends of the spectrum: The weakest axiom we consider, and the one we have most to say about, is T_0. We say less about the T_1 and Hausdorff axioms since these are well-known, and there is also rather little to say about regular spaces. More interesting are complete regularity and normality on which we spend a lot of time: normality because of its importance and particularly the Urysohn lemma, and complete regularity because of its connection with rings of continuous functions.

1.1. Notation

If X and Y are topological spaces, we denote the set of continuous maps $X \to Y$ by $C(X,Y)$. If Y is a real/complex topological vector space, then $C(X,Y)$ is a real/complex vector space when addition and scalar multiplication are defined pointwise. In the case $Y = \mathbb{R}$ we simply write $C(X)$, and this is an algebra over $\mathbb{R}$. We also write $C_b(X)$ for the subalgebra of bounded functions.

1.2. Overview of the axioms

We here summarise the axioms by citing

- T_0: The topology respects the underlying set structure by distinguishing between distinct points.
- T_1: Singletons are closed, or in other words, if $(x_\alpha)_{\alpha \in A}$ is a constant net with $x_\alpha = x$ for all $\alpha \in A$, then x is the unique limit of (x_α).
- Hausdorff: All limits are unique (Proposition 5.2).
- Regularity: Every open neighbourhood contains a closed neighbourhood (Proposition 6.2).
- Complete regularity: Has enough bounded continuous functions to determine its topology (Theorem 7.4).
- Normality: Continuous functions on closed sets can be extended to the entire space (Theorem 8.6). Why should we restrict to *closed* sets? Continuous functions defined on a non-closed set may not even extend to a continuous function on the *closure* of that set: Take for instance the function $x \mapsto 1/x$ on $\mathbb{R} \setminus \{0\}$.

2 • Preliminary definitions and results

If X is a topological space and $A \subseteq X$, then we say that a set $N \subseteq X$ is a *neighbourhood* of A if there is an open set U in X such that $A \subseteq U \subseteq N$. The family of neighbourhoods of a set A is called the *neighbourhood filter of A* and is denoted $\mathcal{N}_A$. If $A = \{x\}$ is a singleton we also write $\mathcal{N}_x$ and call N a neighbourhood of x.

We begin by proving a series of results that do not pertain directly to the separability axioms, but that will be used in the sequel.

2.1. Compactness

If X is a topological space, we say that a collection $\mathcal{U}$ of subsets of X is a *cover* of X if $X = \bigcup_{U \in \mathcal{U}} U$. A *subcover* of a cover $\mathcal{U}$ is a subcollection of $\mathcal{U}$ that itself is a cover. An *open cover* is a cover consisting of open sets.

If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, we say that $\mathcal{U}$ *refines* $\mathcal{V}$ and is a *refinement* of V if each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. Notice that a subcover is in particular a refinement.

A collection $\mathcal{A}$ of subsets of X is called *locally finite* if every point of X has a neighbourhood that intersects finitely many elements in $\mathcal{A}$. It is easy to show that if $\mathcal{A}$ is locally finite, then $\overline{\mathcal{A}} = \{\overline{A} \mid A \in \mathcal{A}\}$ is also locally finite.

Definition 2.1

Let X be a topological space.

(i) X is *compact* if every open cover of X has a finite subcover.

(ii) X is *Lindelöf* if every open cover of X has a countable subcover.

(iii) X is *paracompact* if every open cover of X has an open locally finite refinement.

Clearly a compact space is both Lindelöf and paracompact.

Proposition 2.2

Let X be a topological space. If X is compact/Lindelöf, then every closed subset of X is compact/Lindelöf in the subspace topology.

This result is of course standard for compact spaces, but we include the proof to illustrate that it is identical to the proof for the Lindelöf case.

Proof. Let X be compact/Lindelöf, and let $A \subseteq X$ be closed. If $\mathcal{U}$ is an open cover of A, then there is a collection $\mathcal{V}$ of sets open on X such that $\mathcal{U} = \{V \cap A \mid V \in \mathcal{V}\}$. By adjoining A^c to $\mathcal{V}$ we obtain an open cover of X, so it has a finite/countable subcover $\mathcal{V}'$. Discarding A^c and intersecting every other set in $\mathcal{V}'$ with A yields a finite/countable subcover of $\mathcal{U}$, proving the claim. ☐

Proposition 2.3

The continuous image of a compact/Lindelöf space is compact/Lindelöf.

Proof. Let $f : X \to Y$ be continuous and surjective, and assume that X is compact/Lindelöf. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U})$ is an open cover of X, so it has a finite/countable subcover $f^{-1}(\mathcal{U}')$. But then $\mathcal{U}'$ is a finite/countable subcover of $\mathcal{U}$ as desired. ☐

If X is a topological space, then we say that two disjoint subsets A and B can be *separated* if there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

LEMMA 2.4

Let X be a topological space, and let $A \subseteq X$ be a set that can be separated from points that lie in a set $B \subseteq X$. Then A can be separated from compact sets contained in B.

This is an example of the general principle that compact sets often act like points.

PROOF. Let $K \subseteq X$ be a compact set contained in B. Since A can be separated from points in B, then for every $x \in K$ there are disjoint open sets U_x and V_x such that $x \in U_x$ and $A \subseteq V_x$. The collection $(U_x)_{x \in K}$ is an open cover of K, so there is a finite subcover $(U_{x_i})_{i=1}^n$. Let $U = \bigcup_{i=1}^n U_{x_i}$ and $V = \bigcap_{i=1}^n V_{x_i}$. Then U and V are disjoint open sets containing K and A respectively. □

2.2. *Local compactness*

DEFINITION 2.5: *Local compactness*

A topological space is called *locally compact* if every point has a compact neighbourhood.

There are many different non-equivalent definitions of local compactness, and we have just chosen one for convenience. Locally compact spaces are usually also Hausdorff in which case all the different definitions are equivalent. A locally compact Hausdorff space is also called an *LCH space*. In particular we have the following:

PROPOSITION 2.6

A Hausdorff space is locally compact if and only if every point has a neighbourhood basis of compact sets.

PROOF. Let X be an LCH space and let $x \in U \subseteq X$ with U open. If $\overline{U}$ is not compact, then replace U with $U \cap \operatorname{Int} K$, where K is a compact neighbourhood of x. Then both $\{x\}$ and ∂U are compact, so by Proposition 5.3 there exist disjoint relatively open sets $V, W \subseteq \overline{U}$ such that $x \in V$ and $\partial U \subseteq W$. Then $V \subseteq U$ so V is open in X, and $\overline{V}$ is a closed and thus compact subset of $U \setminus W$.

The converse is obvious. □

COROLLARY 2.7

Let X be a locally compact Hausdorff space. If $K \subseteq U \subseteq X$ with K compact and U open, then K has a compact neighbourhood contained in U.

PROOF. For $x \in K$ we may choose a compact neighbourhood N_x of x contained in U by Proposition 2.6. Then $(\operatorname{Int} N_x)_{x\in X}$ is an open over of K, so there is a finite subcover $(\operatorname{Int} N_{x_i})_{i=1}^n$. Letting $V = \bigcup_{i=1}^n \operatorname{Int} N_{x_i}$ we have $K \subseteq V$, and $\overline{V} = \bigcup_{i=1}^n N_{x_i} \subseteq U$ is compact as desired. □

2.3. Metric spaces

LEMMA 2.8

Let (S,ρ) be a pseudometric space, and let $A \subseteq S$ be nonempty. Define a map $\rho(\cdot,A)\colon S \to [0,\infty)$ by

$$\rho(x,A) = \inf_{a\in A} \rho(x,a).$$

This map has the following properties:

(i) *$\rho(x,A) = 0$ if and only if $x \in \overline{A}$.*

(ii) *If $y \in S$, then $\rho(x,A) \leq \rho(x,y) + \rho(y,A)$.*

(iii) *$\rho(\cdot,A)$ is continuous.*

PROOF. First we prove (i). Notice that $\rho(x,A) = 0$ if and only if for any $r > 0$ there is an $a \in A$ such that $\rho(x,a) < r$. But this is true precisely when any ball $B(x,r)$ intersects A, i.e. when $x \in \overline{A}$.

For any $a \in A$ we have

$$\rho(x,A) \leq \rho(x,a) \leq \rho(x,y) + \rho(y,a),$$

and since this is true for any $a \in A$, (ii) follows.

Finally, (iii) follows immediately from (ii), since

$$\rho(x,A) - \rho(y,A) \leq \rho(x,y)$$

for all $x,y \in S$. □

3 • The T_0 axiom

3.1. Definition and equivalent properties

DEFINITION 3.1

A topological space X satisfies the T_0 axiom and is called a *T_0-space* if, for every pair of distinct points $x, y \in X$, either x has a neighbourhood that does not contain y, or vice versa.

We begin by giving an alternative characterisation of T_0-spaces: Let X be a topological space. We define an ordering on X called the *specialisation preorder* by letting $x \leq y$ if $x \in \overline{\{y\}}$ for $x, y \in X$. It is clear that $\leq$ is in fact a preorder, and so it determines an equivalence relation $\equiv$; that is, $x \equiv y$ if and only if $x \leq y$ and $y \leq x$.

It is easy to show that $x \leq y$ if and only if $\mathcal{N}_x \subseteq \mathcal{N}_y$. If $x \equiv y$, then we say that x and y are *topologically indistinguishable* since then x and y have the same neighbourhoods.

It is clear that X is T_0 if and only if the relation $\equiv$ is trivial, that is if x and y are topologically indistinguishable precisely when $x = y$. The quotient space $X/{\equiv}$ is called the *T_0-identification* or the *Kolmogorov quotient* of X, and it is indeed T_0:

THEOREM 3.2: *The T_0-identification*

Let X be a topological space, and let $q \colon X \to X/{\equiv}$ be the quotient map onto the T_0-identification of X. Then

(i) *open and closed sets are saturated,*

(ii) *q is an open and closed map,*

(iii) *$X/{\equiv}$ is T_0, and*

(iv) *if $\sim$ is an equivalence relation on X such that $X/{\sim}$ is T_0, then ${\equiv} \subseteq {\sim}$.*[1]

Part (iv) expresses the fact that $\equiv$ is the most conservative equivalence relation on X that makes the corresponding quotient a T_0-space.

PROOF. We first show that all open sets are saturated. Let U be an open set of X, and let $x \in U$. If $x \equiv x'$, then U is also a neighbourhood of x', so $x' \in U$. It follows that $q(x) \subseteq U$, and so

$$U = \bigcup_{x \in U} q(x).$$

Hence U is a union of $\equiv$-equivalence classes, and thus it is saturated. The complement U^c is the union of all equivalence classes $q(x)$ for $x \in U^c$, so

[1] Both $\equiv$ and $\sim$ are subsets of $X \times X$, so this inclusion means that if $x \equiv y$ then $x \sim y$ for all $x, y \in X$.

closed sets are also saturated. It follows that $q(U)$ is open and $q(U^c)$ is closed, so q is an open and closed map.

Now we show that $X/{\equiv}$ is T_0. Assume that $x \not\equiv y$. Without loss of generality we may assume the existence of an element $U \in \mathcal{N}_x \setminus \mathcal{N}_y$. Since q is open, $q(U)$ is a neighbourhood of $q(x)$ in $X/{\equiv}$. We claim that $q(y) \notin q(U)$: If not, then since U is a union of equivalence classes there is a $z \in U$ with $y \equiv z$. But then $y \in U$ which is a contradiction. Thus $X/{\equiv}$ is T_0.

Finally, let $X/{\sim}$ be T_0, and let $p\colon X \to X/{\sim}$ be the quotient map. If $x \nsim y$ then $p(x) \neq p(y)$, so without loss of generality we may choose an open set $U \subseteq X/{\sim}$ with $p(x) \in U$ and $p(y) \notin U$. Then $x \in p^{-1}(U)$ and $y \notin p^{-1}(U)$, so $p^{-1}(U)$ is a neighbourhood of x that is not a neighbourhood of y, so $x \not\equiv y$. □

COROLLARY 3.3

Let $(X, \mathcal{T})$ be a topological space, and let $q\colon X \to X/{\equiv}$ be the quotient map onto the T_0-identification of X. Denote the topology on $X/{\equiv}$ by $\mathcal{T}_{\equiv}$. Then q induces a bijection $q_*\colon \mathcal{T} \to \mathcal{T}_{\equiv}$ given by $q_*(U) = q(U)$ whose inverse q^* is given by $q^*(V) = q^{-1}(V)$.

PROOF. Since q is surjective we have $q(q^{-1}(V)) = V$ for all $V \in \mathcal{T}_{\equiv}$, and by Theorem 3.2(i) every $U \in \mathcal{T}$ is saturated, so $q^{-1}(q(U)) = U$. □

This corollary implies that most topological properties are preserved in the T_0-identification. Taking the T_0-identification of a space that is already T_0 leaves the space unchanged but, preempting terminology we will introduce later, the T_0-identification of a regular space is regular, and the same is true for completely regular, normal and paracompact spaces. The proofs are trivial.

One might also expect that continuous functions on a space are unchanged in the T_0-identification, and this is in fact the case:

PROPOSITION 3.4

Let X be a topological space with T_0-identification $X/{\equiv}$. For every T_0-space Y the sets $C(X,Y)$ and $C(X/{\equiv}, Y)$ are in bijection. If Y is a T_0 topological vector space[2]*, then this bijection is a linear isomorphism, and it is a ring isomorphism in the case $Y = \mathbb{R}$.*

PROOF. Let $q\colon X \to X/{\equiv}$ be the quotient map, and let $f \in C(X,Y)$. We claim that if $x \equiv y$ in X, then $f(x) = f(y)$. For if $f(x) \neq f(y)$, then since Y is T_0 we can, without loss of generality, choose a neighbourhood U of $f(x)$ not containing $f(y)$. Then $f^{-1}(U)$ is a neighbourhood of x not containing y, so $x \not\equiv y$.

[2] One can show that a topological group is T_1 if it is T_0, and furthermore is always regular, so a T_0 topological group is T_3.

Thus every $f \in C(X,Y)$ descends to a unique map $\tilde{f} \in C(X/{\equiv},Y)$ with $f = \tilde{f} \circ q$. Conversely, every $g \in C(X/{\equiv},Y)$ uniquely determines a map $g \circ q \in C(X,Y)$. Hence there is a bijection $\Phi\colon C(X/{\equiv},Y) \to C(X,Y)$.

Now let Y be a T_0 topological vector space over $\mathbb{R}$. Let $f,g \in C(X/{\equiv},Y)$ and $\alpha \in \mathbb{R}$. Then

$$\Phi(\alpha f + g) = (\alpha f + g) \circ q = \alpha(f \circ q) + (g \circ q) = \alpha\Phi(f) + \Phi(g).$$

Hence Φ is linear and thus a linear isomorphism.

Finally, if $Y = \mathbb{R}$ then Φ respects multiplication by a similar argument to the above and is thus a ring isomorphism. □

We explore the T_0-identification in the context of metric spaces: Let (S,ρ) be a pseudometric space, and define a relation $\sim$ on S by $x \sim y$ if and only if $\rho(x,y) = 0$. This is clearly an equivalence relation. Let $\tilde{S} = S/{\sim}$ and define a map $\tilde{\rho}\colon \tilde{S} \times \tilde{S} \to [0,\infty)$ by $\tilde{\rho}([x],[y]) = \rho(x,y)$. This is well-defined, since if $x \sim x'$ and $y \sim y'$ then

$$\rho(x,y) \leq \rho(x,x') + \rho(x',y') + \rho(y',y) = \rho(x',y').$$

It is obvious that $\tilde{\rho}$ is then a metric on $\tilde{S}$, and we call $(\tilde{S},\tilde{\rho})$ the *metric identification* of (S,ρ).

Proposition 3.5

Let (S,ρ) be a pseudometric space. Then the relation $\sim$ defined above and the topological indistinguishability relation $\equiv$ coincide.

Proof. It suffices to show that $x \sim y$ if and only if $\mathcal{N}_x = \mathcal{N}_y$ for all $x,y \in S$. Assume that $x \sim y$ and let $U \in \mathcal{N}_x$. Then there is an $r > 0$ such that $B(x,r) \subseteq U$. But we clearly have $y \in B(x,r)$, so $U \in \mathcal{N}_y$. This shows that $\mathcal{N}_x \subseteq \mathcal{N}_y$, and the opposite inclusion follows by symmetry.

Conversely, assume that $x \nsim y$. Then $0 < r < \rho(x,y)$ for some r, and $B(x,r)$ is a neighbourhood of x but not of y. □

3.2. Operations on T_0-spaces

Proposition 3.6

(i) *Any subspace of a T_0-space is T_0.*

(ii) *A nonempty product space is T_0 if and only if every factor is T_0.*

Proof. Let X be a T_0-space and $A \subseteq X$. If $x,y \in A$ and $x \neq y$, then without loss of generality we may choose a neighbourhood U of x in X that does not contain Y. But then $U \cap A$ is a neighbourhood of x in A that doesn't contain y, so A is T_0.

Now let $(X_\alpha)_{\alpha \in A}$ be a collection of topological spaces, and assume that the product $X = \prod_{\alpha \in A} X_\alpha$ is nonempty. Assume that all X_α are T_0, and let $x, y \in X$ be distinct points. Then there is a $\beta \in A$ such that $x_\beta \neq y_\beta$. Without loss of generality we may pick a neighbourhood U of x_β in X_β that does not contain y_β. Then $\pi_\beta^{-1}(U)$ is a neighbourhood of x that does not contain y.

Conversely, assume that the product X is T_0 and let $\beta \in A$. Pick a point $y \in X$, and let

$$Y = \{x \in X \mid x_\alpha = y_\alpha \text{ for } \alpha \neq \beta\}.$$

Then Y is T_0 by (i), so $X_\beta \cong Y$ is also T_0. □

4 • The T_1 axiom

4.1. *Definition and equivalent properties*

Definition 4.1

A topological space X satisfies the T_1 axiom and is called a *T_1-space* if, for every pair of distinct points $x, y \in X$, x has a neighbourhood disjoint from y, and y has a neighbourhood disjoint from x.

Clearly every T_1-space is T_0. We begin by giving some properties of topological spaces that are equivalent to the T_1 axiom:

Proposition 4.2

The following are equivalent for a topological space X:

(i) *X is T_1,*

(ii) *each singleton of X is closed, and*

(iii) *each subset of X is the intersection of all open sets containing it.*

Proof. (i) ⇒ (ii): If X is T_1 and $x \in X$, then every point $y \in X \setminus \{x\}$ has a neighbourhood disjoint from $\{x\}$ so $X \setminus \{x\}$ is open.

(ii) ⇒ (iii): If $A \subseteq X$, then

$$A = \bigcap_{x \notin A} X \setminus \{x\},$$

so A is an intersection of open sets.

(iii) ⇒ (i): If $x, y \in X$ with $x \neq y$, then there is an open subset containing x and not y, and vice versa. □

4.2. Operations on T_1-spaces

Proposition 4.3

(i) *Any subspace of a T_1-space is T_1.*

(ii) *A nonempty product space is T_1 if and only if every factor is T_1.*

(iii) *A quotient space $X/{\sim}$ is T_1 if and only if every $\sim$-equivalence class is closed in X.*

Proof. The proof of the first two claims are almost identical to the proof of Proposition 3.6, so we omit it.

We prove (iii): Fibres of the quotient map are precisely the equivalence classes, so by the definition of the quotient topology, singletons of $X/{\sim}$ are closed if and only if the corresponding equivalence class is closed as a subset of X. □

4.3. Conditions for the T_1 axiom

Proposition 4.4

The closed image[3] *of a T_1-space is T_1.*

Proof. Let $f: X \to Y$ be a closed map from a T_1-space X to a topological space Y, and let $y \in f(X)$. Then there is some $x \in X$ with $f(x) = y$, and since $\{x\}$ is closed in X and f is closed, $\{y\}$ is closed in Y and hence in $f(X)$. □

5 • Hausdorff spaces

5.1. Definition and equivalent properties

Definition 5.1

A topological space X satisfies the T_2 axiom and is called a *T_2-space* or *Hausdorff space* if, for every pair of distinct points $x, y \in X$, x has a neighbourhood U and y a neighbourhood V with $U \cap V = \emptyset$.

Again, a T_2-space is clearly T_1. We give a series of conditions that are equivalent to the T_2 axiom.

[3] By 'closed image' we mean the image of a closed (not necessarily continuous) map.

Proposition 5.2

The following are equivalent for a topological space X:

(i) *X is Hausdorff,*

(ii) *limits of nets (and hence of filters) in X are unique, and*

(iii) *the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. (i) $\Rightarrow$ (ii): Let $(x_\alpha)_{\alpha \in A}$ be a net in X, and assume that $x_\alpha \to x$ and $x_\alpha \to y$. For every pair of neighbourhoods U of x and V of y, (x_α) is eventually in $U \cap V$. Hence x and y have no pair of disjoint neighbourhoods, so $x = y$.

(ii) $\Rightarrow$ (iii): If Δ were not closed, then there would exist a net $(x_\alpha)_{\alpha \in A}$ in X such that $(x_\alpha, x_\alpha) \to (x,y)$ where $x \neq y$, so the limit of (x_α) would not be unique.

(iii) $\Rightarrow$ (i): Let $x,y \in X$ be distinct points so that $(x,y) \notin \Delta$. If Δ is closed, then (x,y) has a neighbourhood $U \times V$ in $X \times X$ disjoint from Δ. But then U and V are disjoint neighbourhoods of x and y respectively, so X is Hausdorff. □

5.2. Further properties of Hausdorff spaces

Proposition 5.3

In a Hausdorff space, disjoint compact sets can be separated.

Proof. Let K_1 and K_2 be disjoint compact sets in a Hausdorff space X, and fix a point $x \in K_1$. Since X is Hausdorff, x can be separated from every $y \in K_2$. It follows from Lemma 2.4 that x can be separated from K_2. But then K_2 can be separated from every point in K_1, so another application of Lemma 2.4 yields the desired claim. □

Proposition 5.4

If $f,g \colon X \to Y$ are continuous and Y is Hausdorff, then the set $\{f = g\} = \{x \in X \mid f(x) = g(x)\}$ is closed. In particular, if f and g agree on a dense subset of X, then $f = g$.

Proof. Let $x \in X$ be such that $f(x) \neq g(x)$. Since Y is Hausdorff, $f(x)$ and $g(x)$ have disjoint neighbourhoods U and V respectively. Then $f^{-1}(U) \cap g^{-1}(V)$ is a neighbourhood of x on which f and g differ. Thus $\{f \neq g\}$ is open which proves the claim.

Alternatively we may argue using nets[4]: Let $(x_\alpha)_{\alpha\in A}$ be a net in $\{f = g\}$ such that $x_\alpha \to x$. By continuity we have $f(x_\alpha) \to f(x)$ and $g(x_\alpha) \to g(x)$, and since limits are unique in Y by Proposition 5.2(ii) and $f(x_\alpha) = g(x_\alpha)$ we have $f(x) = g(x)$. □

6 • Regular and T_3-spaces

6.1. *Definition and equivalent properties*

Definition 6.1

A topological space X is *regular* if, for every point $x \in X$ and closed subset $A \subseteq X$ with $x \notin A$, x has a neighbourhood U and A a neighbourhood V with $U \cap V = \emptyset$.

If furthermore X is T_1, then X is said to satisfy the T_3 axiom and is called a *T_3-space*.

Notice that a regular space is *not* necessarily Hausdorff since singletons are not closed. Of course a T_3-space is Hausdorff.

Proposition 6.2

The following are equivalent for a topological space X:

(i) *X is regular,*

(ii) *if U is an open neighbourhood of $x \in X$, then x has a closed neighbourhood contained in U, and*

(iii) *every $x \in X$ has a neighbourhood basis of closed sets.*

Proof. (i) ⇒ (ii): Assume that X is regular, and let U be an open neighbourhood of $x \in X$. Then U^c is closed, so there exist disjoint open sets V and W with $x \in V$ and $U^c \subseteq W$. Then $x \in V \subseteq W^c \subseteq U$, so W^c is the desired closed neighbourhood.

(ii) ⇒ (iii): This is obvious.

(iii) ⇒ (i): Let $x \in X$ and $A \subseteq X$ closed with $x \notin A$. Then A^c is an open neighbourhood of x, so if (iii) applies then A^c contains a closed neighbourhood B of X. Then $\operatorname{Int} B$ and B^c are disjoint open neighbourhoods of x and A respectively. □

[4] We refrain from using nets (or filters) as far as possible, or at least also provide proofs that do not depend on them. In this case nets do in fact clarify the necessity of the Hausdorff assumption, so we also include a proof using nets.

6.2. *Further properties of regular spaces*

PROPOSITION 6.3

In a regular space, compact sets can be separated from disjoint closed sets.

PROOF. This is a direct consequence of Lemma 2.4. □

PROPOSITION 6.4

A regular Lindelöf space is paracompact.

PROOF. Let X be a regular Lindelöf space, and let $\mathcal{U}$ be an open cover of X. For every $x \in X$ pick a $U_x \in \mathcal{U}$ with $x \in U_x$. By regularity x has a neighbourhood V_x such that $\overline{V}_x \subseteq U_x$. Then $\{V_x \mid x \in X\}$ is also an open cover of X, so it has countable subcover $\{V_{x_n} \mid n \in \mathbb{N}\}$ since X is Lindelöf.

For $n \in \mathbb{N}$ define sets

$$W_n = U_{x_n} \setminus \bigcup_{k<n} \overline{V}_{x_k}.$$

For $x \in X$ there is a smallest $k \in \mathbb{N}$ such that $x \in \overline{V}_{x_k}$, so $x \in W_k$. Hence $\{W_n \mid n \in \mathbb{N}\}$ is also an open cover of X, and it is clearly a refinement of $\mathcal{U}$. It is also locally finite, since $x \in V_{x_k}$ for some $k \in \mathbb{N}$, but V_{x_k} does not intersect W_n for $n > k$. Hence X is paracompact. □

6.3. *Conditions for regularity*

COROLLARY 6.5

Pseudometric spaces are regular.

PROOF. This will follow from Proposition 7.5 since completely regular spaces are regular. □

COROLLARY 6.6

Locally compact Hausdorff spaces are regular.

PROOF. This will follow from Theorem 7.6 since completely regular spaces are regular.

Alternatively, regardless of one's definition of local compactness, every point of an LCH space has a neighbourhood basis of compact (and hence closed) sets, which implies regularity by Proposition 6.2. □

7 • Completely regular and Tychonoff spaces

7.1. Definition and equivalent properties

DEFINITION 7.1

A topological space X is *completely regular* if, for every point $x \in X$ and closed subset $A \subseteq X$ with $x \notin A$, there is a continuous function $f: X \to [0,1]$ with $f(x) = 0$ and $f(A) = 1$. Such a function is said to *separate* x and A.

If furthermore X is T_1, then X is said to satisfy the $T_{3\frac{1}{2}}$-axiom and is called *Tychonoff*.

We now prove that a space is completely regular precisely when the bounded continuous functions on the space induce the topology. Of course, these functions are already continuous, so this says that there are *enough* continuous functions for them to characterise the topology.

To prove this we take a small detour by studying the defining property of completely regular spaces in greater generality. We say that a collection $(f_\alpha)_{\alpha\in A}$ of functions $f_\alpha: X \to X_\alpha$ between topological spaces *separates points from closed sets* if whenever $C \subseteq X$ is closed and $x \notin C$, then $f_\alpha(x) \notin \overline{f_\alpha(C)}$ for some $\alpha \in A$.

PROPOSITION 7.2

A collection $(f_\alpha)_{\alpha\in A}$ of functions $f_\alpha: X \to X_\alpha$ between topological spaces separates points from closed sets if and only if the sets $f_\alpha^{-1}(V)$, for $\alpha \in A$ and $V \subseteq X_\alpha$ open, form a basis for the topology on X.

Hence, if one is interested in studying rings of bounded functions, then one may as well assume that the domain is Tychonoff.

PROOF. First assume that (f_α) separates points from closed sets, let $U \subseteq X$ be open and let $x \in U$. Then U^c is closed, so there is some $\alpha \in A$ such that $f_\alpha(x) \notin \overline{f_\alpha(U^c)}$. Then

$$U^c \subseteq f_\alpha^{-1}\big(f_\alpha(U^c)\big) \subseteq f_\alpha^{-1}\big(\overline{f_\alpha(U^c)}\big).$$

So letting $V = \overline{f_\alpha(U^c)}^c$ we find that $x \in f_\alpha^{-1}(V) \subseteq U$ as desired.

Conversely, assume that the sets $f_\alpha^{-1}(V)$ form a basis for the topology on X. Let $x \in X$ and $C \subseteq X$ closed with $x \notin C$. There is an $\alpha \in A$ and an open $V \subseteq X_\alpha$ such that $x \in f_\alpha^{-1}(V) \subseteq C^c$. Then V is a neighbourhood of $f_\alpha(x)$ disjoint from $f_\alpha(C)$, so $f_\alpha(x) \notin \overline{f_\alpha(C)}$. □

COROLLARY 7.3

If $(f_\alpha)_{\alpha\in A}$ is a collection of functions $f_\alpha\colon X \to X_\alpha$ between topological spaces which separates points from closed sets, then X carries the weak topology induced by the maps f_α.

PROOF. Proposition 7.2 shows that the collection of preimages $f_\alpha^{-1}(V)$, for $\alpha \in A$ and $V \subseteq X_\alpha$ open, forms a basis for the topology on X, so it in particular generates the topology. □

THEOREM 7.4

A topological space X is completely regular if and only if it has the weak topology induced by $C_b(X)$.

PROOF. If X is completely regular, then $C_b(X)$ separates points from closed sets by definition, so Corollary 7.3 shows that X carries the the weak topology induced by $C_b(X)$.

Conversely, suppose that X has the weak topology induced by $C_b(X)$. Let $U \subseteq X$ be open, and let $x \in U$. Then there are functions $f_1,\dots,f_n \in C_b(X)$ and subbasic open sets $V_1,\dots,V_n \subseteq \mathbb{R}$ such that

$$x \in \bigcap_{i=1}^n f_i^{-1}(V_i) \subseteq U.$$

By changing the sign on the f_i if necessary, we may assume that each V_i is on the form (a_i,∞). Define functions $g_i\colon X \to \mathbb{R}$ by $g_i(x) = (f_i(x) - a_i) \vee 0$. Then $g_i^{-1}(0,\infty) = f_i^{-1}(a_i,\infty)$, so

$$x \in \bigcap_{i=1}^n g_i^{-1}(0,\infty) \subseteq U.$$

Let $g = g_1 g_2 \cdots g_n$. Then $g(x) > 0$, so $x \in g^{-1}(0,\infty)$. Furthermore, if $g(y) > 0$ then each $g_i(y) > 0$. it follows that

$$x \in g^{-1}(0,\infty) \subseteq U.$$

Then $g(x) \neq 0$, but $g(U^c) = 0$, so X is completely regular. □

7.2. Conditions for complete regularity

PROPOSITION 7.5

Pseudometric spaces are completely regular.

In Proposition 8.2 we will see that pseudometric spaces are also normal, but since a pseudometric space is not necessarily T_1, this does not imply that it is (completely) regular. Hence the necessity of the present proposition.

PROOF. Let (S,ρ) be a pseudometric space, $x \in S$, and let $A \subseteq S$ be closed with $x \notin A$. Since A is closed, the map $y \mapsto \rho(y,A)$ is zero on A and nonzero at y, and it is continuous by Lemma 2.8. $\square$

We now wish to show that locally compact Hausdorff spaces are completely regular. In the presence of the Hausdorff axiom, complete regularity is weaker than normality, and *compact* spaces are normal, so it is perhaps not surprising that *locally* compact spaces are completely regular.

To show this we will prove a version of Urysohn's lemma for locally compact Hausdorff spaces. This relies on the Urysohn lemma for normal spaces covered in the next section, but we place this discussion here since we are interested in it in the context of completely regular spaces.

THEOREM 7.6: *Urysohn's Lemma, locally compact version*

Let X be a locally compact Hausdorff space, and let $K \subseteq U \subseteq X$ with K compact and U open. Then there exists a continuous function $f\colon X \to [0,1]$ such that $f(K) = 1$ and f vanishes outside a compact subset of U.

PROOF. By Corollary 2.7 there is a precompact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$. Since compact Hausdorff spaces are normal, we can apply Urysohn's lemma for normal spaces to $\overline{V}$: This yields a continuous function $f\colon \overline{V} \to [0,1]$ with $f(K) = 1$ and $f(\partial V) = 0$. Extend f to X by letting $f(\overline{V}^c) = 1$.

We claim that f is continuous on X. Let $B \subseteq [0,1]$ be closed. If $0 \notin B$, then $f^{-1}(B) = (f|_{\overline{V}})^{-1}(B)$ is closed in $\overline{V}$, hence also in X. On the other hand, if $0 \in B$, then

$$f^{-1}(B) = (f|_{\overline{V}})^{-1}(B) \cup \overline{V}^c = (f|_{\overline{V}})^{-1}(B) \cup V^c,$$

where the last equality follows since $\partial V \subseteq (f|_{\overline{V}})^{-1}(B)$. Again $f^{-1}(B)$ is closed, so f is continuous. $\square$

COROLLARY 7.7

Locally compact Hausdorff spaces are completely regular, hence Tychonoff.

PROOF. Let X be a locally compact Hausdorff space, let $x \in X$ and $A \subseteq X$ be a closed subset. In the notation of Urysohn's lemma, let $K = \{x\}$ and $U = A^c$, which yields a continuous function $f\colon X \to [0,1]$ with $f(x) = 1$ and $f(A) = 0$. $\square$

7.3. Further properties of completely regular spaces

Proposition 7.8

Let X be a topological space. There exists a Tychonoff space Y such that $C_b(X)$ and $C_b(Y)$ are isomorphic as rings.

Proof. Let X' be X equipped with the weak topology induced by $C_b(X)$. Then since replacing the topology with a weaker one does not introduce any new continuous functions, we have $C_b(X) = C_b(X')$. Hence X' is completely regular by Theorem 7.4.

Now consider the T_0-identification $X'/\equiv$ of X'. By Proposition 3.4 we have $C(X') \cong C(X'/\equiv)$, and this isomorphism clearly restricts to an isomorphism $C_b(X') \cong C_b(X'/\equiv)$, proving the claim. □

8 • Normal and T_4-spaces

8.1. Definition

Definition 8.1

A topological space X is *normal* if, for every pair of disjoint closed subsets $A, B \subseteq X$, A has a neighbourhood U and B a neighbourhood V with $U \cap V = \emptyset$.

If furthermore X is T_1, then X is said to satisfy the T_4 axiom and is called a *T_4-space*.

We discuss conditions that are equivalent to normality in our discussion of Urysohn's lemma below.

8.2. Conditions for normality

Proposition 8.2

Pseudometric spaces are normal.

Proof. Let (S, ρ) be a pseudometric space, and let $A, B \subseteq S$ be disjoint closed subsets. For $a \in A$ let $r_a = \rho(a, B)/2 > 0$, and for $b \in B$ let $r_b = \rho(b, A)/2 > 0$. Let

$$U = \bigcup_{a \in A} B(a, r_a) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, r_b).$$

We claim that U and V are disjoint. Let $x \in U$ and $y \in V$. Then $x \in B(a, r_a)$ and $y \in B(b, r_b)$ for some $a \in A$ and $b \in B$. Then

$$\rho(a, b) \leq \rho(a, x) + \rho(x, y) + \rho(y, b) < \rho(x, y) + r_a + r_b,$$

which implies that

$$0 \leq \rho(a,b) - r_a - r_b < \rho(x,y),$$

where the first inequality follows since

$$\rho(a,b) = \frac{\rho(a,b)+\rho(a,b)}{2} \geq \frac{\rho(a,B)}{2} + \frac{\rho(b,A)}{2} = r_a + r_b.$$

□

Proposition 8.3

Every paracompact Hausdorff space is normal, hence T_4.

Proof.

Proposition 8.4

A regular Lindelöf space is normal.

Proof. Let X be a regular Lindelöf space, and let $A, B \subseteq X$ be disjoint closed subsets. By regularity, every $a \in A$ has a neighbourhood U_a such that $\overline{U_a} \cap B = \emptyset$. Similarly, every $b \in B$ has a neighbourhood V_b separating it from A. Since A and B are themselves Lindelöf by Proposition 2.2, they are covered by countably many U_a and V_b respectively, say $A \subseteq \bigcup_{n\in\mathbb{N}} U_n$ and $B \subseteq \bigcup_{n\in\mathbb{N}} V_n$.

Now define sequences of sets S_n and T_n by

$$S_n = U_n \setminus \overline{\bigcup_{i<n} T_i} \quad \text{and} \quad T_n = V_n \setminus \overline{\bigcup_{i\leq n} S_i}\ .$$

(Notice the strict and non-strict inequalities.) Define the sets $S = \bigcup_{n\in\mathbb{N}} S_n$ and $T = \bigcup_{n\in\mathbb{N}} T_n$. Clearly S is a neighbourhood of A and T of B. We claim that they are also disjoint: Let $x \in S_n$ for some $n \in \mathbb{N}$. Then $x \notin T_m$ for $m < n$ by the definition of S_n, and $x \notin T_m$ for $m \geq n$ by the definition of T_m. □

8.3. Urysohn's Lemma and related results

If X is a topological space and $A, B \subseteq X$ are closed sets, then a continuous function $f : X \to [0,1]$ with $f(A) = 0$ and $f(B) = 1$ is called a *Urysohn function* for A and B.

Theorem 8.5: *Urysohn's Lemma*

A topological space X is normal if and only if there is a Urysohn function for every pair of closed subsets of X.

PROOF. First assume that X is normal and that $A, B \subseteq X$ are closed. By normality there is an open set $U_{1/2}$ such that

$$A \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq B^c.$$

Then A and $U_{1/2}^c$ are disjoint closed sets, and so are $\overline{U}_{1/2}$ and B. Hence there exist open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq B^c.$$

Let Δ be the set of all dyadic rational numbers in $(0,1)$. We may thus recursively define for every $r \in \Delta$ a set U with the following properties:

(1) $A \subseteq U_r$ and $\overline{U}_r \subseteq B^c$ for each $r \in \Delta$, and

(2) $\overline{U}_r \subseteq U_s$ if $r < s$, for $r, s \in \Delta$.

We furthermore let $U_1 = X$. Then define a function $f : X \to [0,1]$ by $f(x) = \inf\{r \mid x \in U_r\}$. Since $A \subseteq U_r \subseteq B^c$ for all $r \in \Delta$, we clearly have $f(A) = 0$ and $f(B) = 1$, and that $0 \leq f(x) \leq 1$ for all $x \in X$.

It remains to be shown that f is continuous. Let $\alpha \in \mathbb{R}$ and $x \in X$, and notice that $f(x) < \alpha$ if and only if $x \in U_r$ for some $r < \alpha$, which is true just when $x \in \bigcup_{r<\alpha} U_r$. Hence,

$$f^{-1}((-\infty, \alpha)) = \bigcup_{r<\alpha} U_r$$

is open. Similarly $f(x) > \alpha$ if and only if $x \notin U_r$ for some $r > \alpha$, which is equivalent to $x \notin \overline{U}_s$ for some $s > \alpha$ by property Item (2) above. This is the case if and only if $x \in \bigcup_{s>\alpha} (\overline{U}_s)^c$. It follows that

$$f^{-1}((\alpha, \infty)) = \bigcup_{s>\alpha} (\overline{U}_s)^c$$

is also open. Hence f is continuous.

Conversely, assume that f is a Urysohn function for a pair of disjoint closed sets $A, B \subseteq X$. Then $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$ are disjoint neighbourhoods of A and B respectively, so X is normal. □

THEOREM 8.6: *The Tietze extension theorem*

A topological space X is normal if and only if any continuous function $f : A \to \mathbb{R}$ on a closed set $A \subseteq X$ can be extended to a continuous function on all of X, i.e. there exists a continuous $F : X \to \mathbb{R}$ such that $f = F|_A$.

Furthermore, if $a, b \in \mathbb{R}$ and $f(A) \subseteq [a,b]$ then F can be chosen such that $F(X) \subseteq [a,b]$.

PROOF.

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