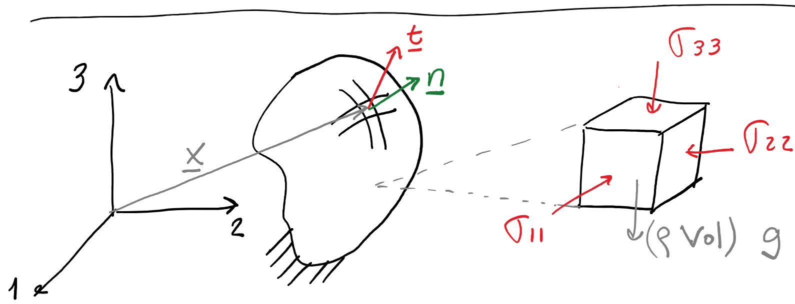
General Solution Cont Mech

Friday, September 11, 2020 10:56 AM

General solution to a continuum mechanics problem



$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \rho\, \underline{a}$$ → Equilibrium (Cauchy's)

$$\underline{\underline{\varepsilon}} = F_1(\underline{u})$$ → Kinematic eq
- small strains ←
- large strains

$$\underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}})$$ → Constitutive equations
- linear isotropic elastic solid ←
- TVI (VTI)
- orthorhombic
- visco-elasticity
- plasticity

$$\nabla \cdot \left[F_2(\underline{\underline{\varepsilon}}) \right] + \underline{f} = \rho\, \underline{a}$$

$$\boxed{\nabla \cdot \left[F_2\left[F_1(\underline{u}) \right] \right] + \underline{f} = \rho\, \underline{a}}$$

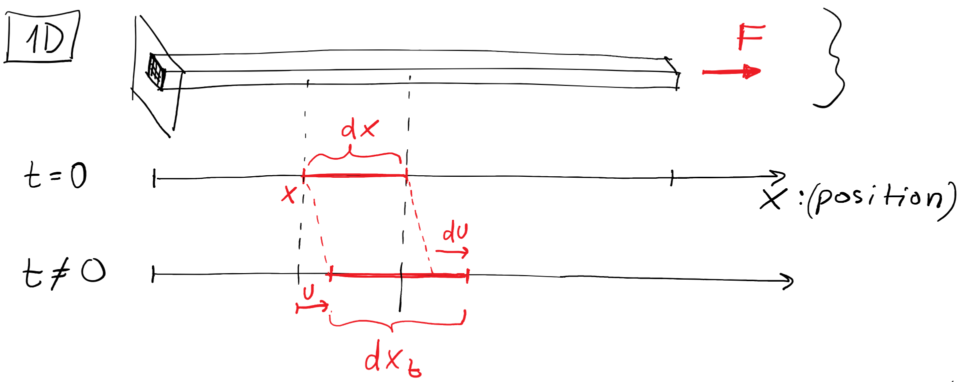
displacement $\underline{u} \rightarrow \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}}$

Kinematic Equations

Monday, August 22, 2022 2:49 PM

Kinematic Equations (small strains) $\boxed{\underline{\underline{\varepsilon}} = F_1(\underline{u})}$

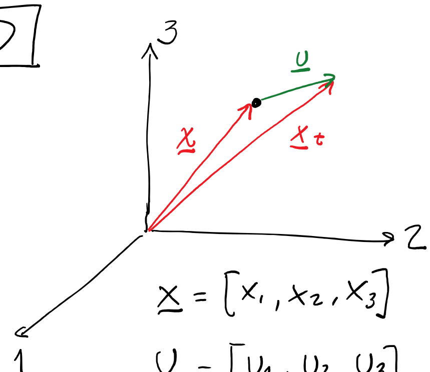
$\boxed{1D}$



$$\varepsilon = \frac{dx_t - dx}{dx} = \frac{[\cancel{x}+\cancel{u}+\cancel{dx}+du - (\cancel{x}+\cancel{u})] - [\cancel{x}+\cancel{dx}-\cancel{x}]}{[\cancel{x}+dx-\cancel{x}]}$$

$$\boxed{\varepsilon = \frac{du}{dx}}$$

$\boxed{3D}$



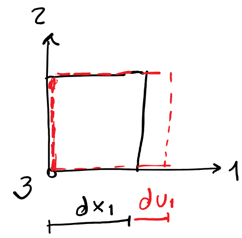
$$\underline{x} = [x_1, x_2, x_3]$$

$$\underline{u} = [u_1, u_2, u_3]$$

Jacobian $\updownarrow$ $\underline{\underline{\varepsilon}}$

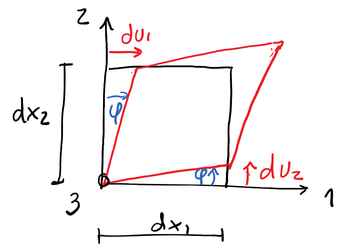
$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$



$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$+ \begin{bmatrix} 0 & \overbrace{\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \cdots & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \cdots & \cdots & 0 \end{bmatrix}$$

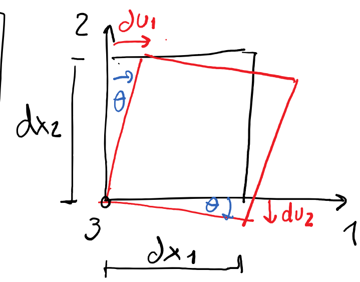


$$\tan \varphi = \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}$$

$$\varepsilon_{12} = \frac{1}{2}(2 \tan \varphi) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\Bigg\} \; \underline{\underline{\varepsilon}}$$

$$+ \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \cdots & \cdots & 0 \end{bmatrix}$$



$$\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right) = \omega_{12}$$

$$\Bigg\} \; \underline{\underline{\omega}}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}\left(\frac{\partial U_3}{\partial x_1}+\frac{\partial U_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial U_3}{\partial x_1}+\frac{\partial U_1}{\partial x_3}\right) & \frac{\partial \cdots}{\partial x_3} \end{bmatrix} \quad \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

↳ symmetric } eigen values → principal strains
↳ real values } eigen vectors → principal directions

$\underbrace{\text{eigen}}_{\text{own}}$

$$I_1(\underline{\underline{\varepsilon}}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{vol}$$

$$\boxed{\varepsilon_{vol} = \frac{Vol(t) - Vol_0}{Vol_0}}$$

Constitutive Equations

$$\underbrace{\underline{\underline{\sigma}}}_{\text{stress}} = \underbrace{F_2}_{} (\underbrace{\underline{\underline{\varepsilon}}}_{\text{strain}})$$

Superposition $\left\langle \begin{array}{l} \text{space} \\ \text{time} \end{array} \right\}$ Green' functions

linear relationships
$$\left\{ \begin{array}{l} F_2(A+B) = F_2(A) + F_2(B) \\ F_2(c \cdot A) = c \cdot F_2(A) \end{array} \right.$$

$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}} \quad \longleftrightarrow \quad \underset{\sim}{y} = \underbrace{b}_{\text{constant}} \cdot \underset{\sim}{x}$$

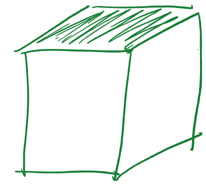
9	81	9
6	36	6

Voigt Notation $\left(\underset{3\times3}{\underline{\underline{\sigma}}} \rightarrow \underset{6\times1}{\underline{\sigma}} \right) \rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$

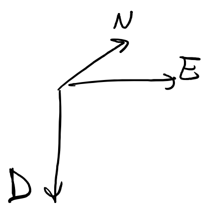
$$\underset{6\times1}{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}} = \underbrace{\underset{6\times6}{\left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot \\ \hline \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ C_{61} & C_{62} & \cdot & \cdot & \cdot & C_{66} \end{array}\right]}}_{\text{Stiffness Matrix}} \underset{6\times1}{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\cdot\varepsilon_{23} \\ 2\cdot\varepsilon_{13} \\ 2\,\varepsilon_{12} \end{bmatrix}}$$

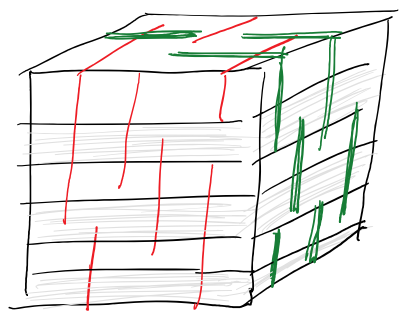
Shear decoupling $\left\langle \begin{array}{l} \text{normal} \leftrightarrow \text{shear} \\ \text{shear} \leftrightarrow \text{shear} \end{array} \right\}$ does not apply for plasticity

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \hline \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ \hline 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{array}\right] \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \hline 2\cdot\varepsilon_{23}\\ 2\cdot\varepsilon_{13}\\ 2\,\varepsilon_{12}\end{bmatrix}$$



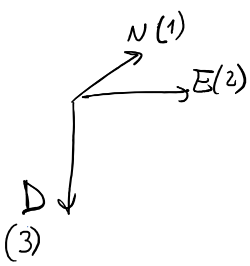
12 coeff $\rightarrow$ $\boxed{\text{9 coeff}}$ $\rightarrow$ Orthorhombic

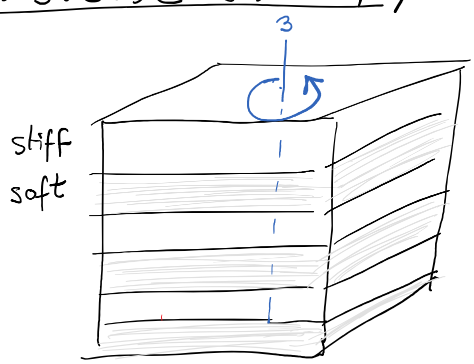




$$E_V \neq E_{H(N\text{-}S)} \neq E_{H(E\text{-}W)}$$

Vertical Transverse Isotropy





$$E_V < E_H$$

$$\rightarrow \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \hline \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \underbrace{\left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ \hline 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{array}\right]}_{\boxed{\text{5 indepent coefficients}}} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \hline 2\,\varepsilon_{23}\\ 2\,\varepsilon_{13}\\ 2\,\varepsilon_{12}\end{bmatrix}$$

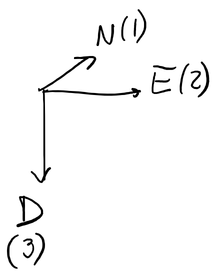
C_{11}, C_{33}

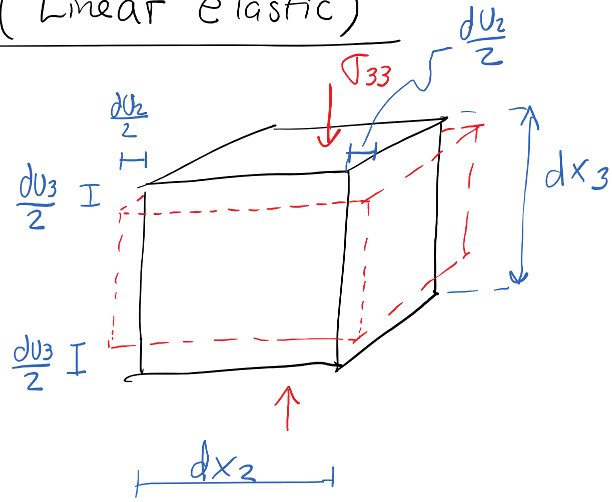
C_{12}, C_{13}

$C_{44}, \cancel{C_{66}}$

$$C_{66} = \frac{C_{11} - C_{12}}{2}$$

Isotropy (Linear elastic)





$$\varepsilon_{33} = \frac{dU_3}{dx_3}$$

$$\varepsilon_{22} = \frac{dU_2}{dx_2}$$

$$\varepsilon_{11} = \frac{dU_1}{dx_2}$$

$$\boxed{E \stackrel{def}{=} \frac{\sigma_{33}}{\varepsilon_{33}}}$$

Young's modulus

$$\boxed{\nu \stackrel{def}{=} -\frac{\varepsilon_{11}}{\varepsilon_{33}} = -\frac{\varepsilon_{22}}{\varepsilon_{33}}}$$

Poisson's ratio

$$\underline{\underline{\sigma}} = \begin{vmatrix} 0 \\ 0 \\ \sigma_{33} \\ \hline 0 \\ 0 \\ 0 \end{vmatrix} \Longrightarrow \underline{\underline{\varepsilon}} = \begin{vmatrix} -(\nu/E)\sigma_{33} \\ -(\nu/E)\sigma_{33} \\ \sigma_{33}/E \\ \hline 0 \\ 0 \\ 0 \end{vmatrix} \leftarrow \begin{matrix} \sigma_{33} \\ \sigma_{22} \\ \sigma_{11} \end{matrix}$$

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{vmatrix} = \begin{vmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{vmatrix}$$

$\leftarrow G = \frac{E}{2(1+\nu)}$ isotropic

$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$\rightarrow \varepsilon_{11} = \frac{1-2\nu}{E}\sigma_m$

$\varepsilon_{vol} = \frac{3(1-2\nu)}{E}\sigma_m \Rightarrow K = \frac{E}{3(1-2\nu)}$

$$\underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}}_{\text{compliance matrix}} \cdot \underline{\underline{\sigma}}$$

E, ν (2 indep. coeff)

$$\underline{\underline{\underline{\underline{D}}}}^{-1} \cdot \underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}^{-1} \cdot \underline{\underline{\underline{\underline{D}}}}} \cdot \underline{\underline{\sigma}}$$

stiffness matrix

stiffness matrix

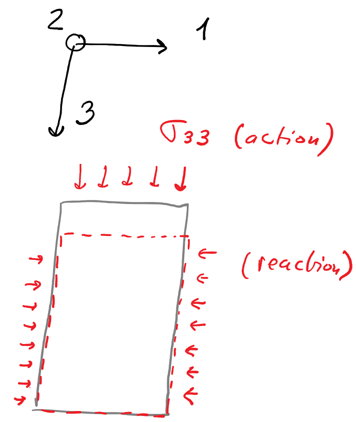
$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \overbrace{\left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{\nu} \end{array}\right]} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

1D Mechanical Earth Model
Monday, August 22, 2022 2:40 PM

Uniaxial-strain loading (stress path)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & 0 & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \cancel{\varepsilon_{11}} \\ \cancel{\varepsilon_{22}} \\ \varepsilon_{33} \\ 2\cancel{\varepsilon_{23}} \\ 2\cancel{\varepsilon_{13}} \\ 2\cancel{\varepsilon_{12}} \end{bmatrix}$$



$$\begin{cases} \varepsilon_{11} = \varepsilon_{22} = 0 \,;\ \varepsilon_{33} \neq 0 \\ \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 \end{cases}$$

$$\boxed{\sigma_{33} = \underbrace{\frac{(1-\nu)\,E}{(1+\nu)(1-2\nu)}}_{} \cdot \varepsilon_{33}}$$

M : constrained modulus
: oedometric modulus
: P-wave modulus

$M \geq E$
for $\nu \geq 0$

$$\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)} \cdot \varepsilon_{33} = \frac{\nu \cancel{E}}{\cancel{(1+\nu)(1-2\nu)}} \cdot \frac{\cancel{(1+\nu)(1-2\nu)}}{(1-\nu)\,\cancel{E}} \cdot \sigma_{33}$$

$$\boxed{\sigma_{11} = \underbrace{\frac{\nu}{1-\nu}}_{} \sigma_{33}}$$

lateral effective stress coefficient
K_0

1D Mechanical Earth Model with tectonic strains

$\varepsilon_{33} \neq 0$; $\varepsilon_{11} \neq 0$; $\varepsilon_{22} \neq 0$; $\varepsilon_{ij} = 0$ for $i \neq j$

K_0

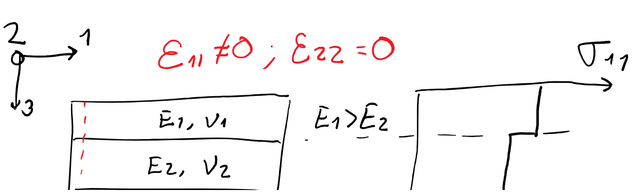
$$\begin{cases} \sigma_{11} = \dfrac{\nu}{1-\nu}\sigma_{33} + \dfrac{E}{1-\nu^2}\varepsilon_{11} + \dfrac{\nu E}{1-\nu^2}\varepsilon_{22} \\ \sigma_{22} = \dfrac{\nu}{1-\nu}\sigma_{33} + \dfrac{\nu E}{1-\nu^2}\varepsilon_{11} + \dfrac{E}{1-\nu^2}\varepsilon_{22} \end{cases}$$

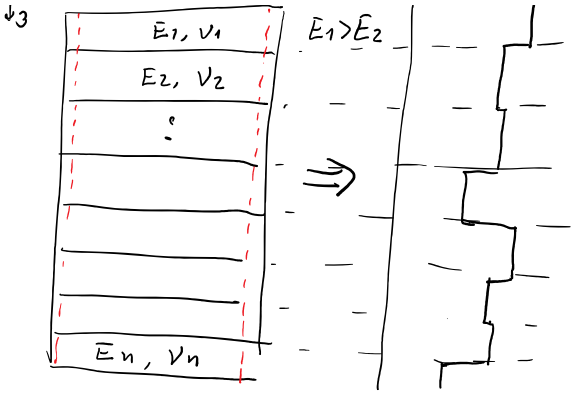
tectonic strains
ε_{11} , ε_{22}

$\sigma_{33} = S_{33} - P_p$

$$E' = \underbrace{\frac{E}{1-\nu^2}}_{}$$

Plane-strain Modulus





Plane-strain Modulus

(E, ν) from field and laboratory data

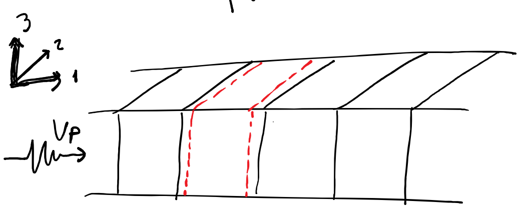
↳ lab: static, dynamic

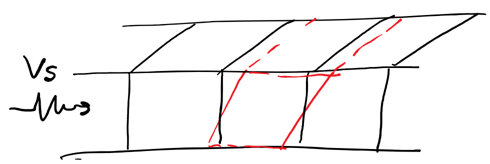
↳ field: dynamic

$(E, \nu) \rightarrow$

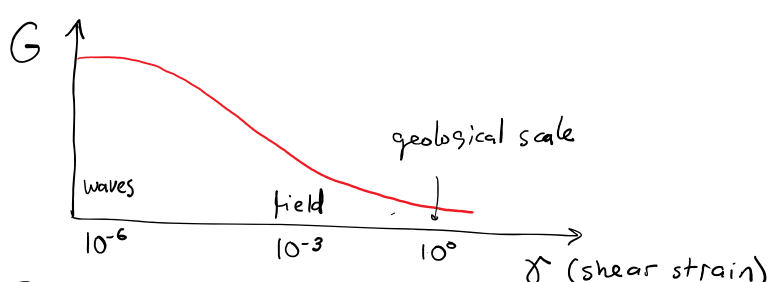
$$V_p = \sqrt{\frac{M}{\rho_{bulk}}} \quad \text{(P-WAVE)}$$

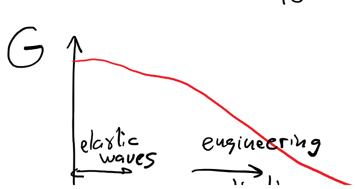
$$V_s = \sqrt{\frac{G}{\rho_{bulk}}} \quad \text{(S-WAVE)}$$

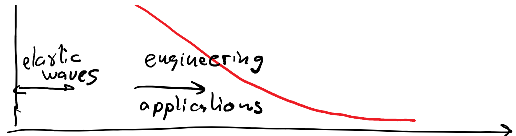




$$\begin{cases} E_{dyn} = \rho_{bulk} V_s^2 \left(\dfrac{3V_p^2 - 4V_s^2}{V_p^2 - V_s^2} \right) \\ \nu_{dyn} = \dfrac{V_p^2 - 2V_s^2}{2(V_p^2 - V_s^2)} \end{cases}$$



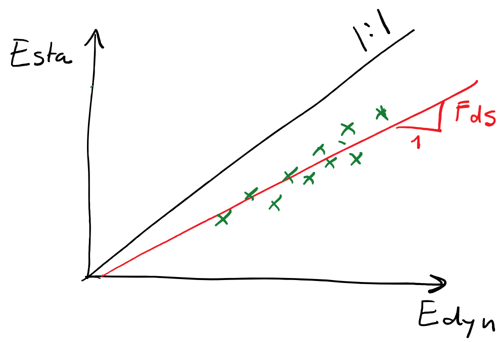




$(1/\dot{\gamma}) \rightarrow \dot{\gamma}$: strain rate

$$\boxed{E_{sta} = F_{ds} \cdot E_{dyn}} \quad ; \quad \boxed{V_{sta} \approx V_{dyn}}$$

	E_{sta}	E_{dyn}
LAB	Triaxial Test	ultrasonic elastic wave velocity
FIELD		acoustic log



$F_{ds} < 1$