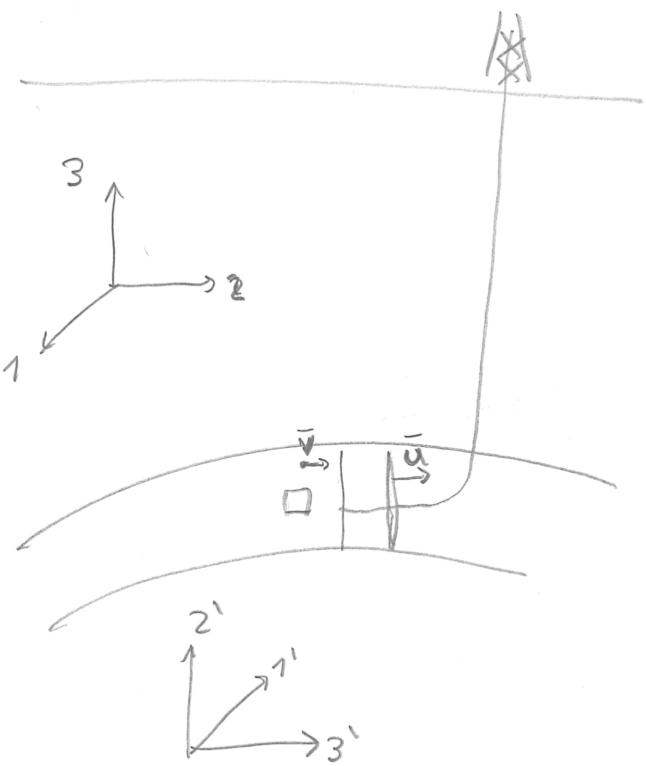
Introduction

$\boxed{2019/9/4}$



scalar = { T, Pp }

vector = { V

$\bar{V} = \{0, 0.1, 0\}$ ft/sec

$\bar{u} = \{0, 0.5, 0\}$ in

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

diagonal

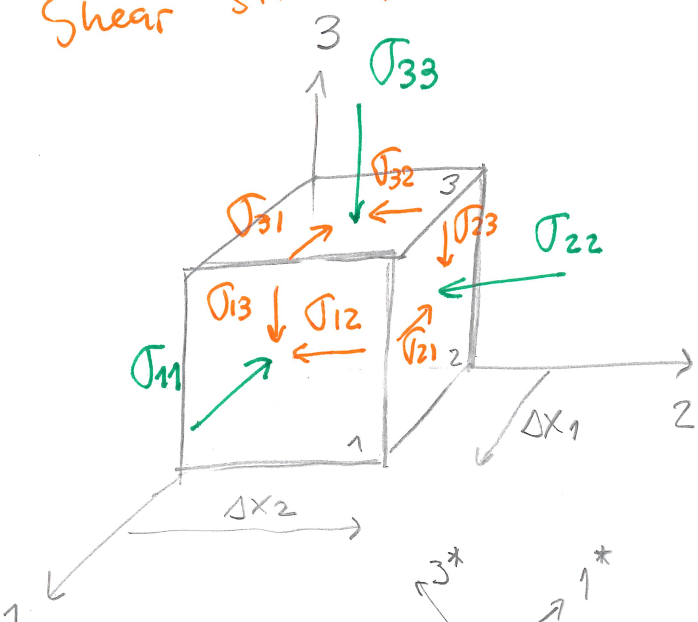
$$\sigma_{ij} \in \mathbb{R}$$

$$\sigma_{ij} = \sigma_{ji}$$

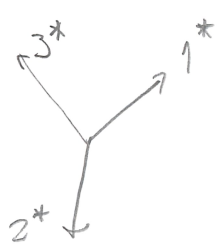
eigenvalues are real numbers

Normal stresses

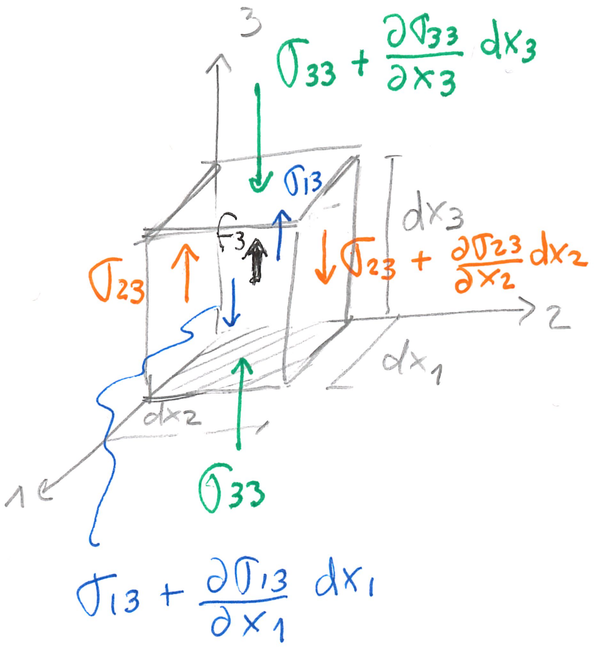
Shear stresses



$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



← eigen vectors



$$\Sigma F_3 = 0 = m \cancel{a}^{0}$$

$$\cancel{\sigma_{33}\, dx_1 dx_2} - \left(\cancel{\sigma_{33}} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1\, dx_2 +$$

$$\cancel{\sigma_{23}\, dx_1 dx_3} - \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_1\, dx_3 +$$

$$\cancel{\sigma_{13}\, dx_2 dx_3} - \left(\cancel{\sigma_{13}} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2\, dx_3 +$$

$$f_3 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{23}}{\partial x_2} dx_2 dx_1 dx_3 +$$

$$\frac{\partial \sigma_{33}}{\partial x_3} dx_3\, dx_1\, dx_2 \quad - f_3 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} - \frac{f_3}{Vol} = 0$$

$$Vol = dx_1 dx_2 dx_3$$

Cauchy's equilibrium equations

$$\left\{ \frac{\partial \sigma_{ij}}{\partial x_j} - b_i \rho = 0 \right.$$

$$\left\{ \underbrace{\nabla \cdot}_{\text{divergence}} \underline{\underline{\sigma}} - \underline{b} \rho = 0 \right.$$

$$\left\{ \begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} - b_1 \rho = 0 \\ &\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} - b_2 \rho = 0 \\ &\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} - \underbrace{b_3}_{\substack{\downarrow \\ g \\ \text{gravity}}} \cdot \rho = 0 \end{aligned} \right.$$

$$\sigma_{ij} = \sigma_{ji}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Convention

S: total stress

σ: effective stress

$$\underline{\sigma_1} > \sigma_2 > \sigma_3$$

$$\sigma_1 \perp \sigma_2 \perp \sigma_3$$

particular case $\downarrow$

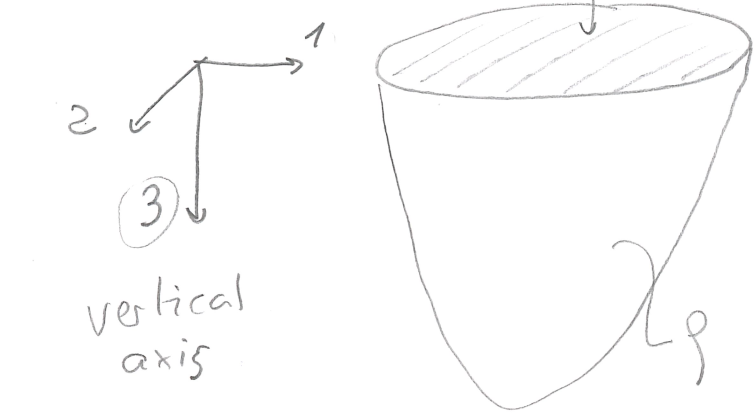
$$\sigma_V \perp \sigma_{Hmax} \perp \sigma_{hmin}$$

$$\underline{\underline{\sigma}} = \left.\begin{bmatrix} \sigma_V & 0 & 0 \\ 0 & \sigma_{Hmax} & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}\right\}$$ in coordinate system of principal stresses

$$\underline{\underline{\sigma}} = \underline{\underline{S}} - P_p \underline{\underline{I}} \qquad \underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{S}} = \begin{bmatrix} S_V & 0 & 0 \\ 0 & S_{Hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix} = \begin{bmatrix} \sigma_V + P_p & 0 & 0 \\ 0 & \sigma_{Hmax} + P_p & 0 \\ 0 & 0 & \sigma_{hmin} + P_p \end{bmatrix}$$

Free surface $\sigma_{31}=\sigma_{32}=0$ (no shear)
$\sigma_{33}=0$



$$\frac{\partial(\,)}{\partial x_1}=\frac{\partial(\,)}{\partial x_2}=0$$

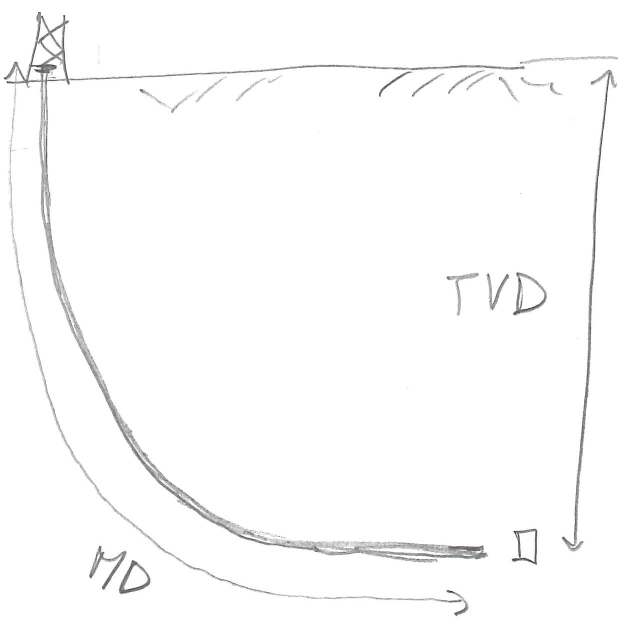
vertical stress gradient

$$\frac{\partial S_v}{\partial x_3}=\rho\cdot g$$

$$\frac{dS_v}{dx_3}\sim 1\frac{psi}{ft}\sim 23\frac{MPa}{km}$$

$$\int_0^{S_v(x_3)} dS_v=\int_0^{x_3}\rho(x_3)\,g\,dx_3$$

$$\boxed{S_v(x_3)=\int_0^{x_3}\rho(x_3)\,g\,dx_3}$$

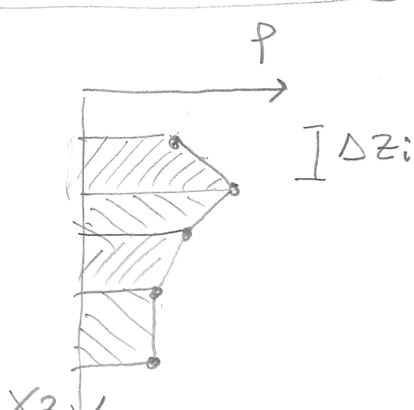


- $\rho(x_3)$: density log (MD) — gamma
- $g=9.8\ m/s^2$
- $x_3 \leftrightarrow \underbrace{TVD=f(MD)}_{\text{deviation survey}}$

$$\boxed{S_v(x_3)=\sum_{i=0}^{i(x_3)}\left[\frac{\rho_i+\rho_{i+1}}{2}\right]\cdot g\cdot\Delta z_i}$$

MD	E-offset	N-offset	TVD

TVDSS	ρ_{bulk}
1000	2.3
1001	2.4
1002	2.35
1003	2.3
1004	2.29



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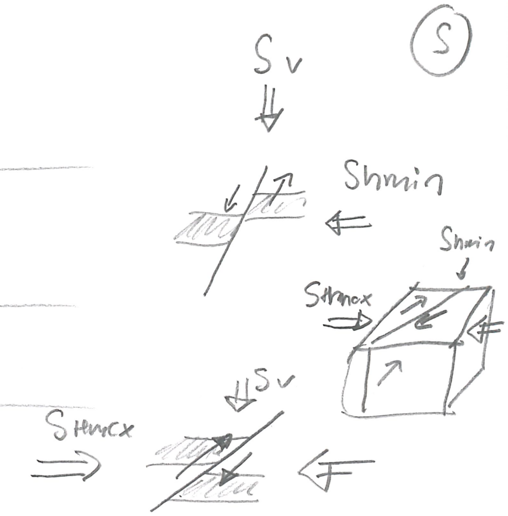
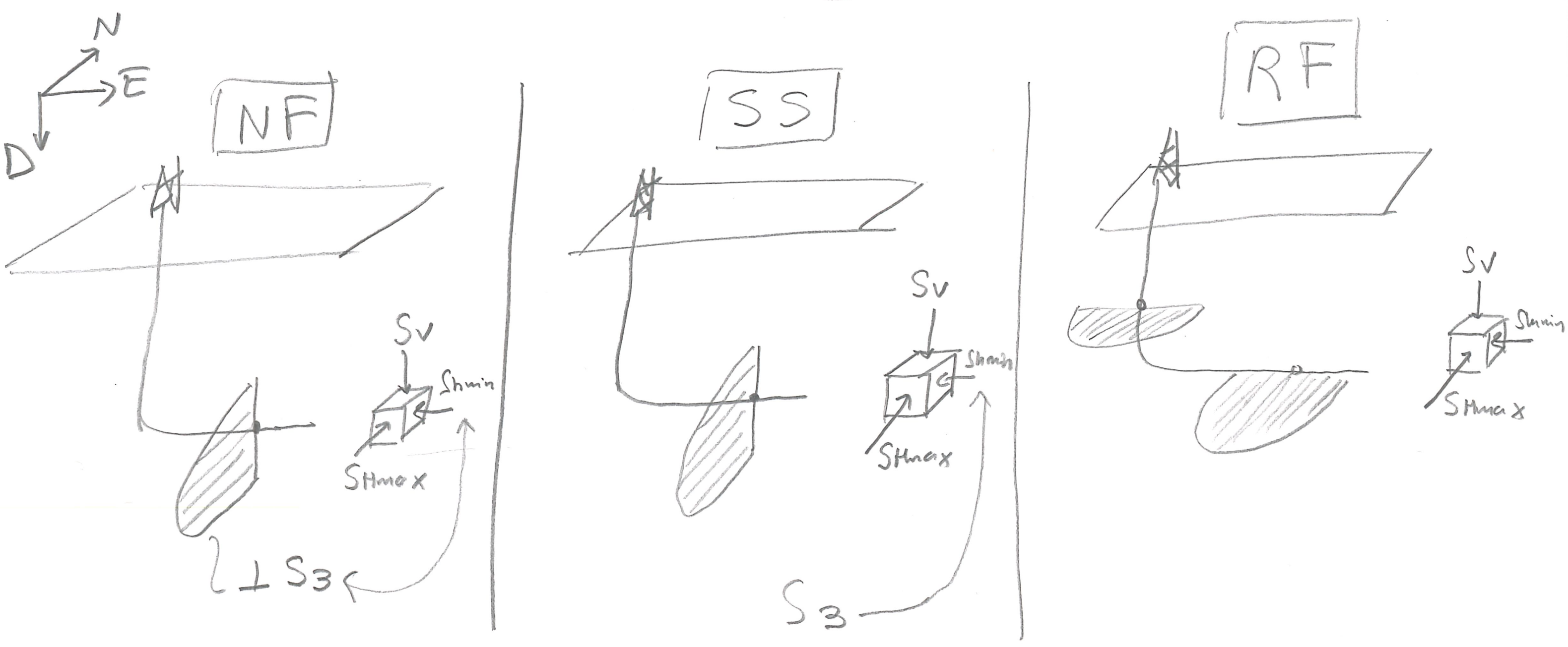
Stress regime	S_1 >	S_2 >	S_3
Normal	S_v	S_{Hmax}	S_{hmin}
Strike-slip	S_{Hmax}	S_v	S_{hmin}
Reverse	S_{Hmax}	S_{hmin}	S_v



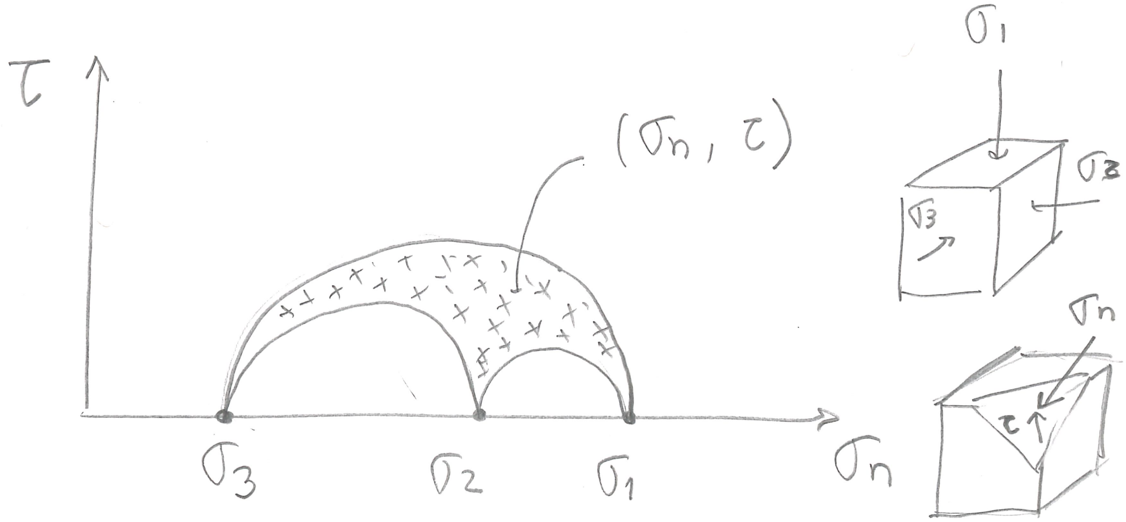
Fault analysis

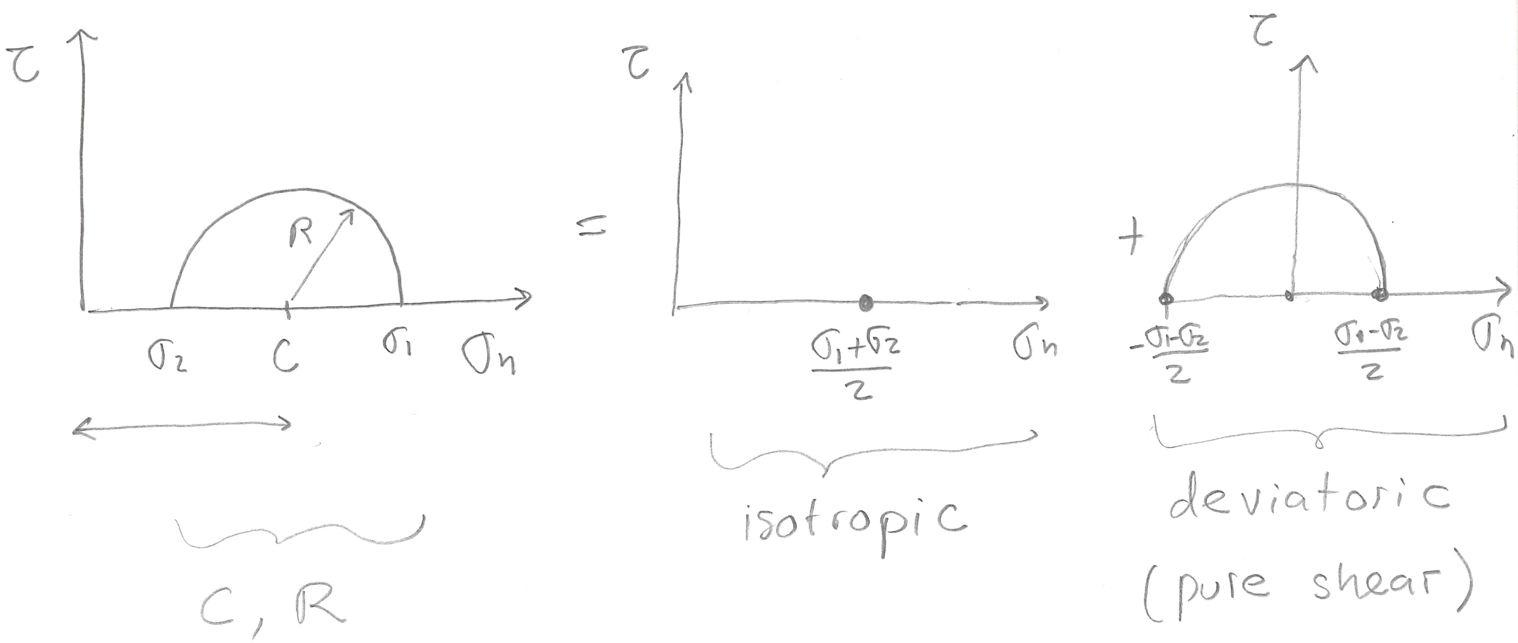
HF ⊥ S_3



3D Mohr Circle

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$





$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{isotropic}} + \underbrace{\begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{deviatoric}}$$

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$$

$$\underline{\underline{\sigma}} = \sigma_m \underline{\underline{I}} + \underline{\underline{S}}_d$$

Invariants (wrt coordinate system)

$$\Rightarrow I_1(\underline{\underline{\sigma}}) = \sigma_{11}+\sigma_{22}+\sigma_{33} = \sigma_1+\sigma_2+\sigma_3 \quad ; \sigma_m = I_1/3$$

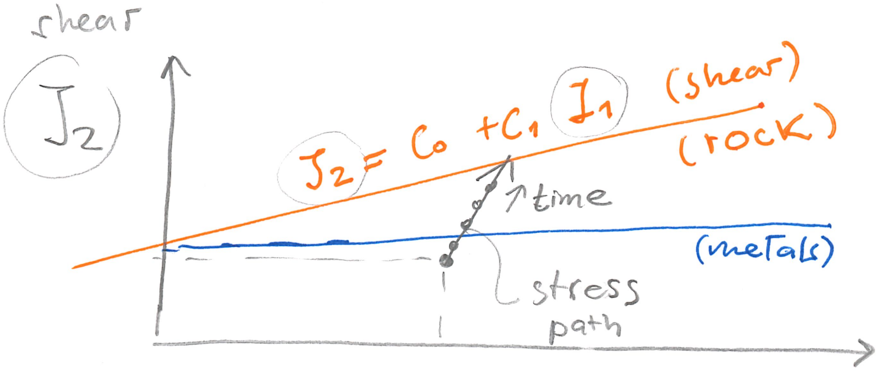
$$I_2(\underline{\underline{\sigma}}) = \sigma_{11}\sigma_{22}+\sigma_{11}\sigma_{33}+\sigma_{22}\sigma_{33}-\sigma_{12}^2-\sigma_{13}^2-\sigma_{23}^2$$

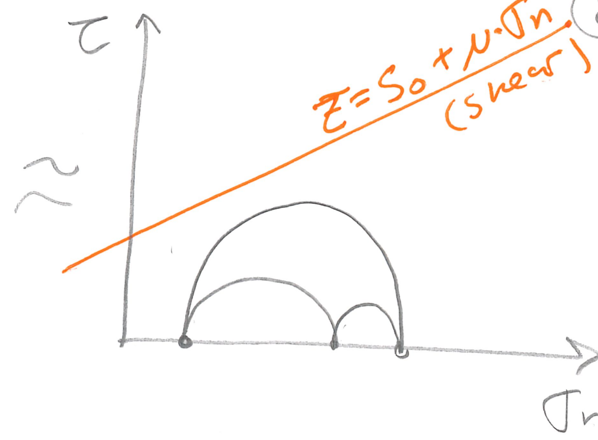
$$I_3(\underline{\underline{\sigma}}) = \det(\underline{\underline{\sigma}}) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$$

$$J_1(\underline{\underline{S}}_d) = \emptyset$$

$$\Rightarrow J_2(\underline{\underline{S}}_d) = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]$$

$$J_3(\underline{\underline{S}}_d) = (\sigma_1-\sigma_m)(\sigma_2-\sigma_m)(\sigma_3-\sigma_m)$$





$\approx$

I_1

↓

Compression

confinement

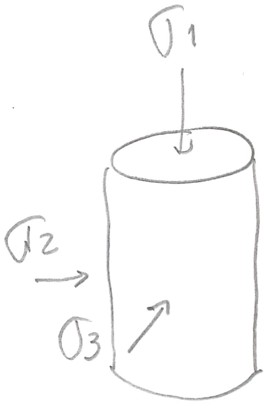
effective mean stress

soil mechanics

rock mechanics

$$p' = \sigma_m = I_1(\underline{\sigma})/3 \quad \Big\} \text{ mean eff stress}$$

$$q = \sqrt{3 J_2}$$



σ_1

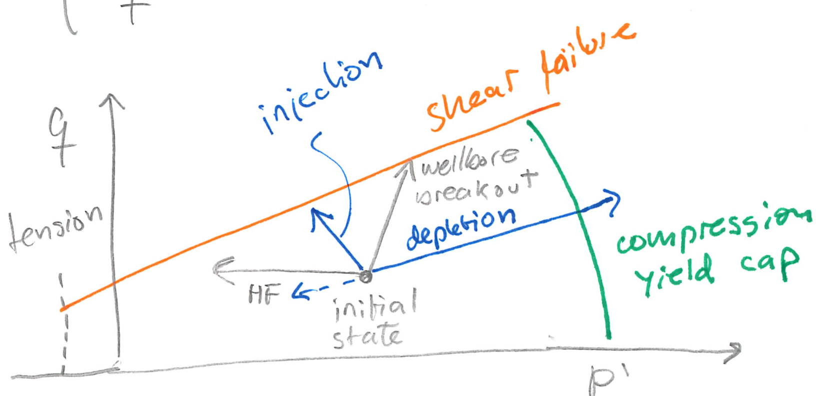
$\sigma_2 = \sigma_3$

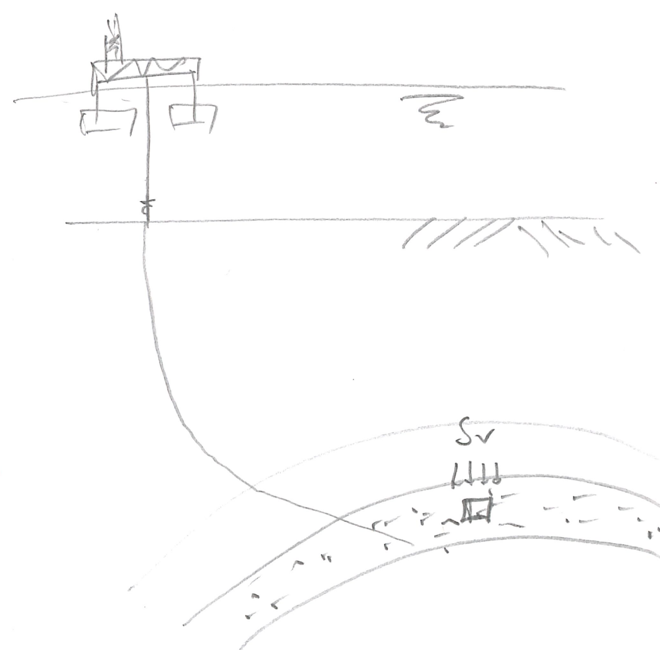
$$q = \sqrt{\cancel{3}\left\{\frac{1}{\cancel{6}_{2}}\left[(\sigma_1-\sigma_3)^2 + (\sigma_1-\sigma_2)^2\right]\right.}$$

$$q = \sigma_1 - \sigma_3 \quad \Big\} \text{ deviatoric stress}$$

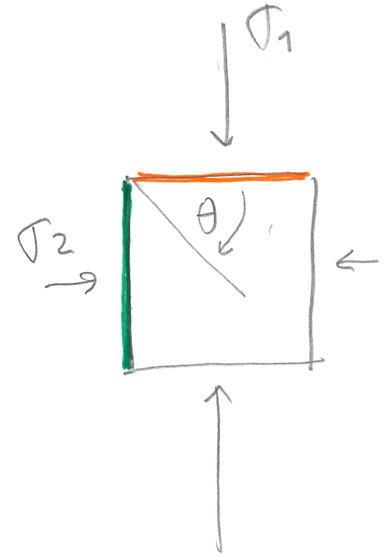
$$p' = \frac{\sigma_1 + 2\sigma_3}{3}$$

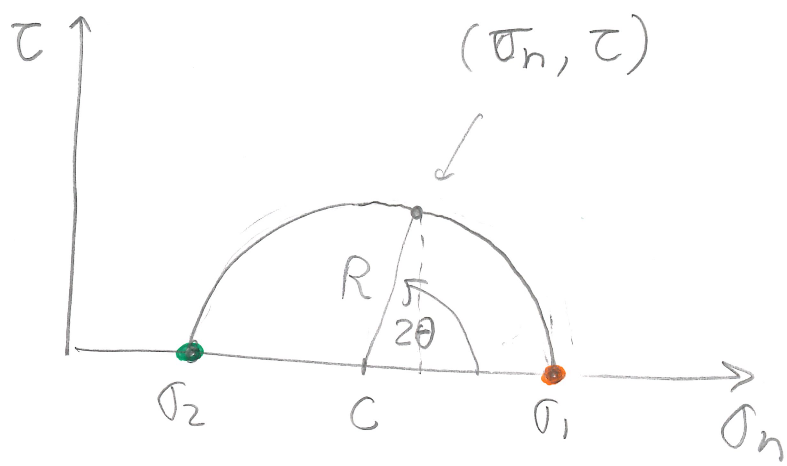
$$q = \sigma_1 - \sigma_3$$

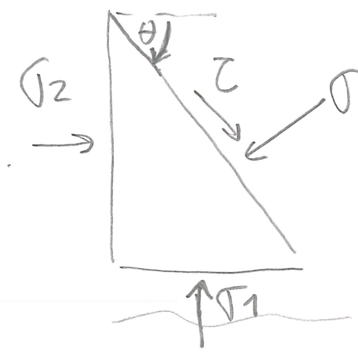




Stress projection on a plane







$$\begin{cases} \sigma_n = C + R \cdot \cos(2\theta) \\ \tau = R \cdot \sin(2\theta) \end{cases}$$

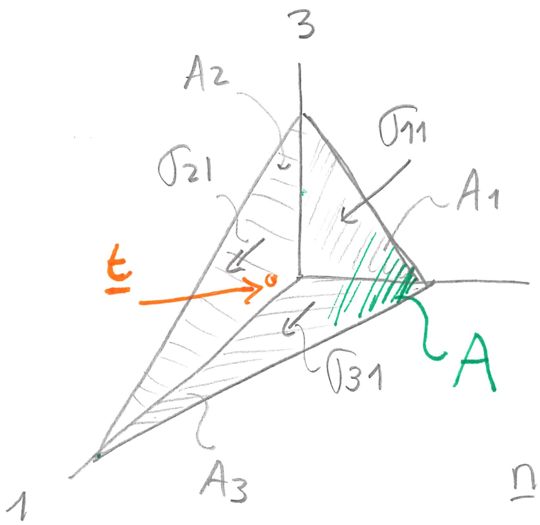
$$C = (\sigma_1 + \sigma_2)/2$$

$$R = (\sigma_1 - \sigma_2)/2$$

$$\frac{\tau}{\sigma_n} \gtrless \underset{\sim}{\mu}$$

friction coeff

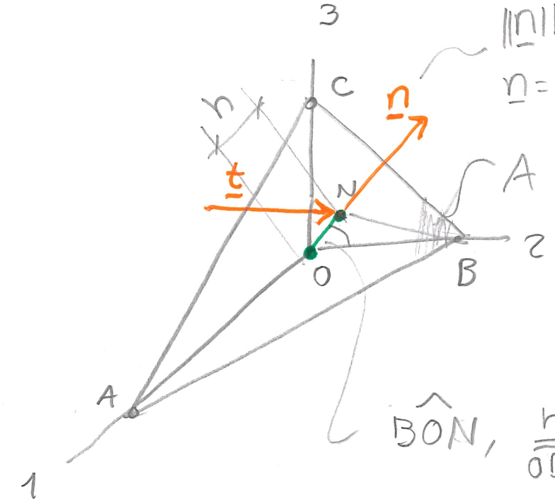
Stress projection on a plane (3D)



$\underline{n} \perp A$

$\|\underline{n}\| = 1$

$\underline{n} = (n_1, n_2, n_3)$



$B\hat{O}N$, $\frac{h}{\overline{OB}} = \cos B\hat{O}N$

$$\boxed{\Sigma F_1 = \emptyset}$$

$$① \left\{ \sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3 = t_1 A \right.$$

$$② \left| \begin{aligned} \text{Vol } \triangle &= \frac{1}{3} A \cdot h \\ &= \frac{1}{3} A_1 \cdot \overline{OA} \end{aligned} \right.$$

$$\cancel{\frac{1}{3}} A \cdot h = \cancel{\frac{1}{3}} A_1 \, \overline{OA}$$

$$A_1 = \frac{h}{\overline{OA}} \cdot A$$

$$A_1 = \cos A\hat{O}N \cdot A$$

$$A_2 = \cos B\hat{O}N \cdot A$$

$$A_3 = \underbrace{\cos C\hat{O}N}_{\substack{\text{cosine} \\ \text{directors}}} \cdot A$$

cosine directors } $\underline{n}$

$$\underline{A_i = n_i A}$$

$$\sigma_{11} n_1 \cancel{A} + \sigma_{21} n_2 \cancel{A} + \sigma_{31} n_3 \cancel{A} = t_1 \cancel{A}$$

$$\underline{t} = \overbrace{\underline{\underline{\sigma}}}^{\text{in-situ stress}} \cdot \overbrace{\underline{n}}^{\text{Fracture}}$$

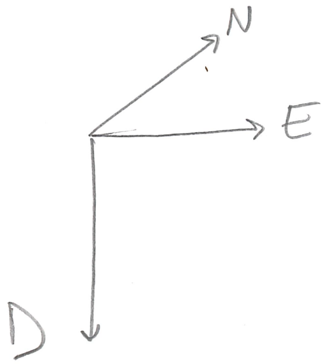
$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

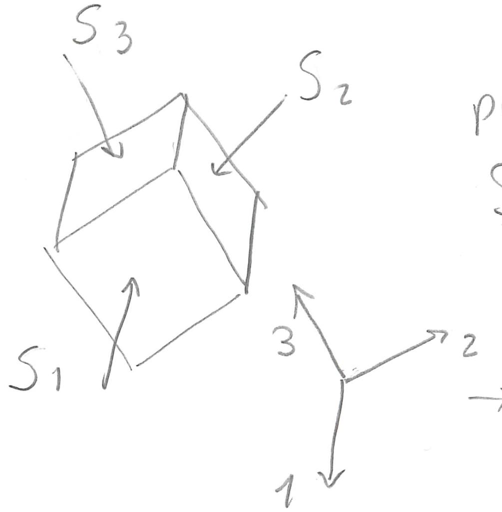
$$\left\{ \begin{aligned} \sigma_n &= \underline{t} \cdot \underline{n} \\ \tau^2 &= \underbrace{\|\underline{t}\|}^2 - \sigma_n^2 \end{aligned} \right.$$

2019/9/11

Geographical coordinate system

N - E - D





$\sigma_1, \sigma_2, \sigma_3$

principal stresses

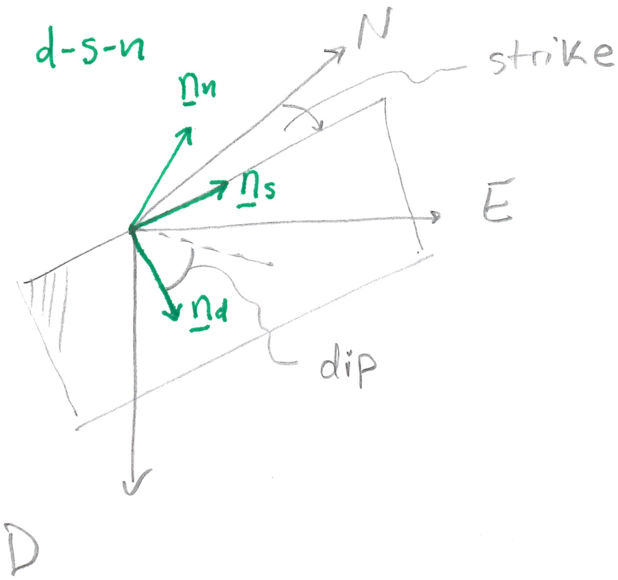
S_1, S_2, S_3

→ principal directions

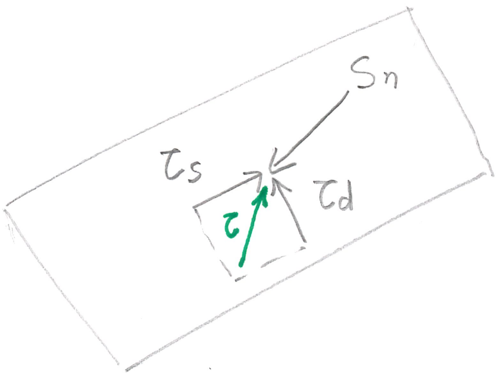
$$\underline{\underline{S}}_G = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix} \qquad \underline{\underline{S}}_P = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

$$\underbrace{\underline{\underline{S}}_G}_{\text{stress tensor in the geo coordinate system}} = R_{PG}^T \, \underline{\underline{S}}_P \, R_{PG} \sim f(\alpha, \beta, \gamma)$$

$$R_{PG} = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$



$$\underline{n}_d = \begin{bmatrix} - \\ - \\ - \end{bmatrix}, \quad \underline{n}_s = \begin{bmatrix} - \\ - \\ - \end{bmatrix}, \quad \underline{n}_n = \begin{bmatrix} - \\ - \\ - \end{bmatrix}$$



$$\underline{t} = \underline{\underline{S}}_G \cdot \underline{n}_n$$

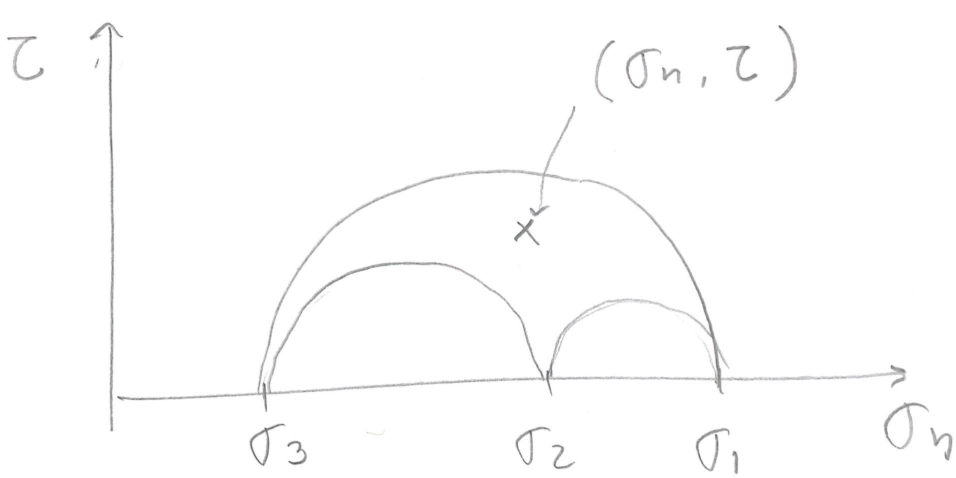
$$S_n = \underline{t} \cdot \underline{n}_n$$

$$\hookrightarrow \sigma_n = S_n - P_p$$

$$\tau_d = \underline{t} \cdot \underline{n}_d$$

$$\tau_s = \underline{t} \cdot \underline{n}_s$$

$$\hookrightarrow \tau^2 = \tau_d^2 + \tau_s^2$$



General solution to a continuum mechanics problem

$$\begin{cases} \nabla\cdot\underline{\underline{\sigma}} + \underline{f} = \overbrace{(\rho\, Vol)}^{m}\, a & \text{(Equilibrium)} \\ \underline{\underline{\varepsilon}} = F_1(\underline{u}) & \leftarrow \text{Kinematic Equations} \\ \underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}}) & \leftarrow \text{Constitutive Equations} \end{cases}$$

Kinematic Equations:
- small strains
- large strains

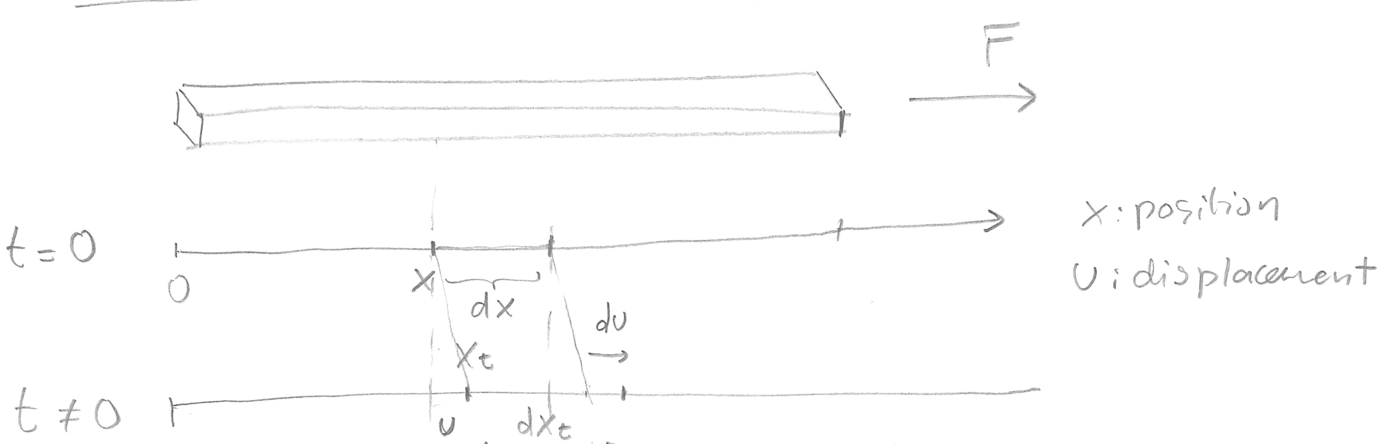
Constitutive Equations:
- → linear isotropic elasticity
- → orthorombic
- → TVI → VTI
- plasticity

$$\nabla\cdot\left[F_2(\underline{\underline{\varepsilon}})\right] + \underline{f} = (\rho\, Vol)\, a$$

$$\nabla\cdot\left[F_2(F_1(\underline{u}))\right] + \underline{f} = (\rho\, Vol)\cdot a$$

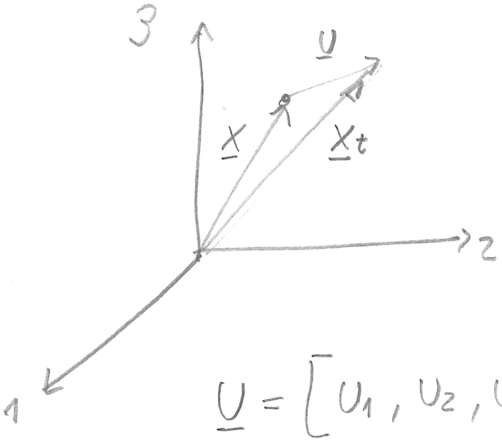
$\underline{u}$ — unknown

Kinematic equations (small strains)



$$\varepsilon = \frac{dx_t - dx}{dx} = \frac{[\cancel{x} + \cancel{u} + \cancel{dx} + du - (\cancel{x}+\cancel{u})] - [\cancel{x} + \cancel{dx} - \cancel{x}]}{[\cancel{x} + dx - \cancel{x}]}$$

$$\boxed{\varepsilon = \frac{du}{dx}}$$



$$\Rightarrow \underline{\underline{\varepsilon}} \Leftarrow \begin{bmatrix} \frac{dU_1}{dX_1} & \frac{dU_1}{dX_2} & \frac{dU_1}{dX_3} \\ \frac{dU_2}{dX_1} & \frac{dU_2}{dX_2} & \frac{dU_2}{dX_3} \\ \frac{dU_3}{dX_1} & \frac{dU_3}{dX_2} & \frac{dU_3}{dX_3} \end{bmatrix}$$

Jacobian

$$\underline{U} = [U_1, U_2, U_3]$$

$$\underline{X} = [X_1, X_2, X_3]$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{dU_1}{dX_1} & \frac{1}{2}\left(\frac{dU_1}{dX_2} + \frac{dU_2}{dX_1}\right) & \frac{1}{2}\left(\frac{dU_1}{dX_3} + \frac{dU_3}{dX_1}\right) \\ \frac{1}{2}\left(\frac{dU_2}{dX_1} + \frac{dU_1}{dX_2}\right) & \frac{dU_2}{dX_2} & \frac{1}{2}\left(\frac{dU_2}{dX_3} + \frac{dU_3}{dX_2}\right) \\ \frac{1}{2}\left(\frac{dU_3}{dX_1} + \frac{dU_1}{dX_3}\right) & \frac{1}{2}\left(\frac{dU_3}{dX_2} + \frac{dU_2}{dX_3}\right) & \frac{dU_3}{dX_2} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

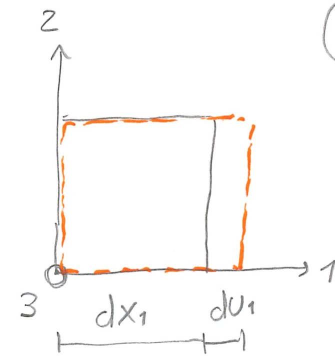
↳ symmetric matrix
↳ real components

→ eigenvalues real numbers
↓
$\varepsilon_1, \varepsilon_2, \varepsilon_3$

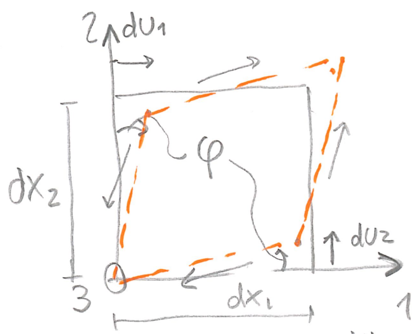
$$I_1(\underline{\underline{\varepsilon}}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \underline{\varepsilon_{vol}}$$

$$\varepsilon_{vol} = \frac{\Delta Vol}{Vol_0}$$

$$\begin{bmatrix} \frac{du_1}{dx_1} & \frac{du_1}{dx_2} & \frac{du_1}{dx_3} \\ \frac{du_2}{dx_1} & \frac{du_2}{dx_2} & \frac{du_2}{dx_3} \\ \frac{du_3}{dx_1} & \frac{du_3}{dx_2} & \frac{du_3}{dx_3} \end{bmatrix} = \begin{bmatrix} \frac{du_1}{dx_1} & 0 & 0 \\ 0 & \frac{du_2}{dx_2} & 0 \\ 0 & 0 & \frac{du_3}{dx_3} \end{bmatrix}$$



$$\varepsilon_{11} = \frac{du_1}{dx_1}$$



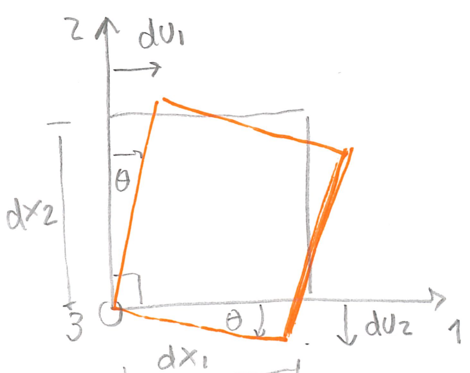
$$tg\,\varphi = \frac{du_2}{dx_1} = \frac{du_1}{dx_2}$$

$$\varepsilon_{12} \cong \frac{1}{2}(2\,tg\,\varphi) = \underbrace{\frac{1}{2}\left(\frac{du_1}{dx_2} + \frac{du_2}{dx_1}\right)}$$

$$+ \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{du_1}{dx_2} + \frac{du_2}{dx_1}\right) & \frac{1}{2}\left(\frac{du_1}{dx_3} + \frac{du_3}{dx_1}\right) \\ — & 0 & \frac{1}{2}\left(\frac{du_2}{dx_3} + \frac{du_3}{dx_2}\right) \\ — & — & 0 \end{bmatrix}$$

$\underline{\underline{\varepsilon}}$

strain tensor



$$\boxed{\frac{du_1}{dx_2} + \frac{du_2}{dx_1} = 0}$$

$$\omega_{12} = \frac{1}{2}\left(\frac{du_1}{dx_2} - \frac{du_2}{dx_1}\right)$$

$$+ \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{du_1}{dx_2} - \frac{du_2}{dx_1}\right) & \frac{1}{2}\left(\frac{du_1}{dx_3} - \frac{du_3}{dx_1}\right) \\ \frac{1}{2}\left(\frac{du_2}{dx_1} - \frac{du_1}{dx_2}\right) & 0 & \frac{1}{2}\left(\frac{du_2}{dx_3} - \frac{du_3}{dx_2}\right) \\ & & 0 \end{bmatrix}$$

$\underline{\underline{\omega}}$

rotation matrix

↓

skew symmetric

Constitutive equations

superposition principle $\left\langle \begin{array}{l} \text{space} \\ \text{time} \end{array} \right.$

$$\begin{cases} F_2(A+B) = F_2(A) + F_2(B) \\ F_2(c \cdot A) = c \cdot F_2(A) \end{cases}$$

$$\underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}})$$

→ linear relationship

$$\underline{\underline{\sigma}} = \underbrace{\underline{\underline{\underline{\underline{C}}}}}_{\text{constant}} \cdot \underline{\underline{\varepsilon}} \longrightarrow (y = b \cdot x)$$

	9	81 coeff	9
independent	6	36 coeff	6

Voigt Notation $\left(\overset{3\times3}{\underline{\underline{\sigma}}} \rightarrow \overset{6\times1}{\underline{\sigma}} \right)$

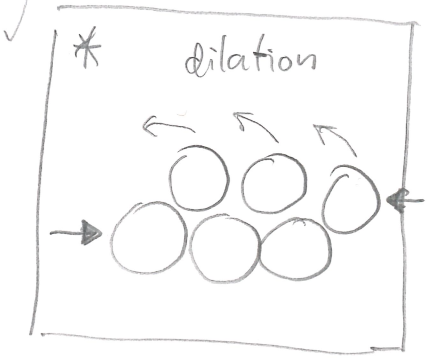
$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \longleftrightarrow \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{vmatrix}$$

$$\begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{vmatrix} = \underbrace{\left| \begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & C_{44} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & C_{55} & \cdot \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{array} \right|}_{\text{stiffness matrix}} \begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2 \cdot \varepsilon_{23} \\ 2 \cdot \varepsilon_{13} \\ 2 \cdot \varepsilon_{12} \end{vmatrix}$$

shear decoupling < normal vs shear ← exception *
shear vs shear

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

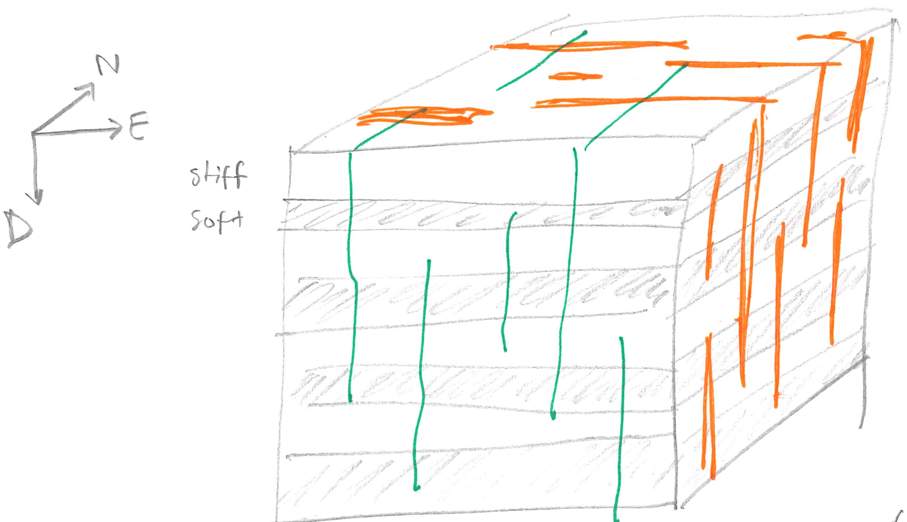
normal, linear strain



$18 \rightarrow 12$ coeff $\longrightarrow$ $\boxed{9 \text{ coeff}}$

symmetry
$C_{ij} = C_{ji}$
$i \neq j$

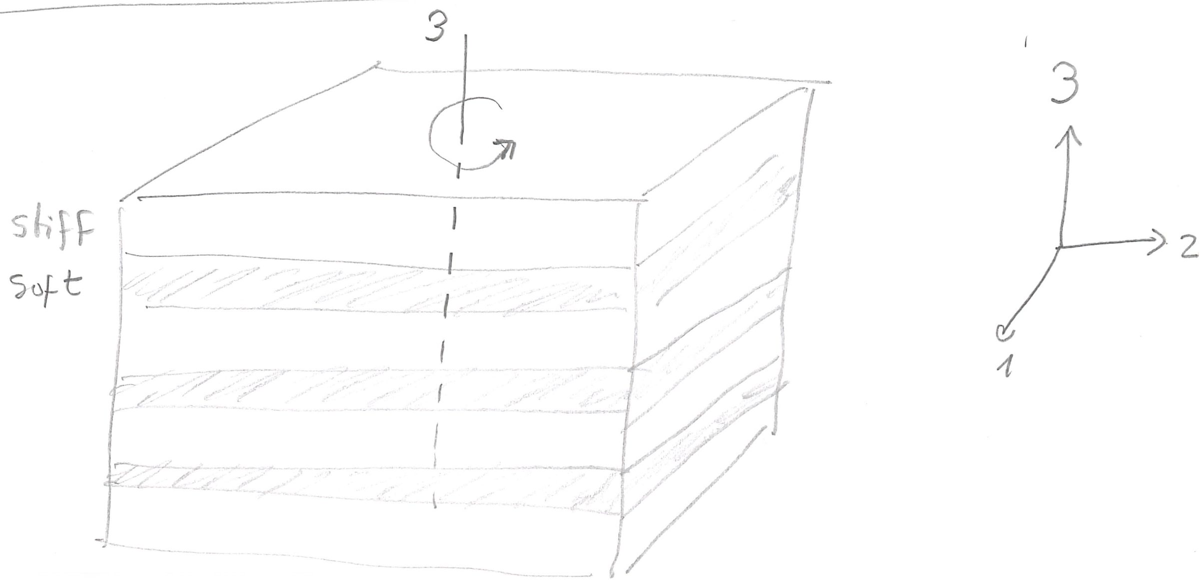
orthorombic



$E_V \neq E_{EW} \neq E_{NS}$

$(E_v < E_h)$ without fractures
horizontal

Vertical Transverse Isotropic (VTI)

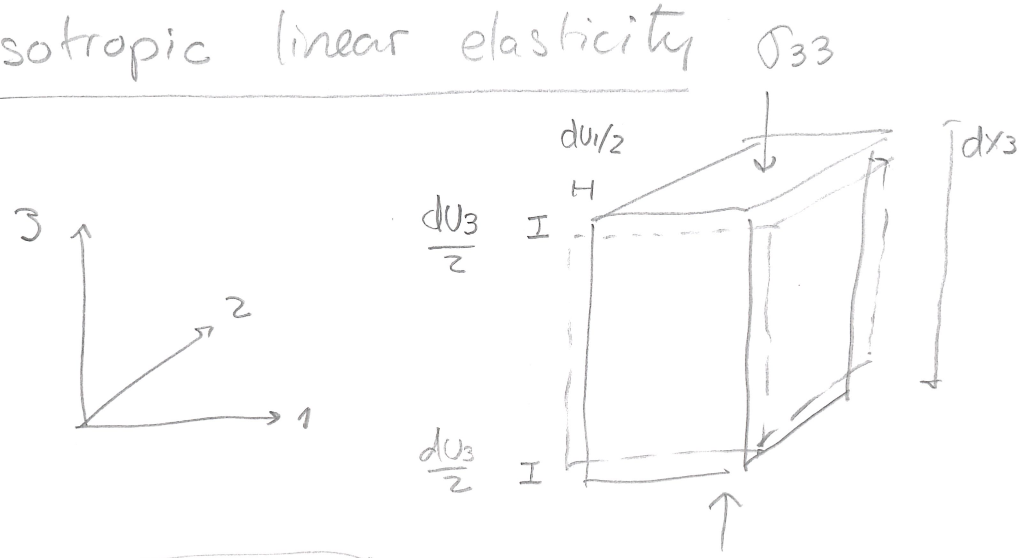


$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

$$C_{66} = \frac{C_{11} - C_{12}}{2}$$

$$\boxed{\text{5 independent coefficients}}$$

Isotropic linear elasticity



$$\underline{\sigma} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{33} \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \quad \underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{E = \frac{\sigma_{33}}{\varepsilon_{33}} \qquad \nu = -\frac{\varepsilon_{11}}{\varepsilon_{33}} = -\frac{\varepsilon_{22}}{\varepsilon_{33}}}$$

$$\varepsilon_{33} = \frac{dU_3}{dX_3}$$

$$\varepsilon_{11} = \frac{dU_1}{dX_1} \quad (= \varepsilon_{22})$$

$$\underline{\varepsilon} = \begin{pmatrix} -\nu \frac{\sigma_{33}}{E} \\ -\nu \sigma_{33}/E \\ \sigma_{33}/E \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- Loading direction 3
- No stress (or constant) stress in directions 1 and 2

$$\underline{\varepsilon} = \underbrace{\underline{\underline{D}}}_{\text{compliance}} \cdot \underline{\sigma}$$

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}$$

demo $\longrightarrow \boxed{G = \frac{E}{2(1+\nu)}}$

2 independent coefficients

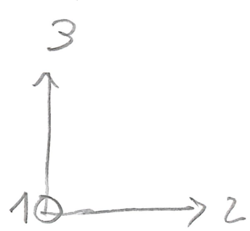
$$\underline{\sigma} = \underset{\equiv}{\underline{C}} \cdot \underline{\varepsilon} \qquad \text{e.g.} \rightarrow C_{11} = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\underset{\equiv}{\underline{C}} \downarrow \underset{\equiv}{\underline{D}}^{-1}$$

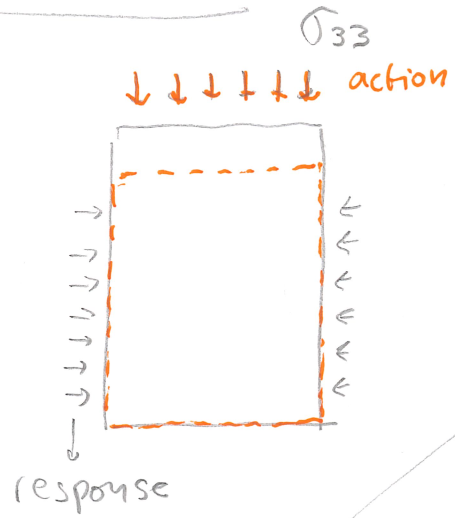
Stiffness matrix

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \underbrace{\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}_{\text{stiffness matrix}} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

Uniaxial strain loading



$$\varepsilon_{22} = \varepsilon_{11} = 0$$



$$\sigma_{33} = \underbrace{\frac{(1-\nu)\,E}{(1+\nu)(1-2\nu)}}_{M} \varepsilon_{33}$$

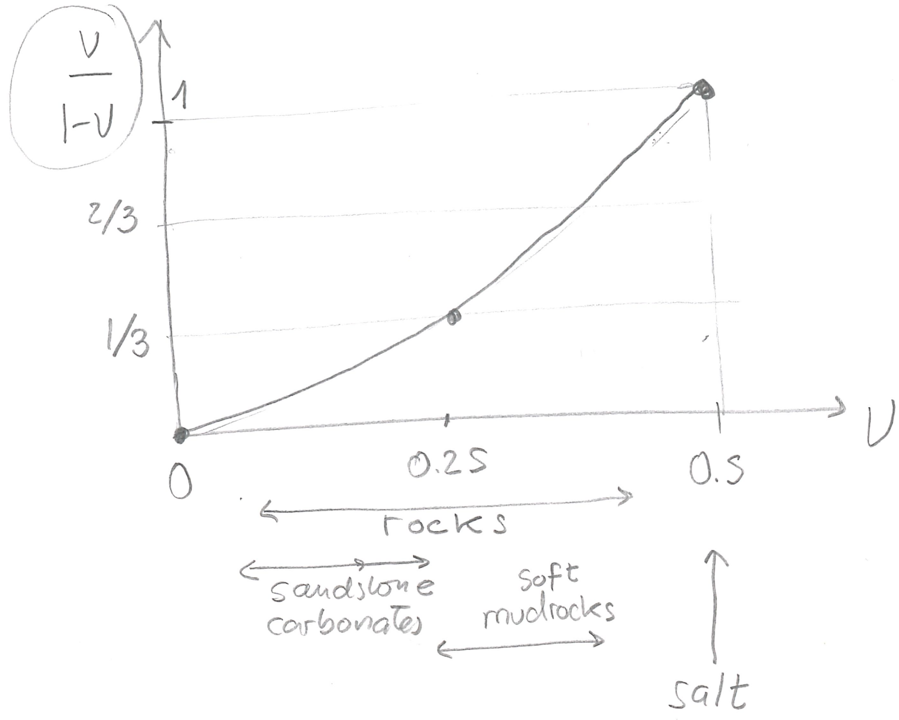
M: constrained modulus
: P-wave mod.
: oedometric mod

$$M \geq E$$

$$\underline{\sigma_{11}} = \frac{\nu \cancel{E}}{\cancel{(1+\nu)}\cancel{(1-2\nu)}} \cdot \frac{\cancel{(1+\nu)}\cancel{(1-2\nu)}}{(1-\nu)\cancel{E}} \sigma_{33} = \frac{\nu}{1-\nu} \sigma_{33}$$

$$\sigma_{22} = \frac{\nu}{1-\nu} \sigma_{33}$$

effective lateral stress coefficient

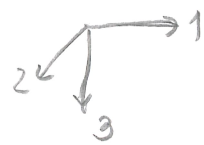


General solution 1D mech earth model (MEM)

$\varepsilon_{11} \neq \varepsilon_{22} \neq \emptyset$, $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$, $\sigma_{33} \Rightarrow$ vertical principal stress

Isotropic media

$$\begin{cases} \sigma_{11} = \dfrac{E}{1-\nu^2}\varepsilon_{11} + \dfrac{\nu E}{1-\nu^2}\varepsilon_{22} + \dfrac{\nu}{1-\nu}\sigma_{33} \\ \sigma_{22} = \dfrac{\nu E}{1-\nu^2}\varepsilon_{11} + \dfrac{E}{1-\nu^2}\varepsilon_{22} + \dfrac{\nu}{1-\nu}\sigma_{33} \\ \sigma_{33} = S_{33} - P_p \end{cases}$$

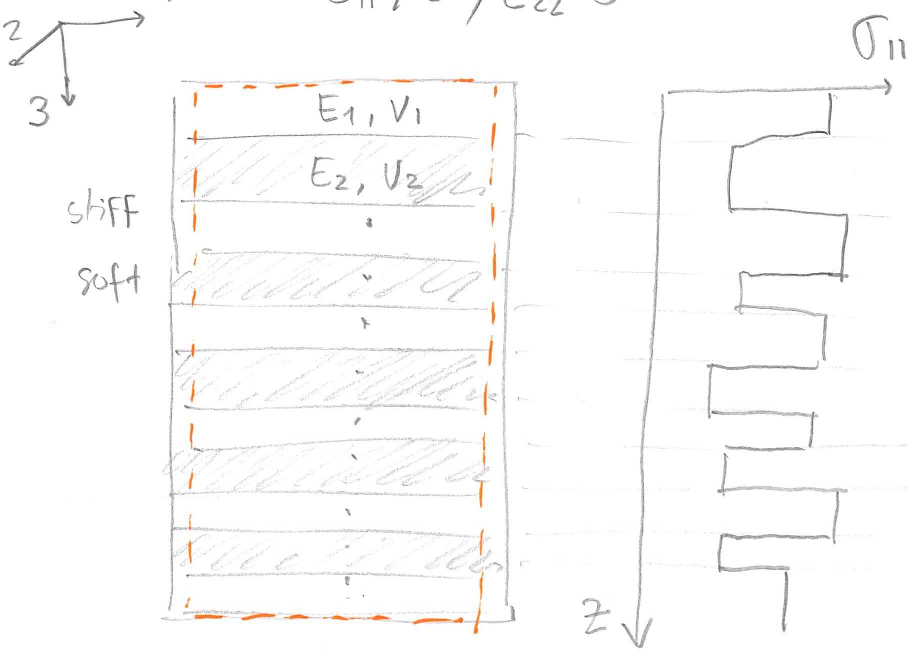


need static values → $\boxed{E' = \dfrac{E}{1-\nu^2}}$ greek letter "nu" ν

Plane strain modulus

ε_{11} , ε_{22} → tectonic strains

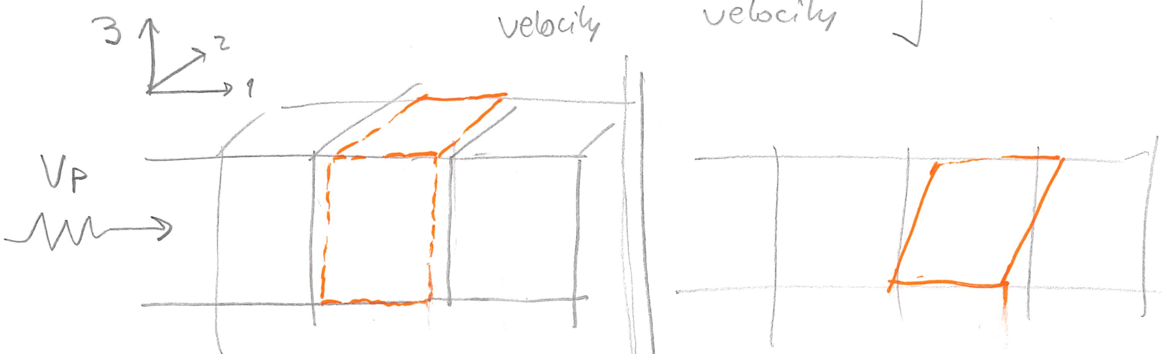
$\varepsilon_{11} \neq 0 \; ; \; \varepsilon_{22} = 0$



$(E, \nu) \leftarrow \underline{V_p}, \underline{V_s}$ Elastic waves

P-wave velocity

Shear-wave velocity

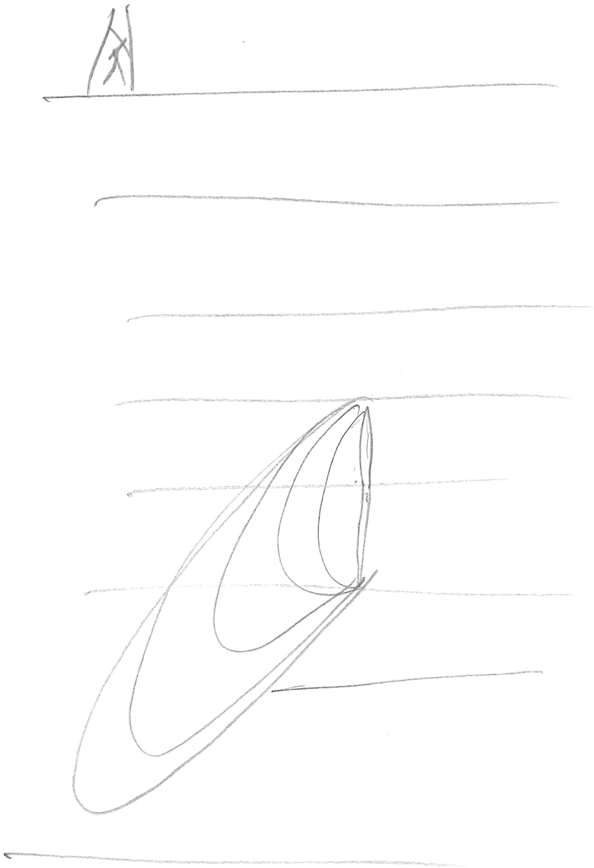


$$V_p = \sqrt{\frac{M}{\rho_{bulk}}} \qquad V_s = \sqrt{\frac{G}{\rho_{bulk}}}$$

conversion dynamic to static

$$E_{dyn} = \rho_{bulk} V_s^2 \left(\frac{3V_p^2 - 4V_s^2}{V_p^2 - V_s^2} \right) \rightarrow E_{st} = \underbrace{F_{sd}}_{\leq 1} E_{dyn}$$

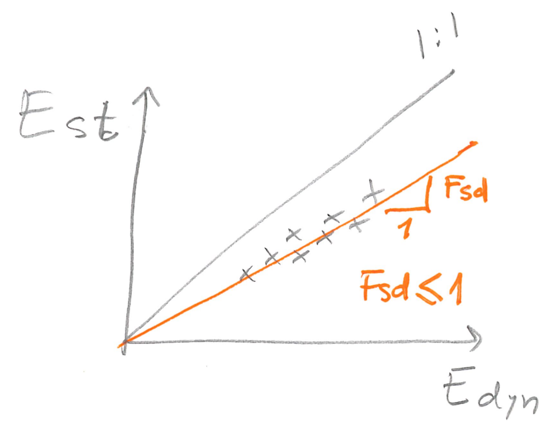
$$\nu_{dyn} = \frac{V_p^2 - 2V_s^2}{2(V_p^2 - V_s^2)} \qquad \nu_{st} \cong \nu_{dyn}$$



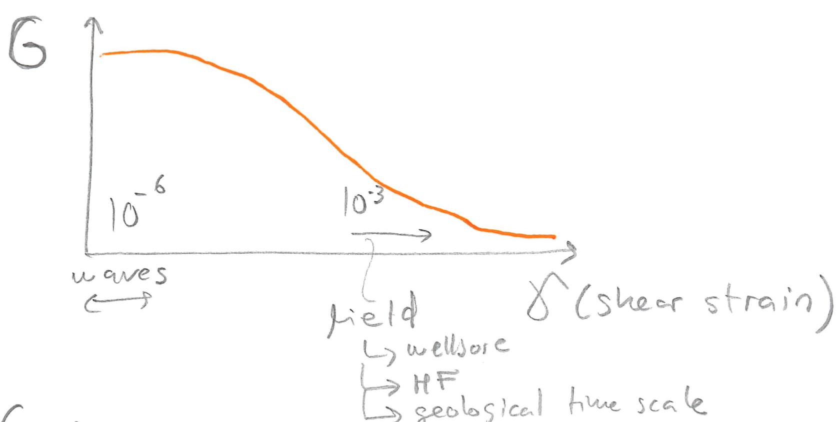
$E_{st} = F_{sd} \, E_{dyn}$ $\Rightarrow$

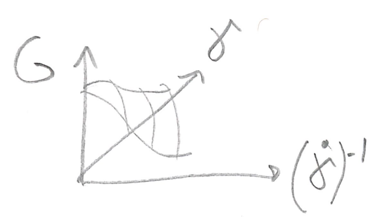
↓ triaxial test ↓ field

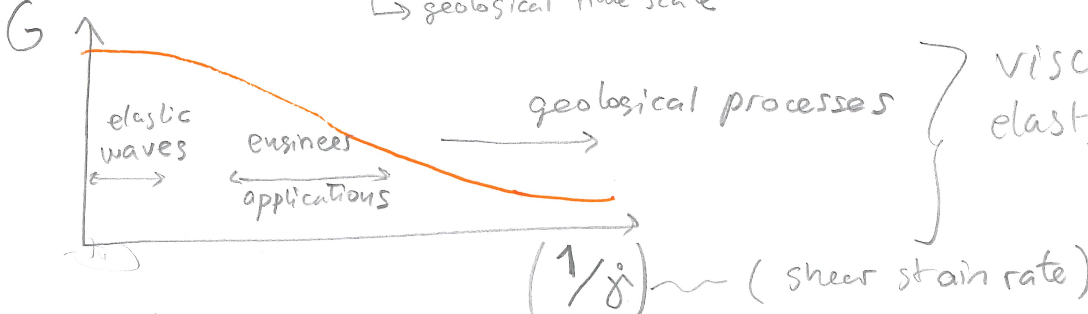
↓ wave propagation



shear modulus





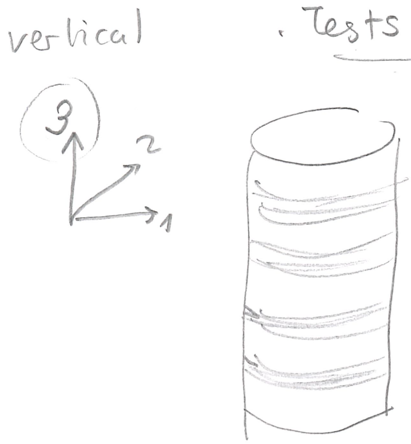


visco-elasticity

Measuring C_{ij} for TVI media

VTI

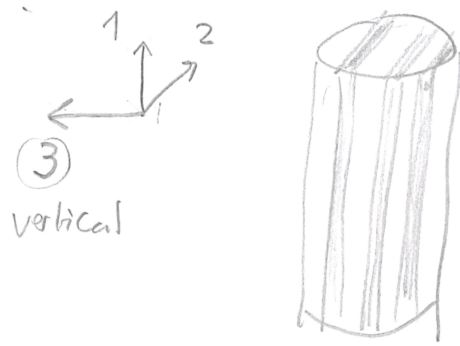
. Tests with contant confining pressure (or unconfined)



$$E_v = \frac{\Delta \sigma_{33}}{\Delta \varepsilon_{33}}\Big|_{\sigma_{11},\sigma_{22}}$$

$$\nu_{31} = -\frac{\Delta \varepsilon_{11}}{\Delta \varepsilon_{33}}\Big|_{\sigma_{11},\sigma_{22}}$$

$$\nu_{31} = \nu_{32}$$



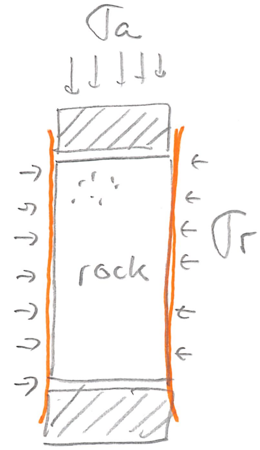
$$E_h = \frac{\Delta \sigma_{11}}{\Delta \varepsilon_{11}}\Big|_{\sigma_{22},\sigma_{33}}$$

$$\nu_{12} = -\frac{\Delta \varepsilon_{22}}{\Delta \varepsilon_{11}}\Big|_{\sigma_{22},\sigma_{33}}$$

usually done in a triaxial cell (axisymmetric)

C_{ij}^{sta}

$\boxed{\text{Static}}$



$\sigma_a \geq \sigma_r$

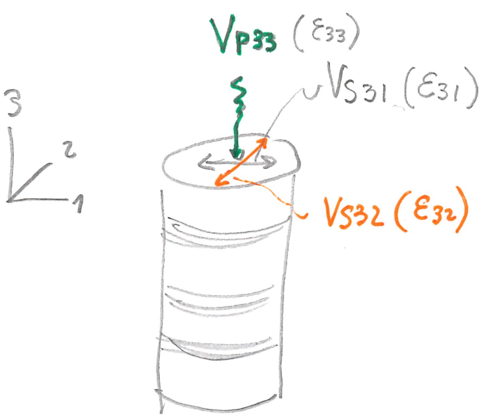
deviatoric tests

↳ $\Delta \sigma_a$

↳ $\sigma_r = cst$

Dynamic

vertical

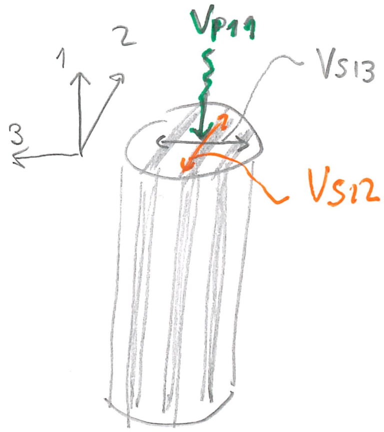


$$C_{33} = \rho (V_{p33})^2$$

$$C_{44} = \rho (V_{s31})^2$$

$$C_{44} = \rho (V_{s32})^2$$

horizontal

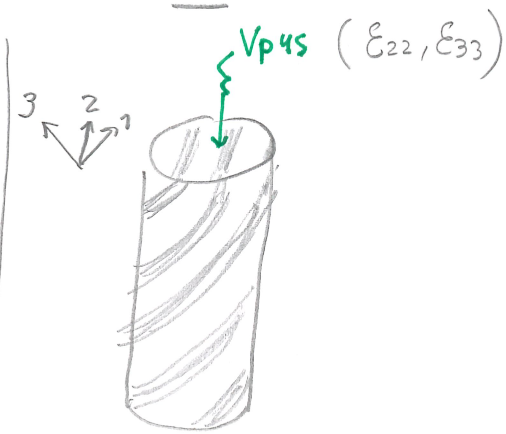


$$C_{11} = \rho (V_{p11})^2$$

$$C_{66} = \rho (V_{s12})^2$$

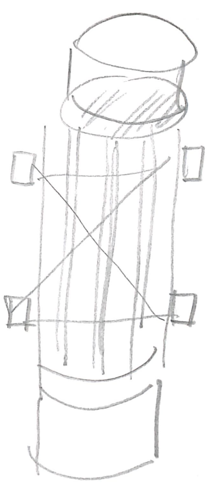
$$C_{12} = C_{11} - 2\,C_{66}$$

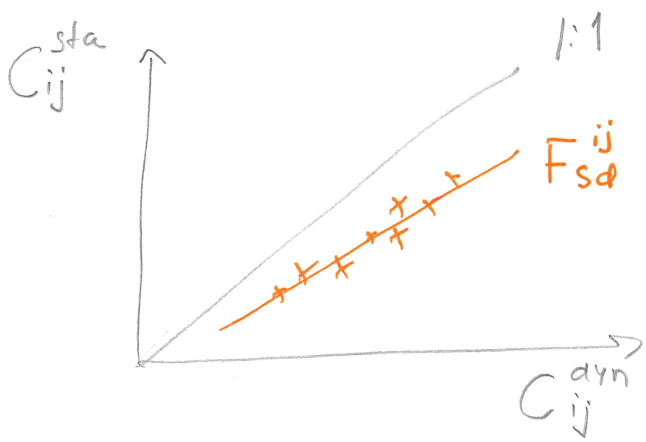
45°



$$\left.C_{13} = f(V_{p45})\right\} C_{ij}^{dyn}$$

Dynamic



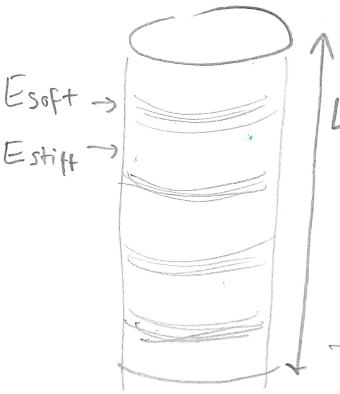


Iso-stress

Reuss Average

Vertical Young Modulus

$\boxed{E_v}$



$$\langle E_v \rangle = \frac{\sigma_v}{\langle \varepsilon_v \rangle}$$

$$\langle \varepsilon_v \rangle = \frac{\sigma_v}{\langle E_v \rangle}$$

$$\langle \varepsilon_v \rangle = \left(\frac{\sigma_v}{E_{stiff}} \cdot L_{stiff} + \frac{\sigma_v}{E_{soft}} \cdot L_{soft} \right) \cdot \frac{1}{L_T}$$

$$\langle \varepsilon_v \rangle = \frac{\sigma_v}{E_{stiff}} \left(\frac{L_{stiff}}{L_T} \right) + \frac{\sigma_v}{E_{soft}} \cdot \left(\frac{L_{soft}}{L_T} \right)$$

$$\frac{\cancel{\sigma_v}}{\langle E_v \rangle} = \frac{\cancel{\sigma_v}}{E_{stiff}} \frac{L_{stiff}}{L_T} + \frac{\cancel{\sigma_v}}{E_{soft}} \frac{L_{soft}}{L_T}$$

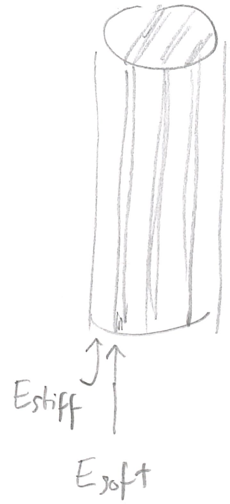
$$\boxed{\langle E_v \rangle = \left(\frac{1}{E_{stiff}} \cdot \frac{L_{stiff}}{L_T} + \frac{1}{E_{soft}} \frac{L_{soft}}{L_T} \right)^{-1}}$$

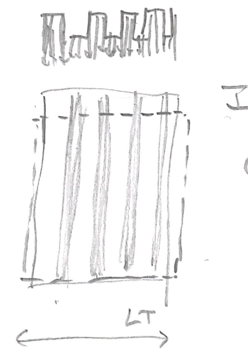
Iso-strain

Voigt Average

Horizontal Young Modulus

$\boxed{E_h}$





$$\varepsilon_h = \frac{\langle \sigma_h \rangle}{\langle E_h \rangle}$$

$$\langle \sigma_h \rangle = \sigma_h^{soft} \cdot \frac{L_{soft}}{L_T} + \sigma_h^{stiff} \cdot \frac{L_{stiff}}{L_T}$$

$$\cancel{\varepsilon_h} \cdot \langle E_h \rangle = \cancel{\varepsilon_h} E_{soft} \frac{L_{soft}}{L_T} + \cancel{\varepsilon_h} E_{stiff} \frac{L_{stiff}}{L_T}$$

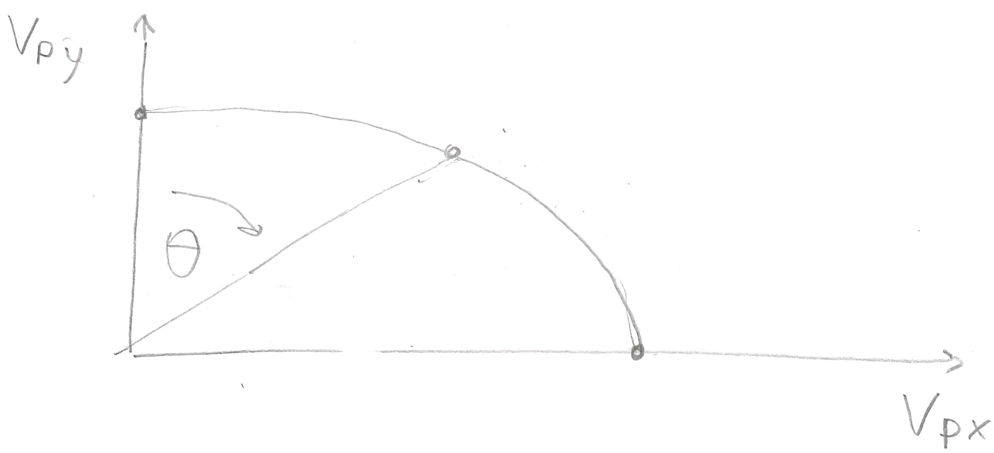
$$\boxed{\langle E_h \rangle = E_{soft} \frac{L_{soft}}{L_T} + E_{stiff} \frac{L_{stiff}}{L_T}}$$

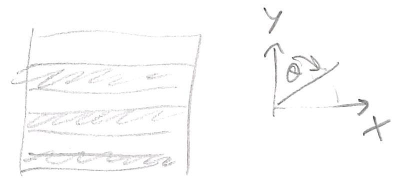
Thomsen parameters

$$\varepsilon = \frac{C_{11} - C_{33}}{2\,C_{33}} = \frac{V_{p11}^2 - V_{p33}^2}{2\,V_{p33}^2}$$

$$\gamma = \frac{C_{66} - C_{44}}{2\,C_{44}}$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2\,C_{33}\,(C_{33} - C_{44})}$$





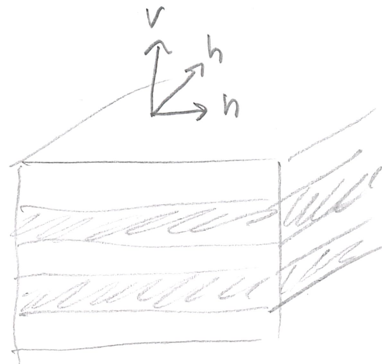
Weak anisotropy

$$\boxed{V_p(\theta) = V_{p33}\left[1 + \delta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta\right]}$$

θ: angle from axis of symmetry

1D Mech Eart Model with anisotropic parameters

$$\begin{bmatrix} \varepsilon_h \\ \varepsilon_H \\ \varepsilon_v \end{bmatrix} = \begin{bmatrix} 1/E_h & -\nu_h/E_h & -\nu_v/E_v \\ -\nu_h/E_h & 1/E_h & -\nu_v/E_v \\ -\nu_v/E_v & -\nu_v/E_v & 1/E_v \end{bmatrix} \begin{bmatrix} \sigma_h \\ \sigma_H \\ \sigma_v \end{bmatrix}$$



$\varepsilon_{ij} = 0$ for $i \neq j$

Solve for $\varepsilon_v \longrightarrow \varepsilon_v = f(\sigma_v)$

$$\alpha_h = 1 - \frac{C_{11} + C_{12} + C_{13}}{3 K_S}$$

$$\alpha_v = 1 - \frac{2 C_{13} + C_{33}}{3 K_S}$$

$$\frac{1}{N} = \frac{(2\alpha_1 + \alpha_3)/3 - \emptyset_0}{K_S}$$

$$\begin{cases} \sigma_h = S_h - \alpha_h P_p \\ \sigma_H = S_H - \alpha_h P_p \\ \sigma_v = S_v - \alpha_v P_p \end{cases}$$

with poroelasticity

α_h, α_v are Biot coefficients

$\alpha \leq 1$

tectonic strains

Total

$$\begin{cases} S_h = \alpha_h P_p + \dfrac{\nu_v}{(1-\nu_h)} \dfrac{E_h}{E_v} (S_v - \alpha_v P_p) + \dfrac{E_h}{1-\nu_h^2} (\varepsilon_h + \nu_h \varepsilon_H) \\ S_H = \alpha_h P_p + \dfrac{\nu_v}{(1-\nu_h)} \dfrac{E_h}{E_v} (S_v - \alpha_v P_p) + \dfrac{E_h}{1-\nu_h^2} (\nu_h \varepsilon_h + \varepsilon_H) \\ S_v = \displaystyle\int_0^z \rho_{bulk}\, g\, dz \end{cases}$$

overburden

$$\sigma_v = S_v - \alpha_v P_p$$

General solution for a continuum mechanics problem

(Linear isotropic elasticity)

① Equil. $\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \underline{0}$

② Kinem. $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^{T}\right)$ [small strains]

③ Constitutive $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$ [linear elasticity]

$\boxed{② \xrightarrow{in} ③} \xrightarrow{in} ①$

Navier's Equation

$$\boxed{(\lambda + \mu)\, \nabla(\nabla \cdot \underline{u}) + \mu\, \nabla^2 \underline{u} + \underline{f} = \underline{0}}$$

λ: 1st Lamé parameter $\lambda = M - 2G = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

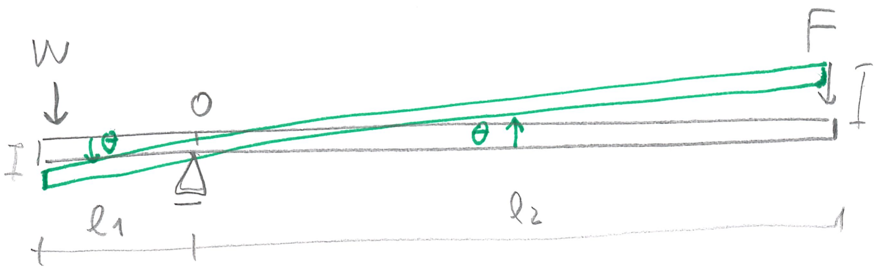
$\mu = G = \dfrac{E}{2(1+\nu)}$

- Analytical solution ⟷ Kirsch, Griffith, Sneddon
  - ↳ Airy's equation

- Numerical solution → Finite differences
  - ↳ Finite elements

Finite Element Method

Equil. Ang Mom

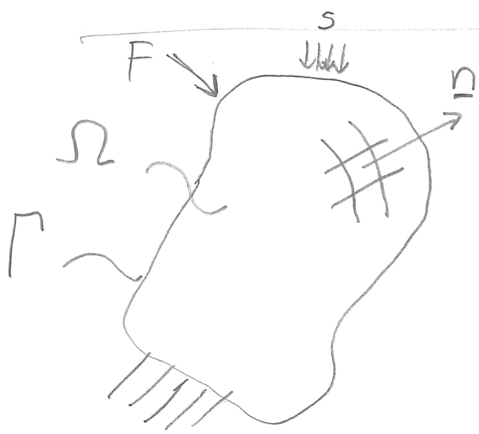


$l_2 F = W \cdot l_1$

$$\boxed{F = \frac{l_1}{l_2} W}$$

Energy conservation $\leftrightarrow$ Principle of <u>virtual</u> <u>work</u>

$$\overbrace{W \cdot (l_1 \underbrace{\tan\theta}_{})}^{\text{Work done by } W} = F\,(l_2 \overbrace{\tan\theta}^{\text{virtual displacement}}) \rightarrow \boxed{F = \frac{l_1}{l_2} W}$$



$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = 0$$

$$-\nabla \cdot \underline{\underline{\sigma}} = \underline{f}$$

$$\underbrace{\int_\Omega \delta \underline{u} \cdot (-\nabla \cdot \underline{\underline{\sigma}})}_{\text{Green's Theorem}} = \int_\Omega \overbrace{\delta \underline{u}}^{\text{virtual displacement}} \cdot \underline{f}$$

$$\int_\Omega \nabla \delta \underset{\sim}{u} \cdot \underline{\underline{\sigma}} - \int_\Gamma \delta \underset{\sim}{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \int_\Omega \delta \underline{u} \cdot \underline{f}$$

variational form $\updownarrow$ weak form

$$\boxed{\underbrace{\int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u})}_{\text{strain energy}} = \underbrace{\int_\Gamma \delta \underline{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n})}_{\text{stress boundary condition}} + \underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}}_{\text{body force}}}$$

unknowns $\rightarrow$ $\underline{u}$, $\underline{\delta u}$

$\underline{u} \rightarrow$ $\boxed{\text{actual displacement}}$

$\underline{\delta u} \rightarrow$ virtual displacement

$$\varepsilon(\nabla \underset{\sim}{\delta u}) : \underline{\underline{\sigma}}(\underline{u}) = \underset{\sim}{\varepsilon}_{11} \cdot \sigma_{11} + \underset{\sim}{\varepsilon}_{22} \cdot \sigma_{22} + \underset{\sim}{\varepsilon}_{33} \cdot \sigma_{33} + \underset{\sim}{\varepsilon}_{12} \cdot \underset{\sim}{\sigma}_{12} + \dots$$

$$E = p \cdot V \quad \text{(Energy)}$$

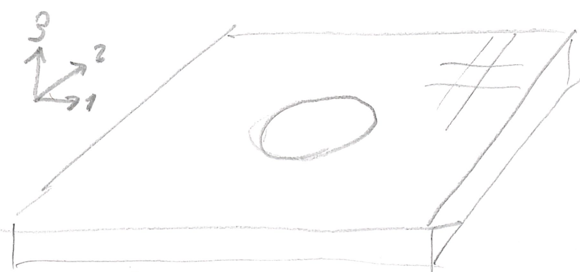
$$\updownarrow$$

$$= \sigma \frac{V}{V} \quad \text{(Energy by unit of volume)}$$

$\rightarrow$ Linear elasticity in 2D

· Plane - strain

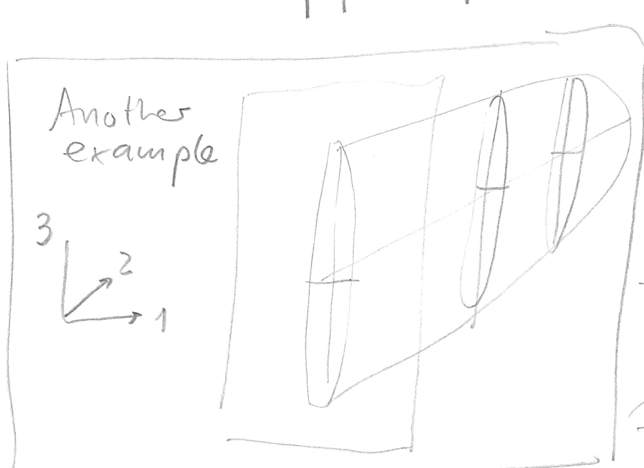
$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \cancel{\varepsilon_{33}} \to 0\\ \cancel{2\cdot\varepsilon_{23}} \to 0\\ \cancel{2\cdot\varepsilon_{13}} \to 0\\ 2\cdot\varepsilon_{12}\end{bmatrix}$$

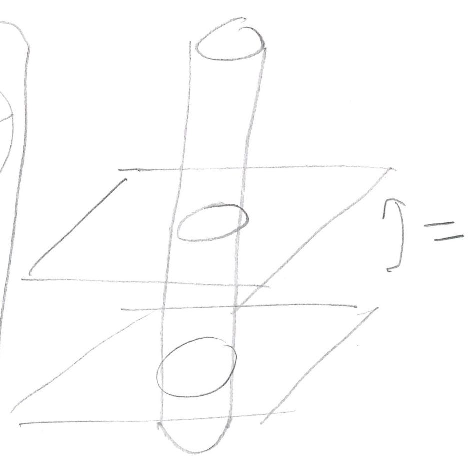


$$\frac{\partial}{\partial x_3} = 0$$

$$\frac{\partial u_3}{\partial x_3} \Rightarrow \varepsilon_{33} = \emptyset$$

$$\varepsilon_{13} = \varepsilon_{23} = \emptyset$$





$\sigma =$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\cdot\varepsilon_{12} \end{bmatrix}$$

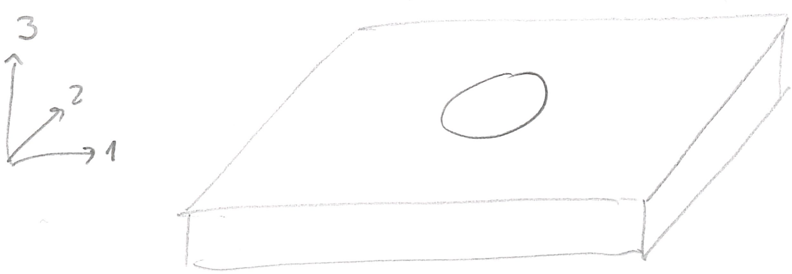
Lamé coefficients

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = G$$

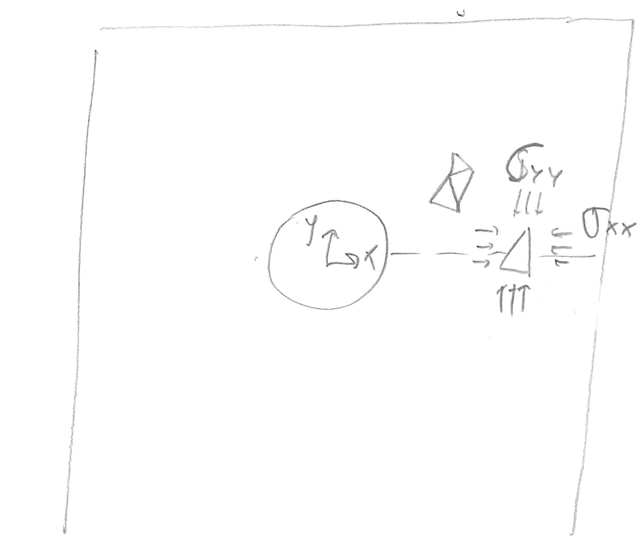
$$\sigma_{33} = \nu\,(\sigma_{11} + \sigma_{22})$$

- Plane stress ($\sigma_{33} = 0$; $\varepsilon_{33} \neq 0$)

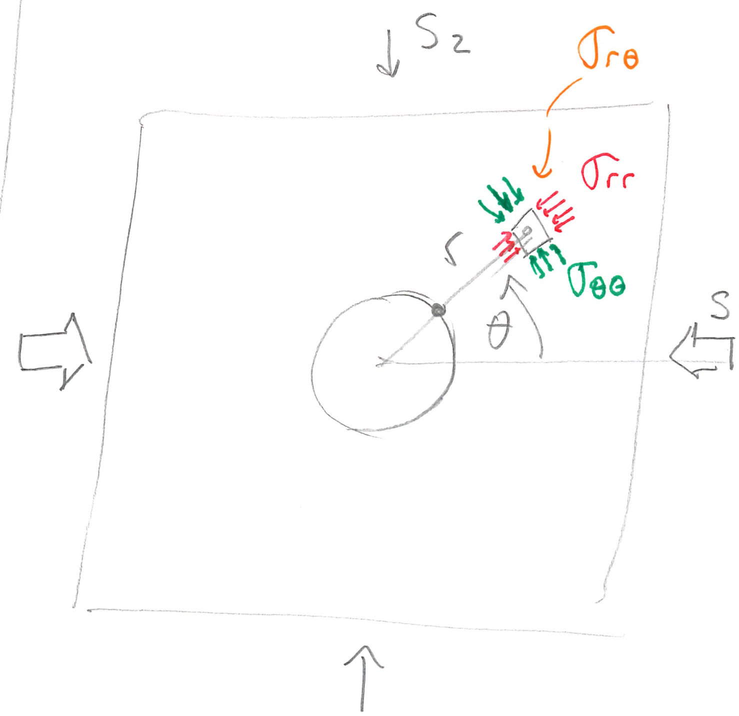


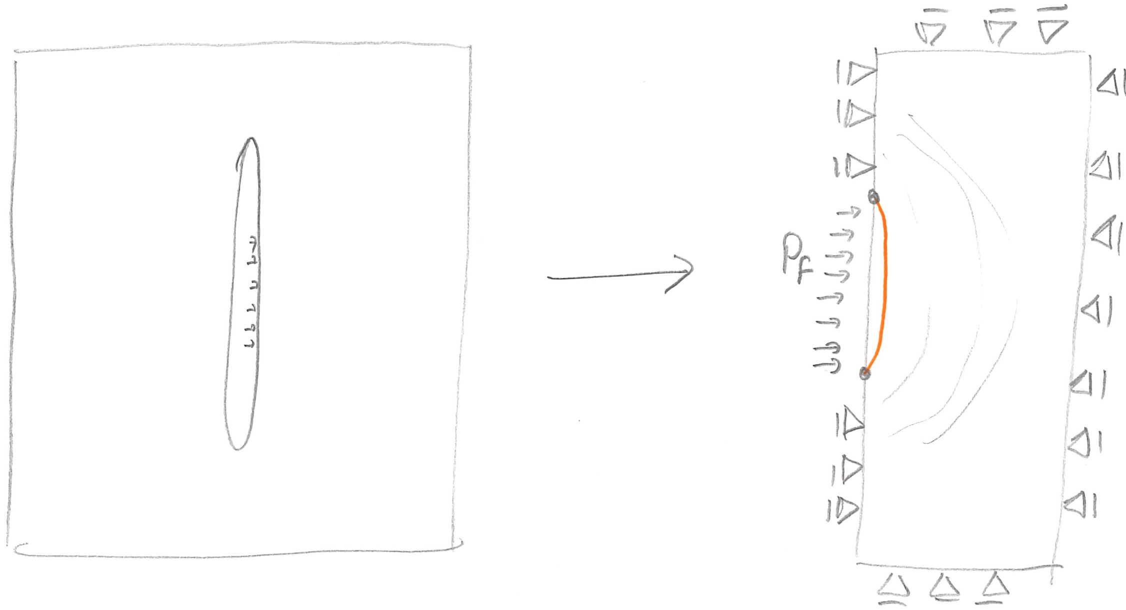
plr2cart

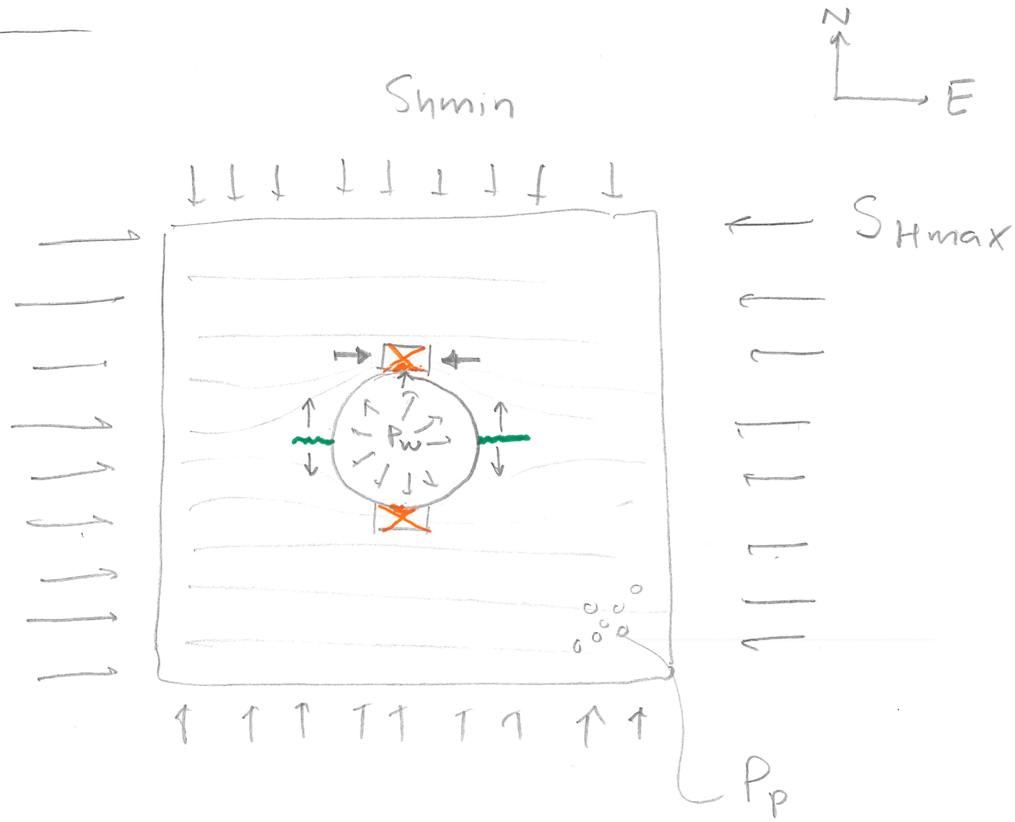
FEM



Kirsch (Analytic)







N-S: shear failure → breakout

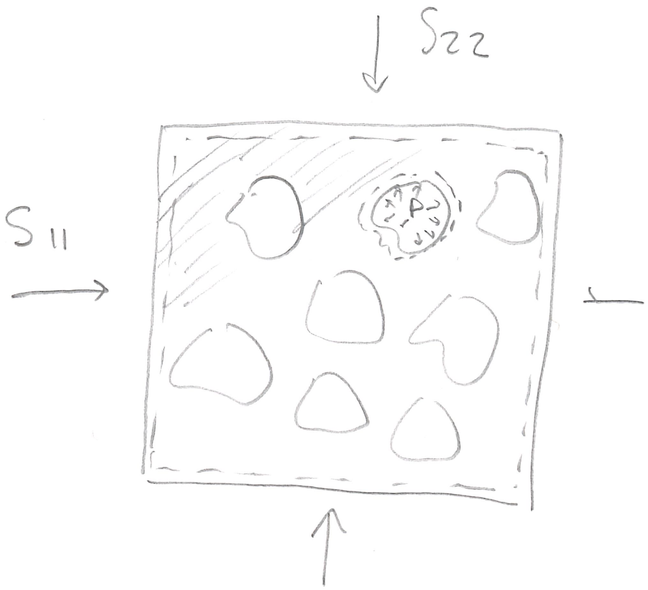
E-W: tensile failure → drilling-induced tensile fracs

WP 4

$P_w - P_p = 0 \rightarrow \sigma_{\theta\theta} \geq UCS$

$\sigma_{\theta\theta} \leq -T_s$

Poroelasticity (Coussy, 2010) Ch 3 - Ch 4



- pores → fluids, pressure
  no shear stress
- deformable pore walls
- bulk volume $d\Omega$

(1) $$\boxed{d\Omega = (1+\underbrace{\epsilon}_{\varepsilon_{vol}})\,\underbrace{d\Omega_0}_{\text{initial bulk volume}}}$$

Porosity

@ time t $\quad n = \dfrac{\square}{\blacksquare + \square}$ } Eulerian porosity

Lagrange

$$n \cdot d\Omega = \phi \cdot d\Omega_0 \Rightarrow \boxed{\phi = n\,\frac{d\Omega}{d\Omega_0}}$$ (2) } Lagrangian Porosity

Solid strain

$$\boxed{d\Omega^s = (1+\epsilon_s)\,d\Omega^s_0}$$ (3)

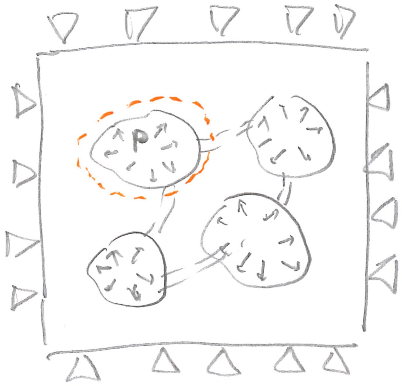
$$\boxed{\varepsilon = \frac{L - L_0}{L_0}}$$

1,2,3 → $$\boxed{\underbrace{\epsilon}_{\text{total bulk strain}} = (1-\phi_0)\underbrace{\epsilon_s}_{\text{solid strain}} + \underbrace{\frac{(\phi - \phi_0)}{\phi_0}}_{\text{Porosity strain}}\,\phi_0}$$ Volume Average

$\phi - \phi_0 = \varphi$

$\boxed{\epsilon = 0}$

1



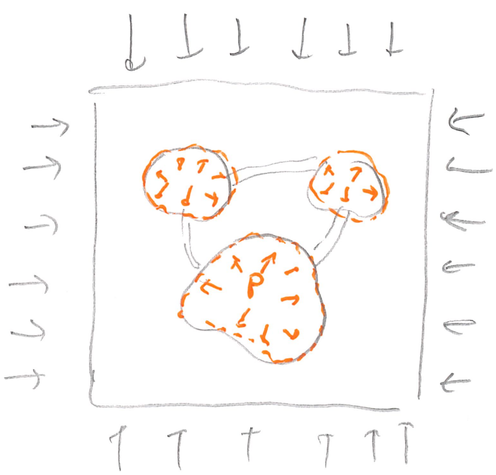
$$0 = (1-\phi_0)\,\epsilon_s + \varphi$$

$$\epsilon_s = -\frac{\varphi}{(1-\phi_0)}$$

$$\text{if } \underbrace{\varphi > 0}_{\text{increase of porosity}} \Rightarrow \underbrace{\epsilon_s < 0}_{\text{contraction}}$$

(Mechanics convention)

2



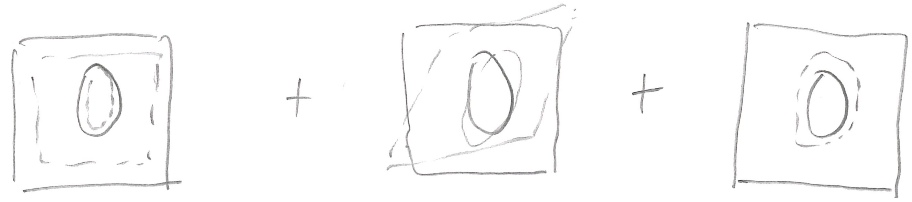
$\boxed{\varphi = 0}$ $\quad \varphi = \phi - \phi_0$

$$\epsilon_s = \frac{\epsilon}{(1-\phi_0)}$$

Free energy of the porous solid

Nonporous solid

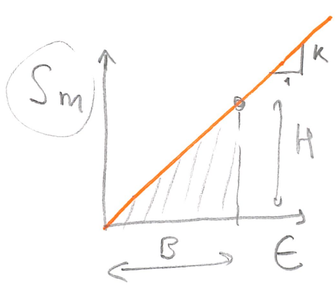
$$dW = \underbrace{S_m}_{S_{11}+S_{22}+S_{33}} \cdot \underbrace{d\epsilon}_{\text{vol strain}} + \underbrace{s_{ij}}_{\substack{\text{shear stress}\\ ij}} \cdot \underbrace{de_{ij}}_{\substack{\text{shear strain}\\ ij}}$$



Porous solid

$$dW = S_m \cdot d\epsilon + s_{ij} \cdot de_{ij} + p \cdot \underbrace{d\phi}_{d\varphi}$$

linear elastic material



$$W = \frac{\epsilon \cdot (K \cdot \epsilon)}{2} = \frac{1}{2} K \epsilon^2$$

$$\boxed{\frac{\partial W}{\partial \epsilon} = K \epsilon = S_m}$$

$$\boxed{\eta_s = \underbrace{\frac{1}{2} K \epsilon^2}_{\text{bulk volum strain}} + \underbrace{G \cdot e_{ij} \cdot e_{ij}}_{\text{shear strain}} - \overset{\text{cst}}{\alpha} \cdot \epsilon \cdot p - \underbrace{\left(\frac{1}{2}\overset{\text{cst}}{\frac{1}{N}}\right) p^2}_{\text{pressure}}}$$

effective stress — pore pressure — total stress

$$\frac{\partial n_s}{\partial \epsilon} = K\,\epsilon - \alpha\, p = S_m \qquad (1)$$

Effective stress equation

Biot coefficient

$\alpha = ?$

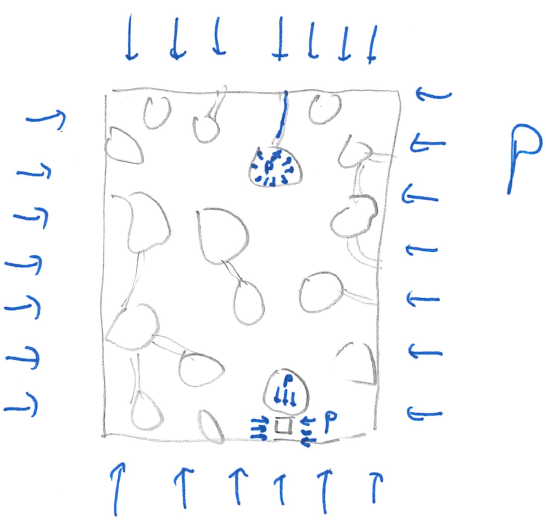
$$\frac{\partial n_s}{\partial e_{ij}} = 2\,G\,e_{ij} = s_{ij} \qquad (2)$$

$$-\frac{\partial n_s}{\partial p} = +\alpha\,\epsilon + \frac{p}{N} = \varphi \qquad (3)$$

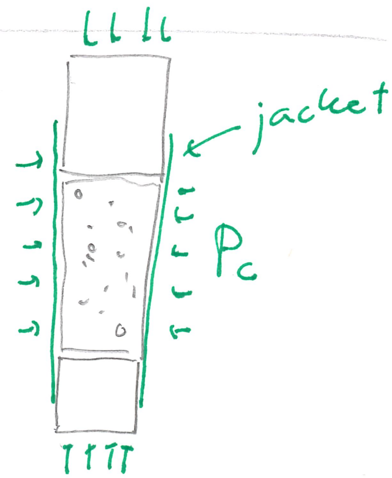
$N = ?$

Poroelastic modulus

Unjacketed loading



Ø all connected



$$(1)\quad K\epsilon - \alpha p = -p$$

$$\epsilon = \frac{p(-1+\alpha)}{K}$$

- $\epsilon = -\dfrac{p\,(1-\alpha)}{K}$ — bulk of porous solid
- $\epsilon_s = -\dfrac{p}{K_s}$

$$\epsilon = \epsilon_s$$

$$+\cancel{p}\,\frac{(1-\alpha)}{K} = +\frac{\cancel{p}}{K_s}$$

$$\boxed{\alpha = 1 - \frac{K}{K_s}}$$

bulk modulus of the porous solid "jacketed" "drained"

bulk modulus of the solid matrix

$$K_s \geq K \Rightarrow \alpha \leq 1$$

3 $$\alpha \epsilon + \frac{p}{N} = \varphi$$

$$\alpha \left(-\frac{p}{K_s} \right) + \frac{p}{N} = \varphi$$

$$\underbrace{\varphi / \phi_0}_{\text{porosity strain}} = -\frac{p}{K_s}$$

$$\alpha \left(-\frac{\cancel{p}}{K_s} \right) + \frac{\cancel{p}}{N} = -\frac{\cancel{p}}{K_s} \cdot \phi_0$$

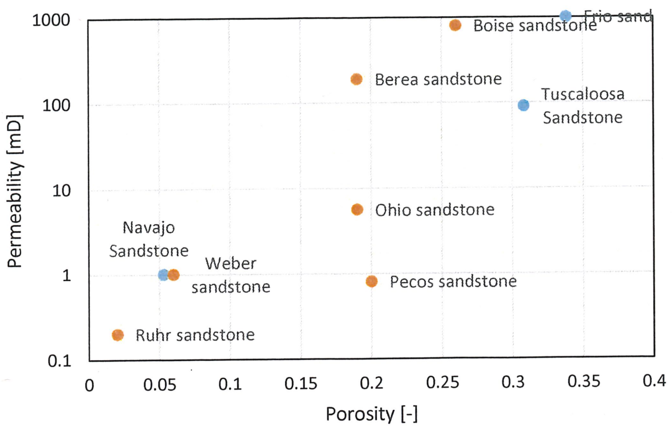
$$\boxed{\frac{1}{N} = \frac{\alpha - \phi_0}{K_s}}$$

$\to$
- second poromechanical modulus
- pore modulus
- $[N] = Pa$
- $N > 0 \Rightarrow \alpha > \phi_0$

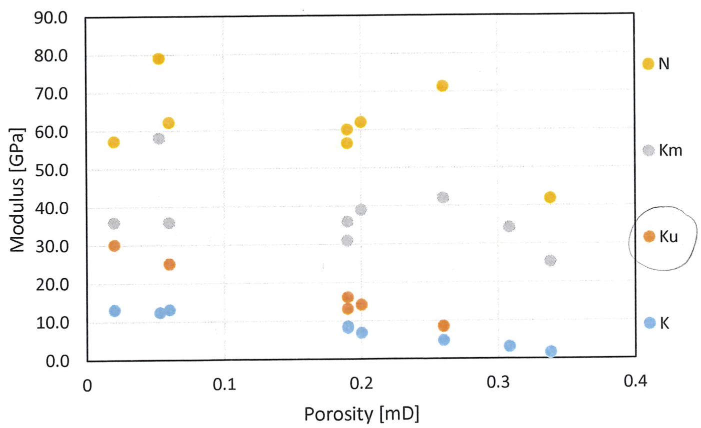
$$\Downarrow$$

$$\underline{\phi_0 \leqslant \alpha \leqslant 1}$$

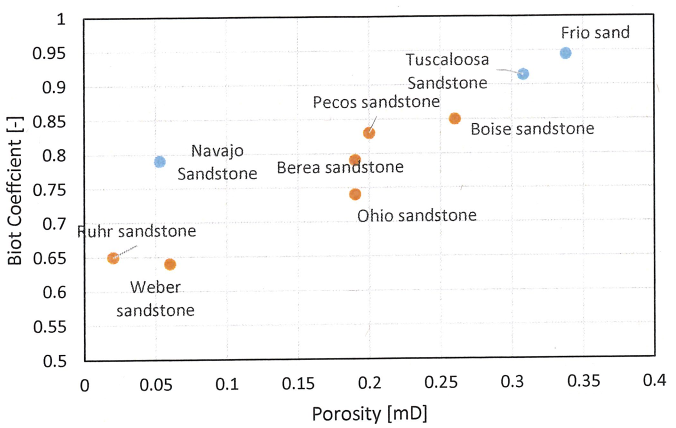
L all ϕ connected



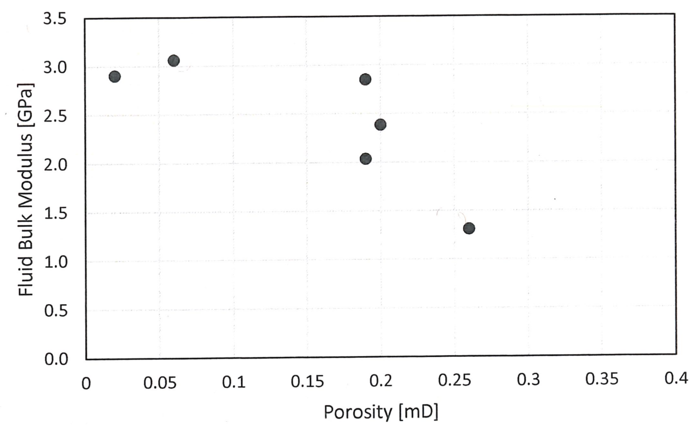
1000
100
10
1
0.1
Permeability [mD]
Boise sandstone
Frio sand
Berea sandstone
Tuscaloosa Sandstone
Ohio sandstone
Navajo Sandstone
Weber sandstone
Pecos sandstone
Ruhr sandstone
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4
Porosity [-]



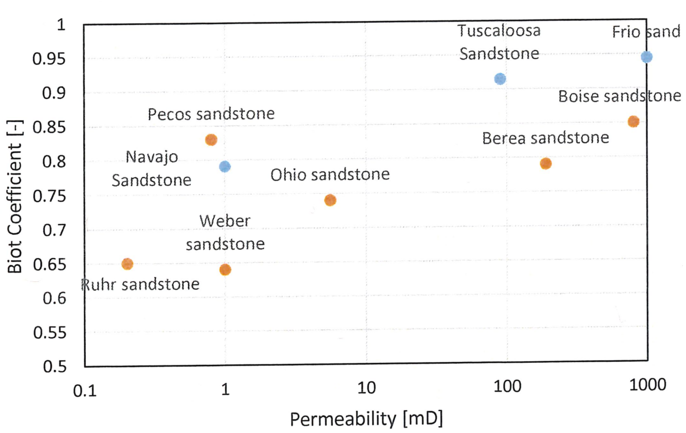
90.0 80.0 70.0 60.0 50.0 40.0 30.0 20.0 10.0 0.0
Modulus [GPa]
N
Km
Ku
K
0 0.1 0.2 0.3 0.4
Porosity [mD]



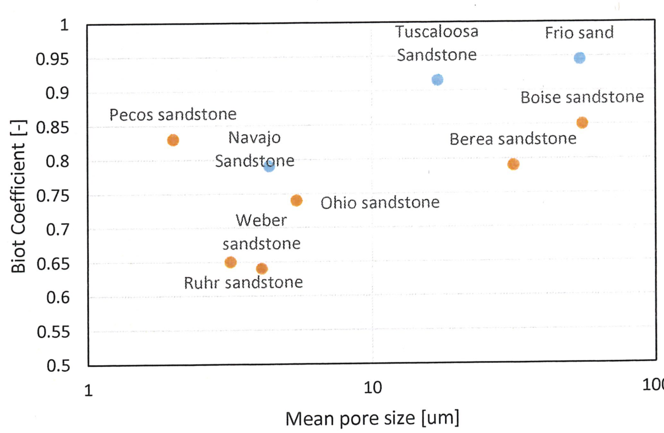
1 0.95 0.9 0.85 0.8 0.75 0.7 0.65 0.6 0.55 0.5
Biot Coeffcient [-]
Frio sand
Tuscaloosa Sandstone
Pecos sandstone
Boise sandstone
Navajo Sandstone
Berea sandstone
Ohio sandstone
Ruhr sandstone
Weber sandstone
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4
Porosity [mD]



3.5 3.0 2.5 2.0 1.5 1.0 0.5 0.0
Fluid Bulk Modulus [GPa]
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4
Porosity [mD]



1 0.95 0.9 0.85 0.8 0.75 0.7 0.65 0.6 0.55 0.5
Biot Coefficient [-]
Tuscaloosa Sandstone
Frio sand
Boise sandstone
Pecos sandstone
Berea sandstone
Navajo Sandstone
Ohio sandstone
Weber sandstone
Ruhr sandstone
0.1 1 10 100 1000
Permeability [mD]



1 0.95 0.9 0.85 0.8 0.75 0.7 0.65 0.6 0.55 0.5
Biot Coefficient [-]
Tuscaloosa Sandstone
Frio sand
Boise sandstone
Pecos sandstone
Navajo Sandstone
Berea sandstone
Ohio sandstone
Weber sandstone
Ruhr sandstone
1 10 100
Mean pore size [um]

Measurement of Biot coefficient

(1) $\alpha = 1 - \dfrac{K}{K_s}$

jacketed experiment
↳ bulk modulus

$K_s \xrightarrow{\text{assume}} K_s = K_{mineral}$ 36 GPa $= K_{SiO_2}$
↳ 100% SiO_2 → K_{SiO_2}

→ wrong if occluded Ø

(2) $\alpha = 1 - \dfrac{K}{K_{unj}}$

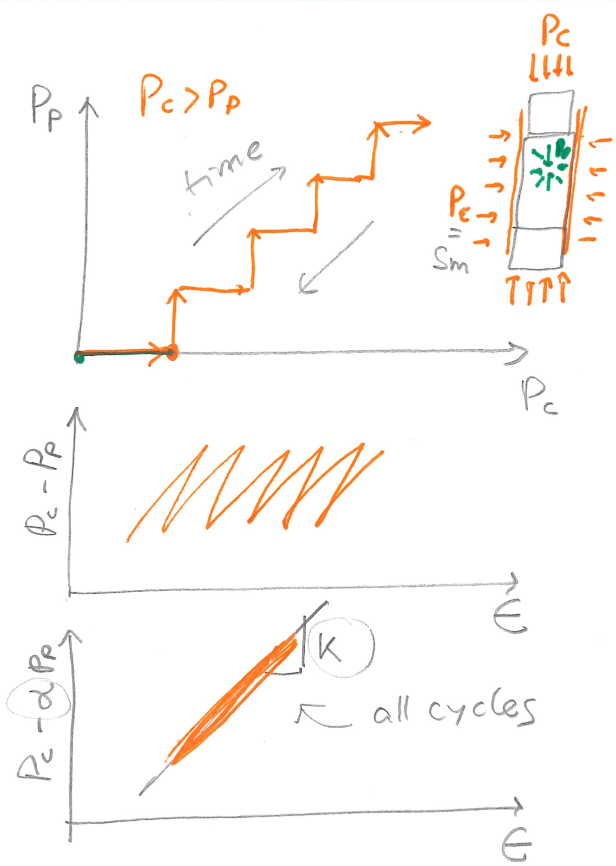
$K_{unj} \xrightarrow{\text{measured}} \dfrac{\Delta P}{\Delta \varepsilon_{vol}}$

- water < clay sensitive; short circuit
- gas < ok if small; not desirable for large volumes
- oil → high viscosity

(3) $S_m = K\epsilon - \alpha P$

$$K\epsilon = \underbrace{S_m + \alpha P}_{\text{Biot's effective stress}}$$

$$\underbrace{\epsilon}_{\text{strain}} = \frac{1}{K}\underbrace{(S_m + \alpha P)}_{\text{stress}}$$



$\underline{\underline{\sigma}} = \underline{\underline{S}} - P_p \underline{\underline{I}}$: Terzaghi's eff. stress

$\underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$: Biot's effective stress

↳ Poroelasticity

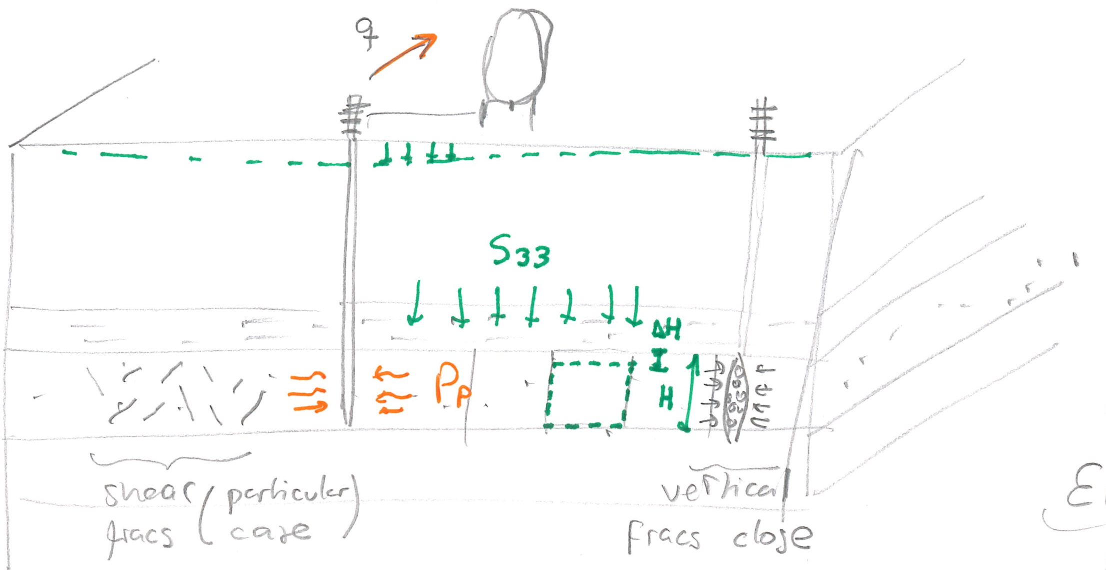
- ↳ φ → chages porosity
- ↳ K_u : undrained bulk modulus
  - ↕
- ↳ undrained loading ↔ Δp
- ↳ "squirt flow" ↔ $\Delta \varepsilon_{vol}$
  - ↳ attenuation of elastic waves
  - ↳ dispersion of elastic wave velocity

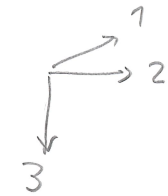
end-members of actual solution

- drained
- undrained

diffusivity eq including poroelasticity

Drained problem of reservoir depletion





$S_{33} = cst$

$\downarrow P_p \Rightarrow \sigma_{33} \uparrow$

$$\underbrace{\varepsilon_{11} = \varepsilon_{22} = \emptyset}_{\text{1D-strain condition}}$$

$$\Downarrow$$

$$S_{11} = S_{22}$$

$$\underline{\sigma} = \underline{\underline{\underline{C}}} \cdot \underline{\varepsilon} \quad ; \quad \underline{\sigma} = \underline{S} - \alpha P_p \underline{\underline{I}}$$

$$\downarrow$$

$$\underline{\varepsilon} = \underline{\underline{\underline{D}}} \cdot \underline{\sigma}$$

variable

$$\begin{bmatrix} \cancel{\varepsilon_{11}} \\ \cancel{\varepsilon_{22}} \\ \varepsilon_{33} \\ \hline \cancel{2\varepsilon_{23}} \\ \cancel{2\varepsilon_{13}} \\ \cancel{2\varepsilon_{12}} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ \hline & & & 1/G & & \\ & 0 & & & 1/G & \\ & & & & & 1/G \end{array}\right] \begin{bmatrix} S_{11} - \alpha P_p \\ S_{22} - \alpha P_p \\ S_{33} - \alpha P_p \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

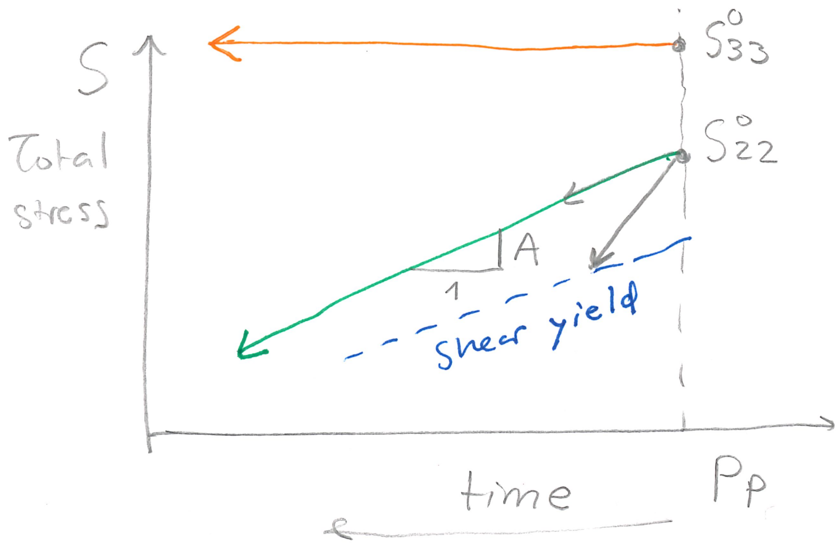
$$\varepsilon_{11} = 0 = \left(\frac{1-\nu}{E}\right)(S_{22} - \alpha P_p) - \frac{\nu}{E}(S_{33} - \alpha P_p)$$

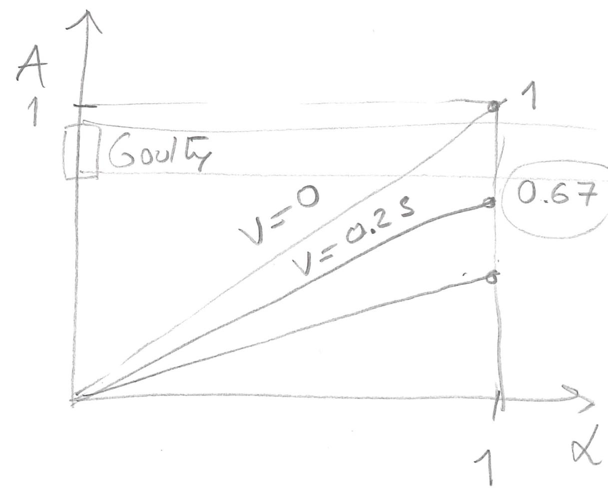
$$\underbrace{(S_{22} - \alpha P_p)}_{\sigma_{22}} = \underbrace{\frac{\nu}{1-\nu}}_{\text{coeff lateral eff. stress}} \underbrace{(S_{33} - \alpha P_p)}_{\sigma_{33}}$$

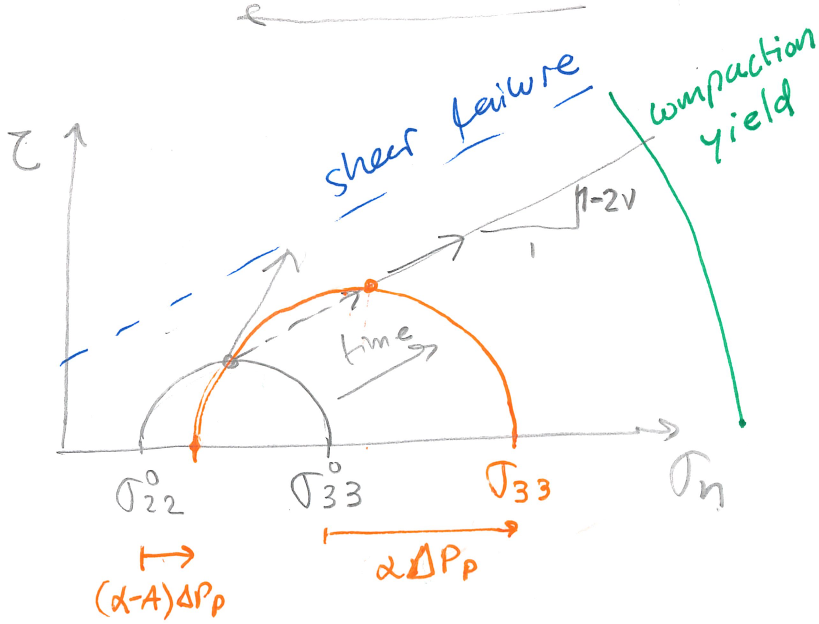
$$S_{22} = \frac{\nu}{1-\nu} S_{33} + \alpha \left(1 - \frac{\nu}{1-\nu}\right) \cdot P_p$$

$$\underbrace{S_{22}}_{\substack{\text{total}\\ \text{Hz stress}}} = \frac{\nu}{1-\nu} \underbrace{S_{33}}_{\substack{\text{total}\\ \text{vert stress}}} + \underbrace{\alpha \left(\frac{1-2\nu}{1-\nu}\right)}_{A} \underbrace{P_p}_{\substack{\text{pore}\\ \text{pressure}}}$$

$$\frac{\partial S_{22}}{\partial P_p} = \alpha \left(\frac{1-2\nu}{1-\nu}\right) = A$$







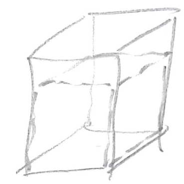
$$\sigma_{33} = S_{33} - \alpha P_p$$

$$\frac{\partial \sigma_{33}}{\partial P_p} = -\alpha$$

$$\frac{\partial \sigma_{22}}{\partial P_p} = -(\alpha - A)$$

$$\varepsilon_{33} = \frac{-2\nu}{E}\left(S_{22} - \alpha P_p\right) + \frac{1}{E}\left(S_{33} - \alpha P_p\right)$$

$$\varepsilon_{33} = \frac{1}{E}\,\frac{(1-2\nu)(1+\nu)}{(1-\nu)}\,\overbrace{\left(S_{33} - \alpha P_p\right)}^{\sigma_{33}}$$



1-D strain condition

$$\frac{\partial \varepsilon_{33}}{\partial P_p} = -\frac{\alpha}{M}$$

constrained modulus

$$\Delta \varepsilon_{33} = -\frac{\alpha}{M}\,\Delta P_p$$

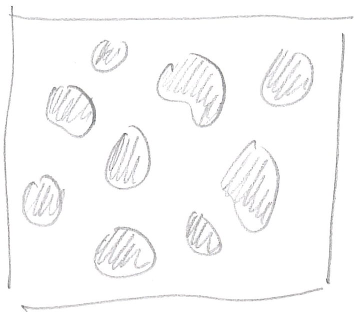
$$\Delta H = \Delta \varepsilon_{33} \cdot H$$

$$\boxed{\Delta H = \left(-\frac{\alpha}{M}\Delta P_p\right) H}$$

→ casing < shearing / buckling

→ subsidende

Undrained Loading (no fluid comes in/out the pore space)



$\underbrace{\qquad\qquad}_{V_b}$

$\dfrac{V_P \; \rho_F}{V_b}$ → $\varnothing$

if undrained

$$d(\varnothing \rho_F) = \rho_F \cdot d\varnothing + \varnothing \cdot d\rho_F = 0$$

$$\frac{d(\varnothing \rho_F)}{\rho_F} = d\varnothing + \varnothing \frac{\partial \rho_F}{\rho_F}$$

$K_F^{-1} = C_F$

$C_F = \dfrac{1}{\rho_F}\dfrac{d\rho_F}{dP}$

$\varphi = \alpha \epsilon + \dfrac{P}{N}$

$$= \left(\alpha\, d\epsilon + \frac{dP}{N}\right) + \varnothing \left(\frac{dP}{K_F}\right)$$

small strains

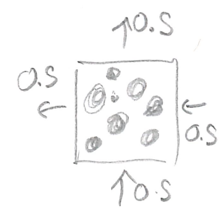
$$\frac{d(\varnothing \rho_F)}{\rho_F} = \alpha \epsilon + \underbrace{\left(\frac{1}{N} + \frac{\varnothing_0}{K_F}\right)}_{\sim cst} dP$$

Biot Modulus

$$\boxed{\frac{1}{M^*} = \left(\frac{\alpha - \varnothing_0}{K_S} + \frac{\varnothing_0}{K_F}\right)}$$

Continuity equation

$\underline{q} = -\dfrac{K_h}{\mu} \nabla P$



$$\frac{\partial(\rho_F \varnothing)}{dt} + \nabla \cdot (\rho_F \underline{q}) = 0$$

compressibility + Biot

darcy, homogeneous

$\nabla^2 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)$

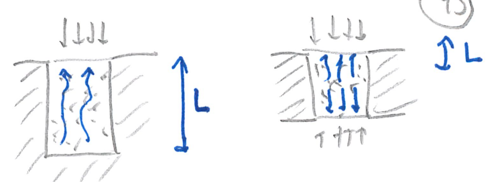
$$\boxed{\frac{dP}{dt} = \frac{K_h M^*}{\mu} \nabla^2 P - \alpha M^* \frac{d\epsilon}{dt}}$$

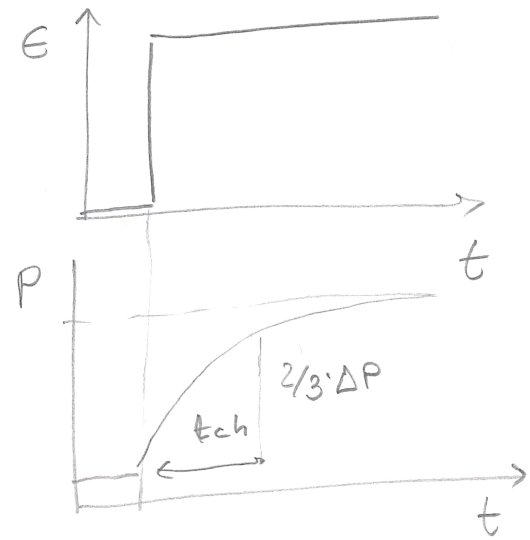
Diffusivity equation coupled with linear Poroelasticity

$$\frac{dP}{dt} \sim \underbrace{\frac{k_h M^*}{\mu}} \nabla^2 P$$

$$\left[\frac{m^2}{s}\right] \leftarrow \boxed{D = \frac{k_h M^*}{\mu}}$$

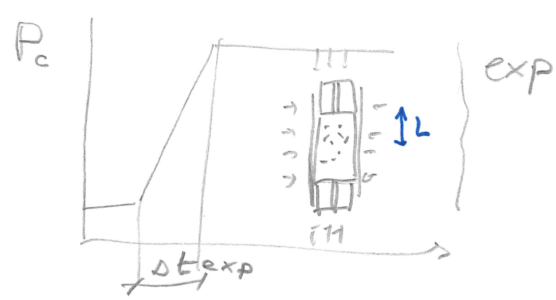
$$t_{ch} = \frac{L^2}{D}$$

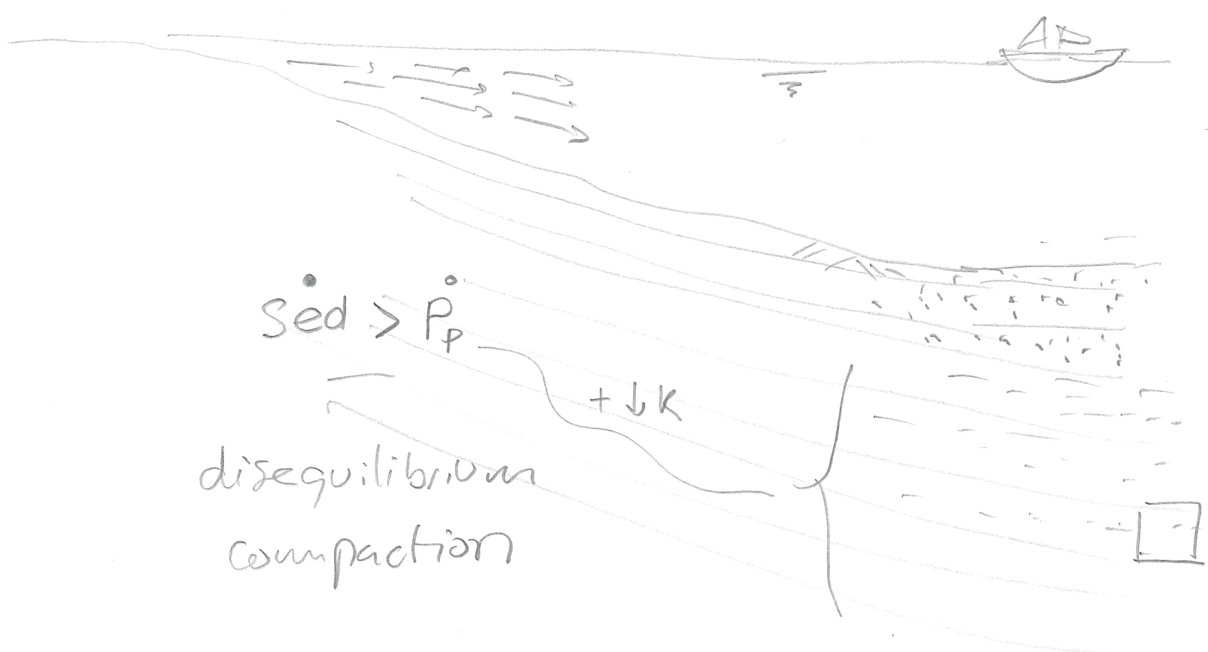




$t_{ch} \ll \Delta t_{exp} \Rightarrow$ drained

$t_{ch} \gg \Delta t_{exp} \Rightarrow$ undrained





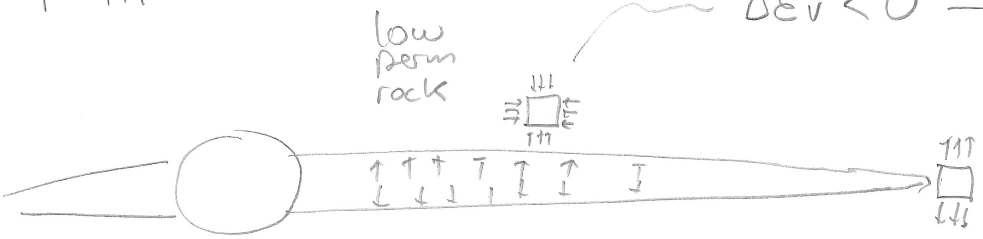
$\dot{Sed} > \dot{P}_p$

disequilibrium compaction

Examples of undrained processes

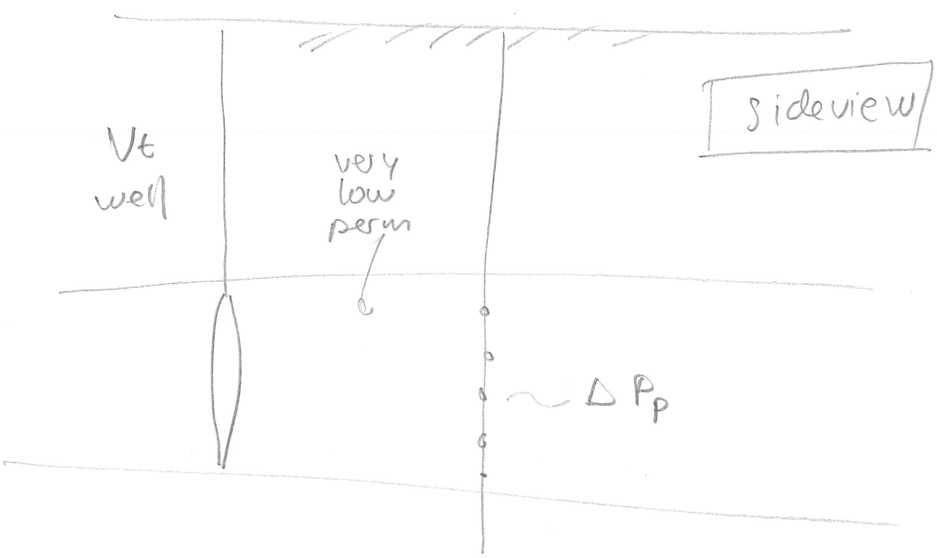
1 - HF

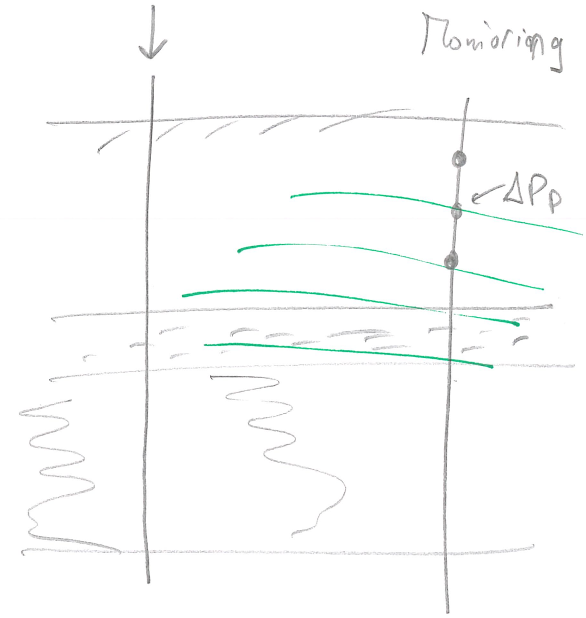
$\Delta \varepsilon_v < 0 \Rightarrow \Delta P_p > 0 \Rightarrow \downarrow$ Eff stress



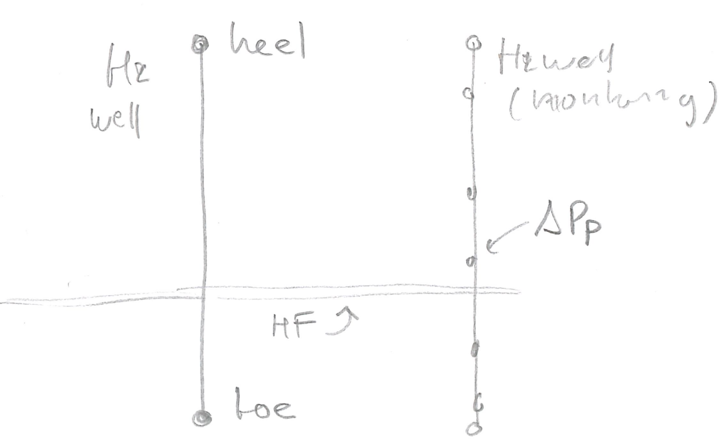
$\Delta \varepsilon_v > 0 \Rightarrow \Delta P_p < 0$

2. Poroelastic interference



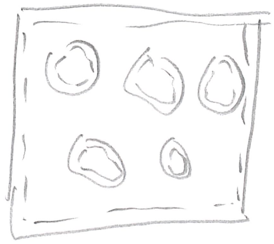


top view



(1) Pore pressure change due to volumetric strain (undrained loading)

$$\left.\frac{\partial P}{\partial \epsilon}\right|_{d(\rho_F \emptyset)=0} = -\alpha M^* = -\alpha \left(\frac{\alpha - \emptyset_0}{K_S} + \frac{\emptyset_0}{K_F} \right)^{-1}$$



(2) Undrained bulk modulus

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{drained} = K$$

Bulk modulus
↳ drained
↳ dry

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{d(\rho_F \emptyset)=0} = K_U$$

$$S_m = K\epsilon - \alpha P$$

$$\frac{\partial S_m}{\partial \epsilon} = K - \alpha \frac{\partial P}{\partial \epsilon} \Rightarrow \left.\frac{\partial S_m}{\partial \epsilon}\right|_{d(\rho_F \emptyset)=0} = K - \alpha(-\alpha M^*)$$

$$\boxed{K_U = K + \alpha^2 M^*}$$

3 Skempton's parameter

$$\left.\frac{\partial P}{\partial S_m}\right|_{\partial(\rho_F \phi)=0} = -B$$

$$S_m = K\epsilon - \alpha P$$

$$\frac{\partial S_m}{\partial P} = K\frac{\partial \epsilon}{\partial P} - \alpha$$

$$\frac{\partial S_m}{\partial P} = K\left(-\frac{1}{\alpha \Pi^*}\right) - \alpha$$

$$\boxed{B^{-1} = \frac{1}{\alpha}\left(\frac{K}{\Pi^*} + \alpha^2\right)}$$

$B \to \sim 1 \to$ liquid saturated