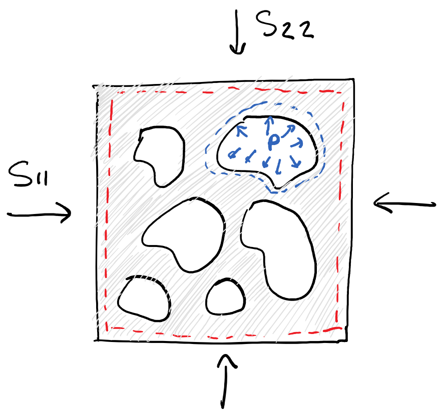
Poroelasticity (Coussy, 2010 - Chapters 3 and 4)

Friday, October 2, 2020 5:45 PM



- pores → filled with fluids ↔ pressure
  ↳ no shear stress
- deformable pore walls

Volumetric deformation

$$\varepsilon_{vol} = \epsilon \overset{def}{=} \frac{d\Omega - d\Omega_0}{d\Omega_0}$$

$$\hookrightarrow \underbrace{d\Omega}_{\text{bulk volume}} = (1+\epsilon)\cdot \underbrace{d\Omega_0}_{\text{initial bulk volume}} \quad (1)$$

Porosity

@ time t $\quad n \overset{def}{=} \dfrac{\square}{\blacksquare + \square}$ } Eulerian porosity

$$n \cdot d\Omega \overset{def}{=} \phi \cdot d\Omega_0 \rightarrow \underbrace{\phi}_{\text{Lagrangian Porosity}} = n \frac{d\Omega}{d\Omega_0} \quad (2)$$

Solid strain

$$\epsilon_s \overset{def}{=} \frac{d\Omega^s - d\Omega_0^s}{d\Omega_0^s} \Rightarrow d\Omega^s = (1+\epsilon_s)\, d\Omega_0^s \quad (3)$$

Combination of (1), (2), (3)

$$d\Omega^s = (1+\epsilon_s)\, d\Omega_0^s$$

$$(1-n)\, d\Omega = (1+\epsilon_s)(1-\phi_0)\, d\Omega_0$$

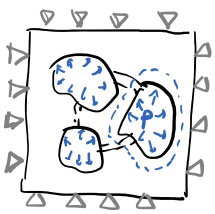
$$\boxed{\underbrace{\epsilon}_{\text{Bulk Vol. Strain}} = (1-\phi_0)\underbrace{\epsilon_s}_{\text{Solid Vol. Strain}} + (\phi_0)\underbrace{\left(\frac{\phi - \phi_0}{\phi_0}\right)}_{\text{Porosity strain}}}$$

↳ Micromechanical Eq.
↳ Volume average

$$\boxed{\epsilon = (1-\phi_0)\,\epsilon_s + \varphi}$$

$\varphi \overset{\text{def}}{=} \phi - \phi_0$: change of porosity

Test 1



$\boxed{\epsilon = 0}$

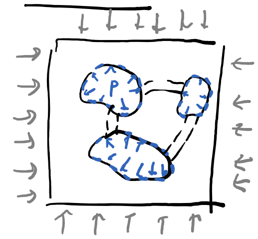
$$0 = (1-\phi_0)\,\epsilon_s + \varphi$$

$$\epsilon_s = -\frac{\varphi}{(1-\phi_0)}$$

if $\varphi > 0 \Rightarrow \underbrace{\epsilon_s < 0}_{\text{contraction}}$

Mechanics convention

Test 2

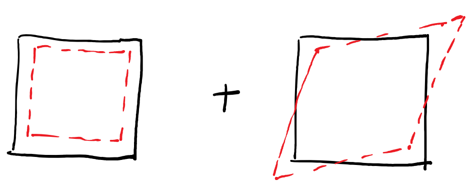


$\boxed{\varphi = 0}$

$$\epsilon_s = \frac{\epsilon}{1-\phi_0}$$

Free energy of the porous solid

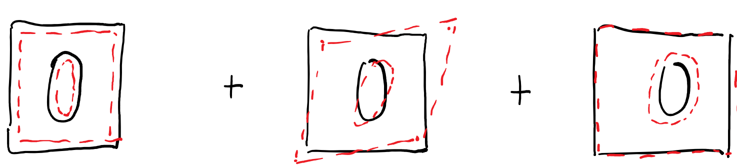
Non-porous solid



$$dW = \underset{\frac{S_{11}+S_{22}+S_{33}}{3}}{\underbrace{S_m}} \cdot d\epsilon + \underbrace{S_{ij}} \cdot de_{ij}$$

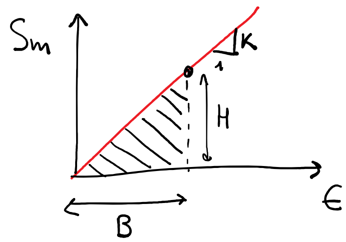
$S_{12} \leftrightarrow e_{12}$
$S_{23} \leftrightarrow e_{23}$
$S_{13} \leftrightarrow e_{13}$

Porous solid



$$dW = S_m \cdot d\epsilon + S_{ij} \cdot de_{ij} + p\,\underbrace{d\phi}$$

$$d\,\bar{\bar{\varphi}}$$



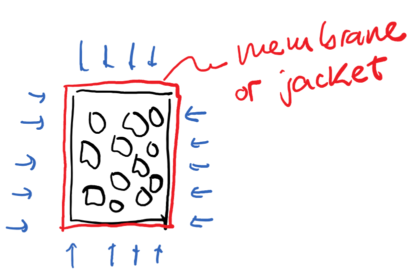
$$W = \frac{\epsilon \cdot (\epsilon \cdot K)}{2} \; ; \; K \overset{\text{def}}{=} \frac{dS_m}{d\epsilon} \; : \text{Bulk modulus}$$

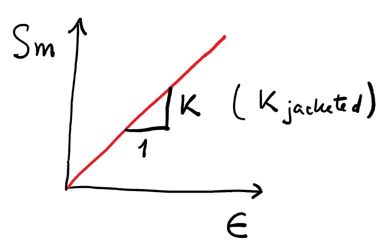
$$W = \frac{1}{2} K \epsilon^2 \longleftrightarrow \frac{\partial W}{\partial \epsilon} = K\epsilon = S_m$$

$$\Psi_s = \underbrace{\frac{1}{2} K \epsilon^2}_{\text{Bulk vol. strain}} + \underbrace{G\, e_{ij} \cdot e_{ij}}_{\text{Shear strains}} \underbrace{- \underbrace{\alpha}_{\text{cst1}} \cdot \epsilon \cdot p - \underbrace{\left(\frac{1}{2}\,\frac{1}{N}\right)}_{\text{cst 2}} p^2}_{\text{pressure}}$$

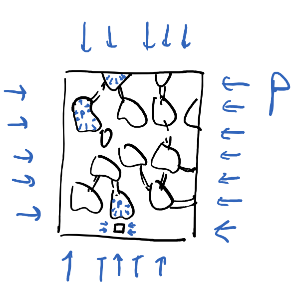
$$\begin{cases} \dfrac{\partial \Psi_s}{\partial \epsilon} = \underbrace{K\epsilon}_{\text{effective stress}} - \underbrace{\alpha}_{\text{Biot coefficient}} \underbrace{p}_{\text{pressure}} = \underbrace{S_m}_{\text{total stress}} & (1) \text{ Biot effective stress equation} \\[2ex] \dfrac{\partial \Psi_s}{\partial e_{ij}} = 2G\, e_{ij} = s_{ij} & (2) \quad \alpha = \,? \\[2ex] -\dfrac{\partial \Psi_s}{\partial p} = \alpha \epsilon + \underbrace{\dfrac{p}{N}}_{N:\ \text{Poroelastic Modulus}} = \varphi & (3) \quad N = \,? \end{cases}$$

Jacketed loading





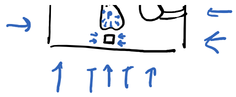
Unjacketed loading



$$(1) \quad K\epsilon - \alpha p = S_m = -p$$

$$\epsilon = \frac{p(-1+\alpha)}{K}$$

$$\bullet \quad \epsilon = -\frac{p(1-\alpha)}{K}$$



- isotropic elastic solid
- connected porosity

- $\epsilon = - \frac{\quad}{K}$
- $\epsilon_s = - \frac{p}{K_s}$

$\epsilon = \epsilon_s$

$$\frac{\cancel{p}(1-\alpha)}{K} = \frac{\cancel{p}}{K_s} \Rightarrow \boxed{\alpha = 1 - \frac{K}{K_s}}$$

bulk modulus porous solid = jacketed =

bulk modulus solid skeleton

$\alpha \leq 1$ for $K_s \geq K$

③ $\alpha \epsilon + \frac{p}{N} = \varphi$

$$\begin{cases} \alpha\left(-\frac{p}{K_s}\right) + \frac{p}{N} = \varphi \\ \frac{\varphi}{\phi_0} = -\frac{p}{K_s} \end{cases}$$

$$\alpha\left(-\frac{\cancel{p}}{K_s}\right) + \frac{\cancel{p}}{N} = -\frac{\cancel{p}}{K_s} \cdot \phi_0$$

$$\boxed{\frac{1}{N} = \frac{\alpha - \phi_0}{K_s}}$$

- second poromechanical modulus
- $N > 0 \Rightarrow \alpha > \phi_0$
- $\phi_0 \leq \alpha \leq 1$

Measurement of poroelastic parameters

METHOD 1

- Measure K
- Assume or estimate K_s
  - mono-mineral*
  - multimineral
    - ↳ volume average
      - Reuss
      - Voigt
      - Hashin-Shtrikman

$$\boxed{\alpha = 1 - \frac{K}{K_s}}$$

$$\boxed{N = \frac{K_s}{\alpha - \phi_0}}$$

jacketed bulk modulus

dry rock, E, ν

↳ $K = \dfrac{E}{3(1-2\nu)}$

↳ $V_p, V_s, \rho \Rightarrow E, \nu, K$

$*\ K_s = \underbrace{K_{mineral} = K_{SiO_2}}$

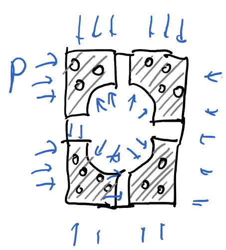
Sandstone 100% SiO_2

$K_{SiO_2} = 36$ GPa

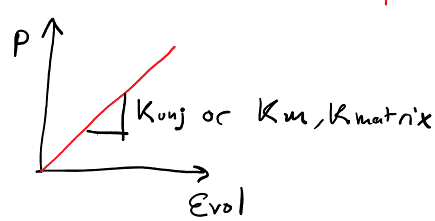
→ wrong if occluded ϕ

METHOD 2

- Unjacketed test



$$\boxed{\alpha = 1 - \frac{K}{K_{unj}}}$$



- water < clay sensitive / shortcircuit
- oil - high viscosity
- gas → small vessel
  - ↳ not desirable for large vessels
  - ↳ non-sorptive

- Increase P_c and P_p simultaneously

constant Terzaghi's effective stress $= P_c - P_p \neq 0$

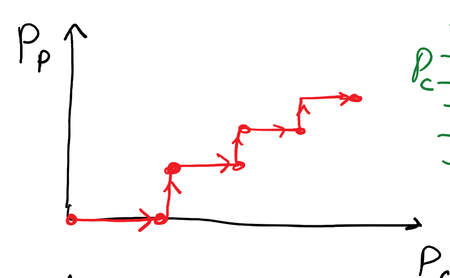
METHOD 3

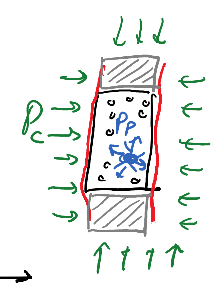
$$S_m = K\epsilon - \alpha P$$

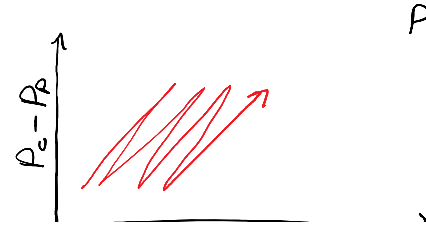
$$K\epsilon = S_m + \alpha P$$

$$\epsilon = \frac{1}{K}\underbrace{(S_m + \alpha P)}_{\text{Biot effective stress}}$$

$(K, \alpha) \leftarrow$ error minimization



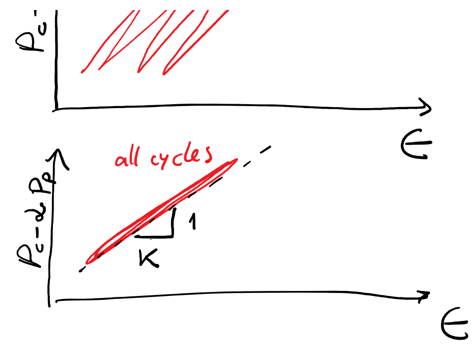




stress

$(K, \alpha) \leftarrow$ error minimization

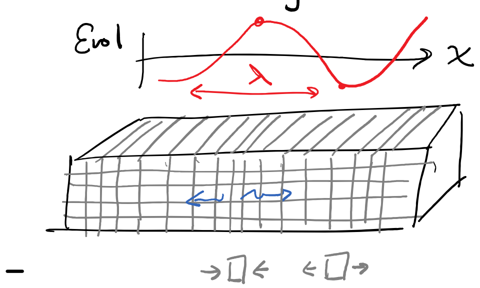
$$\underbrace{K_m}_{\text{matrix}} = \frac{K}{1-\alpha}$$



Drained and Undrained Problems in Poromechanics

Terzaghi's effective stress : $\underline{\underline{\sigma}} = \underline{\underline{S}} - P_p \underline{\underline{I}}$

Poroelasticity



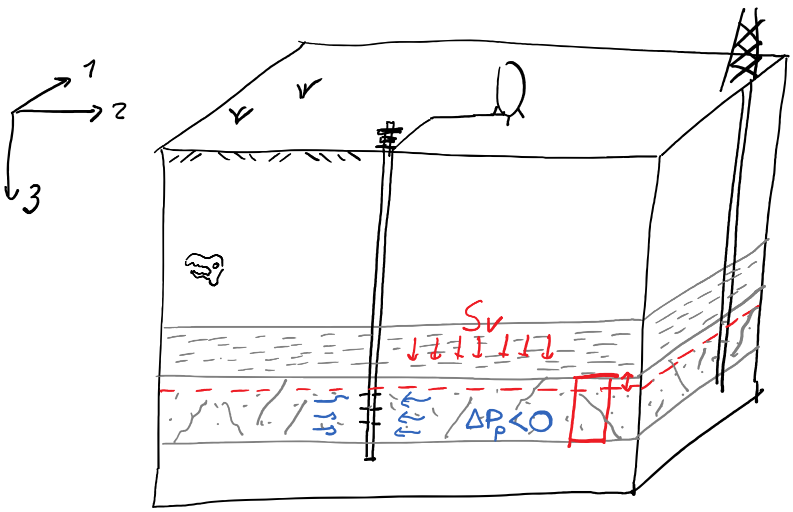
- $\underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$: Biot's effective stress
- $\varphi \rightarrow$ changes of porosity
- K_u : undrained bulk modulus
- undrained loading $\leftrightarrow \Delta P_p$
  $\updownarrow$
- "squirt flow" $\leftrightarrow \Delta \varepsilon_{vol}$
  - $\rightarrow$ attenuation of elastic waves
  - $\rightarrow$ dispersion of " waves

End-members of actual solution $\left\langle \begin{matrix} \text{drained} \\ \updownarrow \\ \text{undrained} \end{matrix} \right] \rightarrow$ Diffusivity Eq for poroelasticity

Drained solution of change of horizontal stress with depletion/injection

Wednesday, September 28, 2022 11:30 AM

(10 years later)



1) Stresses in-situ $(\varepsilon_{11} = \varepsilon_{22} = 0)$

2) Stresses in-situ $(\varepsilon_{11} \neq \varepsilon_{22} \neq 0)$

3) $\Delta \underline{\sigma}$ with change in P_p ?

4) Extend to VTI rock

1) $\underline{\sigma} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$; $\underline{\sigma} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$

$\hookrightarrow$ $\underline{\underline{\varepsilon}} = \underline{\underline{\underline{\underline{D}}}} \cdot \underline{\underline{\sigma}}$

$$\begin{bmatrix} \cancel{\varepsilon_{11}} \\ \cancel{\varepsilon_{22}} \\ \underline{\varepsilon_{33}} \\ \cancel{2\varepsilon_{23}} \\ \cancel{2\varepsilon_{13}} \\ \cancel{2\varepsilon_{12}} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ \hline & & & 1/G & 0 & 0 \\ & 0 & & 0 & 1/G & 0 \\ & & & 0 & 0 & 1/G \end{array}\right] \begin{bmatrix} S_{11} - \alpha P_p \\ S_{22} - \alpha P_p \\ \underline{S_{33} - \alpha P_p} \\ \cancel{S_{23}} \\ \cancel{S_{13}} \\ \cancel{S_{12}} \end{bmatrix}$$

Passive tectonic env

$\varepsilon_{11} = \varepsilon_{22} = \emptyset$

$\varepsilon_{ij} = 0$ for $i \neq j$

$\downarrow$

$S_{11} = S_{22}$

$$\varepsilon_{11} = 0 = \frac{1}{E}(S_{11} - \alpha P_p) - \frac{\nu}{E}(S_{22} - \alpha P_p) - \frac{\nu}{E}(S_{33} - \alpha P_p) \quad ;$$

$$\left(\frac{1-\nu}{\cancel{E}}\right) \cdot (S_{22} - \alpha P_p) = \frac{\nu}{\cancel{E}}(S_{33} - \alpha P_p)$$

$$\underbrace{(S_{22} - \alpha P_p)}_{\sigma_{22}} = \underbrace{\frac{\nu}{1-\nu}}_{} \underbrace{(S_{33} - \alpha P_p)}_{\sigma_{33}}$$

$$S_{hmin} = S_{Hmax} = S_{22} = \alpha P_p + \frac{\nu}{1-\nu}\big(\underbrace{S_{33}}_{\downarrow\, S_v} - \alpha P_p\big)$$

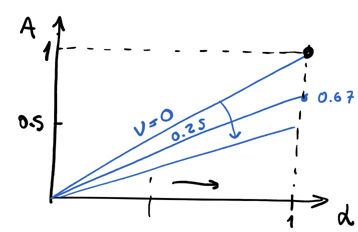
2

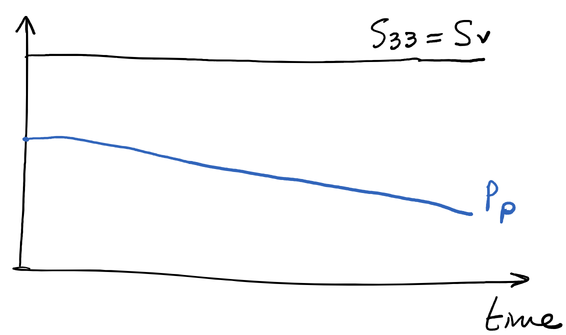
$$\begin{cases} S_{hmin} = \alpha P_p + \dfrac{\nu}{1-\nu}\left(S_v - \alpha P_p\right) + \dfrac{E}{1-\nu}\left(\varepsilon_{hmin} + \nu\, \varepsilon_{Hmax}\right) \\ S_{Hmax} = \alpha P_p + \dfrac{\nu}{1-\nu}\left(S_v - \alpha P_p\right) + \dfrac{E}{1-\nu}\left(\nu\, \varepsilon_{hmin} + \varepsilon_{Hmax}\right) \\ S_v = \displaystyle\int_0^z \rho_{bulk}\, g\, dz \end{cases}$$

3

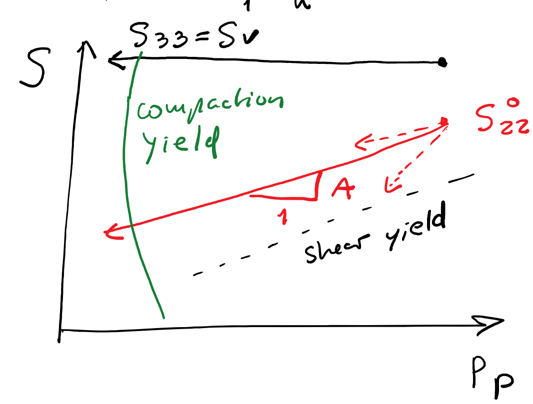
$$\underbrace{S_{22}}_{\text{Total Hz stress}} = \underbrace{\frac{\nu}{1-\nu} S_{33}}_{\text{Contrib Total vert stress}} + \underbrace{\alpha\left(\frac{1-2\nu}{1-\nu}\right)}_{A} \underbrace{P_p}_{\text{Pore pressure}}$$

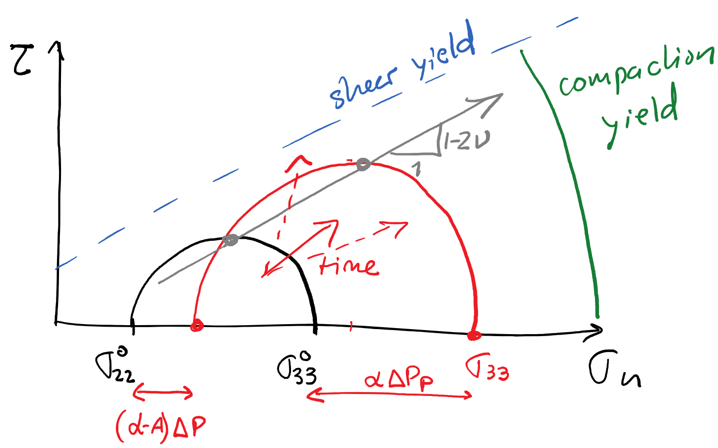
$$\frac{\partial S_{22}}{\partial P_p} = \alpha\left(\frac{1-2\nu}{1-\nu}\right) = A \rightarrow$$





$\Rightarrow$





$$\sigma_{33} = S_{33} - \alpha P_p$$

$$\frac{\partial \sigma_{33}}{\partial P_p} = -\alpha$$

$$\frac{\partial \sigma_{22}}{\partial P_p} = -(\alpha - A) = -\alpha\left(\frac{\nu}{1-\nu}\right)$$

$$\varepsilon_{33} = \frac{-2\nu}{E}(S_{22} - \alpha P_p) + \frac{1}{E}(S_{33} - \alpha P_p)$$

$$\varepsilon_{33} = \left(\frac{1}{E}\frac{(1-2\nu)(1+\nu)}{(1-\nu)}\right)\overbrace{(S_{33} - \alpha P_p)}^{\sigma_{33}}$$

1-D strain condition

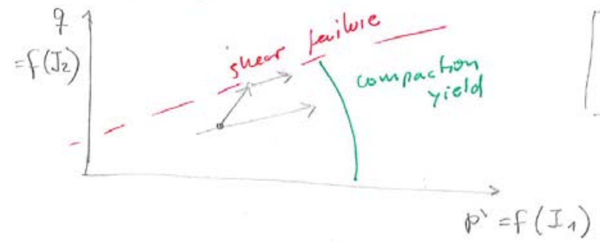
$$\frac{\partial \varepsilon_{33}}{\partial P_p} = -\frac{\alpha}{M}$$ constrained modulus

$$\Delta\varepsilon_{33} = -\frac{\alpha}{M}\Delta P_p$$

$$\Delta H = \Delta\varepsilon_{33} \cdot H$$

$$\boxed{\Delta H = \left(-\frac{\alpha}{M}\Delta P_p\right) H}$$

→ casing (shearing, buckling)
→ subsidence



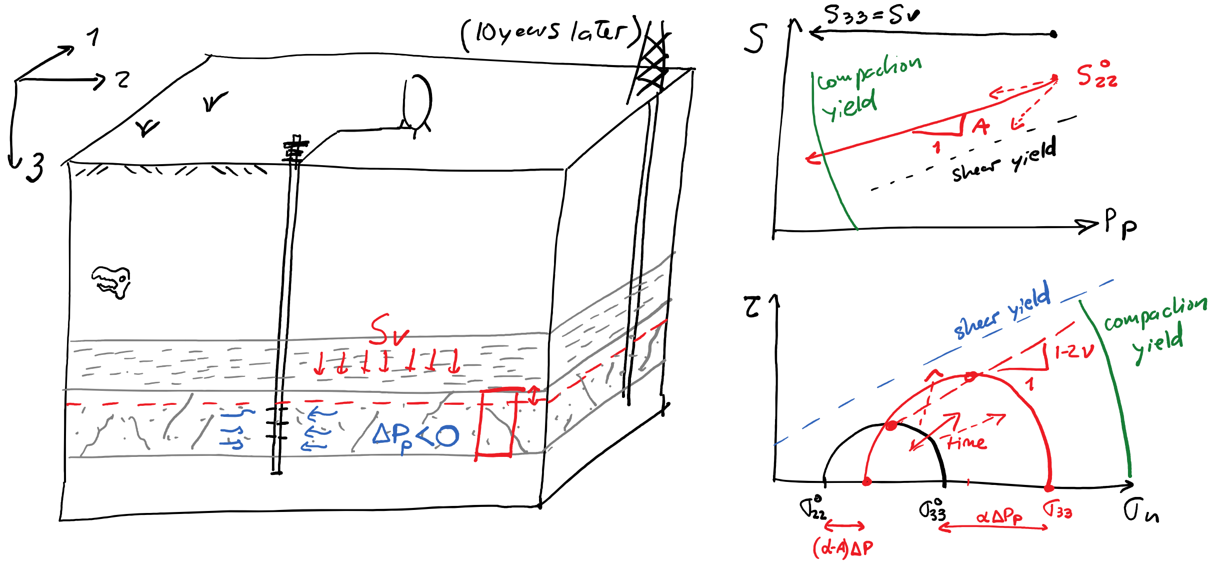
4

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} +\frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_h}{E_h} & +\frac{1}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & +\frac{1}{E_v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_h} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{cases} S_{hmin} = \alpha_h P_p + \frac{\nu_v}{1-\nu_h}\frac{E_h}{E_v}(S_v - \alpha_v P_p) + \frac{E_h}{1-\nu_h^2}(\varepsilon_{hmin} + \nu_h \varepsilon_{Hmax}) \\ S_{Hmax} = \alpha_h P_p + \frac{\nu_v}{1-\nu_h}\frac{E_h}{E_v}(S_v - \alpha_v P_p) + \frac{E_h}{1-\nu_h^2}(\nu_h \varepsilon_{hmin} + \varepsilon_{Hmax}) \\ S_v = \int_0^z \rho_{bulk}\, g\, dz \end{cases}$$

$$\begin{cases} \sigma_{hmin} = S_{hmin} - \alpha_h P_p \\ \sigma_{Hmax} = S_{Hmax} - \alpha_h P_p \\ \sigma_v = S_v - \alpha_v P_p \end{cases}$$

$$\begin{cases} \alpha_h = 1 - \frac{C_{11} + C_{12} + C_{13}}{3 K_m} \\ \alpha_v = 1 - \frac{2C_{13} + C_{33}}{3 K_m} \\ \frac{1}{N} = \frac{(2\alpha_h + \alpha_v)/3 - \phi_0}{K_m} \end{cases}$$



1
2
3
(10 years later)
Sv
ΔPp<0
S
S33=Sv
compaction yield
S°22
A
1
shear yield
Pp
τ
shear yield
compaction yield
1-2ν
1
time
σ°22
σ°33
αΔPp
σ33
σn
(α-A)ΔP

Pore pressure diffusivity equation coupled with poroelasticity

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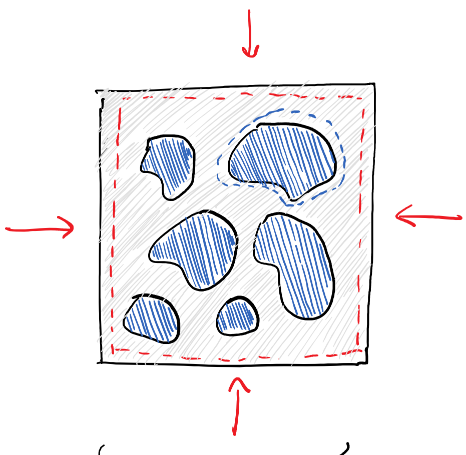


$$\phi = V_p / V_b^\circ$$

$$\frac{\text{fluid mass}}{V_b^\circ} = \frac{V_p \, \rho_F}{V_b^\circ} = \phi \rho_F$$

$$d(\phi \rho_F) = \rho_F \, d\phi + \phi \, d\rho_F \quad \begin{cases} = 0 \to \text{undrained loading} \\ \neq 0 \end{cases}$$

$$\frac{d(\phi \rho_F)}{\rho_F} = d\phi + \phi \frac{d\rho_F}{\rho_F} \sim C_F = \frac{1}{\rho_F}\frac{d\rho_F}{dP} \; ; \; K_F = C_F^{-1}$$

$\varphi = \alpha d\epsilon + \frac{P}{N}$

$$\frac{d(\phi \rho_F)}{\rho_F} = \left(\alpha d\epsilon + \frac{dp}{N}\right) + \phi \frac{dP}{K_F}$$

$d\phi << \phi_0$ (small strains)

$$\frac{d(\phi \rho_F)}{\rho_F} = \alpha d\epsilon + \underbrace{\left(\frac{1}{N} + \frac{\phi_0}{K_F}\right)}_{\sim cst} dP$$

Biot Modulus $\boxed{\frac{1}{M^*} \overset{def}{=} \left(\frac{\alpha - \phi_0}{K_s} + \frac{\phi_0}{K_F}\right)}$; $\frac{1}{N} = \frac{\alpha - \phi_0}{K_s}$

Continuity equation

(Mass conservation)

$$\frac{d(\rho_F \phi)}{dt} + \nabla \cdot (\rho_F \underline{q}) = 0$$

isotropic Darcy's

$$\frac{1}{dt}\left(\cancel{\rho_F}\left[\alpha d\epsilon + \frac{1}{M^*} dP\right]\right) + \nabla \cdot \left[\cancel{\rho_F}\left(-\frac{k}{\mu} \nabla P\right)\right] = 0$$

homogeneous porous solid

$$\alpha \frac{d\epsilon}{dt} + \frac{1}{M^*}\frac{dP}{dt} + \left(-\frac{k}{\mu} \nabla^2 P\right) = 0$$

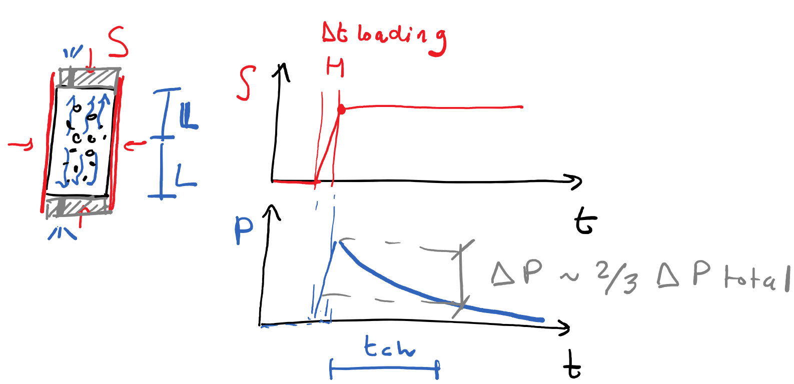
$$\boxed{\frac{dP}{dt} = \frac{k M^*}{\mu} \nabla^2 P - \alpha M^* \frac{d\epsilon}{dt}}$$

Diff. Eq. coupled with poroelasticity

$$\frac{dP}{dt} \sim \underset{\sim}{\mathbb{D}} \nabla^2 P$$

$\hookrightarrow \boxed{t_{ch} = \frac{L^2}{\mathbb{D}}}$

$$\mathbb{D} \sim \frac{k M^*}{\mu}$$



| Δt loading << t_{ch} ⇒ undrained loading

$$\begin{cases} \Delta t_{loading} \ll t_{ch} \Rightarrow \text{undrained loading} \\ \Delta t_{loading} \gg t_{ch} \Rightarrow \text{drained loading} \end{cases}$$

Undrained Loading

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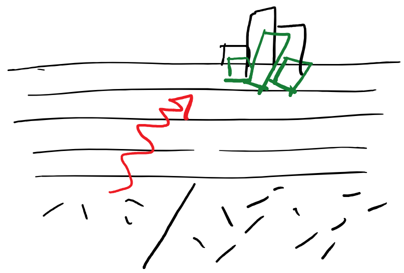
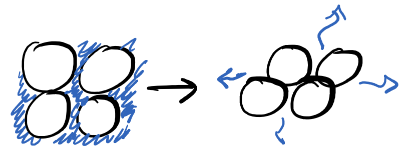
Undrained loading examples

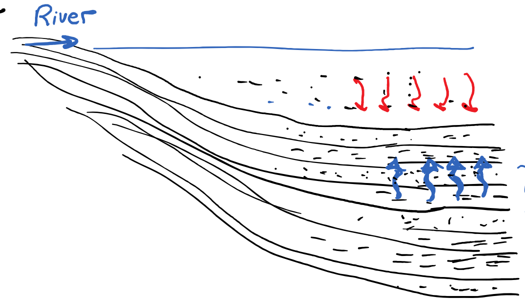
1 Liquefaction





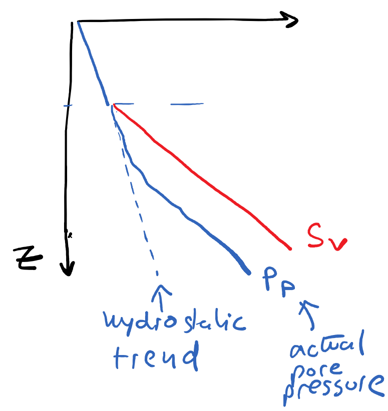
$\uparrow P_p \Rightarrow \downarrow \sigma' \Rightarrow \downarrow$ sheer strength

2 Dis-equilibrium compaction

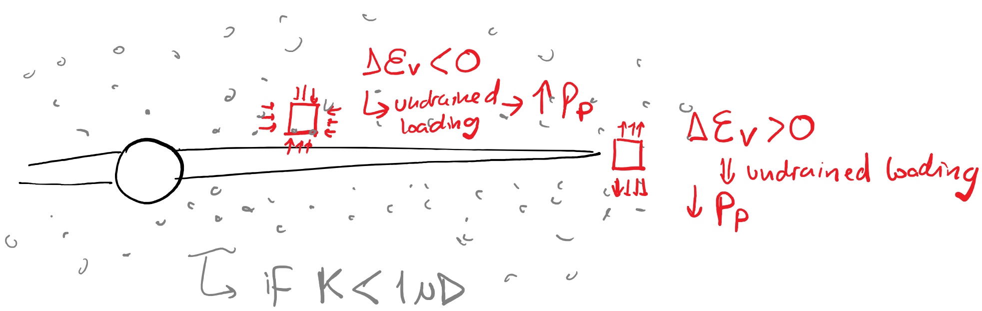


rate of sedimentation

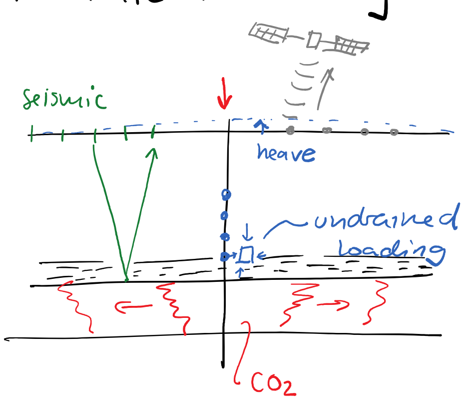
$$\frac{dm}{dt} \gg \frac{dP}{dt}$$



3 Hydraulic fracture

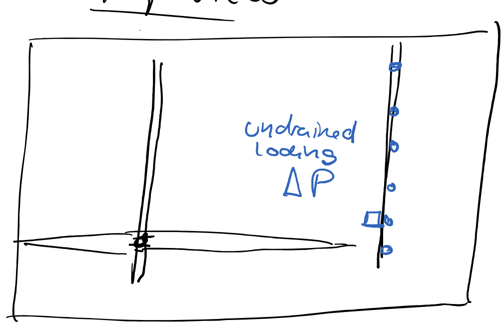


4 Poroelastic monitoring



ΔP

top view

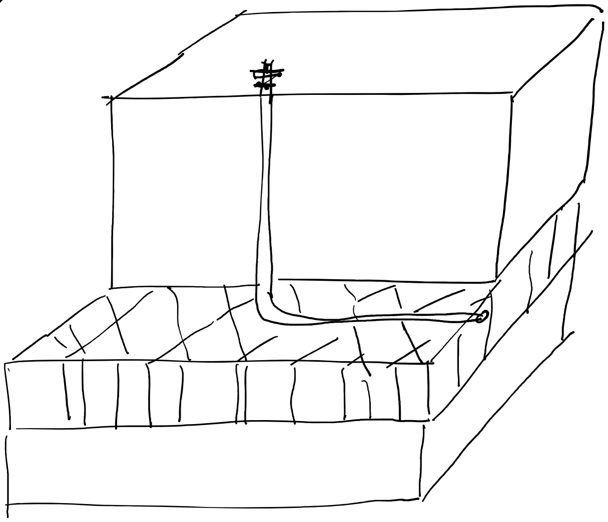


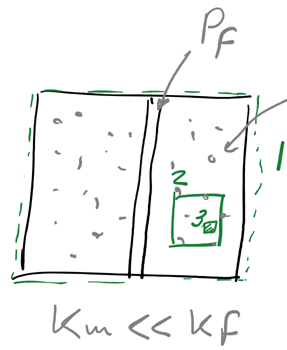
Rousell

ConocoPhillips

5

5





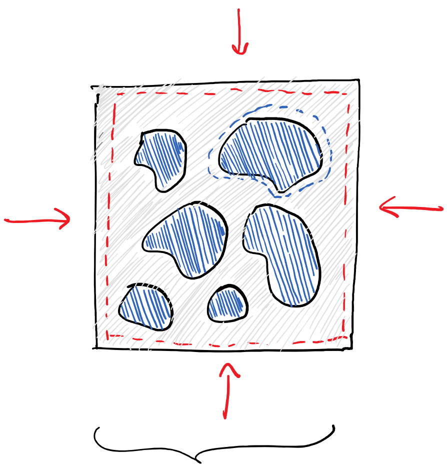
undrained loading of the matrix

During depletion

$$P_F < P_m$$

$$\alpha = 1 - \frac{\langle K \rangle}{\langle K_m \rangle}$$

$\langle \; \rangle$ volume average



$\phi = V_p / V_b^\circ$

$$\frac{\text{fluid mass}}{V_b^\circ} = \frac{V_p \cdot \rho_F}{V_b^\circ} = \phi \rho_F$$

Biot Modulus

$$\boxed{\frac{1}{M^*} \overset{def}{=} \left(\frac{\alpha - \phi_0}{K_s} + \frac{\phi_0}{K_F} \right)}$$

Diff. Eq. coupled with poroelasticity

$$\boxed{\frac{dP}{dt} = \frac{k M^*}{\mu} \nabla^2 p - \alpha M^* \frac{d\epsilon}{dt}}$$

Undrained loading parameters

1 Pore pressure change due to volumetric strain

Diff. Eq. coupled with poroelasticity

$$\frac{dP}{\cancel{dt}} = \cancel{\frac{k M^*}{\mu} \nabla^2 p} - \alpha M^* \frac{d\epsilon}{\cancel{dt}}$$

$$\left.\frac{dP}{d\epsilon}\right|_{d(\rho_F \phi)=0} = -\alpha M^* = -\alpha \left(\frac{\alpha - \phi_0}{K_m} + \frac{\phi_0}{K_F} \right)^{-1}$$

$$\Delta p = (-\alpha M^*)\,\Delta \epsilon$$

② Undrained Bulk Modulus

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{\text{drained, dry}} = K$$

Bulk modulus
- drained
- dry

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{\partial(\rho_F \phi)=0} \stackrel{\text{def}}{=} K_U$$

Undrained Bulk modulus

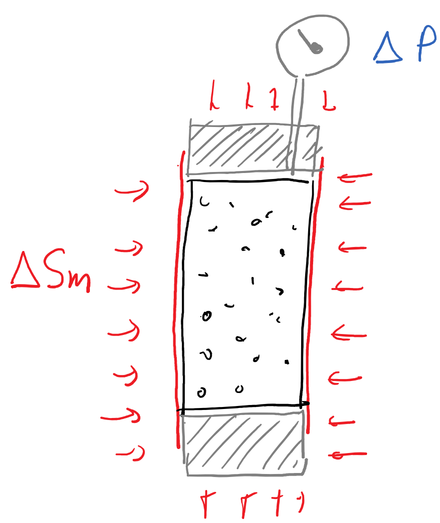
$$S_m = K\epsilon - \alpha P$$

$$\frac{\partial S_m}{\partial \epsilon} = K - \alpha \frac{\partial P}{\partial \epsilon} \Rightarrow \left.\frac{\partial S_m}{\partial \epsilon}\right|_{\partial(\rho_F \phi)=0} = K - \alpha(-\alpha M^*)$$

$$\boxed{K_U = K + \alpha^2 M^*}$$

③ Skempton's parameter

$$B \stackrel{\text{def}}{=} -\left.\frac{\partial P}{\partial S_m}\right|_{\partial(\rho_F \phi)=0}$$



$$S_m = K\epsilon - \alpha P$$

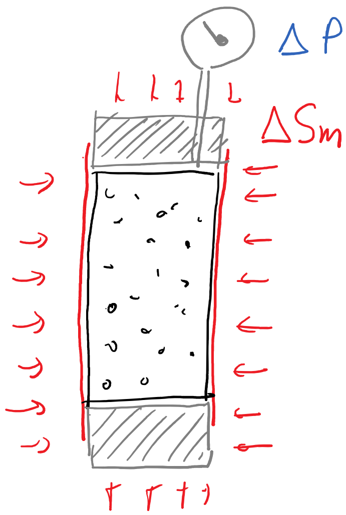
$$\frac{\partial S_m}{\partial P} = K\frac{\partial \epsilon}{\partial P} - \alpha$$

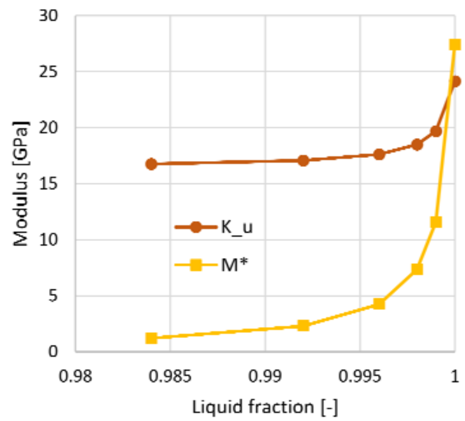
$$\frac{\partial S_m}{\partial P} = K\left(-\frac{1}{\alpha M^*}\right) - \alpha$$

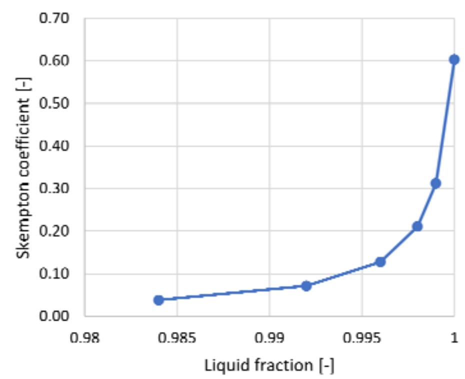
$$\boxed{B^{-1} = \frac{1}{\alpha}\left(\frac{K}{M^*} + \alpha^2\right)}$$

$$\boxed{K_u = K + \alpha^2 M^*}$$

$$\boxed{B^{-1} = \frac{1}{\alpha}\left(\frac{K}{M^*} + \alpha^2\right)}$$







$$\rightarrow K_F^{-1} = \left(\frac{1-S_w}{K_g} + \frac{S_w}{K_w}\right)$$