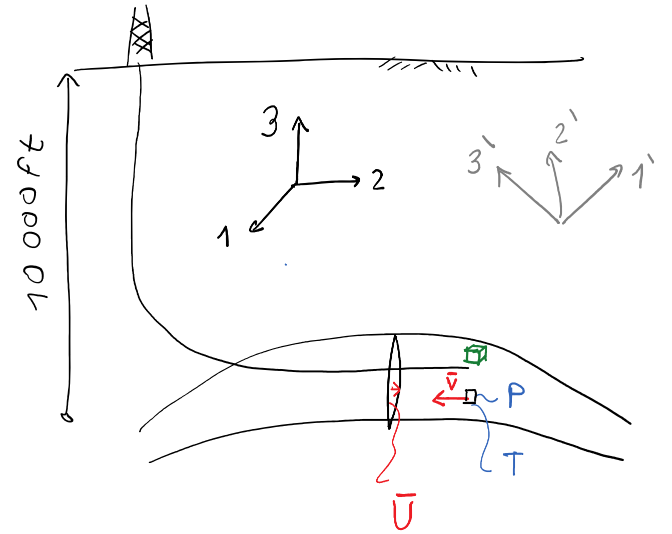
The stress tensor, Cauchy's equilibrium eqs., and principal stresses

Wednesday, August 26, 2020 5:05 PM

2020/8/26



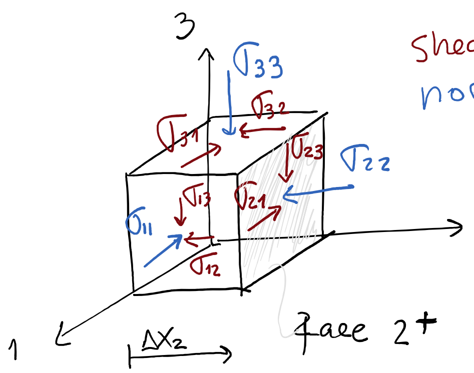
scalar : P , T

vector : $\bar{V} = [0, -0.1, 0] \frac{m}{day}$

$\bar{U} = [0, 1, 0]$ cm

(2nd order) tensor :

stress $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$



shear stresses

normal stresses

σ_{ij} — i: face, j: direction

$\sigma_{ij} = \sigma_{ji}$

symmetric

$\sigma_{ij} \in \mathbb{R}$

eigenvalues $\in \mathbb{R}$

$$\underline{\underline{S}} = \begin{bmatrix} 7000 & 0 & 0 \\ 0 & 6500 & 0 \\ 0 & 0 & 10000 \end{bmatrix} \text{psi}$$

in coord system 1,2,3

principal stresses $\rightarrow \underline{\underline{\sigma}}^{P} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

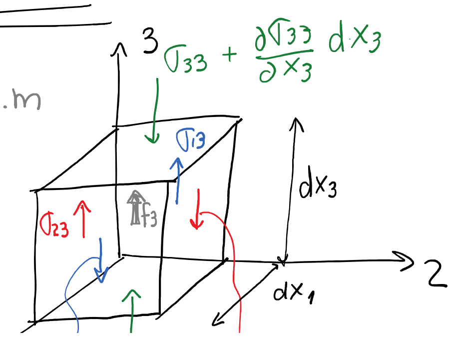
$\updownarrow$

$\perp \leftarrow$ principal directions

$\sigma_1 \neq \sigma_2 \neq \sigma_3$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

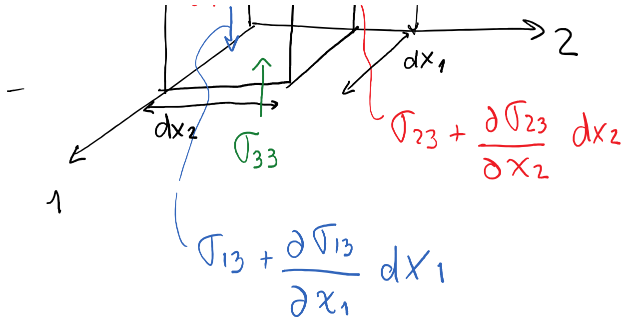
$f_3 = b_3 \cdot m$



$$\sum F_3 = m \cancel{a_3}^{0} = 0$$

$$\cancel{\sigma_{33}}\, dx_1 dx_2 - \left(\cancel{\sigma_{33}} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1 dx_2 +$$

$$\cancel{\sigma_{23}}\, dx_1 dx_3 - \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_1 dx_3 +$$



$$\text{Vol} = dx_1\, dx_2\, dx_3$$

$$\cancel{\sigma_{23}}\, dx_1\, dx_3 - \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_1 dx_3 +$$

$$\cancel{\sigma_{13}}\, dx_2\, dx_3 - \left(\cancel{\sigma_{13}} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2\, dx_3 +$$

$$F_3 = 0$$

$$\frac{\partial \sigma_{33}}{\partial x_3} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{23}}{\partial x_2} dx_1 dx_2 dx_3 +$$

$$\frac{\partial \sigma_{13}}{\partial x_1} dx_1\, dx_2\, dx_3 + b_3\, m = 0$$

$$\Sigma F_3 \rightarrow \frac{\partial \sigma_{33}}{\partial x_3} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_1} + b_3 \frac{m}{\text{Vol}} = 0$$

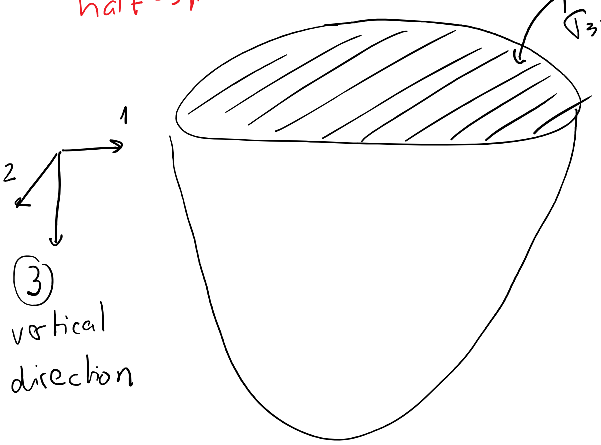
Cauchy's equilibrium equations

- $\frac{\partial \sigma_{ij}}{\partial x_j} + b_i \rho = 0$
- $\nabla \cdot \underline{\underline{\sigma}} + \underline{b}\, \rho = 0$

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 \rho = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 \rho = 0 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \underbrace{b_3}_{\substack{\text{gravity in 3} \\ b_3 = -g}}\, \underbrace{\rho}_{\frac{m}{\text{Vol}}} = 0 \end{cases}$$

half-space $\rightsquigarrow \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = 0$

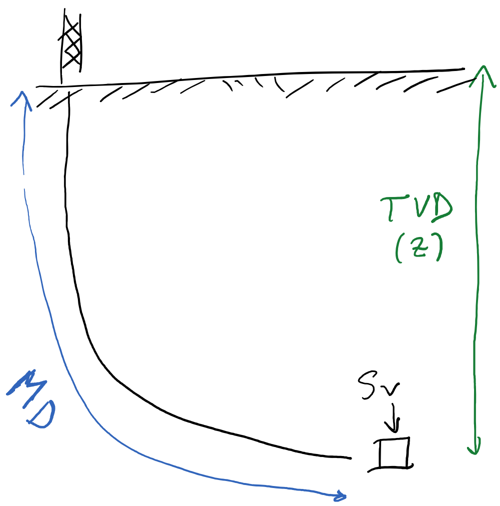
free surface $\sigma_{33}(x_3 = 0) = 0$



$$\begin{cases} \partial \sigma_{11} \quad \frac{}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 \rho = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 \rho = 0 \\ \cancel{\frac{\partial \sigma_{31}}{\partial x_1}}^{0} + \cancel{\frac{\partial \sigma_{32}}{\partial x_2}}^{0} + \boxed{\frac{\partial \sigma_{33}}{\partial x_3}} + b_3\, \rho = 0 \end{cases}$$

$$\frac{\partial \sigma_{33}}{\partial x_3} = g \cdot \rho$$

$$\int^{\sigma_{33}(x_3)} d\sigma_{33} = \int^{x_3} g \cdot \rho \cdot dx_3$$



$$\int_{\sigma_{33}(x_3=0)}^{\sigma_{33}(x_3)} d\sigma_{33} = \int_{x_3=0}^{x_3} g\cdot\rho\cdot dx_3$$

$$\sigma_{33}(x_3) = \int_0^{x_3} g\cdot\rho(x_3)\,dx_3$$

vertical depth

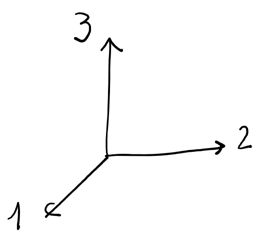
$$\boxed{S_v(z) = \int_0^{z} g\cdot\rho_{bulk}(z)\,dz}$$

$\rho_{bulk}(z) = \rho_{bulk}$ $\qquad S_v(z) = \rho_{bulk}\cdot g\cdot z$

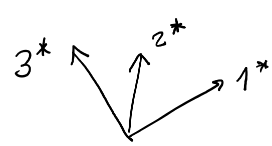
- - - - - - - - -

$$\frac{dS_v}{dz} = \underbrace{\rho_{bulk}\cdot g}_{2300\,\frac{kg}{m^3}\cdot 9.8\,\frac{m}{s^2}} \begin{cases} \underline{23\ MPa/km} \\ \underline{1\ psi/ft} \end{cases}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



$\Longrightarrow$



$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\sigma_1 \perp \sigma_2 \perp \sigma_3$$

σ_v is a principal stress:

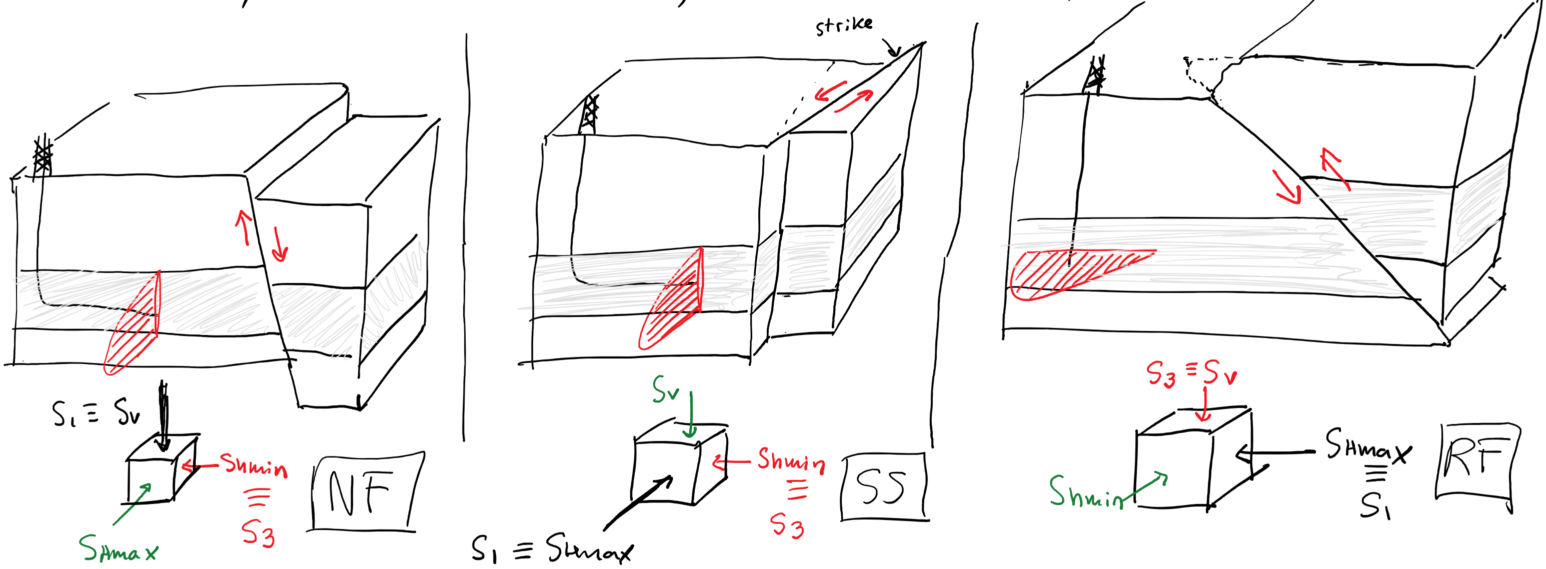
$\hookrightarrow \quad \sigma_v \perp \sigma_{Hmax} \perp \sigma_{hmin}$

NORMAL FAULTING	STRIKE-SLIP FAULTING	REVERSE F.
$\sigma_v \geq \sigma_{Hmax} \geq \sigma_{hmin}$	$\sigma_{Hmax} \geq \sigma_v \geq \sigma_{hmin}$	$\sigma_{Hmax} \geq \sigma_{hmin} \geq \sigma_v$
$\begin{bmatrix} \sigma_v & 0 & 0 \\ 0 & \sigma_{Hmax} & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$	$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_v & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$	$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_{hmin} & 0 \\ 0 & 0 & \sigma_v \end{bmatrix}$

Effective stresses : $\underline{\underline{\sigma}}$; Total stresses $\underline{\underline{S}} = \underline{\underline{\sigma}} + P_p \underline{\underline{I}}$ (*)

$$\begin{bmatrix} S_v & 0 & 0 \\ 0 & S_{Hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix} \quad \begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_v & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix} \quad \begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_{hmin} & 0 \\ 0 & 0 & S_v \end{bmatrix}$$

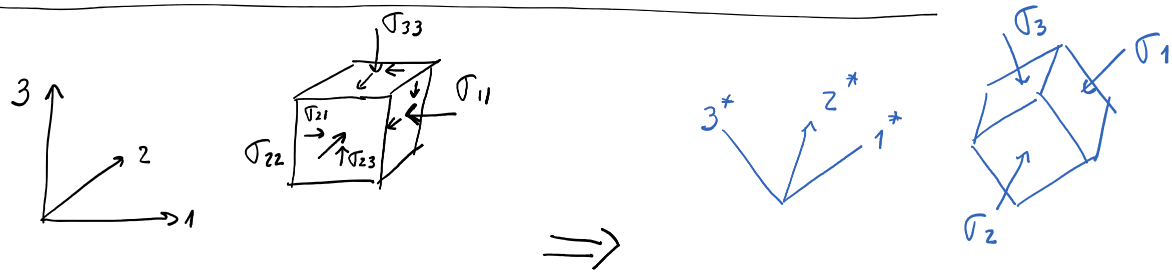
$S_v = \sigma_v + P_p$; $S_{Hmax} = \sigma_{Hmax} + P_p$; $S_{hmin} = \sigma_{hmin} + P_p$



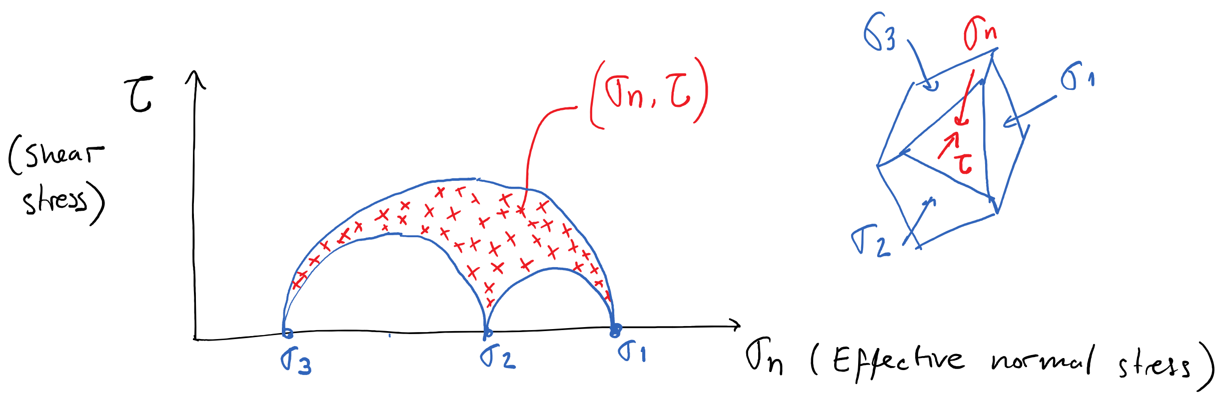
(*)

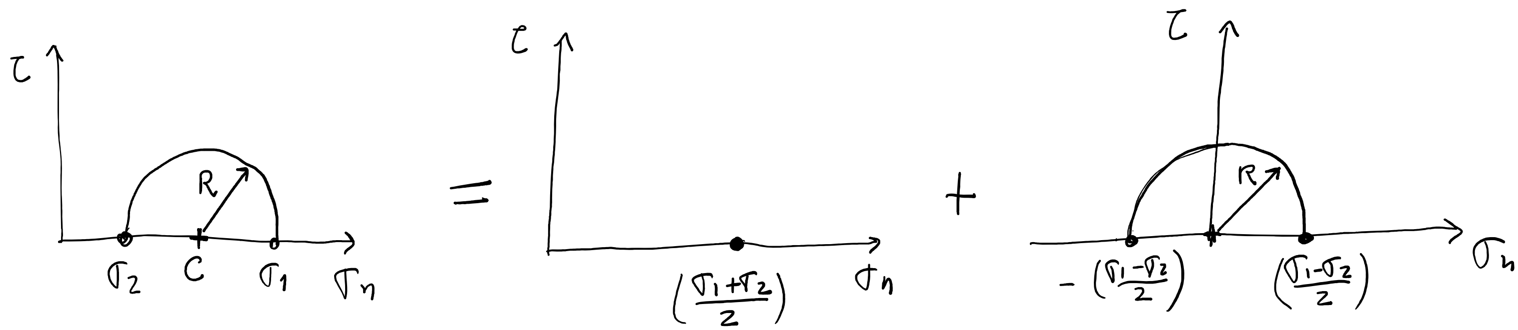
$$\underline{\underline{\sigma}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} - \begin{bmatrix} P_p & 0 & 0 \\ 0 & P_p & 0 \\ 0 & 0 & P_p \end{bmatrix}$$

Stress Invariants and graphical representation



$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$





isotropic

deviatoric (pure shear)

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{isotropic}} + \underbrace{\begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{deviatoric}}$$

$$\underbrace{[\ \dots\ ]}_{\text{isotropic}} \qquad \underbrace{\begin{bmatrix} \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{deviatoric}}$$

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$$

$$\boxed{\underline{\underline{\sigma}} = \sigma_m \underline{\underline{I}} + \underline{\underline{S}}_d}$$

Invariants (do not change wrt coord system)

$$\Rightarrow I_1(\underline{\underline{\sigma}}) = \sigma_{11}+\sigma_{22}+\sigma_{33} = \sigma_1+\sigma_2+\sigma_3 \qquad \Rightarrow \sigma_m = \frac{I_1(\underline{\underline{\sigma}})}{3}$$

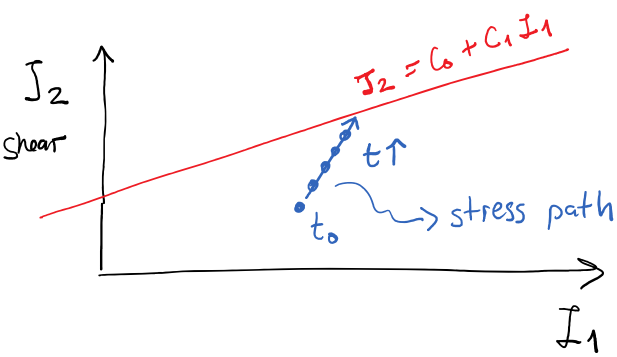
$$I_2(\underline{\underline{\sigma}}) = \sigma_{11}\sigma_{22} + \sigma_{11}\cdot\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$I_3(\underline{\underline{\sigma}}) = \det(\underline{\underline{\sigma}}) = \sigma_1\cdot\sigma_2\cdot\sigma_3 \leftarrow$$

$$J_1(\underline{\underline{S}}_d) = 0$$

$$\Rightarrow J_2(\underline{\underline{S}}_d) = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]$$

$$J_3(\underline{\underline{S}}_d) = (\sigma_1-\sigma_m)\cdot(\sigma_2-\sigma_m)\cdot(\sigma_3-\sigma_m)$$



(compression)

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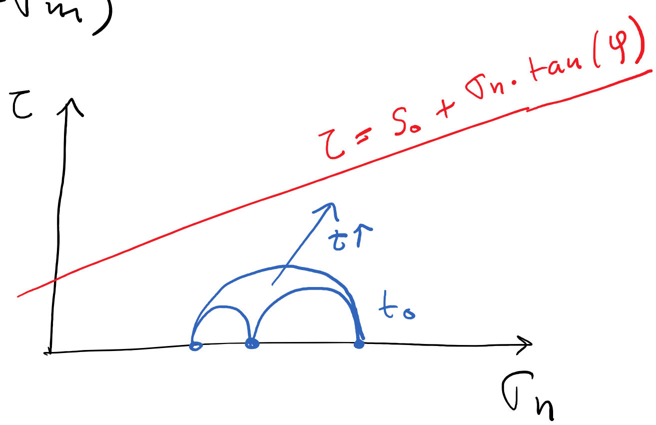
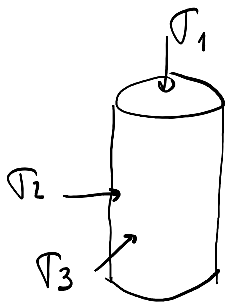


soft sediments (soil mechanics)

$$\begin{cases} p' = \sigma_m^{\text{effective}} = I_1(\underline{\underline{\sigma}})/3 \\ q = \sqrt{3 J_2} \end{cases}$$

σ_1

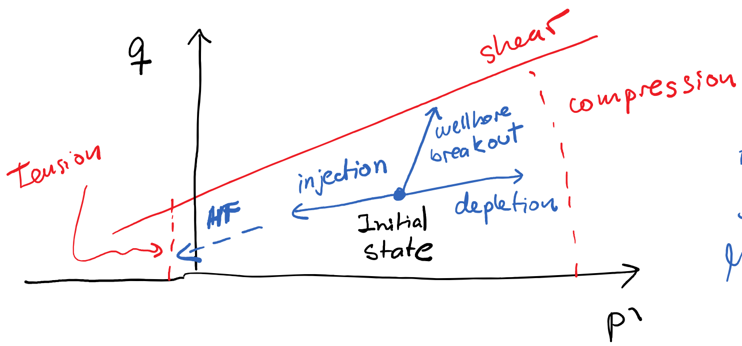


$\sigma_2 = \sigma_3$

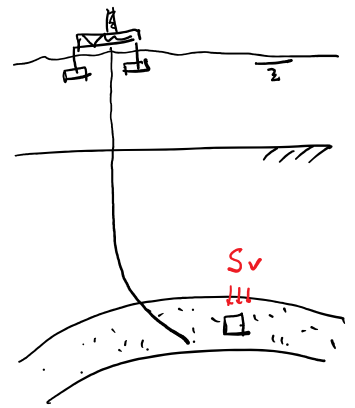
$\Rightarrow$ Axisymmetric stress conditions

$$p' = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_1 + 2\sigma_3}{3}$$

$$q = \sqrt{\cancel{3}\left\{\frac{1}{\cancel{6}_{\cancel{3}}}\left[(\sigma_1-\sigma_3)^2 + \cancel{(\sigma_1-\sigma_3)^2}\right]\right\}} = \sigma_1 - \sigma_3$$



stress path
$\Downarrow$
track state of stress as a function of time

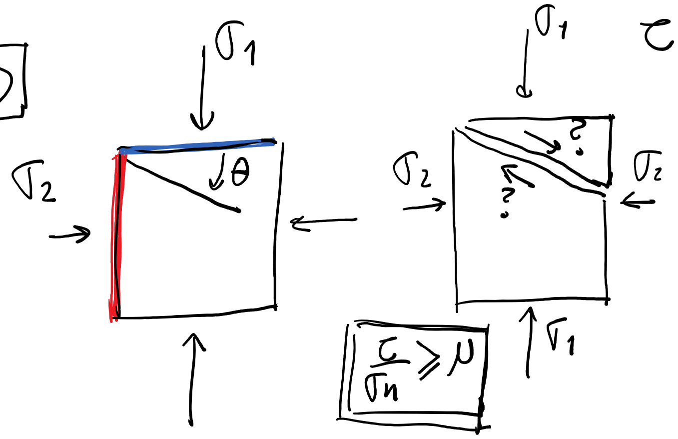


Stress projection on a plane

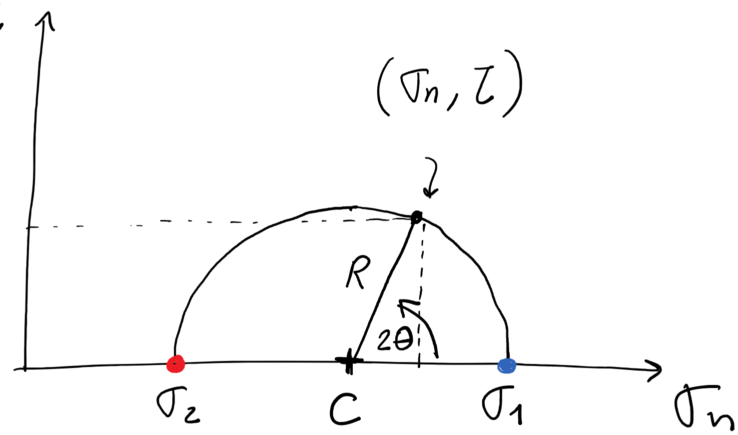
Monday, August 22, 2022 12:17 PM

Stress projection on a plane

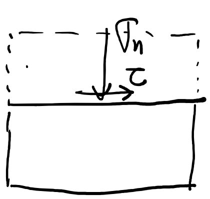
2D



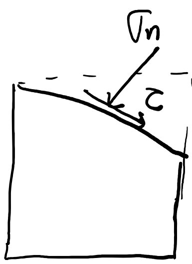
$$\boxed{\frac{\tau}{\sigma_n} \geq \mu}$$



$\theta = 0$

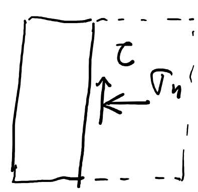


$\underline{\sigma_n = \sigma_1}$; $\tau = 0$



σ_n, τ

$\theta = 90°$



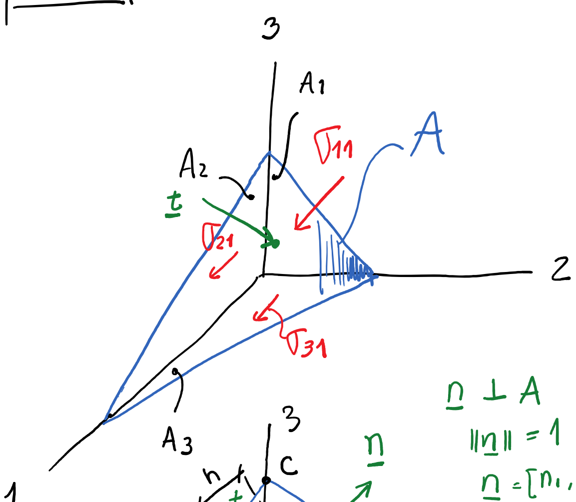
$\sigma_n = \sigma_2$; $\tau = 0$

$$\begin{cases} \sigma_n = C + R \cdot \cos 2\theta \\ \tau = R \cdot \sin 2\theta \end{cases}$$

$$C = (\sigma_1 + \sigma_2)/2$$

$$R = (\sigma_1 - \sigma_2)/2$$

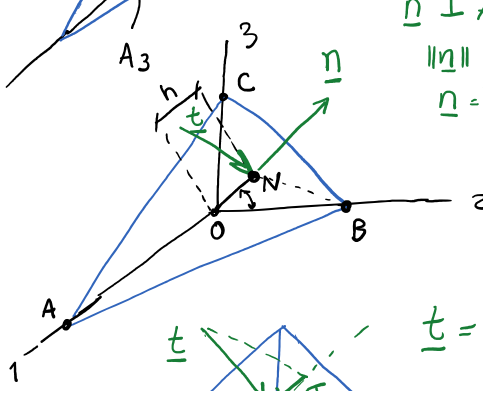
3D



$\underline{n} \perp A$

$\|\underline{n}\| = 1$

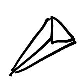
$\underline{n} = [n_1, n_2, n_3]$



$\underline{t} = \underline{\sigma_n} + \underline{\tau}$

$$\Sigma F_1 = 0$$

① $\sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3 = t_1 A$

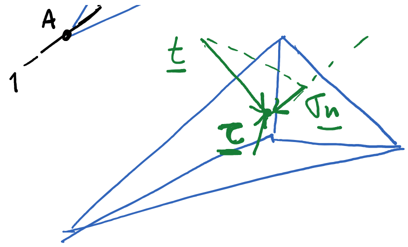
② Vol  $= \frac{1}{3} A \cdot h$

$$= \underbrace{\frac{1}{3} A_1 \cdot \overline{OA}}$$

$$\cancel{\tfrac{1}{3}} A h = \cancel{\tfrac{1}{3}} A_1 \overline{OA}$$

$$A_1 = \frac{h}{\overline{OA}} A$$

$$\begin{cases} A_1 = \cos A\hat{O}N \cdot A = n_1 A \\ A_2 = \cos B\hat{O}N \cdot A = n_2 A \\ A_3 = \cos C\hat{O}N \cdot A = n_3 A \end{cases}$$



$\underline{t} = \underline{\sigma_n} + \underline{\tau}$

$$\begin{cases} A_2 = \cos B\hat{O}N \cdot A = n_2 A \\ A_3 = \underbrace{\cos C\hat{O}N}_{\text{cosine directors}} \cdot A = n_3 A \end{cases}$$

$$\boxed{A_i = n_i A}$$

→ ① + ② $\quad \sigma_{11} n_1 \cancel{A} + \underbrace{\sigma_{21}}_{=\sigma_{12}} n_2 \cancel{A} + \underbrace{\sigma_{31}}_{=\sigma_{13}} n_3 \cancel{A} = t_1 \cancel{A}$

$\Downarrow$ $\sigma_{21} = \sigma_{12}$ (angular momentum equil.)

→
$$\begin{bmatrix} \underline{t_1} \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \underline{\sigma_{11}} & \underline{\sigma_{12}} & \underline{\sigma_{13}} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \underline{n_1} \\ \underline{n_2} \\ \underline{n_3} \end{bmatrix}$$

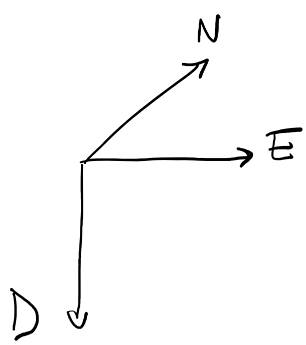
row × column

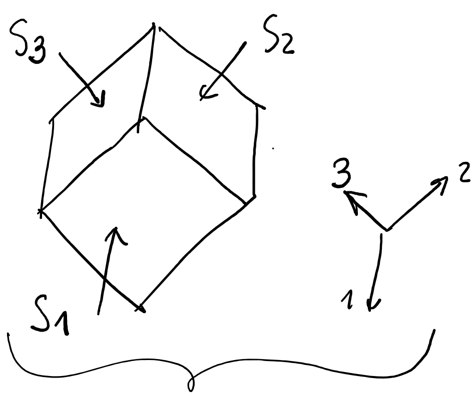
$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\begin{cases} \sigma_n = \underline{t} \cdot \underline{n} & \text{proyection of } \sigma_n \text{ on } \underline{n} \\ \tau = \sqrt{\|\underline{t}\|^2 - \|\sigma_n\|^2} & \Longleftarrow \quad \|\underline{t}\|^2 = \|\sigma_n\|^2 + \|\tau\|^2 \end{cases}$$

Geographical coordinate system

N - E - D





Principal stresses and direction

$$\underline{\underline{S}}_P = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

$$\underline{\underline{S}}_G = \begin{bmatrix} S_{NN} & S_{NE} & S_{ND} \\ S_{EN} & S_{EE} & S_{ED} \\ S_{DN} & S_{DE} & S_{DD} \end{bmatrix}$$

$$= \begin{bmatrix} \\ S_{DN} & S_{DE} & S_{DD} \end{bmatrix}$$

$$\boxed{\underline{\underline{S}}_G = R_{PG}^T \, \underline{\underline{S}}_P \, R_{PG}}$$

$$R_{PG} = \underbrace{\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}}_{f(\alpha, \beta, \gamma)}$$