Ullrich, *Complex Made Simple*

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0 • Differentiability and the Cauchy–Riemann Equations

Exercise 0.3

Suppose that $T\colon \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear. Show that T is $\mathbb{C}$-linear if and only if $T(\mathrm{i}z) = \mathrm{i}Tz$ for all z.

Solution. The 'if' part is obvious, so we prove the 'only if' part. If $T(\mathrm{i}z) = \mathrm{i}Tz$ for all z, then for $a, b \in \mathbb{R}$ we have

$$T((a+\mathrm{i}b)z) = T(az + \mathrm{i}bz) = aTz + \mathrm{i}T(bz) = aTz + \mathrm{i}bTz = (a+\mathrm{i}b)Tz,$$

as desired. □

Exercise 0.4

Suppose that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the $\mathbb{R}$-linear mapping defined by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that T is $\mathbb{C}$-linear if and only if $a = d$ and $b = -c$.

Solution. Notice that, for $a, b, x, y \in \mathbb{R}$,

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot_{\mathbb{C}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is, multiplication by a complex number $a + \mathrm{i}b$ is the same as ordinary matrix multiplication by the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Since matrix multiplication is associative, by [TODO ref] the requirement is that T commutes with the matrix representing the imaginary unit i. This means that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \quad \text{should equal} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

And this is the case if and only if $a = d$ and $b = -c$ as claimed.

Note: The subring of $\mathrm{Mat}_2(\mathbb{R})$ representing $\mathbb{C}$-linear maps is precisely the centraliser of the matrix of i. □

1 • Power Series

2 • Preliminary Results on Holomorphic Functions

3 • Elementary Results on Holomorphic Functions

REMARK 3.1: *Topological lemmas.*
We give alternative proofs of Lemmas 3.3 and 3.4, using the covering definition of compactness.

For Lemma 3.3, let d denote the Euclidean metric on $\mathbb{C}$, and write $d(z, A)$ for the distance from $z \in \mathbb{C}$ to $A \subseteq \mathbb{C}$. It is well-known that this is continuous in z for a fixed A, and that $d(z, A) = 0$ if and only if $z \in \overline{A}$. Since K is compact, this obtains its minimum at some $z_0 \in K$. But then we must have $d(z_0, V^c) > 0$ since V^c is closed. Then choose $\rho = d(z_0, V^c)/2$.

For Lemma 3.4 we use the Heine–Borel theorem. Since K' is obviously bounded, it suffices to show that it is closed. To this end, let $w \in \mathbb{C} \setminus K'$. The map $z \mapsto d(w, z)$ obtains its minimum $d(w, K)$ on K at a point z_0. Notice that $d(w, K) > \rho$, since otherwise we would have $w \in \overline{D}(z_0, \rho) \subseteq K'$. Hence $D(w, d(w, K) - \rho)$ is disjoint from K', so $\mathbb{C} \setminus K'$ is open. ⌟

REMARK 3.2: *Order of zeros.*
We show that the implication in Lemma 3.9 is an equivalence, i.e. that

> *A function $f \in H(V)$ has a zero of order N at $z \in V$ if and only if there exists a $g \in H(V)$ with $g(z) \neq 0$ such that*
>
> $$f(w) = (w - z)^N g(w)$$
>
> *for all $w \in V$.*

The 'only if' direction is just Lemma 3.9, so we prove the converse. In this case, the Leibniz differentiation rule implies that

$$f^{(n)}(w) = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)}(w) \frac{\mathrm{d}^k}{\mathrm{d}w^k}(w-z)^N$$

for all $n \in \mathbb{N}_0$. If $n < N$, then the nth derivative of $(w-z)^N$ is zero at z, so $f^{(n)}(z) = 0$. But if $n = N$, then all terms except the term $k = n$ vanish, so

$$f^{(N)}(w) = g(w)\frac{\mathrm{d}^N}{\mathrm{d}w^N}(w-z)^N = g(w)N!,$$

so $f^{(N)}(z) \neq 0$. Hence f has a zero of order N at z. ⌟

EXERCISE 3.9

Suppose that V is a connected open set, $f \in H(V)$, and $f'(z) = 0$ for all $z \in V$. Show that f is constant.

SOLUTION. Let $D(z_0, r) \subseteq V$. For $a, b \in D(z_0, r)$ we have

$$f(b) - f(a) = \int_{[a,b]} f'(z)\,\mathrm{d}z = 0,$$

so f is constant on $D(z_0, r)$. Hence all derivatives of f vanish at z_0, so Corollary 3.8 implies that f is constant. □

EXERCISE 3.19

Suppose that $f \in H(D'(z_0, r))$. Show that f has a pole of order N at z_0 if and only if $1/f$ has a zero of order N at z_0.

SOLUTION. First assume that f has a pole of order N at z_0. Then f is on the form

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-z_0)^n = (z-z_0)^{-N} \sum_{n=0}^{\infty} c_{n-N}(z-z_0)^n = (z-z_0)^{-N} h(z)$$

for $z \in D'(z_0, r)$, where $h(z) = \sum_{n=0}^{\infty} c_{n-N}(z-z_0)^n$ is holomorphic in $D(z_0, r)$ (it is holomorphic in the punctured disk, but it is a power series with a positive radius of convergence, hence is differentiable at 0). Notice that $h(z_0) = c_{-N} \neq 0$. But then

$$\frac{1}{f(z)} = (z-z_0)^N \frac{1}{h(z)}$$

in a neighbourhood of z_0 (since $1/f$ has a removable singularity at z_0 by Lemma 3.13). And since $h(z_0) \neq 0$, $1/h$ is holomorphic in a neighbourhood of z_0. The claim now follows from Remark 3.2.

Now assume that $1/f$ has a zero of order N at z_0. Then there is a $g \in H(D'(z_0, r))$ with $g(z_0) \neq 0$ such that

$$\frac{1}{f(w)} = (z-w)^N g(w),$$

by Lemma 3.9. But then

$$f(w) = (z-w)^{-N} \frac{1}{g(w)},$$

and $1/g$ is holomorphic near z_0 since it is nonzero there. Hence it is given by a power series, and the claim follows. □

Exercise 3.20

Suppose we agree that a pole of order N is a zero of order $-N$, and that if f is holomorphic near z_0 and $f(z_0) \neq 0$ then f has a zero of order 0 at z_0. Suppose that f and g are holomorphic in $D'(z_0, r)$ and have zeroes of order $n, m \in \mathbb{N}$ at z_0, respectively. Show that fg has a zero of order $n+m$ at z_0.

Solution. If $n, m \geq 0$, then the claim follows easily from Remark 3.2. By [TODO ref] Exercise 3.19 we only need to consider the case where $n > 0$ and $m \leq 0$. We thus have

$$f(z) = (z-z_0)^n h(z) \quad \text{and} \quad g(z) = \sum_{k=m}^{\infty} c_k (z-z_0)^k,$$

where h is holomorphic near z_0 and $h(z_0) \neq 0$, and $c_m \neq 0$. But then

$$f(z)g(z) = (z-z_0)^{n+m} h(z) \sum_{k=0}^{\infty} c_{k+m} (z-z_0)^k,$$

and since $h(z) \sum_{k=0}^{\infty} c_{k+m}(z-z_0)^k$ is holomorphic and nonzero at $z = z_0$ (since $h(z_0)$ and c_m are nonzero), it follows from Remark 3.2 that fg has a zero of order $n+m > 0$ at z_0. □

4 • Logarithms, Winding Numbers and Cauchy's Theorem

Exercise 4.1

Show directly from the definition that $\operatorname{Ind}(\partial D(a,r),a) = 1$ for any $a \in \mathbb{C}$ and $r > 0$.

Solution. Since $\partial D(a,r)$ is smooth we have

$$\operatorname{Ind}(\partial D(a,r),a) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D(a,r)} \frac{\mathrm{d}z}{z-a} = \frac{1}{2\pi \mathrm{i}} \int_0^1 \frac{2\pi \mathrm{i}\, \mathrm{e}^{2\pi \mathrm{i} t}}{\mathrm{e}^{2\pi \mathrm{i} t}} \,\mathrm{d}t = 1. \qquad \square$$

Exercise 4.2

Suppose $a \in \mathbb{C}$ and $r > 0$. Show that

$$\operatorname{Ind}(\partial D(a,r),z) = \begin{cases} 1, & |z-a| < r, \\ 0, & |z-a| > r. \end{cases}$$

Solution. By Proposition 4.7, the index is constant on the components of $\mathbb{C} \setminus \partial D(a,r)$ and 0 on the unbounded component. The claim then follows from Exercise 4.1. $\square$

Exercise 4.7

(i) Suppose that f and g are holomorphic near z_0 and f has a simple zero at z_0. Find an expression for the residue of g/f at z_0.

(ii) Suppose that f has a simple pole at z_0 and g is holomorphic near z_0. Show that

$$\operatorname{Res}(fg,z_0) = g(z_0)\operatorname{Res}(f,z_0).$$

(iii) Suppose that f is holomorphic in a neighborhood of z_0, set $g(z) = f(z)/(z-z_0)^n$, and show that

$$\operatorname{Res}(g,z_0) = f^{(n-1)}(z_0)/(n-1)!.$$

Solution. (i) By [TODO ref] Exercise 3.20 g/f has a zero of order $N \geq -1$ at z_0. If $N = -1$, the residue is then, by the calculations on p. 71,

$$\operatorname{Res}(g/f,z_0) = \lim_{z \to z_0} (z-z_0)\frac{g(z)}{f(z)} = \lim_{z \to z_0} g(z)\frac{z-z_0}{f(z)-f(z_0)} = \frac{g(z_0)}{f'(z_0)}.$$

If instead $N \geq 0$, then g has a zero of order at least 1 at z_0, in which case $g(z_0)$. But in this case g/f has a removable singularity at z_0, so its residue at z_0 is 0, so the formula also holds in this case.

(ii) Notice that $1/f$ has a simple zero at z_0 by Exercise 3.20, so part (i) implies that

$$\begin{aligned}\operatorname{Res}(fg,z_0) &= \frac{g(z_0)}{(1/f)'(z_0)}\\ &= g(z_0)\frac{1}{(1/f)'(z_0)}\\ &= g(z_0)\operatorname{Res}(1/(1/f),z_0)\\ &= g(z_0)\operatorname{Res}(f,z_0).\end{aligned}$$

(iii) This is obvious from the calculations on p. 71. □

EXERCISE 4.8

Suppose that f has an isolated singularity at z_0. Fix $r>0$ and $n\in\mathbb{Z}$, and define $\gamma\colon[0,2\pi]\to\mathbb{C}$ by

$$\gamma(t)=z_0+r\,\mathrm{e}^{\mathrm{i}nt}.$$

Show that $\operatorname{Ind}(\gamma,z_0)=n$ and that

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)\,\mathrm{d}z=\operatorname{Ind}(\gamma,z_0)\operatorname{Res}(f,z_0)$$

if $r>0$ is small enough.

SOLUTION. Both sides are zero if $n=0$, so we may assume that $n\neq 0$. Since γ is smooth we have

$$\operatorname{Ind}(\gamma,z_0)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{\mathrm{d}z}{z-a}=\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{\mathrm{i}n\,\mathrm{e}^{\mathrm{i}nt}}{\mathrm{e}^{\mathrm{i}nt}}\,\mathrm{d}t=n.$$

Assume that $n>0$ and notice that γ is an n-fold sum

$$\gamma=\partial D(z_0,r)\dotplus\cdots\dotplus\partial D(z_0,r),$$

so that

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\gamma f(z)\,\mathrm{d}z &= \frac{n}{2\pi\mathrm{i}}\int_{\partial D(z_0,r)}f(z)\,\mathrm{d}z\\ &= n\operatorname{Res}(f,z_0)\\ &= \operatorname{Ind}(\gamma,z_0)\operatorname{Res}(f,z_0),\end{aligned}$$

where the second equality follows from Theorem 4.11 as on p. 72. Here we choose r such that $r<R$ and f is holomorphic on $D'(z_0,R)$ for some R. Substituting $-n$ for n changes sign on the integral on the left-hand side and on $\operatorname{Ind}(\gamma,z_0)$, proving the claim. □

8 • Conformal Mappings

EXERCISE 8.7

Suppose that $\psi \in \mathrm{Aut}(\Pi^+)$. Show that there exist $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$, such that

$$\psi(z) = \frac{az+b}{cz+d}$$

for all $z \in \Pi^+$. Show that a, b, c, d are not unique but are almost unique.

SOLUTION. First notice that $\varphi_C^{-1} \circ \psi \circ \varphi_C \in \mathrm{Aut}(\mathbb{D})$, where $\varphi_C \colon \mathbb{D} \to \Pi^+$ is the Cayley transform. Hence this is a Möbius transformation by Theorem 8.4.3, so ψ is as well.

Since ψ is also a homeomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$, it preserves the boundary of the upper hemisphere, which is precisely the one-point compactification $\mathbb{R}^*$ of the real line. For $i = 1, 2, 3$ choose distinct $z_i \in \mathbb{R} \setminus \{d/c\}$ such that $w_i := \psi(z_i) \in \mathbb{R}$. Then the w_i are also distinct, so Theorem 8.2.2 implies that ψ is unique among Möbius transformations with this property. Now recall that the coefficients a, b, c, d are unique up to multiplication by a nonzero scalar, so we may assume that $b \in \{0, 1\}$. Given b the remaining coefficients are thus uniquely determined. We claim that they are also real.

Consider the linear system of equations

$$\begin{pmatrix} -z_1 & z_1 w_1 & w_1 \\ -z_2 & z_2 w_2 & w_2 \\ -z_3 & z_3 w_3 & w_3 \end{pmatrix} \begin{pmatrix} a \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$$

in the variables a, c, d. This has a unique solution by uniqueness of ψ, so the matrix on the left-hand side is invertible. But it has real entries, and so does the vector on the right-hand side, so a, c, d must also be real.

Next notice that if $z \in \Pi^+$ then

$$\frac{az+b}{cz+d} = \frac{ac|z|^2 + bd + adz + bc\bar{z}}{|cz+d|^2},$$

so

$$\mathrm{Im}\, \frac{az+b}{cz+d} = \frac{(ad-bc)\,\mathrm{Im}\, z}{|cz+d|^2},$$

and for this to lie in Π^+ we must have $ad - bc > 0$. Hence by dividing each coefficient by $\sqrt{ad-bc}$ we may assume that $ad - bc = 1$.

Furthermore, this argument shows that any Möbius transformation φ with real coefficients and determinant 1 maps Π^+ into itself. But its inverse as a map $\mathbb{C}_\infty \to \mathbb{C}_\infty$ is also a Möbius transformation with real coefficients and determinant 1, so this is also an endomorphism on Π^+. Hence φ is in fact an automorphism on Π^+.

Finally, consider the (well-defined) homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\Pi^+)$ given by $A \mapsto \varphi_A$. Its kernel is the subgroup of real scalar multiples of the identity matrix I whose determinant is 1. But this is just $\{I, -I\}$, so we find that $\mathrm{Aut}(\Pi^+) \cong \mathrm{SL}_2(\mathbb{R})/\{I, -I\} = \mathrm{PSL}_2(\mathbb{R})$. □

9 • Normal Families and the Riemann Mapping Theorem

REMARK 9.1: *Equivalent quasi-metrics.*
Let (X, d) be a metric space, and let ψ be as in Lemma 9.1.2. We claim that d and $\tilde{d} := \psi \circ d$ are equivalent. It suffices to show that any d-ball centred at $x \in X$ contains a $\tilde{d}$-ball centred at x, and vice-versa.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|t| < \delta$ implies $|\psi(t)| < \varepsilon$ for all $t \in [0, \infty)$. If $d(y, x) < \delta$ then $\tilde{d}(y, x) < \varepsilon$, so $B_d(x, \delta) \subseteq B_{\tilde{d}}(x, \varepsilon)$. Conversely, let $\varepsilon > 0$ and put $\delta = \psi(\varepsilon) > 0$. Then if

$$\psi(d(y, x)) = \tilde{d}(y, x) < \delta = \psi(\varepsilon),$$

then since ψ is increasing we have $d(y, x) < \varepsilon$. Hence $B_{\tilde{d}}(x, \delta) \subseteq B_d(x, \varepsilon)$. Notice that in both cases, δ only depends on ε and not on x.

We next show that Cauchy sequences with respect to d and $\tilde{d}$ agree. If $(x_n)_{n \in \mathbb{N}}$ is a d-Cauchy sequence, then continuity of ψ at 0 implies that (x_n) is also a $\tilde{d}$-Cauchy sequence. Conversely, assume that (x_n) is not a d-Cauchy sequence. Then there exists an $\varepsilon > 0$ such that given any $N \in \mathbb{N}$ there are $m, n \geq N$ with $d(x_m, x_n) \geq \varepsilon$. But then $\tilde{d}(x_m, x_n) \geq \psi(\varepsilon) > 0$, so (x_n) is not a $\tilde{d}$-Cauchy sequence. ⌟

REMARK 9.2: *Product metrics??.*
Let $(d_j)_{j \in \mathbb{N}}$ be a family of quasi-metrics on X, and let $\tilde{d}_j = \psi \circ d_j$ be an equivalent metric with $\tilde{d}_j \leq 1$ for all $j \in \mathbb{N}$. The formula

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \tilde{d}_j(x, y)$$

then defines a quasi-metric on X as in Lemma 9.1.3. It is clear that a sequence converges with respect to d if and only if it converges with respect to each $\tilde{d}_j$, which by the above remark is equivalent to convergence with respect to each d_j. Furthermore, if (x_n) is d-Cauchy then it is clearly d_j-Cauchy (in particular, if each d_j is complete then d is also complete), but it is not clear that the converse holds.

Next notice that $\tilde{d}_j \leq d$, so $B_d(x, r) \subseteq B_{\tilde{d}_j}(x, r)$. By the previous remark, for every $\varepsilon > 0$ there is an $r_j > 0$ such that

$$B_d(x, r_j) \subseteq B_{\tilde{d}_j}(x, r_j) \subseteq B_{d_j}(x, \varepsilon).$$

If $N \in \mathbb{N}$ and $r = \min\{r_1, \ldots, r_N\}$, then $B_d(x,r) \subseteq B_{d_j}(x,\varepsilon)$. It follows that

$$B_d(x,r) \subseteq B_{d_1}(x,\varepsilon) \cap \cdots \cap B_{d_N}(x,\varepsilon) = B_N(x,\varepsilon),$$

proving Lemma 9.1.5(iii). This also implies, as in the previous remark, that $B_N(x,\varepsilon)$ is d-open, proving (i).

As far as I can tell, part (ii) does not hold for all functions ψ. But assume that $\psi(t) \le t$ for all $t \in [0,\infty)$ (notice that both $t \mapsto t \wedge 1$ and $t \mapsto t/(1+t)$ have this property). Then Ullrich's argument goes through as stated. ⌟