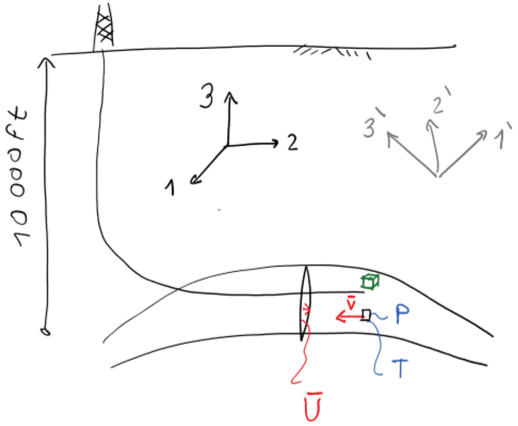
p'

q

$$P_{mud} = P_W$$

S_{Hmax}



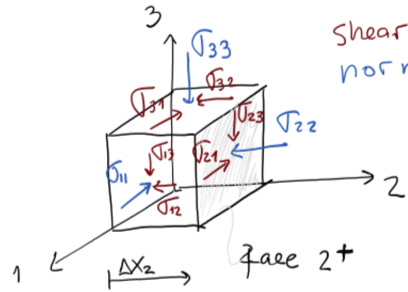
scalar : P , T

vector : $\bar{v} = [0, -0.1, 0]\ \frac{m}{day}$

$\bar{U} = [0, 1, 0]$ cm

(2nd order) tensor :

stress $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$



shear stresses
normal stresses

$\sigma_{\underset{\text{face}}{i}\,\underset{\text{direction}}{j}}$

$\sigma_{ij} = \sigma_{ji}$ symmetric

$\sigma_{ij} \in \mathbb{R}$

eigenvalues $\in \mathbb{R}$

$$\underline{\underline{S}} = \begin{bmatrix} 7000 & 0 & 0 \\ 0 & 6500 & 0 \\ 0 & 0 & 10000 \end{bmatrix} \text{psi}$$

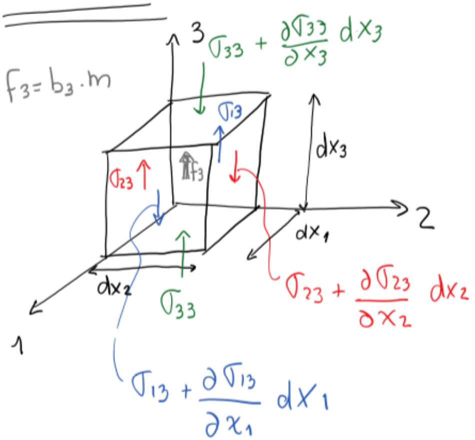
in coord system 1,2,3

principal stresses $\Updownarrow$ principal directions $\rightarrow \underline{\underline{\sigma}}^P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$\perp \leftarrow$ principal directions

$\sigma_1 \neq \sigma_2 \neq \sigma_3$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$



$Vol = dx_1\, dx_2\, dx_3$

$$\sum F_3 = m \cancel{a_3}^{\,0} = 0$$

$$\cancel{\sigma_{33}}\, dx_1 dx_2 - \left(\cancel{\sigma_{33}} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1 dx_2 +$$

$$\cancel{\sigma_{23}}\, dx_1 dx_3 - \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_1 dx_3 +$$

$$\cancel{\sigma_{13}}\, dx_2 dx_3 - \left(\cancel{\sigma_{13}} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2 dx_3 +$$

$$F_3 = 0$$

$$\frac{\partial \sigma_{33}}{\partial x_3} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{23}}{\partial x_2} dx_1 dx_2 dx_3 +$$

$$\frac{\partial \sigma_{13}}{\partial x_1} dx_1 dx_2 dx_3 + b_3 m = 0$$

$$\underline{\underline{\sum F_3}} \longrightarrow \frac{\partial \sigma_{33}}{\partial x_3} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_1} + b_3 \frac{m}{Vol} = 0$$

Cauchy's equilibrium equations

- $\frac{\partial \sigma_{ij}}{\partial x_j} + b_i \rho = 0$
- $\nabla \cdot \underline{\underline{\sigma}} + \underline{b}\, \rho = 0$

$$\begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 \rho = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 \rho = 0 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 \rho = 0 \end{cases}$$

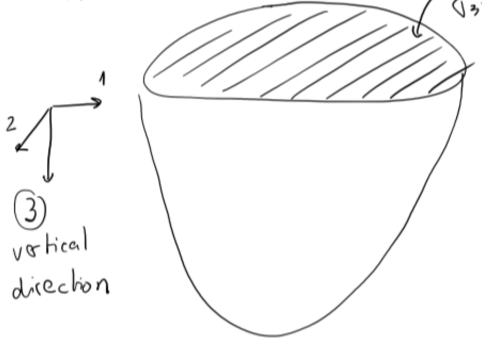
gravity in 3

$b_3 = -g$

$\frac{m}{Vol}$

half-space $\rightsquigarrow \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = 0$

free surface $\sigma_{33}(x_3=0)=0$

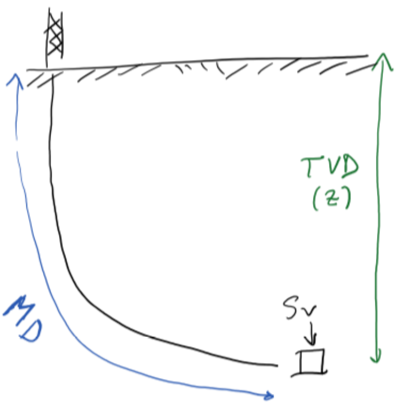


$$\begin{cases} \partial\sigma_{11} \quad \partial x_2 + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 \rho = 0 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 \rho = 0 \\ \cancel{\frac{\partial \sigma_{31}}{\partial x_1}}^{\,0} + \cancel{\frac{\partial \sigma_{32}}{\partial x_2}}^{\,0} + \boxed{\frac{\partial \sigma_{33}}{\partial x_3}} + b_3 \rho = 0 \end{cases}$$

$$\frac{\partial \sigma_{33}}{\partial x_3} = g \cdot \rho$$

$$\int_{\sigma_{33}(x_3=0)}^{\sigma_{33}(x_3)} d\sigma_{33} = \int_{x_3=0}^{x_3} g \cdot \rho \cdot dx_3$$

$$\sigma_{33}(x_3) = \int_0^{x_3} g \cdot \rho(x_3)\, dx_3$$



$$\boxed{S_v(z) = \int_0^{z} g \cdot \rho_{bulk}(z)\, dz}$$

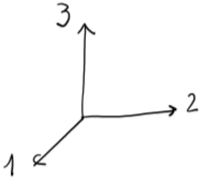
vertical depth

$\rho_{bulk}(z) = \rho_{bulk}$ $\qquad S_v(z) = \rho_{bulk} \cdot g \cdot z$

$$\frac{dS_v}{dz} = \rho_{bulk} \cdot g \begin{cases} 23 \text{ MPa/km} \\ 1 \text{ psi/ft} \end{cases}$$

$2300 \frac{kg}{m^3} \cdot 9.8 \frac{m}{s^2}$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



$\Longrightarrow$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\sigma_1 \perp \sigma_2 \perp \sigma_3$$

σ_v is a principal stress:

↳ $\sigma_v \perp \sigma_{Hmax} \perp \sigma_{hmin}$

(*)

$$\underline{\underline{\sigma}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} - \begin{bmatrix} P_p & 0 & 0 \\ 0 & P_p & 0 \\ 0 & 0 & P_p \end{bmatrix}$$

NORMAL FAULTING

$$\sigma_v \geq \sigma_{Hmax} \geq \sigma_{hmin}$$

$$\begin{bmatrix} \sigma_v & 0 & 0 \\ 0 & \sigma_{Hmax} & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$$

STRIKE-SLIP FAULTING

$$\sigma_{Hmax} \geq \sigma_v \geq \sigma_{hmin}$$

$$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_v & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$$

REVERSE F.

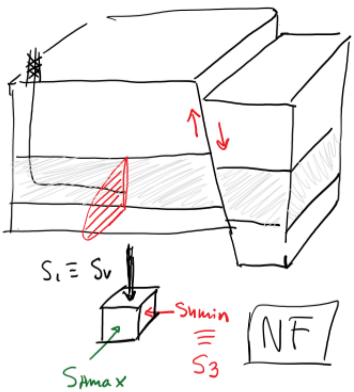
$$\sigma_{Hmax} \geq \sigma_{hmin} \geq \sigma_v$$

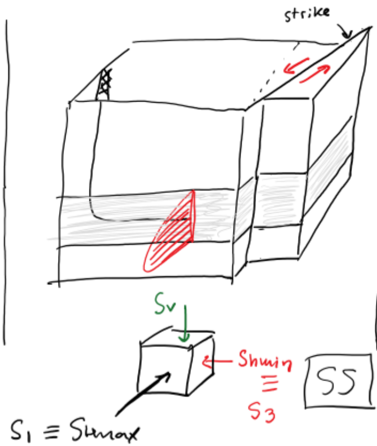
$$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_{hmin} & 0 \\ 0 & 0 & \sigma_v \end{bmatrix}$$

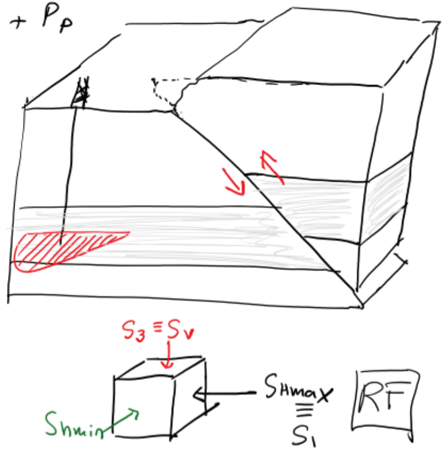
Effective stresses : $\underline{\underline{\sigma}}$; Total stresses $\underline{\underline{S}} = \underline{\underline{\sigma}} + P_p \underline{\underline{I}}$ [*]

$$\begin{bmatrix} S_v & 0 & 0 \\ 0 & S_{Hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix} \qquad \begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_v & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix} \qquad \begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_{hmin} & 0 \\ 0 & 0 & S_v \end{bmatrix}$$

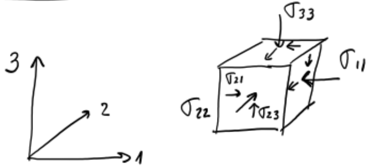
$S_v = \sigma_v + P_p$; $S_{Hmax} = \sigma_{Hmax} + P_p$; $S_{hmin} = \sigma_{hmin} + P_p$

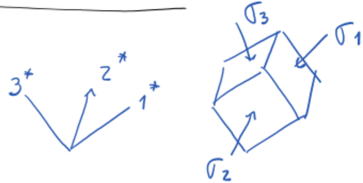




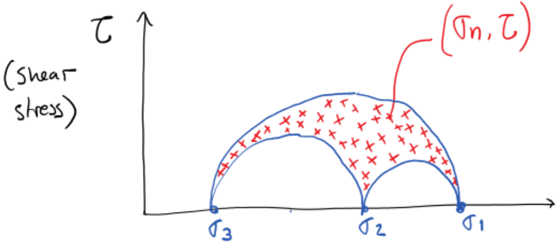


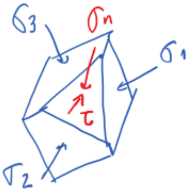
Stress Invariants and graphical representation

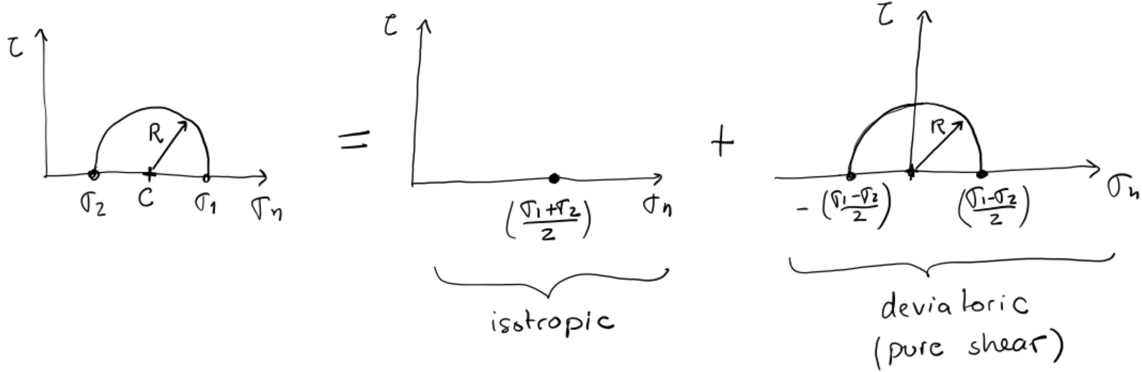




$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$







$$
\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}
=
\underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{isotropic}}
+
\underbrace{\begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{deviatoric}}
$$

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$$

$$\boxed{\underline{\underline{\sigma}} = \sigma_m \underline{\underline{I}} + \underline{\underline{S}}_d}$$

Invariants (do not change wrt coord system)

$$\Rightarrow I_1(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3 \quad \Longrightarrow \quad \sigma_m = \frac{I_1(\underline{\underline{\sigma}})}{3}$$

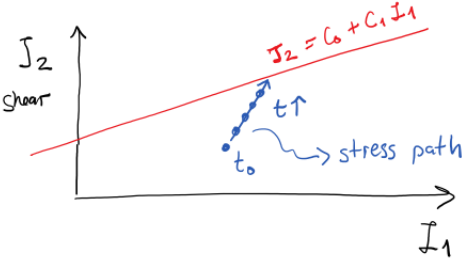
$$I_2(\underline{\underline{\sigma}}) = \sigma_{11}\sigma_{22} + \sigma_{11}\cdot\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$I_3(\underline{\underline{\sigma}}) = \det(\underline{\underline{\sigma}}) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \leftarrow$$

$$J_1(\underline{\underline{s}}_d) = 0$$

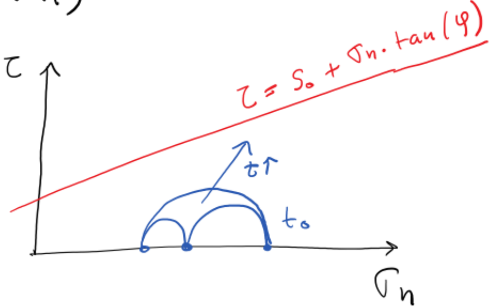
$$\Rightarrow J_2(\underline{\underline{s}}_d) = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]$$

$$J_3(\underline{\underline{s}}_d) = (\sigma_1-\sigma_m)\cdot(\sigma_2-\sigma_m)\cdot(\sigma_3-\sigma_m)$$



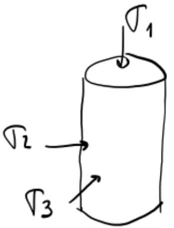
(compression)

=



soft sediments
(soil mechanics)

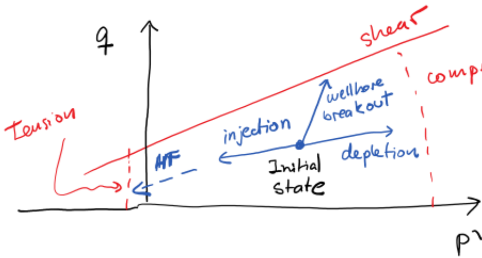
$$\begin{cases} p' = \sigma_m^{\text{ effective}} = I_1(\underline{\sigma})/3 \\ q = \sqrt{3 J_2} \end{cases}$$



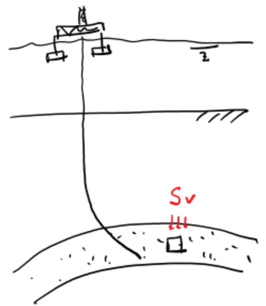
$\Rightarrow$ Axisymmetric stress conditions

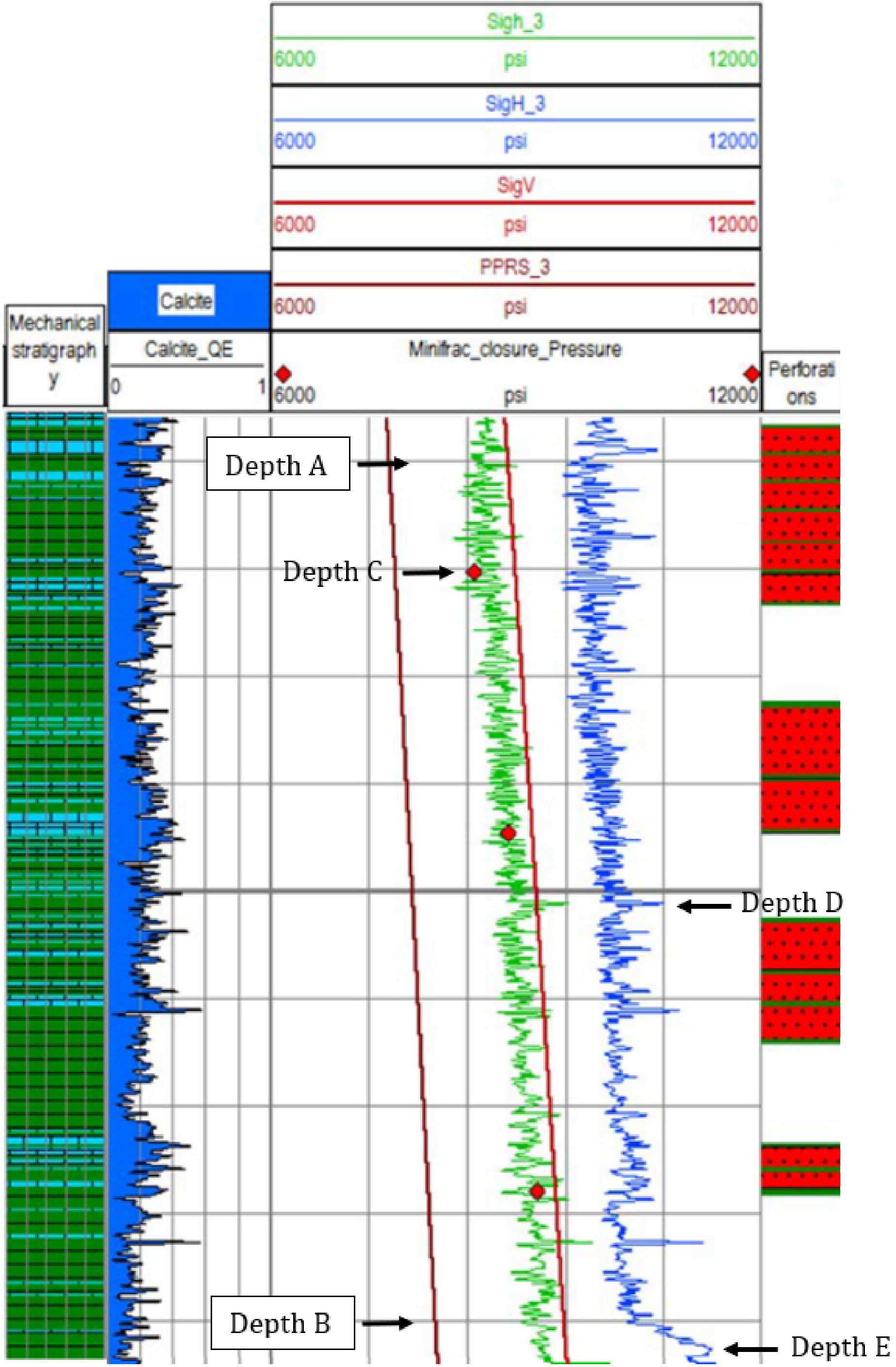
$\sigma_2 = \sigma_3$

$$\begin{cases} p' = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \dfrac{\sigma_1 + 2\sigma_3}{3} \\ q = \sqrt{\cancel{3}\left\{\dfrac{1}{\cancel{6}_{\cancel{3}}}\left[(\sigma_1-\sigma_3)^2 + \cancel{(\sigma_1-\sigma_3)^2}\right]\right\}} = \sigma_1 - \sigma_3 \end{cases}$$



stress path
$\Downarrow$
track state of stress as a function of time





Sigh_3
6000
psi
12000
SigH_3
6000
psi
12000
SigV
6000
psi
12000
PPRS_3
6000
psi
12000
Calcite
Mechanical stratigraphy
Calcite_QE
0
1
Minifrac_closure_Pressure
6000
psi
12000
Perforations
Depth A
Depth C
Depth D
Depth B
Depth E

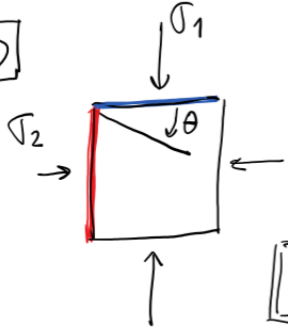
p

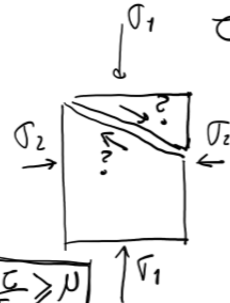
I_1

J_2

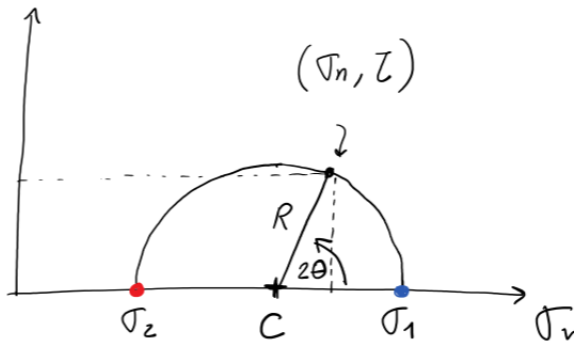
Stress projection on a plane

$\boxed{2D}$

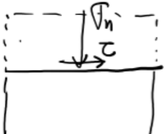




$$\boxed{\frac{\tau}{\sigma_n} \geq \mu}$$



$\theta = 0$



$\sigma_n = \sigma_1\ ;\ \tau = 0$

σ_n, τ

$\theta = 90°$

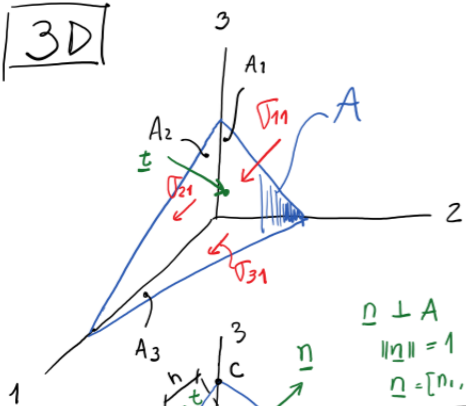
$\sigma_n = \sigma_2\ ;\ \tau = 0$

$$\begin{cases} \sigma_n = C + R \cdot \cos 2\theta \\ \tau = R \cdot \sin 2\theta \end{cases}$$

$$C = (\sigma_1 + \sigma_2)/2$$

$$R = (\sigma_1 - \sigma_2)/2$$

$\boxed{3D}$



$$\sum F_1 = 0$$

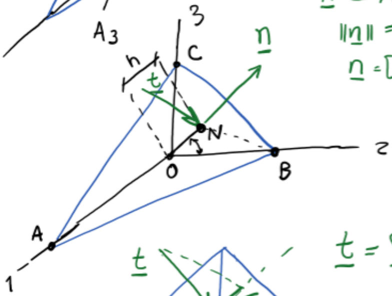
① $\sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3 = t_1 A$

② Vol  $= \frac{1}{3} A \cdot h$

$$\underbrace{= \frac{1}{3} A_1 \cdot \overline{OA}}$$

$$\cancel{\tfrac{1}{3}} A h = \cancel{\tfrac{1}{3}} A_1 \overline{OA}$$

$$A_1 = \frac{h}{\overline{OA}} A$$



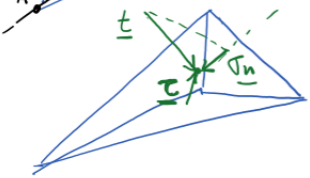
$\underline{n} \perp A$

$\|\underline{n}\| = 1$

$\underline{n} = [n_1, n_2, n_3]$

$$\begin{cases} A_1 = \cos A\hat{O}N \cdot A = n_1 A \\ A_2 = \cos B\hat{O}N \cdot A = n_2 A \\ A_3 = \underbrace{\cos C\hat{O}N}_{\text{cosine directors}} \cdot A = n_3 A \end{cases}$$

$$\boxed{A_i = n_i A}$$



$\underline{t} = \underline{\sigma_n} + \underline{\tau}$

→ ①+② $\sigma_{11} n_1 \cancel{A} + \overset{=\sigma_{12}}{\sigma_{21}} n_2 \cancel{A} + \overset{=\sigma_{13}}{\sigma_{31}} n_3 \cancel{A} = t_1 \cancel{A}$

$\Downarrow$ $\sigma_{21} = \sigma_{12}$ (angular momentum equil.)

→ $$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

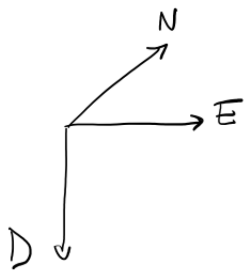
row × column

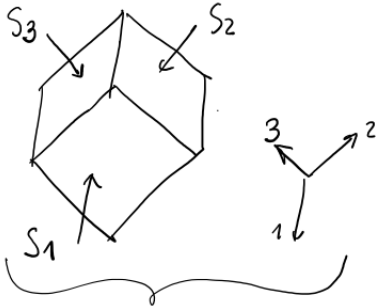
$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}$$

$$\begin{cases} \sigma_n = \underline{t} \cdot \underline{n} & \Big\} \text{ proyection of } \underline{\sigma_n} \text{ on } \underline{n} \\ \tau = \sqrt{\|\underline{t}\|^2 - \|\sigma_n\|^2} & \Big\} \Longleftarrow \|\underline{t}\|^2 = \|\sigma_n\|^2 + \|\tau\|^2 \end{cases}$$

Geographical coordinate system

N - E - D





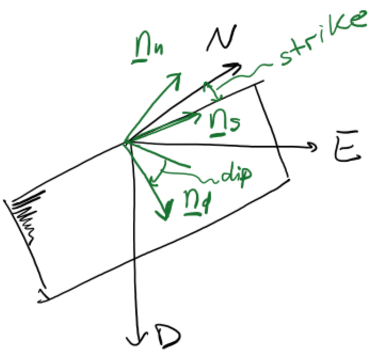
Principal stresses and direction

$$\underline{\underline{S}}_P = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

$$\underline{\underline{S}}_G = \begin{bmatrix} S_{NN} & S_{NE} & S_{ND} \\ S_{EN} & S_{EE} & S_{ED} \\ S_{DN} & S_{DE} & S_{DD} \end{bmatrix}$$

$$\boxed{\underline{\underline{S}}_G = R_{PG}^T \, \underline{\underline{S}}_P \, R_{PG}}$$

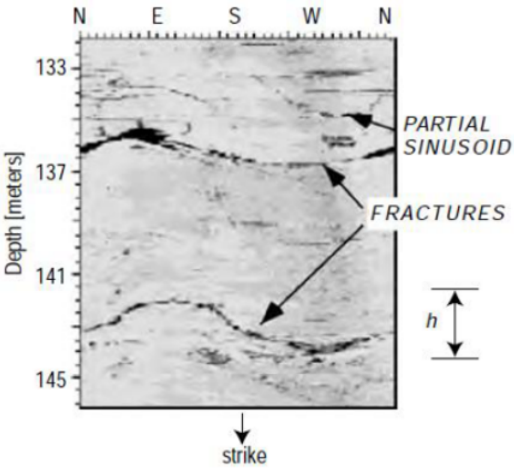
$$R_{PG} = \underbrace{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}_{f(\alpha, \beta, \gamma)}$$



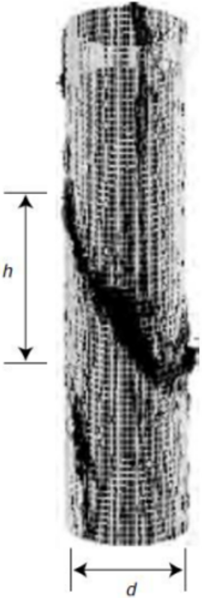
τ

σ_n

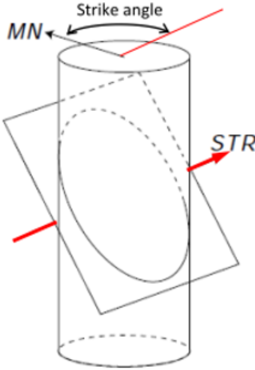
Un-wrapped image
(ultrasonic)



3D-representation



Interpretation



Dip = $\tan^{-1}(h/d)$

(Zoback 2013, RM, Ch 5.3)

○

$$(\sigma_n, \tau)$$

τ/σ_n

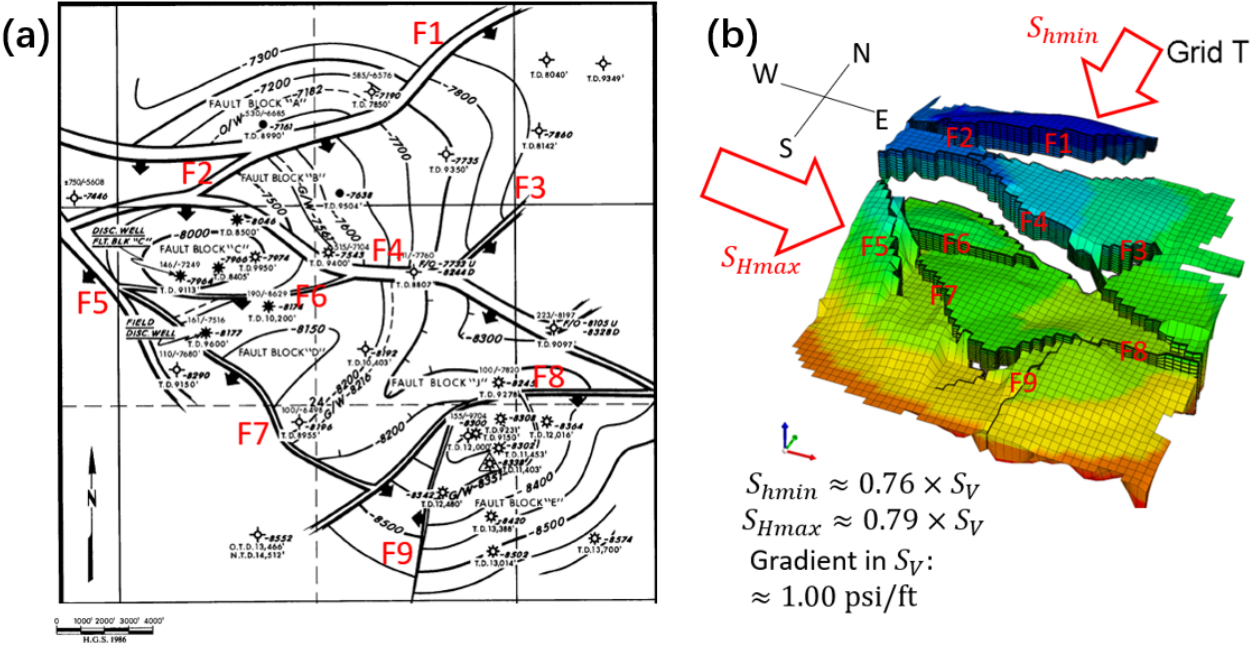
$$\mu = 0.5 \pm 0.1$$

$$60^\circ \pm 10^\circ$$

$$80^\circ \pm 5^\circ$$

$$10^\circ \pm 10^\circ$$

$$60^\circ \pm 5^\circ$$


(a)
F1
F2
F3
F4
F5
F6
F7
F8
F9
-7300
-7200
-7182
FAULT BLOCK "A"
O/W
FAULT BLOCK "B"
FAULT BLOCK "C"
FAULT BLOCK "D"
FAULT BLOCK "J"
FAULT BLOCK "E"
-7800
-7700
-7600
-7500
-8000
-8150
-8200
-8300
-8400
-8500
DISC. WELL
FLT. BLK "C"
FIELD
DISC. WELL
N
0 1000' 2000' 3000' 4000'
H.G.S. 1986
(b)
S_{hmin}
Grid T
W
N
E
S
S_{Hmax}
F1
F2
F3
F4
F5
F6
F7
F8
F9
$S_{hmin} \approx 0.76 \times S_V$
$S_{Hmax} \approx 0.79 \times S_V$
Gradient in S_V:
≈ 1.00 psi/ft

S_{hmin}

σ_n/σ_v

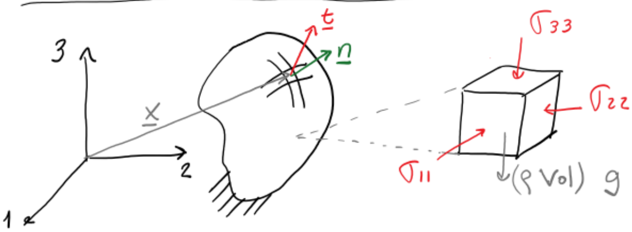
τ / σ_v

$P_p/$

$$z = 0.44$$

$$\mu = 0.6$$

General solution to a continuum mechanics problem



$$\left\{\begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \rho\, \underline{a} & \rightarrow \text{Equilibrium (Cauchy's)} \\ \underline{\underline{\varepsilon}} = F_1(\underline{u}) & \rightarrow \text{Kinematic eq} \\ \underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}}) & \rightarrow \text{Constitutive equations} \end{array}\right.$$

Kinematic eq:
- small strains ←
- large strains

Constitutive equations:
- linear isotropic elastic solid ←
- TVI (VTI)
- orthorhombic
- visco-elasticity
- plasticity

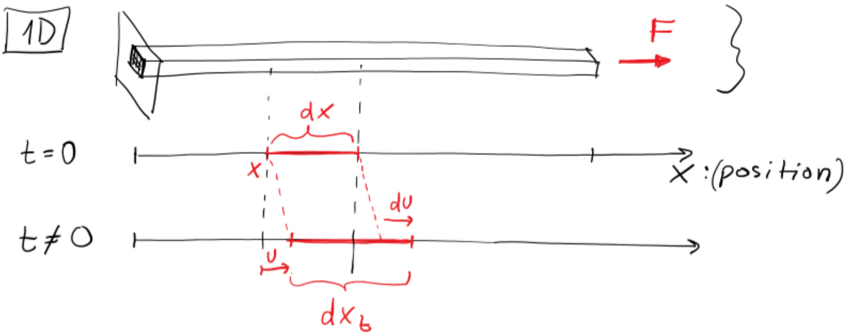
$$\nabla \cdot \left[F_2(\underline{\underline{\varepsilon}}) \right] + \underline{f} = \rho\, \underline{a}$$

$$\boxed{\nabla \cdot \left[F_2 \left[F_1 (\underline{u}) \right] \right] + \underline{f} = \rho\, \underline{a}}$$

displacement $\underline{u} \rightarrow \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}}$

Kinematic Equations (small strains) $\boxed{\underline{\underline{\varepsilon}} = F_1(\underline{u})}$

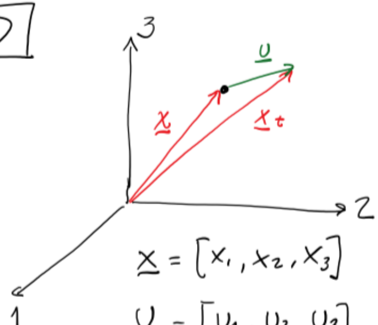
1D



$$\varepsilon = \frac{dx_t - dx}{dx} = \frac{[\cancel{x}+\cancel{u}+\cancel{dx}+du-(\cancel{x}+\cancel{u})]-[\cancel{x}+\cancel{dx}-\cancel{x}]}{[\cancel{x}+dx-\cancel{x}]}$$

$$\boxed{\varepsilon = \frac{du}{dx}}$$

3D



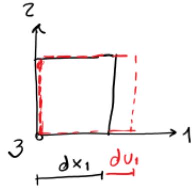
$\underline{x} = [x_1, x_2, x_3]$

$\underline{u} = [u_1, u_2, u_3]$

Jacobian $\updownarrow$ $\underline{\underline{\varepsilon}}$

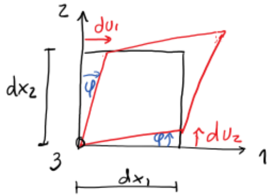
$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial u_2}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$



$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$+ \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \cdots & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \cdots & \cdots & 0 \end{bmatrix}$$

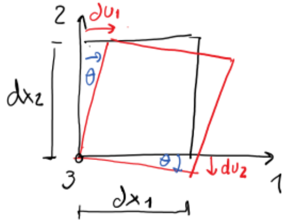


$$\tan\varphi = \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}$$

$$\varepsilon_{12} = \frac{1}{2}(2 \cdot \tan\varphi) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$\underline{\underline{\varepsilon}}$

$$+ \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \cdots & \cdots & 0 \end{bmatrix}$$



$\underline{\underline{\omega}}$

$$\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right) = \omega_{12}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$\hookrightarrow$ symmetric
$\hookrightarrow$ real values

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ eigenvalues $\Rightarrow$ principal strains $\Updownarrow$

$\underbrace{\text{eigen}}_{\text{own}}$ vectors $\rightarrow$ principal directions

$$I_1(\underline{\underline{\varepsilon}}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{vol}$$

$$\boxed{\varepsilon_{vol} = \frac{Vol(t) - Vol_0}{Vol_0}}$$

$$\underbrace{\underline{\underline{\sigma}}}_{\text{stress}} \underbrace{=F_2}\left(\underbrace{\underline{\underline{\varepsilon}}}_{\text{strain}}\right)$$

Superposition $\left\{\begin{array}{l}\text{space}\\ \text{time}\end{array}\right\}$ Green's functions

linear relationships $\left\{\begin{array}{l} F_2(A+B) = F_2(A) + F_2(B) \\ F_2(c \cdot A) = c \cdot F_2(A) \end{array}\right.$

$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}} \longleftrightarrow \underset{\sim}{y} = \underbrace{b}_{\text{constant}} \cdot \underset{\sim}{x}$$

9	81	9
6	36	6

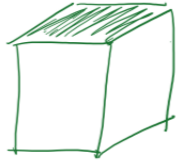
Voigt Notation

$$\left(\underset{3\times 3}{\underline{\underline{\sigma}}} \rightarrow \underset{6\times 1}{\underline{\sigma}} \right) \rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

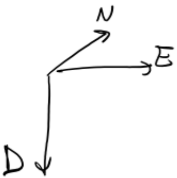
$$\underset{6\times 1}{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}} = \underbrace{\underset{6\times 6}{\left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot \\ \hline \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ C_{61} & C_{62} & \cdot & \cdot & \cdot & C_{66} \end{array}\right]}}_{\text{Stiffness Matrix}} \underset{6\times 1}{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\cdot\varepsilon_{23} \\ 2\cdot\varepsilon_{13} \\ 2\ \varepsilon_{12} \end{bmatrix}}$$

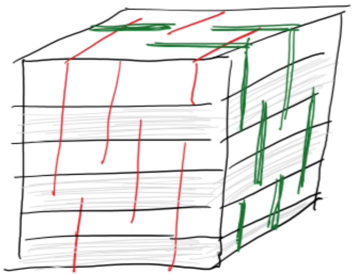
Shear decoupling $\left\langle \begin{array}{l} \text{normal} \leftrightarrow \text{shear} \\ \text{shear} \leftrightarrow \text{shear} \end{array} \right\}$ does not apply for plasticity

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{array}\right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2 \cdot \varepsilon_{23} \\ 2 \cdot \varepsilon_{13} \\ 2\,\varepsilon_{12} \end{bmatrix}$$



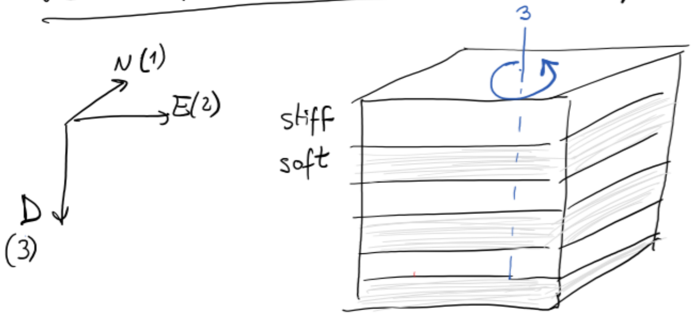
12 coeff → $\boxed{\text{9 coeff}}$ → Orthorhombic





$$E_V \neq E_{H(N\text{-}S)} \neq E_{H(E\text{-}W)}$$

Vertical Transverse Isotropy



$$E_V < E_H$$

$$\rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}}_{\boxed{\text{5 indepent coefficients}}}$$

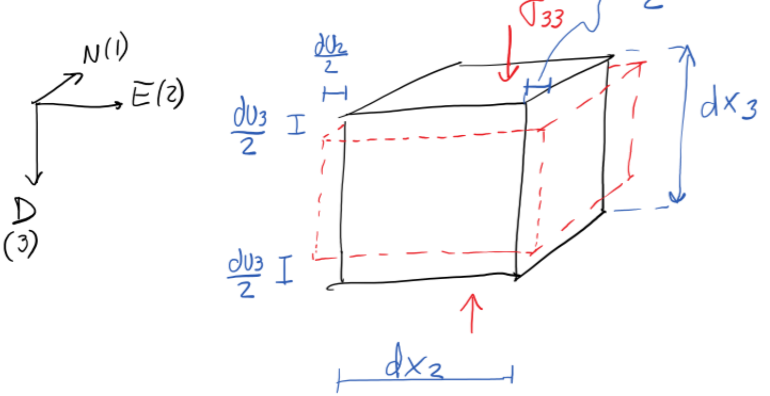
C_{11}, C_{33}

C_{12}, C_{13}

$C_{44}, \cancel{C_{66}}$

$$C_{66} = \frac{C_{11} - C_{12}}{2}$$

Isotropy (Linear elastic)



$$\varepsilon_{33} = \frac{dU_3}{dx_3}$$

$$\varepsilon_{22} = \frac{dU_2}{dx_2}$$

$$\varepsilon_{11} = \frac{dU_1}{dx_2}$$

$$\boxed{E \stackrel{\text{def}}{=} \frac{\sigma_{33}}{\varepsilon_{33}}}$$

Young's modulus

$$\boxed{\nu \stackrel{\text{def}}{=} -\frac{\varepsilon_{11}}{\varepsilon_{33}} = -\frac{\varepsilon_{22}}{\varepsilon_{33}}}$$

Poisson's ratio

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 \\ 0 \\ \sigma_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \underline{\underline{\varepsilon}} = \begin{bmatrix} -(\nu/E)\sigma_{33} \\ -(\nu/E)\sigma_{33} \\ \sigma_{33}/E \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \begin{array}{l} \sigma_{33} \\ \sigma_{22} \\ \sigma_{11} \end{array}$$

$$\downarrow$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \leftarrow G = \frac{E}{2(1+\nu)}$$

Isotropic loading $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$$\rightarrow \varepsilon_{11} = \frac{1-2\nu}{E}\sigma_m$$

$$\varepsilon_{vol} = \frac{3(1-2\nu)}{E}\sigma_m \Rightarrow K = \frac{E}{3(1-2\nu)}$$

$$\underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}}_{\text{compliance matrix}} \cdot \underline{\underline{\sigma}}$$

E, ν (2 indep. coeff)

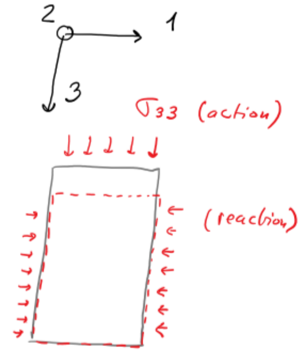
$$\underbrace{\underline{\underline{\underline{\underline{D}}}}^{-1}} \cdot \underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}^{-1} \cdot \underline{\underline{\underline{\underline{D}}}}} \cdot \underline{\underline{\sigma}}$$

stiffness matrix

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$$

$$\left[\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{array}\right] = \frac{E}{(1+\nu)(1-2\nu)} \overbrace{\left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{2} \end{array}\right]} \left[\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{array}\right]$$

Uniaxial-strain loading (stress path)



$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\left[\begin{array}{ccc|ccc}
1-\nu & \nu & \nu & & & \\
\nu & 1-\nu & \nu & & 0 & \\
\nu & \nu & 1-\nu & & & \\
\hline
 & & & \frac{1-2\nu}{2} & 0 & 0 \\
 & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\
 & & & 0 & 0 & \frac{1-2\nu}{\nu}
\end{array}\right]
\begin{bmatrix} \cancel{\varepsilon_{11}} \\ \cancel{\varepsilon_{22}} \\ \varepsilon_{33} \\ \hline \cancel{2\varepsilon_{23}} \\ \cancel{2\varepsilon_{13}} \\ \cancel{2\varepsilon_{12}} \end{bmatrix}
$$

$$
\begin{cases} \varepsilon_{11} = \varepsilon_{22} = 0 \; ; \; \varepsilon_{33} \neq 0 \\ \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 \end{cases}
$$

$$
\boxed{\sigma_{33} = \underbrace{\frac{(1-\nu)\,E}{(1+\nu)(1-2\nu)}}_{M} \cdot \varepsilon_{33}}
$$

M : constrained modulus
: oedometric modulus
: P-wave modulus

$M \geq E$ for $\nu \geq 0$

$$
\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)} \cdot \varepsilon_{33} = \frac{\nu \cancel{E}}{\cancel{(1+\nu)(1-2\nu)}} \cdot \frac{\cancel{(1+\nu)(1-2\nu)}}{(1-\nu)\,\cancel{E}} \cdot \sigma_{33}
$$

$$
\boxed{\sigma_{11} = \underbrace{\frac{\nu}{1-\nu}}_{K_0} \, \sigma_{33}}
$$

lateral effective stress coefficient

K_0

1D Mechanical Earth Model with tectonic strains

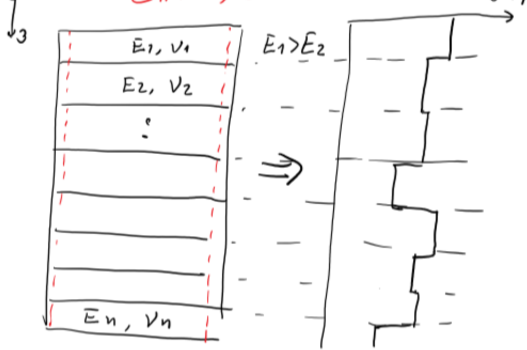
$\varepsilon_{33} \neq 0$; $\varepsilon_{11} \neq 0$; $\varepsilon_{22} \neq 0$; $\varepsilon_{ij} = 0$ for $i \neq j$

$$\begin{cases} \sigma_{11} = \overbrace{\dfrac{\nu}{1-\nu}}^{K_0}\sigma_{33} + \dfrac{E}{1-\nu^2}\varepsilon_{11} + \dfrac{\nu E}{1-\nu^2}\varepsilon_{22} \\ \sigma_{22} = \dfrac{\nu}{1-\nu}\sigma_{33} + \dfrac{\nu E}{1-\nu^2}\varepsilon_{11} + \dfrac{E}{1-\nu^2}\varepsilon_{22} \end{cases} \rightarrow \text{tectonic strains } \varepsilon_{11},\ \varepsilon_{22}$$

$\sigma_{33} = S_{33} - P_p$

$$E' = \underbrace{\frac{E}{1-\nu^2}}_{\text{Plane-strain Modulus}}$$

$\varepsilon_{11} \neq 0$; $\varepsilon_{22} = 0$



(E, ν) from field and laboratory data

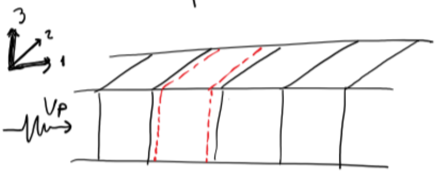
↳ lab: static, dynamic

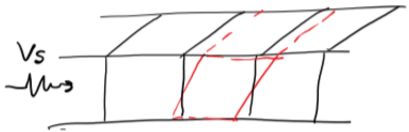
↳ field: dynamic

$(E, \nu) \rightarrow$

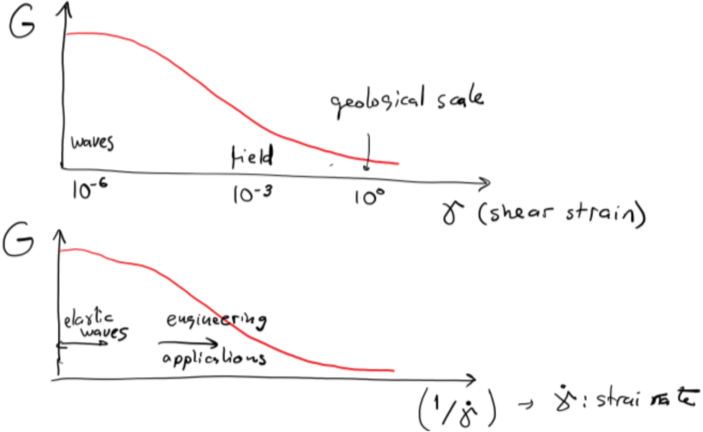
$$V_P = \sqrt{\frac{M}{\rho_{bulk}}} \quad \text{(P-WAVE)}$$

$$V_S = \sqrt{\frac{G}{\rho_{bulk}}} \quad \text{(S-WAVE)}$$



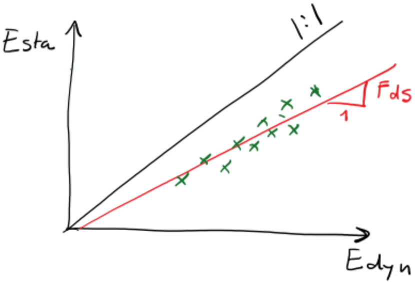


$$\begin{cases} E_{dyn} = \rho_{bulk} V_S^2 \left(\dfrac{3V_P^2 - 4V_S^2}{V_P^2 - V_S^2} \right) \\[2ex] \nu_{dyn} = \dfrac{V_P^2 - 2 \cdot V_S^2}{2(V_P^2 - V_S^2)} \end{cases}$$

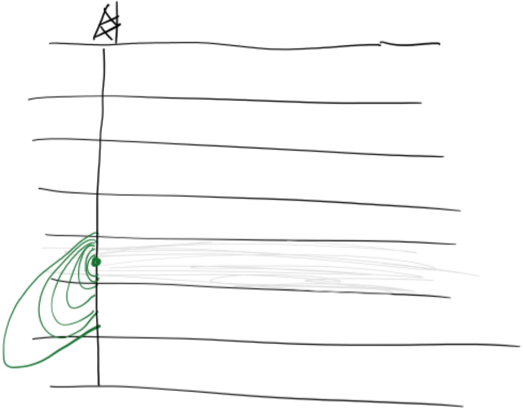

G
geological scale
waves
field
10^{-6}
10^{-3}
10^{0}
γ (shear strain)
G
elastic waves
engineering applications
$(1/\dot{\gamma}) \to \dot{\gamma}$: strain rate

$$\boxed{E_{sta} = F_{ds} \cdot E_{dyn}} \quad ; \quad \boxed{V_{sta} \approx V_{dyn}}$$

	E_{sta}	E_{dyn}
LAB	Triaxial Test	ultrasonic elastic wave velocity
FIELD	—	acoustic log



$F_{ds} < 1$



S_3

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} +\dfrac{1}{E_h} & -\dfrac{\nu_h}{E_h} & -\dfrac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\dfrac{\nu_h}{E_h} & +\dfrac{1}{E_h} & -\dfrac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\dfrac{\nu_v}{E_v} & -\dfrac{\nu_v}{E_v} & +\dfrac{1}{E_v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{G_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{G_v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{G_h} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
$$

$$G_h = \frac{E_h}{2(1 + \nu_h)}$$

G_v

$$E_h = \frac{(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2\,C_{13}^2\right]}{C_{11}C_{33} - C_{13}^2}$$

$$E_v = C_{33} - \frac{2\,C_{13}^2}{C_{11} + C_{12}}$$

$$\nu_h = \frac{C_{12}C_{33} - C_{13}^2}{C_{11}C_{33} - C_{13}^2}$$

$$\nu_v = \frac{C_{13}}{C_{11} + C_{12}}$$

$$G_v = C_{44}$$

$$G_h = C_{66} = \frac{C_{11} - C_{12}}{2}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

$$C_{11} = \left[\frac{1}{(1 - \nu_h)E_v - 2\nu_v^2 E_h} \right] \left(\frac{E_h E_v - \nu_v^2 E_h^2}{1 + \nu_h} \right)$$

$$C_{33} = \left[\frac{1}{(1-\nu_h)E_v - 2\nu_v^2 E_h}\right](E_v^2 - \nu_h E_v^2)$$

$$C_{12} = \left[\frac{1}{(1-\nu_h)E_v - 2\nu_v^2 E_h}\right]\left(\frac{\nu_v^2 E_h^2 + \nu_h E_h E_v}{1+\nu_h}\right)$$

$$C_{13} = \left[\frac{1}{(1 - \nu_h)E_v - 2\nu_v^2 E_h} \right] (\nu_v E_h E_v)$$

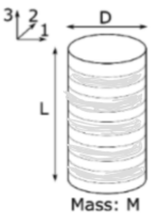
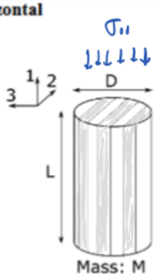
$$C_{66} = \frac{C_{11} - C_{12}}{2} = G_h = \frac{E_h}{2(1 + \nu_h)}$$

C_{44}

VTI Static Elastic Properties

- Conventional Method
  - ↳ Axisymmetric Triaxial Cell
  - ↳ Deviatoric Loading Stress Path
    - ↳ $\sigma_r = cst$
    - ↳ $\Delta(\sigma_a - \sigma_r) = \Delta\sigma_a \quad , \quad \sigma_a \geqslant \sigma_r$

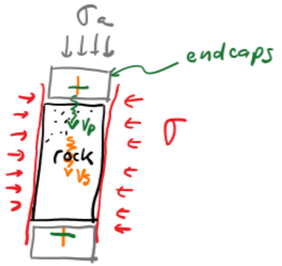
 →

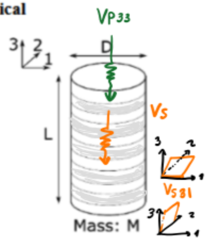
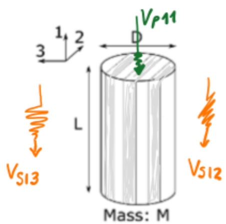
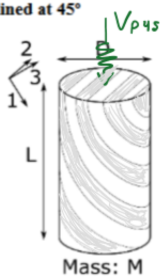
Sample	Quasi-static
Vertical 	Vertical Young modulus $$E_v = \frac{\Delta\sigma_{33}}{\Delta\varepsilon_{33}}\Big\vert_{\sigma_{11},\sigma_{22}}$$ Vertical Poisson ratio $$\nu_v = -\frac{1}{2}\left(\frac{\Delta\varepsilon_{11}}{\Delta\varepsilon_{33}} + \frac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{33}}\right) = -\frac{\overbrace{\Delta\varepsilon_{11}}^{\nu_{13}}}{\Delta\varepsilon_{33}} = -\frac{\overbrace{\Delta\varepsilon_{22}}^{\nu_{23}}}{\Delta\varepsilon_{33}}$$
Horizontal 	Horizontal Young modulus $$E_h = \frac{\Delta\sigma_{11}}{\Delta\varepsilon_{11}}$$ Vertical Poisson ratio $$\nu_v = -\frac{\Delta\varepsilon_{33}}{\Delta\varepsilon_{11}} = \nu_{31} \leftarrow = \nu_{13}$$ Horizontal Poisson ratio $$\nu_h = -\frac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{11}} = \nu_{21} \Rightarrow = \nu_{12}$$

} 4 parameters

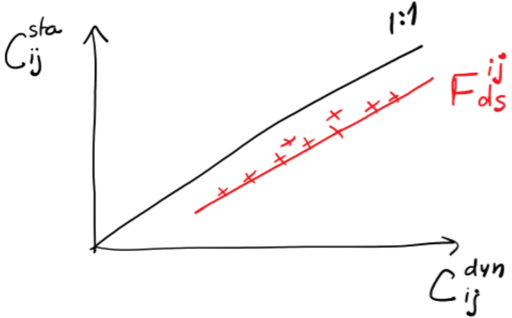
VTI Dynamic Elastic Properties

- Conventional Method



Sample	Dynamic
Vertical  	P-wave stiffness perpendicular to bedding $C_{33} = \rho(V_{p33})^2$ S-wave stiffness perpendicular to bedding $C_{44} = \frac{1}{2}[\rho(V_{s31})^2 + \rho(V_{s32})^2]$ $C_{44} = \rho(V_{s31})^2 = \rho(V_{s32})^2$
Horizontal  	P-wave stiffness parallel to bedding $C_{11} = \rho(V_{p11})^2$ S-wave stiffness perpendicular to bedding $C_{44} = \rho(V_{s13})^2$ ✓ S-wave stiffness in the plane of bedding $C_{66} = \rho(V_{s12})^2$
Inclined at 45°  	Off-diagonal stiffness $C_{13} = -C_{44} + [4\rho^2 V_{p45}^4$ $-2\rho V_{p45}^2(C_{11} + C_{33} + 2C_{44})$ $+(C_{11} + C_{44})(C_{33} + C_{44})]^{1/2}$

Dynamic to static conversion



Quantification of anisotropy

Static

Young modulus anisotropy

$$\frac{E_h}{E_v} \; ; \; E_h > E_v$$

Poisson's ratio anisotropy

$$\frac{\nu_v}{1-\nu_h}$$ } effective lateral stress coefficient

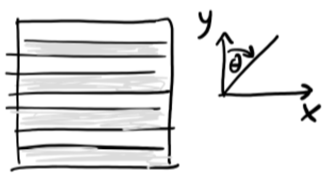
Dynamic

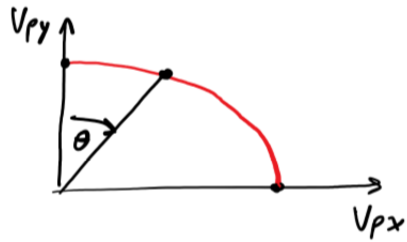
Thomsen Parameters :

$$\varepsilon = \frac{V_{p11}^2 - V_{p33}^2}{2\,V_{p33}^2} = \frac{C_{11} - C_{33}}{2\,C_{33}}$$

$$\gamma = \frac{C_{66} - C_{44}}{2\,C_{44}}$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2\,C_{33}\,(C_{33} - C_{44})}$$

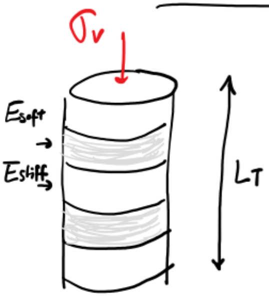




Weak anisotropy

$$V_p(\theta) = V_{p33}\left[1 + \delta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta\right]$$

Iso-stress (Reuss Average)



$$\langle E_v \rangle = \frac{\sigma_v}{\langle \varepsilon_v \rangle}$$

$$\hookrightarrow \langle \varepsilon_v \rangle = \frac{\sigma_v}{\langle E_v \rangle} \quad (1)$$

L_{soft}, L_{stiff}

$$(2) \quad \langle \varepsilon_v \rangle = \left(\frac{\sigma_v}{E_{stiff}} \cdot L_{stiff} + \frac{\sigma_v}{E_{soft}} \cdot L_{soft} \right) \cdot \frac{1}{L_T}$$

$$= \frac{\sigma_v}{E_{stiff}} \cdot \underbrace{\frac{L_{stiff}}{L_T}}_{f_{stiff}} + \frac{\sigma_v}{E_{soft}} \cdot \underbrace{\frac{L_{soft}}{L_T}}_{f_{soft}}$$

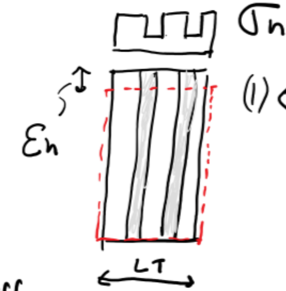
$(1,2) \rightarrow$

$$\frac{\cancel{\sigma_v}}{\langle E_v \rangle} = \frac{\cancel{\sigma_v}}{E_{stiff}} \cdot f_{stiff} + \frac{\cancel{\sigma_v}}{E_{soft}} \cdot f_{soft}$$

$$\boxed{\langle E_v \rangle = \left(\frac{f_{stiff}}{E_{stiff}} + \frac{f_{soft}}{E_{soft}} \right)^{-1}}$$

Iso-strain (Voigt Average)

L_{soft}, L_{stiff}



$$(1) \quad \langle E_h \rangle \; \frac{\langle \sigma_n \rangle}{\varepsilon_n}$$

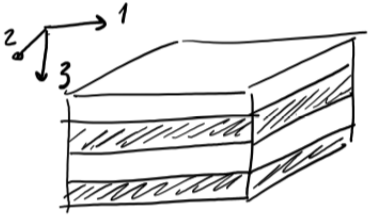
$$(2) \quad \langle \sigma_h \rangle = \sigma_n^{soft} \cdot \underbrace{\frac{L_{soft}}{L_T}} + \sigma_n^{stiff} \cdot \underbrace{\frac{L_{stiff}}{L_T}}$$

$(1,2)$

$$\cancel{\varepsilon_h} \langle E_h \rangle = \cancel{\varepsilon_h} E_{soft} \cdot f_{soft} + \cancel{\varepsilon_h} \cdot E_{stiff} \cdot f$$

$$\boxed{\langle E_h \rangle = E_{soft} \cdot f_{soft} + E_{stiff} \cdot f_{stiff}}$$

VTI stiffness matrix



$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} \\ -\frac{\nu_h}{E_h} & \frac{1}{E_h} & -\frac{\nu_v}{E_v} \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & \frac{1}{E_v} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix}$$

For isotropic loading $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$$\varepsilon_{11} = \varepsilon_{22} = \left(\frac{1-\nu_h}{E_h} - \frac{\nu_v}{E_v} \right) \sigma_m$$

$$\varepsilon_{33} = \left(-\frac{2\nu_v}{E_v} + \frac{1}{E_v} \right) \sigma_m$$

$$\varepsilon_{vol} = \left[2\left(\frac{1-\nu_h}{E_h} - \frac{\nu_v}{E_v} \right) + \left(\frac{1-2\nu_v}{E_v} \right) \right] \sigma_m$$

$$\longrightarrow \boxed{K_{VTI} = \left[\frac{2(1-\nu_h)}{E_h} + \frac{1-4\nu_v}{E_v} \right]^{-1}}$$

if $\nu_h = \nu_v$; $E_h = E_v$

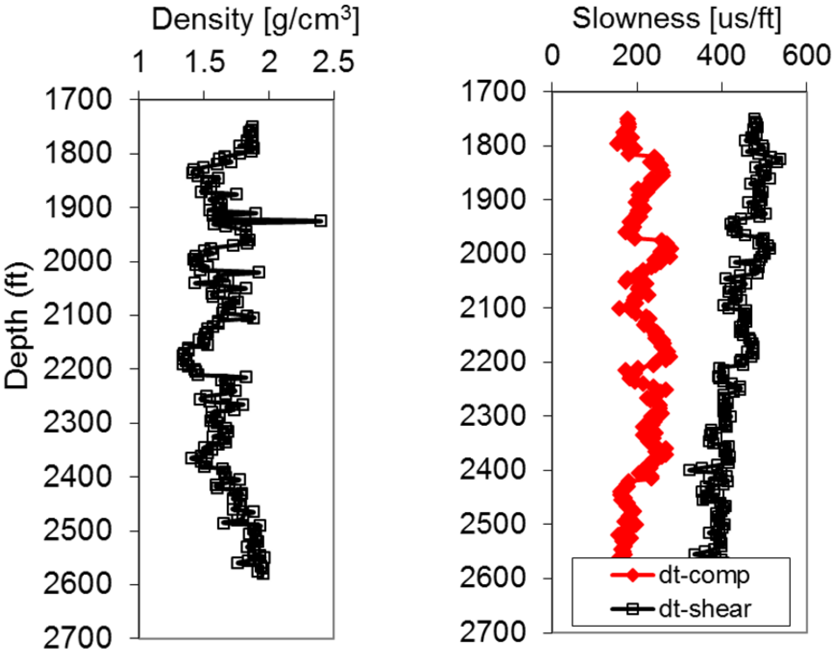
$$K = \left[\frac{2-2\nu+1-4\nu}{E} \right]^{-1} = \frac{E}{3(1-2\nu)} \quad \checkmark$$

$$E_{static} = 0.65 \cdot E_{dynamic}$$

$$E'_{static} = E_{static}/(1-\nu^2)$$

$$\varepsilon_{Hmax} = 0.0015$$

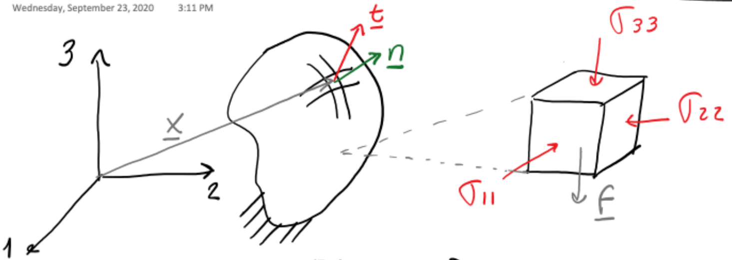
$$\varepsilon_{hmin} = 0$$



Density [g/cm3]
1
1.5
2
2.5
Depth (ft)
1700
1800
1900
2000
2100
2200
2300
2400
2500
2600
2700
Slowness [us/ft]
0
200
400
600
dt-comp
dt-shear

2- General solution for a continuum mechanics problem

Wednesday, September 23, 2020 3:11 PM



① Equilibrium : $\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = 0$

② Kinematic : $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^{T}\right)$ → small strains

③ Constitutive : $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$ → linear elasticity

Plan : ① ← ③ ← ②

① , Coordinate 1: $\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$

$$\underline{\sigma} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} \Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} ; \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} = G$$

$$\frac{\partial}{\partial x_1}\left(\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\,\varepsilon_{11}\right) + \frac{\partial}{\partial x_2}\left(2\mu\,\varepsilon_{12}\right) + \frac{\partial}{\partial x_3}\left(2\mu\,\varepsilon_{13}\right) + f_1 = 0$$

$$\frac{\partial}{\partial x_1}\left(\lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\cancel{2}\mu\,\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right) + \frac{\partial}{\partial x_3}\left(\cancel{2}\mu\,\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\right) + f_1 = 0$$

$$\lambda\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\right] + 2\mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{\partial^2 u_1}{\partial x_2^2} + \mu\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}\right) + \mu\frac{\partial^2 u_1}{\partial x_3^2} + \mu\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + f_1 = 0$$

$$\leftarrow \frac{\partial^2 f(x,y)}{\partial x\,\partial y} = \frac{\partial^2 f(x}{\partial y\,\partial}$$

$$\lambda\left[\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}}\right] + \mu\underbrace{\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)}_{\nabla^2 u_1} + \mu\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}} + f_1 = 0$$

$$\lambda\frac{\partial}{\partial x_{\underline{1}}}(\nabla\cdot\underline{u}) + \mu\frac{\partial}{\partial x_{\underline{1}}}(\nabla\cdot\underline{u}) + \mu\nabla^2 u_{\underline{1}} + f_{\underline{1}} = 0 \Rightarrow \text{Coord. } 1$$

$$\boxed{(\lambda+\mu)\,\nabla(\nabla\cdot\underline{u}) + \mu\nabla^2\underline{u} + \underline{f} = \underline{0}} \qquad \underline{u} : \text{unknown}$$

Navier's Equation

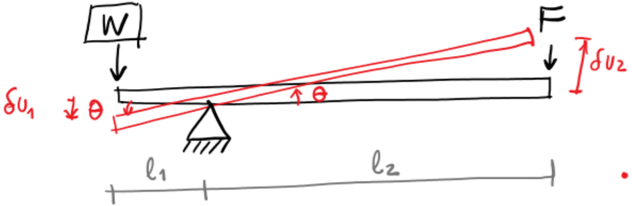
$\boxed{\text{Solution}}$

- Analytical
  ↓
  simple boundary conditions
  - Beltrami-Mitchell (strain compatibility)
    - $\underline{u} \rightarrow \underline{\underline{\varepsilon}}$ ✓
    - $\underline{\underline{\varepsilon}} \rightarrow u$ ✗
  - Airy's functions

    $$\nabla^4 \varphi = 0 \; ; \quad \sigma_{11} = \frac{\partial^2 \varphi}{\partial x_1^2}$$
  - Green functions (convolution) : Boussinesq
  - Examples: Kirsch, Griffith, Sneddon
- Numerical
  - Finite differences ; FLAC-Itasca
  - Finite Element Method (FEM) : Abaqus, FreeFEM++, FEniCS, Comsol

Weak formulation of continuum mechanics equations

Analogy with virtual work



- Solution #1 : Angular Momentum (Equil)

$$W \cdot l_1 = F \cdot l_2$$

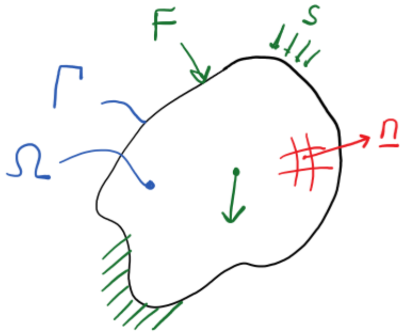
$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$

- Solution #2 : Energy Conserv $\Leftrightarrow$ Principle of Virtual Work

$$W \cdot \delta u_1 = F \cdot \delta u_2$$

$$W \cdot (l_1 \cdot \cancel{\tan\theta}) = F \cdot (l_2 \cancel{\tan\theta})$$

$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$



Equil: $\nabla \cdot \underline{\underline{\sigma}} + \underline{F} = \underline{0}$

$$-\nabla \cdot \underline{\underline{\sigma}} = \underline{F}$$

virtual displacement

$$\underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \left(-\nabla \cdot \underline{\underline{\sigma}}\right)}_{\text{Green's Theorem}} = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{F}$$

$$\int_\Omega \nabla \delta \underset{\sim}{u} \cdot \underline{\underline{\sigma}} - \int_\Gamma \delta \underset{\sim}{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{F}$$

- Variational form
- Weak form

$$\boxed{\underbrace{\int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u})}_{\text{strain energy}} = \underbrace{\int_\Gamma \delta \underset{\sim}{u} \cdot \left(\underline{\underline{\sigma}}(\underline{u}) \cdot \underline{n}\right)}_{\text{stress boundary condition}} + \underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \underline{F}}_{\text{body force}}}$$

unknowns : $\underset{\text{actual displacement}}{\underline{u}}$; $\underset{\text{virtual displacement}}{\delta \underset{\sim}{u}}$

$$\varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \underset{\sim}{\varepsilon}_{11} \cdot \sigma_{11} + \underset{\sim}{\varepsilon}_{22} \cdot \sigma_{22} + \underset{\sim}{\varepsilon}_{33} \cdot \sigma_{33} + \underset{\sim}{\varepsilon}_{12} \cdot \sigma_{12} + \ldots$$

$$\rightarrow E = P \cdot V \quad \text{(Energy)}$$

$$E = \sigma \cdot \underbrace{\frac{dV}{V}}_{\varepsilon} \quad \text{(Energy per unit of volume)}$$

$$\int_{\Omega} \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \int_{\Omega} \varepsilon(\nabla \delta \underset{\sim}{u}) : \left[\underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}(\underline{u}) \right]$$

constitutive equation

σ_{xx}

σ_{yy}

E

ν

UCS

T_s

σ_{rr}

$\sigma_{\theta\theta}$

$\sigma_{r\theta}$

x

y

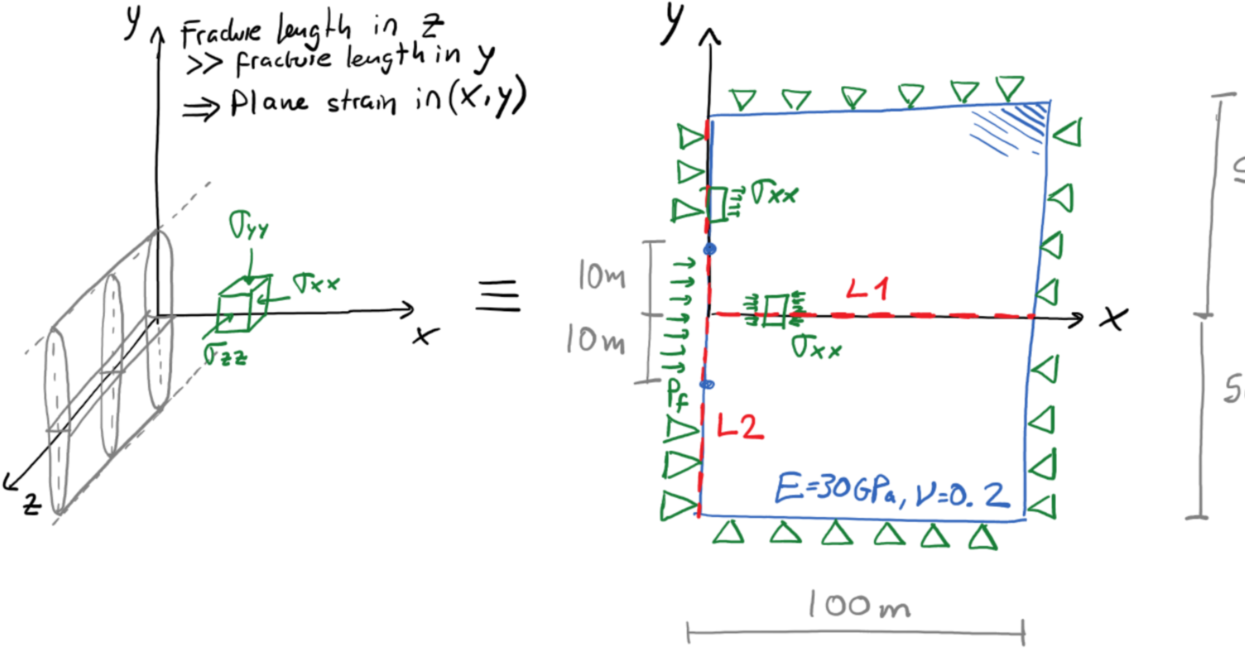
(r, θ)

r

σ_{xy}

c

$(x, y) =$



Fracture length in z
>> fracture length in y
⇒ Plane strain in (x,y)
σyy
σxx
σzz
x
y
z
≡
10m
10m
σxx
L1
L2
σxx
Pf
E=30GPa, ν=0.2
100m

$y =$

P_c

$$P_p$$

ε_{vol}

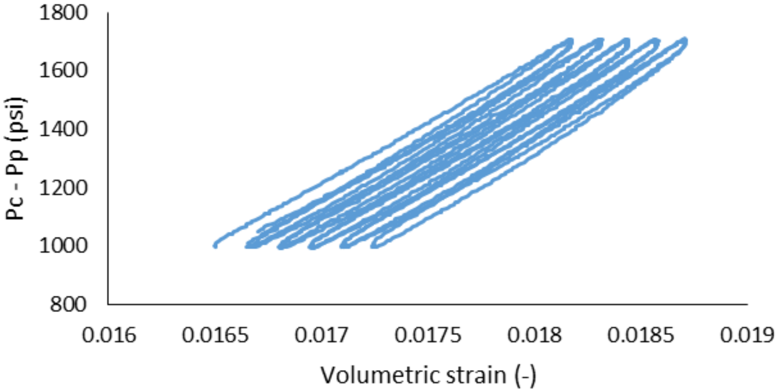
$$\sigma_{mean}$$

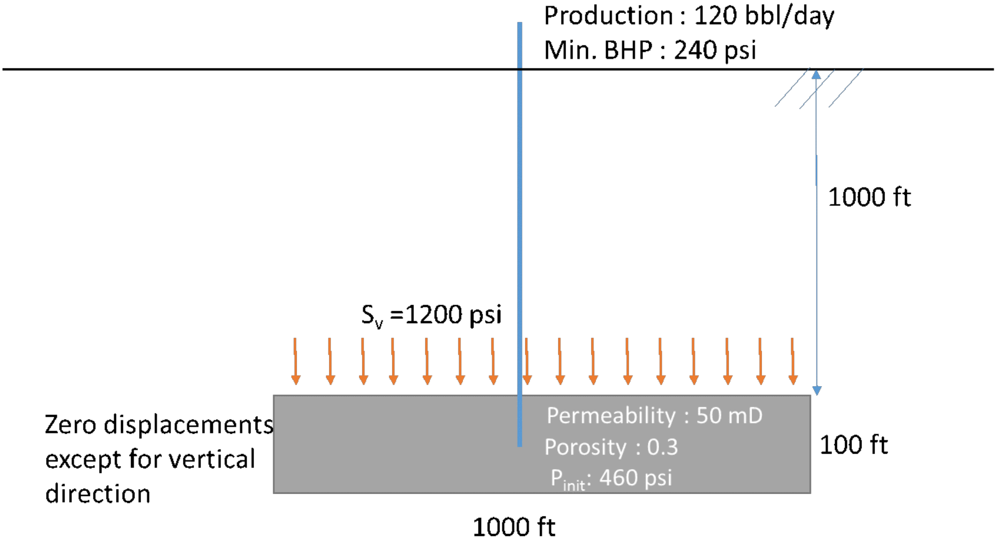
α

T_{ch}

α_h

$$\alpha_v$$


1800
1600
1400
1200
1000
800
Pc - Pp (psi)
0.016
0.0165
0.017
0.0175
0.018
0.0185
0.019
Volumetric strain (-)



Production : 120 bbl/day
Min. BHP : 240 psi
1000 ft
S_v =1200 psi
Zero displacements except for vertical direction
Permeability : 50 mD
Porosity : 0.3
P_{init}: 460 psi
100 ft
1000 ft

$$\alpha \frac{(1-2\nu)}{(1-\nu)}$$

(p', q)

$$
\begin{cases}
\nabla \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{Equilibrium} \\
\underline{\underline{\varepsilon}} = \dfrac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) & \text{Kinematic} \\
\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}} + 3\alpha_L K \theta \underline{\underline{I}}; \quad \theta = T - T_0 & \text{Constitutive} \\
\dfrac{\partial \theta}{\partial t} = \dfrac{k_T}{\rho c_v} \nabla^2 \theta + \dfrac{3\beta K T_0}{\rho c_v} \dfrac{\partial \varepsilon_{vol}}{\partial t} & \text{Diffusivity}
\end{cases}
$$

$$\begin{cases} \dfrac{\partial p}{\partial t} - \dfrac{k}{\mu} M \nabla^2 p = -\alpha M \dfrac{\partial \varepsilon_{vol}}{\partial t} + \beta_e M \dfrac{\partial T}{\partial t} & \text{Pore pressure diffusivity} \\ \dfrac{\partial T}{\partial t} - \kappa_T \nabla^2 T = -\dfrac{\alpha_d}{m_d} \dfrac{\partial \varepsilon_{vol}}{\partial t} + \dfrac{\beta_e}{m_d} \dfrac{\partial p}{\partial t} & \text{Temperature diffusivity} \end{cases}$$

$$\beta_e = \alpha\beta_d + \phi(\beta_f - \beta_s)$$

$$\beta_f > \beta_s$$

$$\beta_d = \beta_s;$$

K

$$\alpha_d = K\ \beta_d$$

$$m_d = \frac{c_d}{T_0}$$

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}} + 3\alpha_L K\theta \underline{\underline{I}}; \;\; \theta = T - T_0$$

$$\varepsilon_{11} = \varepsilon_{22} = 0$$

$\sigma_{33} =$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{33} \end{bmatrix} + \begin{bmatrix} 3\alpha_L K\theta \\ 3\alpha_L K\theta \\ 3\alpha_L K\theta \end{bmatrix}$$

σ_{11}

σ_{33}

$$\begin{cases} \sigma_{11} = \dfrac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{33} + 3\alpha_L K\theta \\ \sigma_{33} = \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{33} + 3\alpha_L K\theta \end{cases}$$

ε_{33}

$$\sigma_{11} = \left(\frac{\nu}{1-\nu}\right)\sigma_{33} + \left(\frac{1-2\nu}{1-\nu}\right)3\alpha_L K\theta$$

$$K = E/[3(1 - 2\nu)]$$

$$\sigma_{11} = \left(\frac{\nu}{1-\nu} \right) \sigma_{33} + \frac{\alpha_L E}{1-\nu} \theta$$

$$\frac{\partial \sigma_{11}}{\partial \theta} = +\frac{\alpha_L E}{1-\nu}$$

$$\frac{\partial \sigma_{11}}{\partial \theta} = 0.1 \frac{\mathrm{MPa}}{{}^{\circ}\mathrm{C}}$$

$$\alpha_L = 10^{-5}$$

$1/^\circ$

$$E = 10$$

$\nu = 0.2$

S_1

S_3

$$P_{pore\ fluid}$$

$$\sigma_1 = UCS + q\sigma_3$$

S_0

φ

$$(J_2)^{1/2}$$

$$P_W = P_p$$

$$w_{BO}$$

$$0^\circ < w_{BO} < 60^\circ$$

$$w_{BO} > 60^\circ$$

P_W

$$dP_p/dz$$

$$dS_v/dz$$

$$\phi = \phi_0 \exp\left(-\beta \sigma_v\right)$$

$$e = e_0 - C_c \ln\left(\frac{\sigma_v}{1\ \mathrm{MPa}}\right)$$

λ_p

S_v

$$dq = 3dp'$$

G

p'_o

$$\phi_{CS} = 24^\circ$$

λ

κ

e_o

ε_q

e

$$dq = 0.9\, dp'$$

$$p' = 400$$

$$d\varepsilon_{p'}^{e} = \frac{\kappa}{\upsilon}\frac{dp'}{p'};\; d\varepsilon_{q}^{e} = \frac{dq}{3G}$$

$$\begin{bmatrix} d\varepsilon_{p'}^{p} \\ d\varepsilon_{q}^{p} \end{bmatrix} = \frac{\lambda-\kappa}{\upsilon p'(M^2+\eta^2)} \begin{bmatrix} M^2-\eta^2 & 2\eta \\ 2\eta & \frac{4\eta^2}{M^2-\eta^2} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$v = 1 + e$$

$$\eta = q/p'$$

$$de = -\upsilon d\varepsilon_p$$

$$dp'_o = d\varepsilon^p_{p'} \frac{\upsilon}{\lambda - \kappa} p'_o$$

$$\Delta S_{yy} = \nu(\Delta S_{xx} + \Delta S_{zz})$$

h_f

$$E' = 8.9 \times 10^6$$

$$x_f$$

$$w_{w,0}$$

p_n

p_n

-6

$$p' - q$$

10^{-6}

$$E_{sta}/E_{dyn}$$