Hopfield Network

Implementation Notes

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1 Introduction

This document outlines the mathematical formulae implemented in the Hopfield network library.

2 Definitions

Let N be the number of neurons in the network. Let S be the state vector of the network, $S = (S_1, S_2, \ldots, S_N)$, where each $S_i \in \{-1, +1\}$ represents the state of neuron i. Let W be the weight matrix, where W_{ij} is the connection strength from neuron j to neuron i. In this implementation, $W_{ii} = 0$. Let $\xi^p = (\xi_1^p, \xi_2^p, \ldots, \xi_N^p)$ be the p-th pattern to be stored, where each $\xi_i^p \in \{-1, +1\}$.

3 Learning Rule (Hebbian)

The weight matrix W is calculated based on the patterns ξ^p to be stored using the Hebbian rule (specifically, the outer product rule):

$$W_{ij} = \begin{cases} \sum_p \xi_i^p \xi_j^p & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

where the sum is over all patterns p provided during training.

4 Update Rules

4.1 Synchronous Update

All neurons update their state simultaneously based on the state at the previous time step t. The state of neuron i at time $t+1$ is determined by:

$$S_i(t+1) = \operatorname{sgn}\left(\sum_{j=1}^{N} W_{ij} S_j(t)\right)$$

where the sign function is defined as:

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ S_i(t) & \text{if } x = 0 \text{ (no change)} \end{cases}$$

4.2 Asynchronous Update

A single neuron k is chosen (typically randomly) at each step. Its state is updated based on the current states of all other neurons:

$$S_k(\text{new}) = \operatorname{sgn}\left(\sum_{j=1}^{N} W_{kj} S_j(\text{current})\right)$$

The sign function definition is the same as for the synchronous update, using $S_k(\text{current})$ if the sum is zero. An *iteration* or *sweep* in the asynchronous mode consists of N such single-neuron updates.

5 Energy Function (Lyapunov Function)

The energy of a given network state S is defined as:

$$E = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} W_{ij} S_i S_j$$

Since $W_{ii} = 0$, this is equivalent to:

$$E = -\frac{1}{2}\sum_{i \neq j} W_{ij} S_i S_j$$

The network dynamics (both synchronous and asynchronous) tend to evolve the state S towards configurations that minimize this energy function.