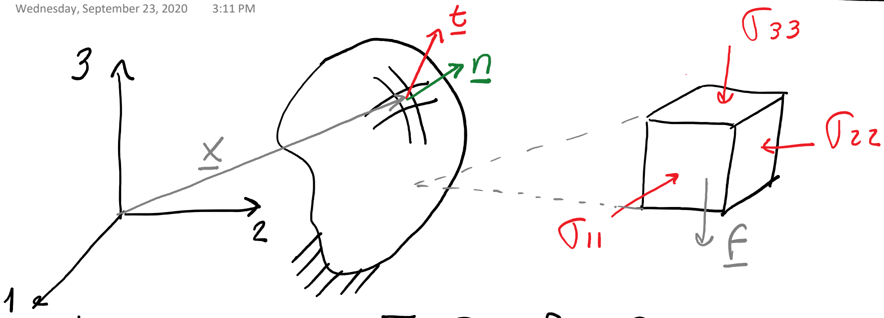
General solution for a continuum mechanics problem

Wednesday, September 23, 2020 3:11 PM



$$\left\{\begin{array}{lll} ① \text{ Equilibrium:} & \nabla\cdot\underline{\underline{\sigma}} + \underline{f} = 0 & \\ ② \text{ Kinematic:} & \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla\underline{u} + \nabla\underline{u}^T\right) & \rightarrow \text{small strains} \\ ③ \text{ Constitutive:} & \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\cdot\underline{\underline{\varepsilon}} & \rightarrow \text{linear elasticity} \end{array}\right.$$

Plan: $① \leftarrow ③ \leftarrow ②$

①, Coordinate 1: $$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\cdot\underline{\underline{\varepsilon}} \Rightarrow \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \left[\begin{array}{ccc|ccc} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ \hline & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{array}\right] \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix} ; \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} ; \quad \mu = \frac{E}{2(1+\nu)} = G$$

$$\frac{\partial}{\partial x_1}\left(\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\varepsilon_{11}\right) + \frac{\partial}{\partial x_2}\left(2\mu\varepsilon_{12}\right) + \frac{\partial}{\partial x_3}\left(2\mu\varepsilon_{13}\right) + f_1 = 0$$

$$\frac{\partial}{\partial x_1}\left(\lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\cancel{2}\mu\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right) + \frac{\partial}{\partial x_3}\left(\cancel{2}\mu\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\right) + f_1 = 0$$

$$\lambda\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\right] + 2\mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{\partial^2 u_1}{\partial x_2^2} + \mu\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}\right) + \mu\frac{\partial^2 u_1}{\partial x_3^2} + \mu\frac{\partial^2 u_3}{\partial x_3\partial x_1} + f_1 = 0$$

$$\leftarrow \frac{\partial^2 f(x,y)}{\partial x\,\partial y} = \frac{\partial^2 f(x,y)}{\partial y\,\partial x}$$

$$\lambda\left[\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}}\right] + \mu\underbrace{\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)}_{\nabla^2 u_1} + \mu\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}} + f_1 = 0$$

$$\lambda\frac{\partial}{\partial x_1}(\nabla\cdot\underline{u}) + \mu\frac{\partial}{\partial x_1}(\nabla\cdot\underline{u}) + \mu\nabla^2 u_1 + f_1 = 0 \Rightarrow \text{Coord. 1}$$

$$\lambda \frac{\partial}{\partial x_1}(\nabla\cdot\underline{U}) + \mu \frac{\partial}{\partial x_1}(\nabla\cdot\underline{U}) + \mu \nabla^2 U_1 + f_1 = 0 \Rightarrow \text{Coord. } 1$$

$$\boxed{(\lambda+\mu)\,\nabla(\nabla\cdot\underline{U}) + \mu\nabla^2\underline{U} + \underline{F} = \underline{0}}$$

$\underline{U}$: unknown

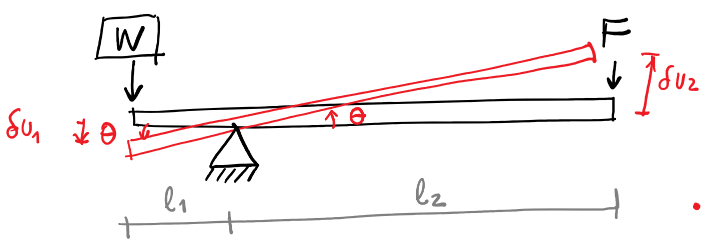
Navier's Equation

Solution

- Analytical → simple boundary conditions
  - Beltrami-Mitchell (strain compatibility)
    - $\underline{U} \rightarrow \underline{\underline{\varepsilon}}$ ✓
    - $\underline{\underline{\varepsilon}} \rightarrow U$ ✗
  - Airy's functions
    $$\nabla^4 \varphi = 0 \; ; \quad \sigma_{11} = \frac{\partial^2 \varphi}{\partial x_1^2}$$
  - Green functions (convolution) : Boussinesq
  - Examples: Kirsch, Griffith, Sneddon
- Numerical
  - Finite differences ; FLAC-Itasca
  - Finite Element Method (FEM) : Abaqus, FreeFEM++, FEniCS, Comsol

Weak formulation of continuum mechanics equations

Analogy with virtual work



- Solution #1 : Angular Momentum (Equil)

$$W \cdot l_1 = F \cdot l_2$$

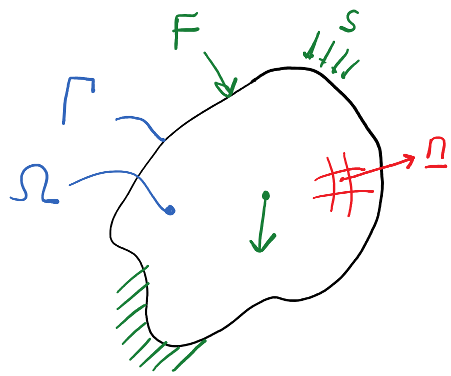
$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$

- Solution #2 : Energy Conserv ⇔ Principle of Virtual Work

$$W \cdot \delta U_1 = F \cdot \delta U_2$$

$$W \cdot (l_1 \cdot \cancel{\tan\theta}) = F \cdot (l_2 \cancel{\tan\theta})$$

$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$



Equil: $\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \underline{0}$

$$-\nabla \cdot \underline{\underline{\sigma}} = \underline{f}$$

virtual displacement

$$\underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \left(-\nabla \cdot \underline{\underline{\sigma}}\right)}_{\text{Green's Theorem}} = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}$$

$$\int_\Omega \nabla \delta \underset{\sim}{u} \cdot \underline{\underline{\sigma}} - \int_\Gamma \delta \underset{\sim}{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}$$

- Variational form
- Weak form

$$\boxed{\underbrace{\int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u})}_{\text{strain energy}} = \underbrace{\int_\Gamma \delta \underset{\sim}{u} \cdot (\underline{\underline{\sigma}}(\underline{u}) \cdot \underline{n})}_{\text{stress boundary condition}} + \underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}}_{\text{body force}}}$$

unknowns : $\underline{u}$; $\delta \underset{\sim}{u}$

actual displacement ; virtual displacement

$$\varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \underset{\sim}{\varepsilon}_{11} \cdot \sigma_{11} + \underset{\sim}{\varepsilon}_{22} \cdot \sigma_{22} + \underset{\sim}{\varepsilon}_{33} \cdot \sigma_{33} + \underset{\sim}{\varepsilon}_{12} \cdot \sigma_{12} + \ldots$$

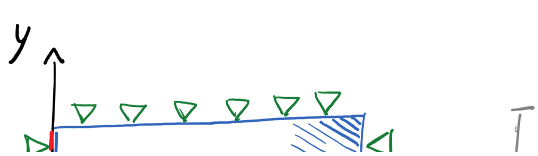
$\rightarrow E = p \cdot V$ (Energy)

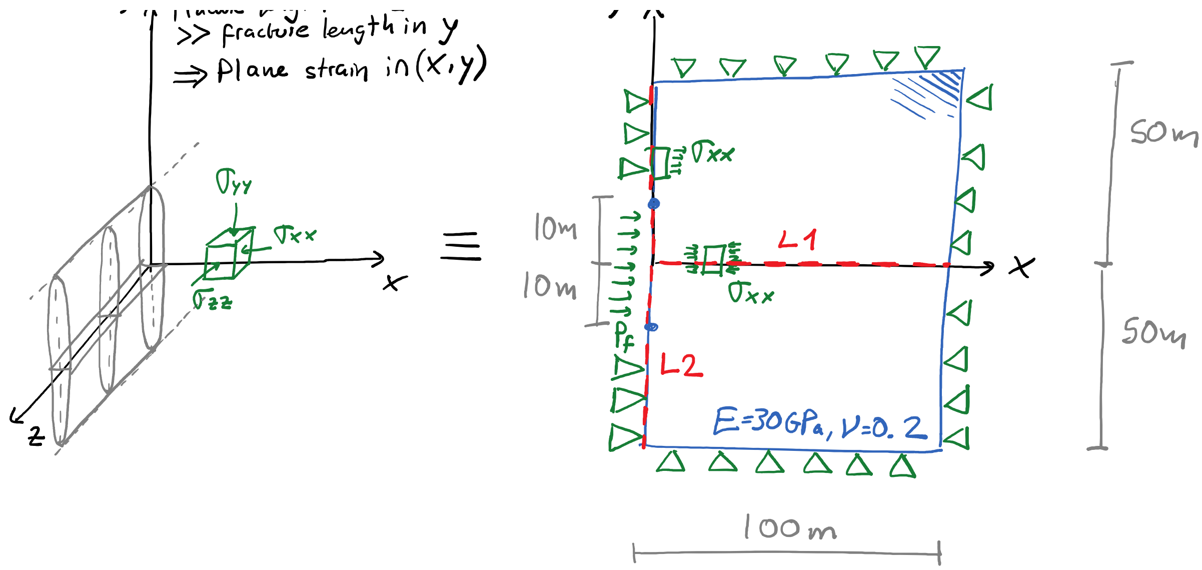
$E = \sigma \cdot \underbrace{\frac{dV}{V}}_{\varepsilon}$ (Energy per unit of volume)

$$\int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \left[\underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}(\underline{u})\right]$$

constitutive equation

y ↑ Fracture length in z
>> fracture length in y
$\Rightarrow$ Plane strain in (x, y)





>> fracture length in y
⇒ Plane strain in (x,y)
σyy
σxx
σzz
x
z
≡
σxx
10m
10m
Pf
L1
L2
σxx
x
E=30GPa, ν=0.2
50m
50m
100m