

scalar :  $P, T$   
 vector :  $\bar{v} = [0, -0.1, 0] \frac{m}{day}$   
 $\bar{U} = [0, 1, 0] \text{ cm}$   
 tensor :  $(2\text{nd order})$

stress  $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

$$\sigma_{ij} = \sigma_j i$$

$\sigma_{ij}$   
 face direction  
 symmetric  
 $\sigma_{ij} \in \mathbb{R}$   
 eigenvalues  $\in \mathbb{R}$

$$\underline{\underline{\sigma}} = \begin{bmatrix} 7000 & 0 & 0 \\ 0 & 6500 & 0 \\ 0 & 0 & 10000 \end{bmatrix} \text{ psi}$$

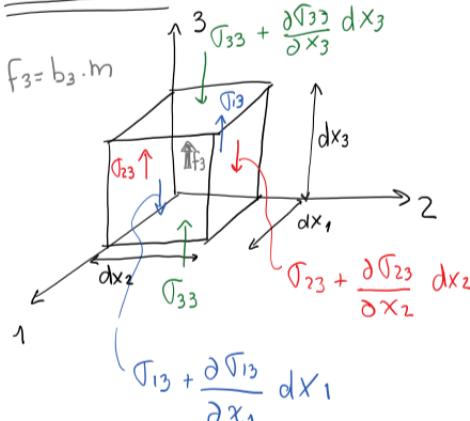
in coord system 1,2,3

principal stresses  $\uparrow$   $\rightarrow \underline{\underline{\sigma}}^P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

principal directions  $\leftarrow$

$$\sigma_1 \neq \sigma_2 \neq \sigma_3$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$



$$V_01 = dx_1 dx_2 dx_3$$

$$\sum F_3 = m \cancel{a}_3^0 = 0$$

$$\cancel{T_{33}} dx_1 dx_2 - \left( \cancel{T_{33}} + \frac{\partial \cancel{T_{33}}}{\partial x_3} dx_3 \right) dx_1 dx_2 +$$

$$\cancel{T_{23}} dx_1 dx_3 - \left( \cancel{T_{23}} + \frac{\partial \cancel{T_{23}}}{\partial x_2} dx_2 \right) dx_1 dx_3 +$$

$$\cancel{T_{13}} dx_2 dx_3 - \left( \cancel{T_{13}} + \frac{\partial \cancel{T_{13}}}{\partial x_1} dx_1 \right) dx_2 dx_3 +$$

$$\underline{F_3} = 0$$

$$\frac{\partial \cancel{T_{33}}}{\partial x_3} dx_1 dx_2 dx_3 + \frac{\partial \cancel{T_{23}}}{\partial x_2} dx_1 dx_2 dx_3 +$$

$$\frac{\partial \cancel{T_{13}}}{\partial x_1} dx_1 dx_2 dx_3 + b_3 m = 0$$

$$\underline{\underline{\sum F_3}} \rightarrow \frac{\partial \cancel{T_{33}}}{\partial x_3} + \frac{\partial \cancel{T_{23}}}{\partial x_2} + \frac{\partial \cancel{T_{13}}}{\partial x_1} + b_3 \frac{m}{V_01} = 0$$

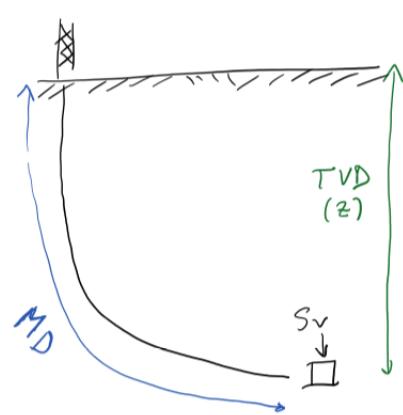
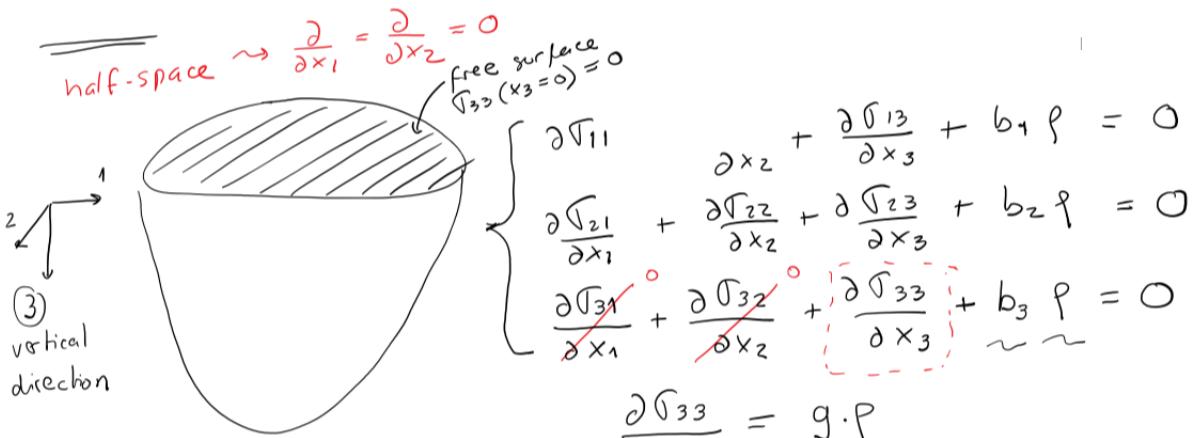
Cauchy's equilibrium equations

$$\bullet \frac{\partial \underline{\underline{T}}_{ij}}{\partial x_j} + b_i \rho = 0$$

$$\bullet \nabla \cdot \underline{\underline{T}} + b \rho = 0$$

$$\left\{ \begin{array}{l} \frac{\partial \underline{\underline{T}}_{11}}{\partial x_1} + \frac{\partial \underline{\underline{T}}_{12}}{\partial x_2} + \frac{\partial \underline{\underline{T}}_{13}}{\partial x_3} + b_1 \rho = 0 \\ \frac{\partial \underline{\underline{T}}_{21}}{\partial x_1} + \frac{\partial \underline{\underline{T}}_{22}}{\partial x_2} + \frac{\partial \underline{\underline{T}}_{23}}{\partial x_3} + b_2 \rho = 0 \\ \frac{\partial \underline{\underline{T}}_{31}}{\partial x_1} + \frac{\partial \underline{\underline{T}}_{32}}{\partial x_2} + \frac{\partial \underline{\underline{T}}_{33}}{\partial x_3} + b_3 \rho = 0 \end{array} \right.$$

gravity in 3  $\downarrow$   
 $b_3 = -g$   $\frac{m}{V_01}$



$$\frac{\partial \sigma_{33}}{\partial x_3} = g \cdot \rho$$

$$\int_{\sigma_{33}(x_3=0)}^{\sigma_{33}(x_3)} d\sigma_{33} = \int_{x_3=0}^{x_3} g \cdot \rho \cdot dx_3$$

$$\sigma_{33}(x_3) = \int_0^{x_3} g \cdot \rho(x_3) dx_3$$

$$S_v(z) = \int_0^z g \cdot \rho_{bulk}(z) dz$$

vertical depth

$$\rho(z) = \rho_{bulk}$$

$$S_v(z) = \rho_{bulk} \cdot g \cdot z$$

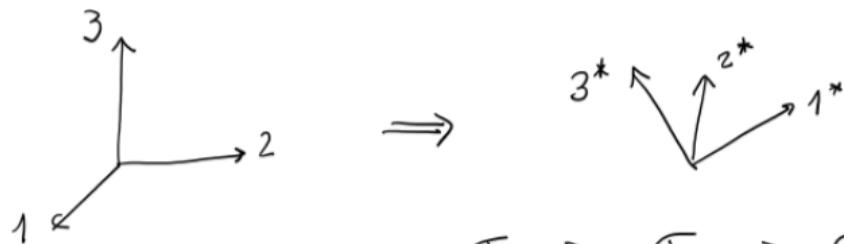
$$\frac{dS_v}{dz} = \rho_{bulk} \cdot g$$

$$23 \text{ MPa/km}$$

$$1 \text{ psi/ft}$$

$$2300 \frac{\text{kg}}{\text{m}^3} \cdot 9.8 \frac{\text{m}}{\text{s}^2}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\sigma_1 \perp \sigma_2 \perp \sigma_3$$

$\sigma_v$  is a principal stress:

$$\hookrightarrow \sigma_v \perp \sigma_{H\max} \perp \sigma_{h\min}$$

(\*)

$$\hat{0} \equiv \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} - \begin{bmatrix} P_p & & \\ & P_p & \\ & & P_p \end{bmatrix}$$

### NORMAL FAULTING

$$\sigma_v \geq \sigma_{hmax} \geq \sigma_{hmin}$$

$$\begin{bmatrix} \sigma_v & 0 & 0 \\ 0 & \sigma_{hmax} & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$$

### STRIKE-SLIP FAULTING

$$\sigma_{hmax} \geq \sigma_v \geq \sigma_{hmin}$$

$$\begin{bmatrix} \sigma_{hmax} & 0 & 0 \\ 0 & \sigma_v & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$$

### REVERSE F.

$$\sigma_{hmax} \geq \sigma_{hmin} \geq \sigma_v$$

$$\begin{bmatrix} \sigma_{hmax} & 0 & 0 \\ 0 & \sigma_{hmin} & 0 \\ 0 & 0 & \sigma_v \end{bmatrix}$$

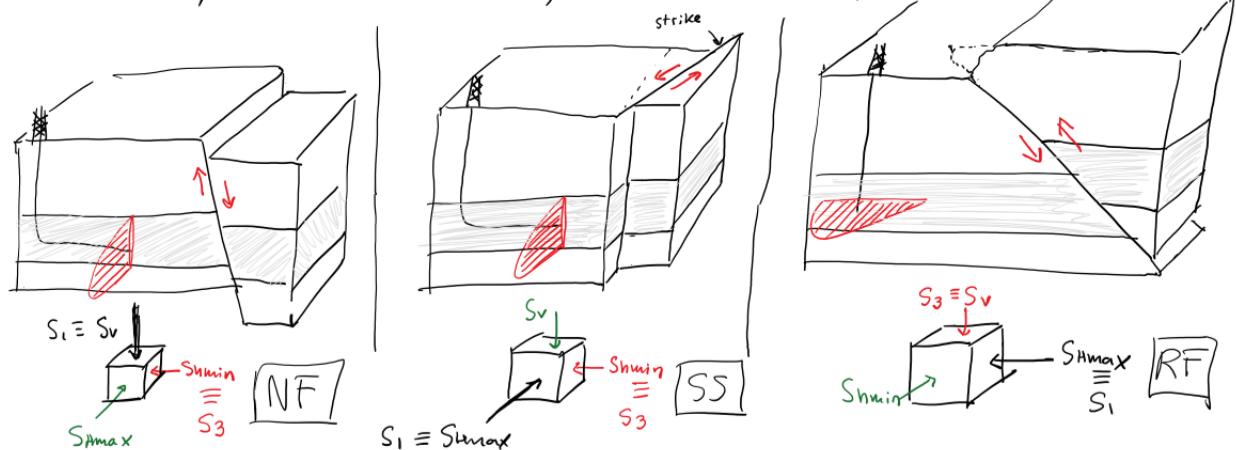
Effective stresses :  $\underline{\sigma}$ ; Total stresses  $\underline{S} = \underline{\sigma} + P_p \equiv$  (X)

$$\begin{bmatrix} S_v & 0 & 0 \\ 0 & S_{hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix}$$

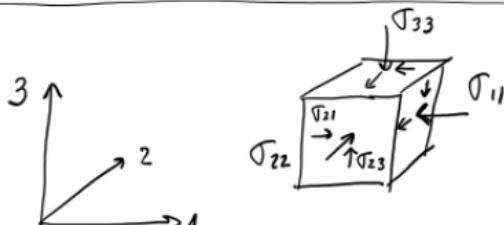
$$\begin{bmatrix} S_{hmax} & 0 & 0 \\ 0 & S_v & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix}$$

$$\begin{bmatrix} S_{hmax} & 0 & 0 \\ 0 & S_{hmin} & 0 \\ 0 & 0 & S_v \end{bmatrix}$$

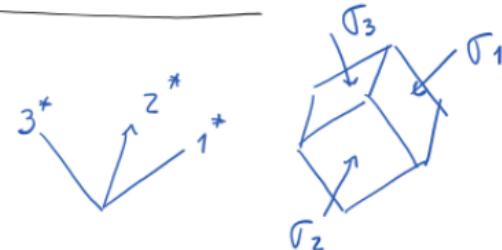
$$S_v = \sigma_v + P_p; S_{hmax} = \sigma_{hmax} + P_p; S_{hmin} = \sigma_{hmin} + P_p$$



# Stress Invariants and graphical representation

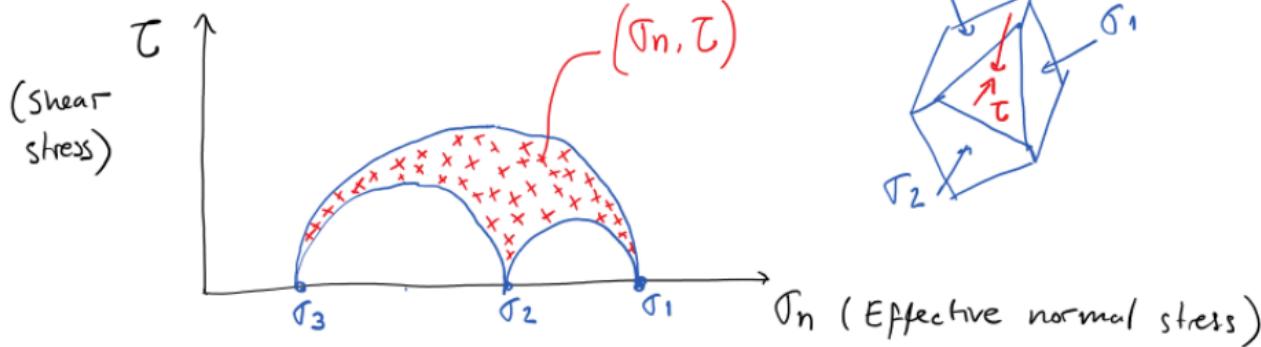


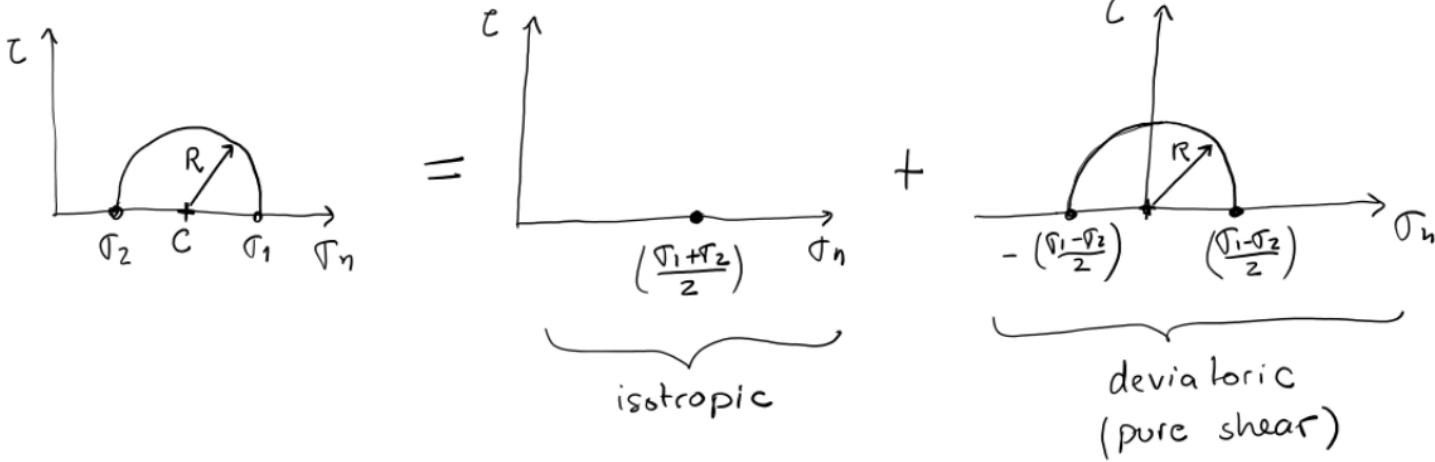
$\Rightarrow$



$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$





$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{isotropic}} + \underbrace{\begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{deviatoric}}$$

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

$\underline{\sigma} = \sigma_m \underline{\mathbb{I}} + \underline{\underline{\sigma}}_d$

Invariants (do not change wrt coord system)

$$\Rightarrow J_1(\underline{\underline{\tau}}) = \tau_{11} + \tau_{22} + \tau_{33} = \tau_1 + \tau_2 + \tau_3 \quad \Rightarrow \tau_m = \frac{J_1(\underline{\underline{\tau}})}{3}$$

$$J_2(\underline{\underline{\tau}}) = \tau_{11}\tau_{22} + \tau_{11}\cdot\tau_{33} + \tau_{22}\tau_{33} - \tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2$$

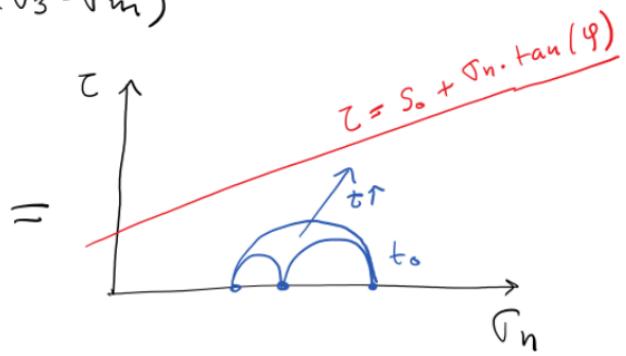
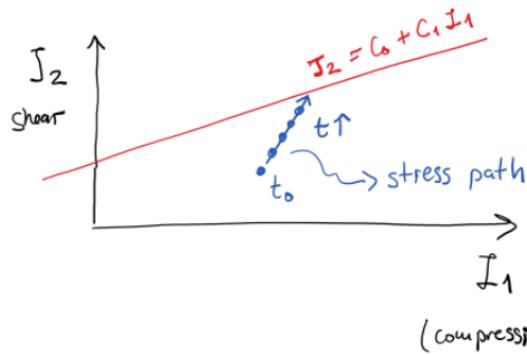
$$J_3(\underline{\underline{\tau}}) = \det(\underline{\underline{\tau}}) = \tau_1 \cdot \tau_2 \cdot \tau_3 \leftarrow$$

=

$$J_1(\underline{\underline{s}}_d) = \emptyset$$

$$\Rightarrow J_2(\underline{\underline{s}}_d) = \frac{1}{6} \left[ (\tau_1 - \tau_2)^2 + (\tau_1 - \tau_3)^2 + (\tau_2 - \tau_3)^2 \right]$$

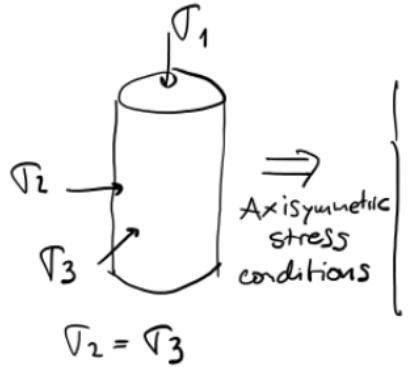
$$J_3(\underline{\underline{s}}_d) = (\tau_1 - \tau_m) \cdot (\tau_2 - \tau_m) \cdot (\tau_3 - \tau_m)$$



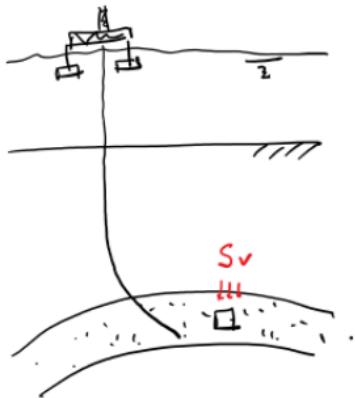
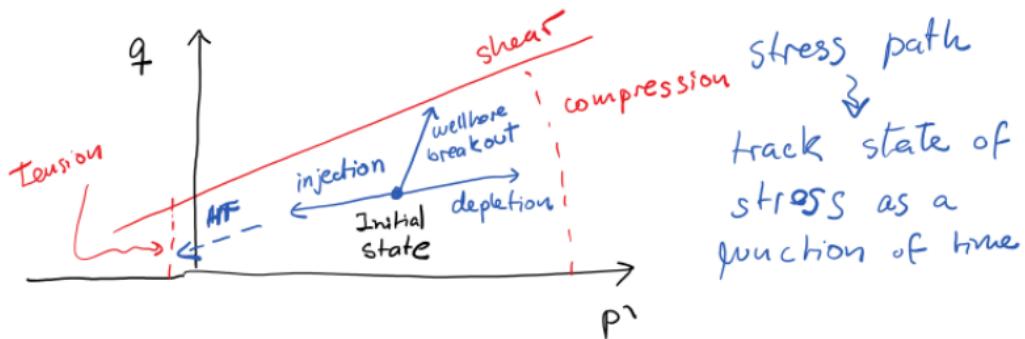
(compression)

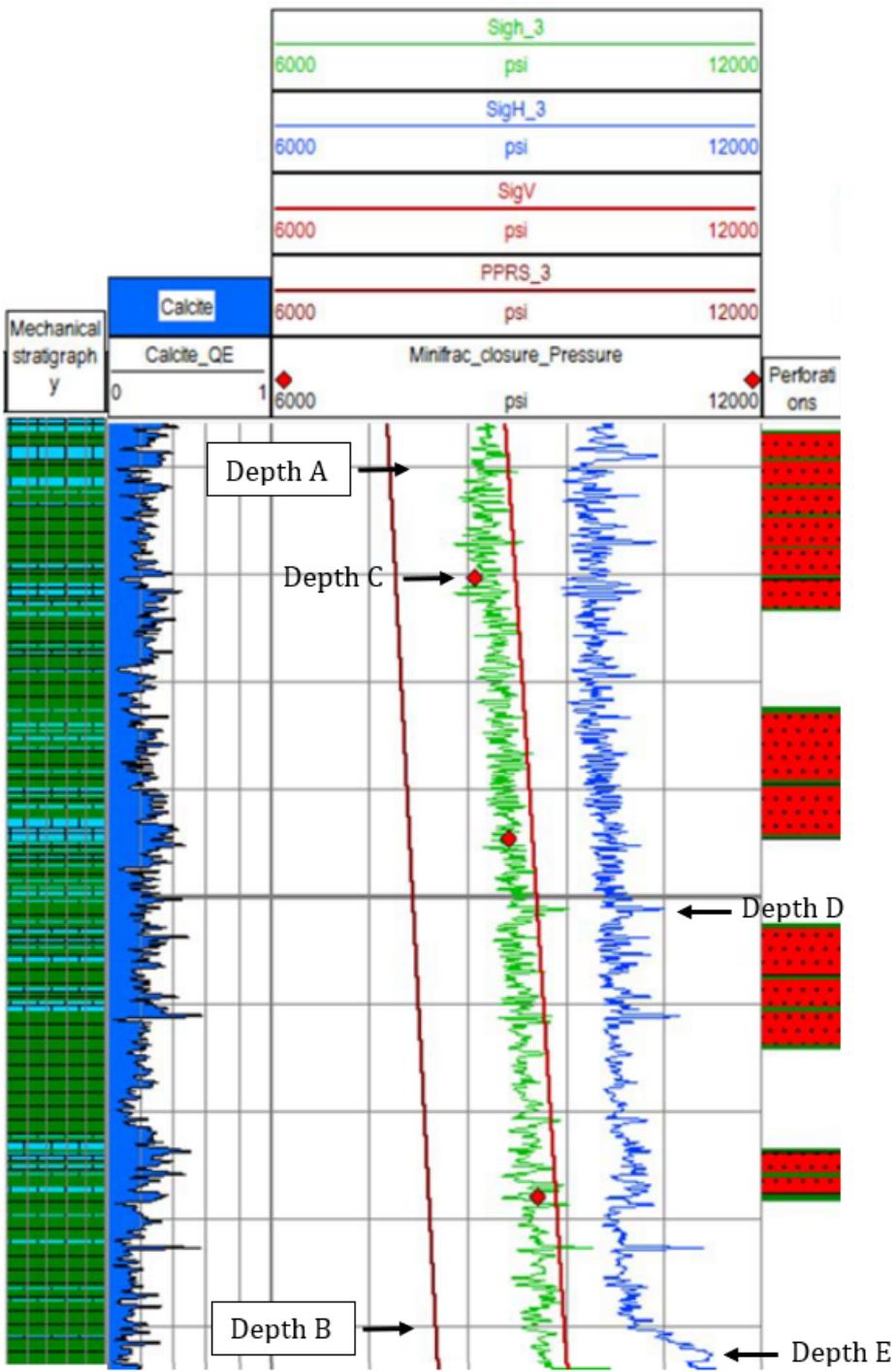
soft sediments  
(soil mechanics)

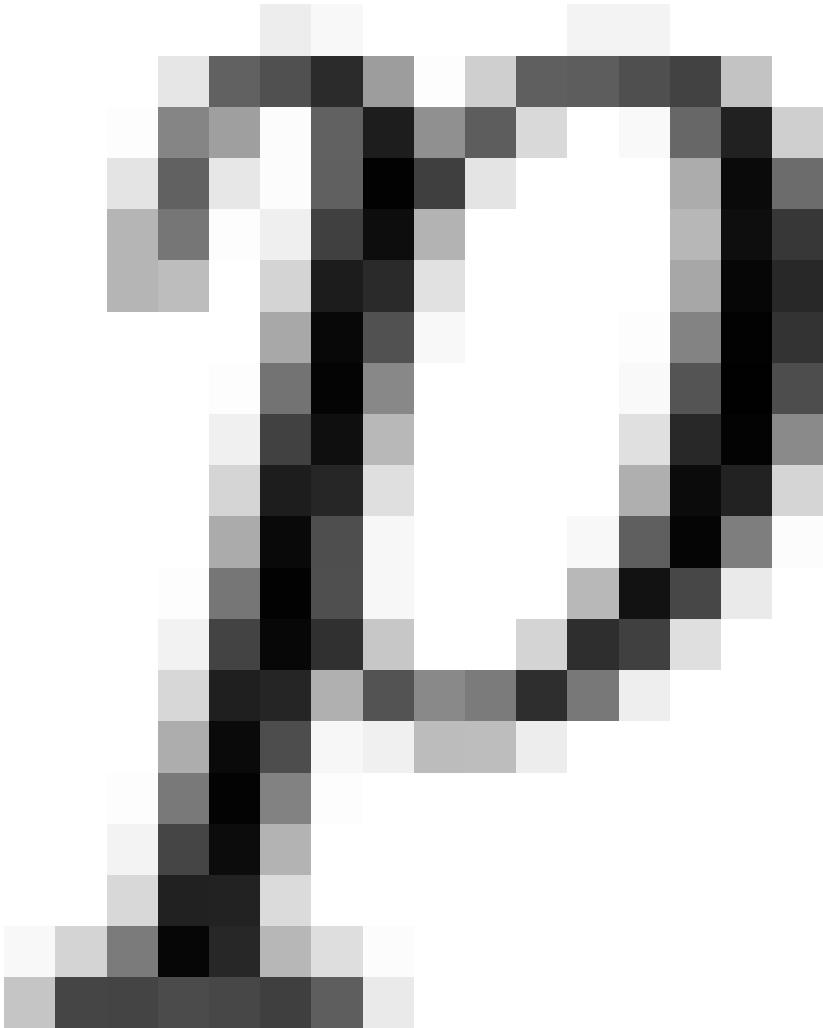
$$\left\{ \begin{array}{l} P' = \sigma_m^{\text{effective}} = J_1(\Sigma) / 3 \\ q = \sqrt{3 J_2} \end{array} \right.$$

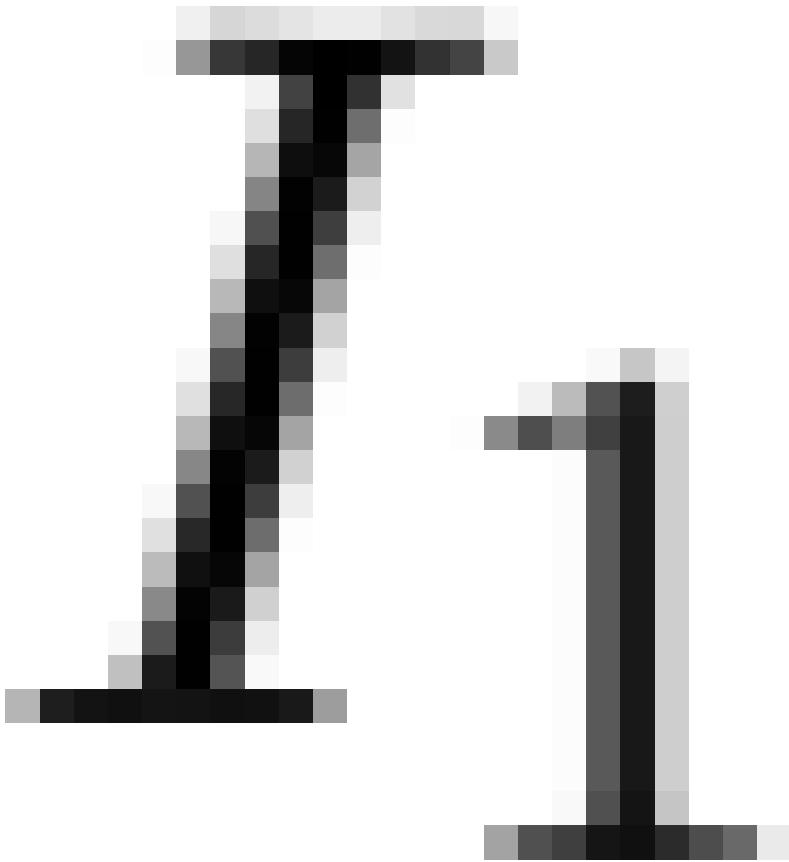


$$\left\{ \begin{array}{l} P' = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_1 + 2\sigma_3}{3} \\ q = \sqrt{3 \left\{ \frac{1}{8} \left[ (\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right] \right\}} = \sigma_1 - \sigma_3 \end{array} \right.$$





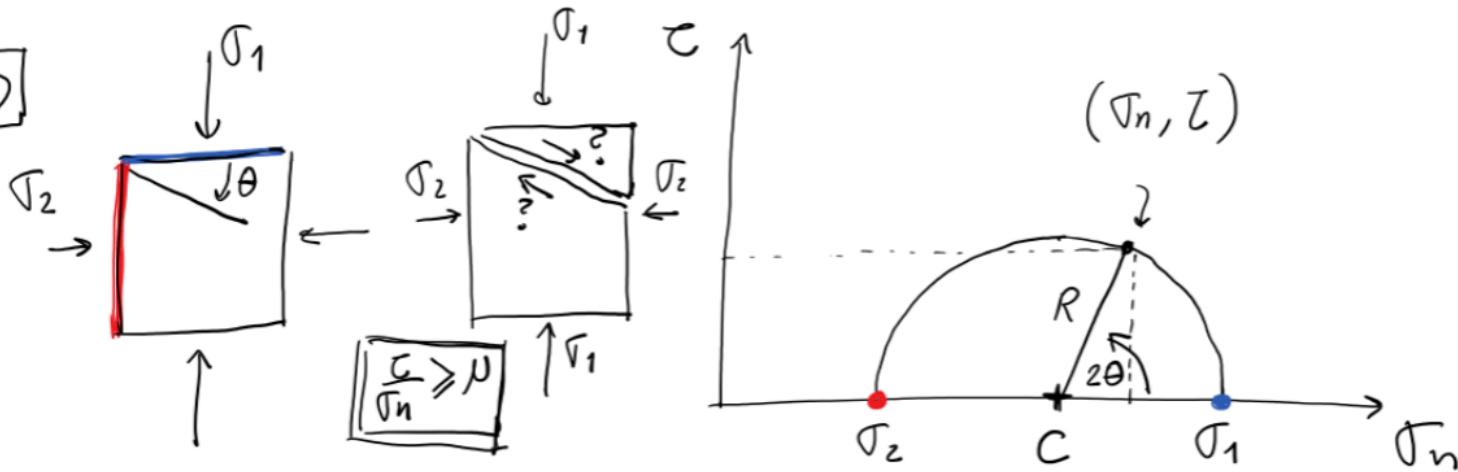




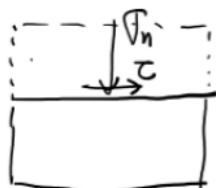


# Stress projection on a plane

2D



$$\theta = 0$$

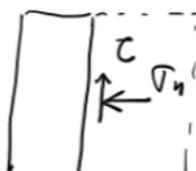


$$\sigma_n = \sigma_1; \tau = 0$$



$$\sigma_n, \tau$$

$$\theta = 90^\circ$$



$$\sigma_n = \sigma_2; \tau = 0$$

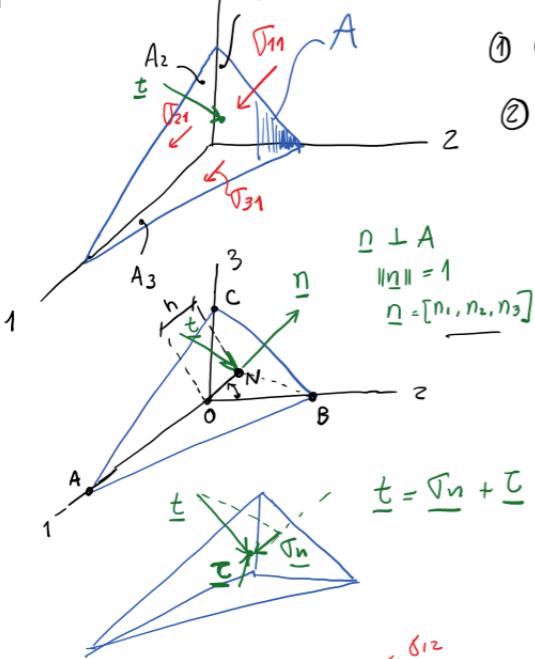
$$\begin{cases} \sigma_n = C + R \cdot \cos 2\theta \\ \tau = R \cdot \sin 2\theta \end{cases}$$

$$C = (\sigma_1 + \sigma_2)/2$$

$$R = (\sigma_1 - \sigma_2)/2$$

3D

$$\sum F_1 = 0$$



$$\textcircled{1} \quad \sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3 = t_1 A$$

$$\textcircled{2} \quad \text{Vol } \Delta = \frac{1}{3} A \cdot h$$

$$= \frac{1}{3} A_1 \cdot \overline{OA}$$

$$\frac{1}{3} Ah = \frac{1}{3} A_1 \overline{OA}$$

$$A_1 = \frac{h}{\overline{OA}} A$$

$$\begin{cases} A_1 = \cos A \hat{o} N \cdot A = n_1 A \\ A_2 = \cos B \hat{o} N \cdot A = n_2 A \\ A_3 = \cos C \hat{o} N \cdot A = n_3 A \end{cases}$$

cosine directors

$$[A_i = n_i A]$$

$$\rightarrow \textcircled{1} + \textcircled{2} \quad \sigma_{11} n_1 A + \sigma_{21} n_2 A + \sigma_{31} n_3 A = t_1 A$$

$\Downarrow \sigma_{21} = \sigma_{12}$  (angular momentum equil.)

$$\rightarrow \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

row x column

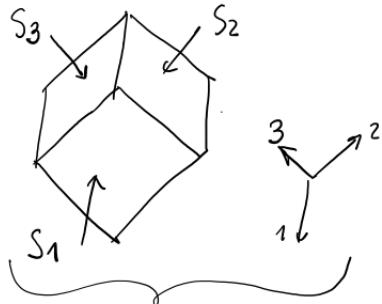
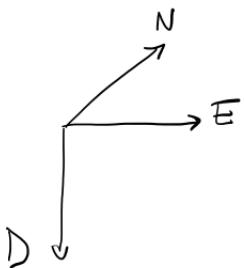
$$\underline{t} = \underline{\sigma} \cdot \underline{n}$$

$$\left\{ \begin{array}{l} \sigma_n = \underline{t} \cdot \underline{n} \\ \sigma_n \text{ on } \underline{n} \end{array} \right\} \text{ projection of } \underline{t} \text{ on } \underline{n}$$

$$\left\{ \begin{array}{l} C = \sqrt{\|\underline{t}\|^2 - \|\sigma_n\|^2} \\ \Leftrightarrow \|\underline{t}\|^2 = \|\sigma_n\|^2 + \|C\|^2 \end{array} \right.$$

## Geographical coordinate system

N - E - D



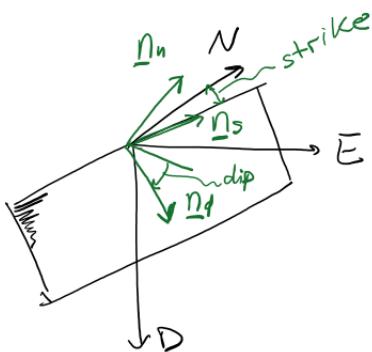
Principal  
stresses  
and  
direction

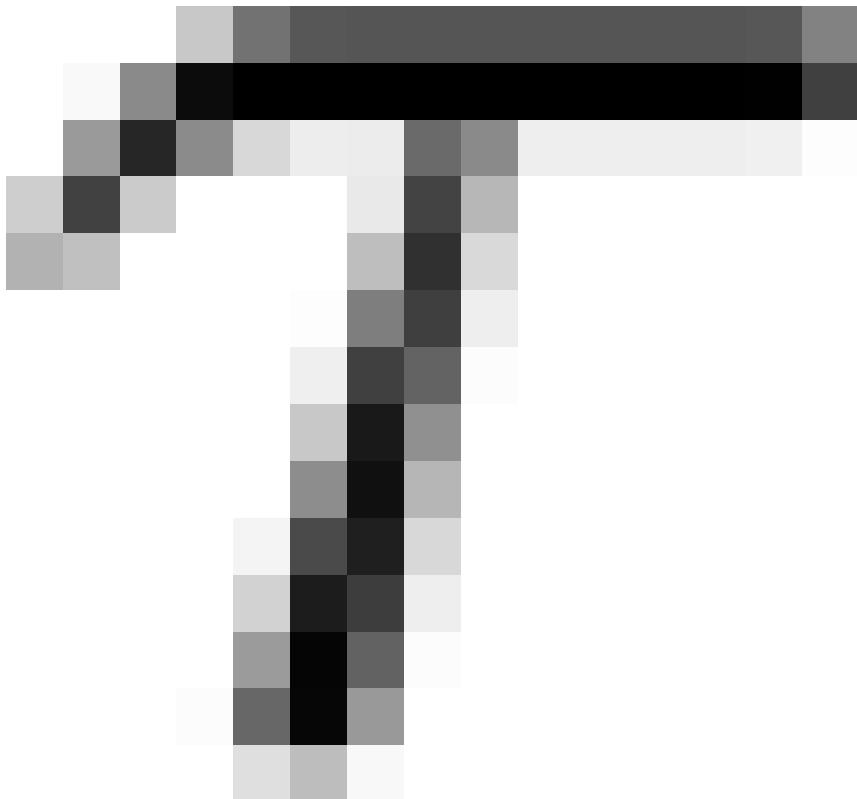
$$\underline{\underline{S}}_G = \begin{bmatrix} S_{NN} & S_{NE} & S_{ND} \\ S_{EN} & S_{EE} & S_{ED} \\ S_{DN} & S_{DE} & S_{DD} \end{bmatrix} \quad \underline{\underline{S}}_P = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

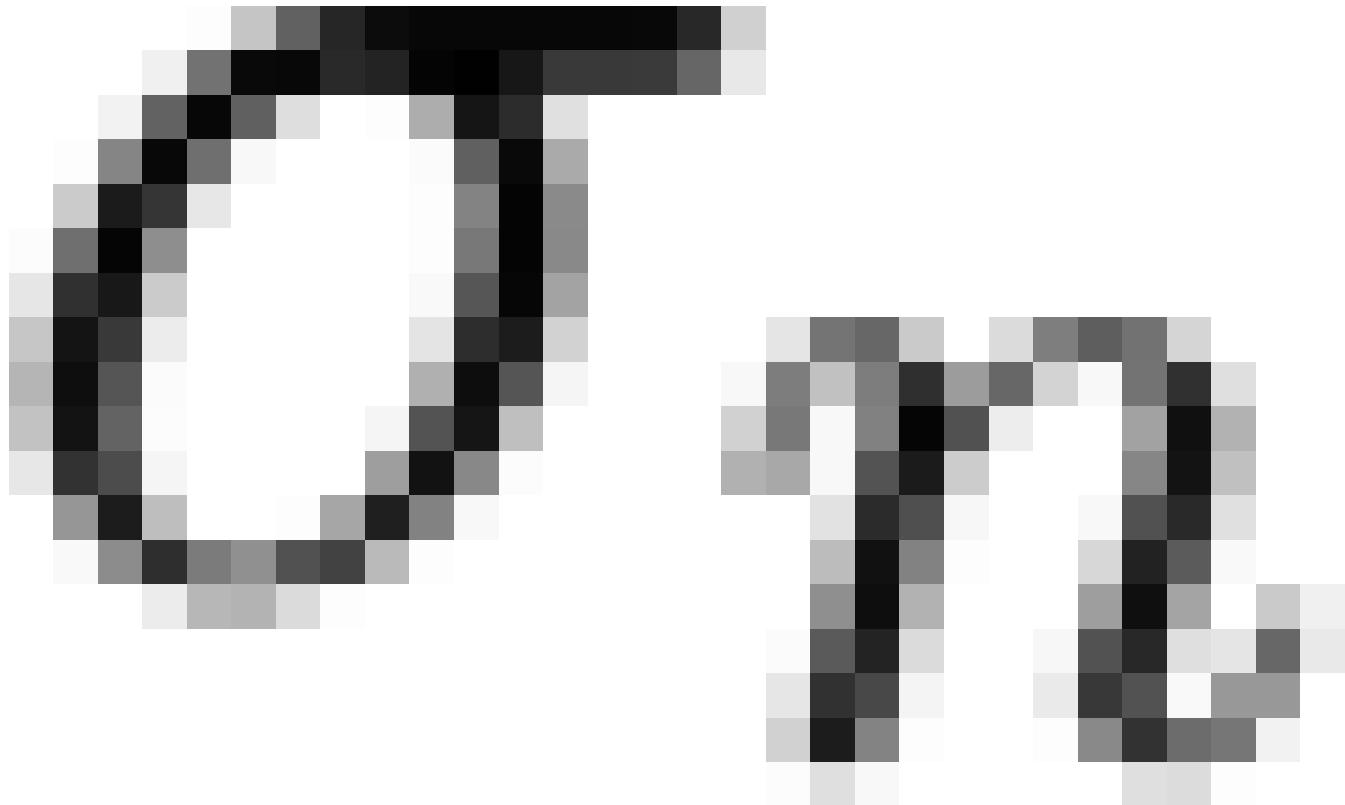
$$\underline{\underline{S}}_G = R_{PG}^T \underline{\underline{S}}_P R_{PG}$$

$$R_{PG} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

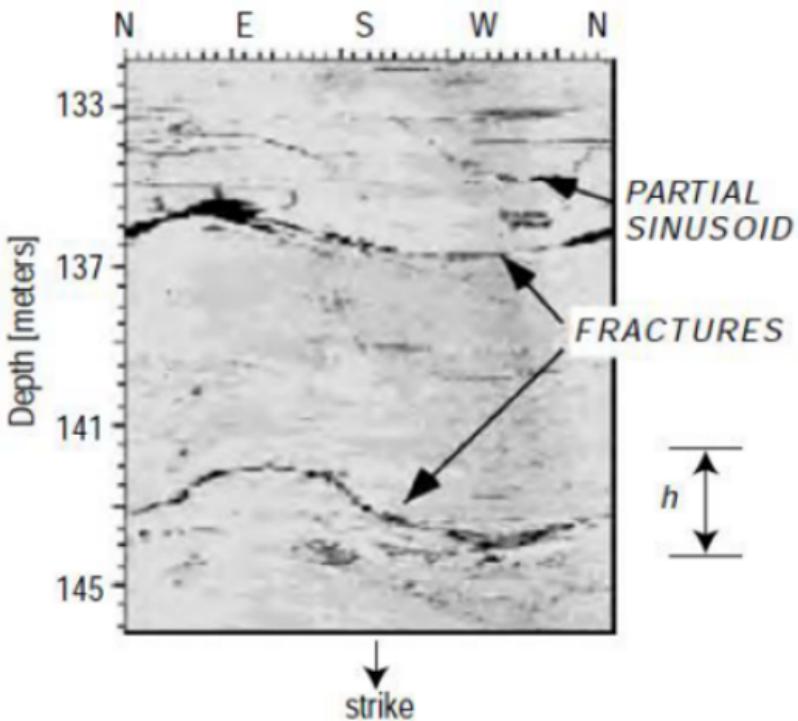
$$f(\alpha, \beta, \gamma)$$



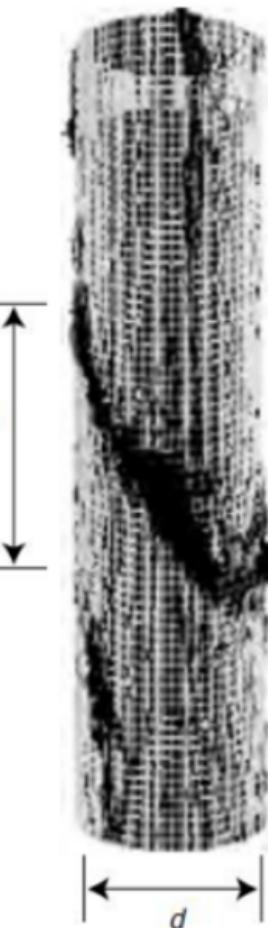




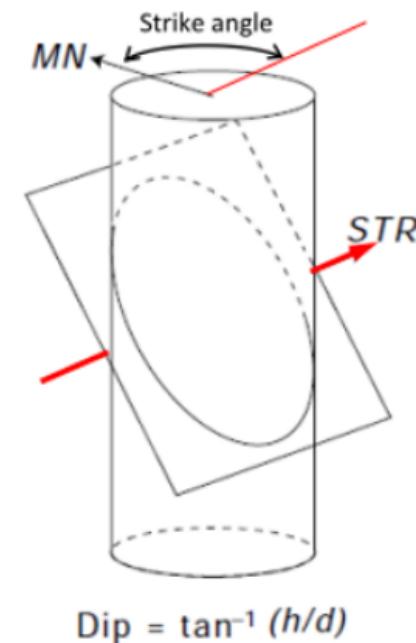
## Un-wrapped image (ultrasonic)



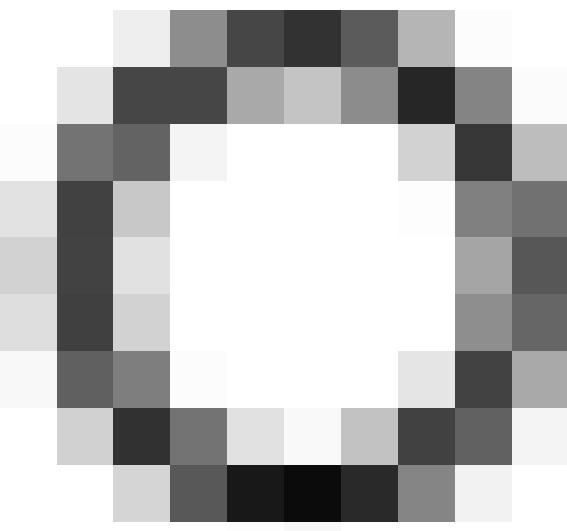
## 3D-representation

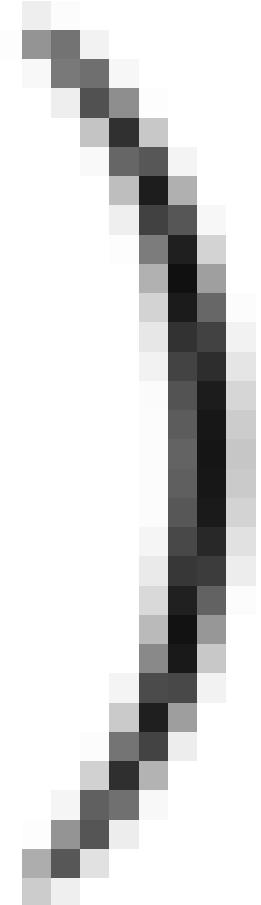
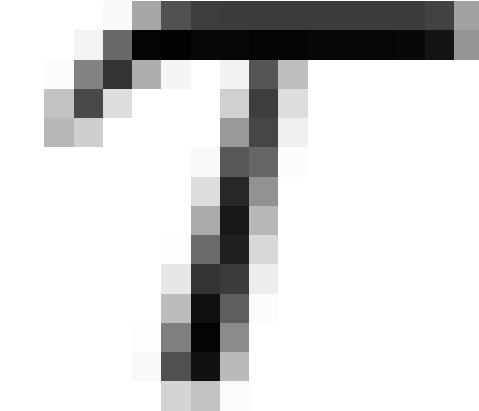
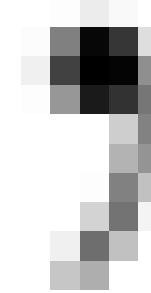
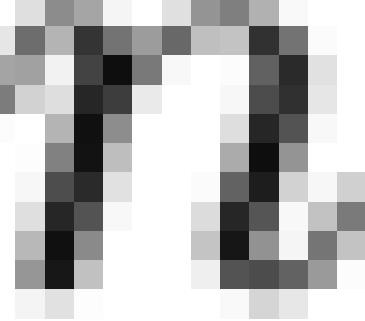
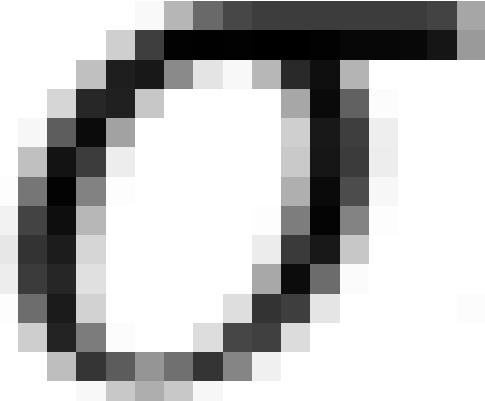
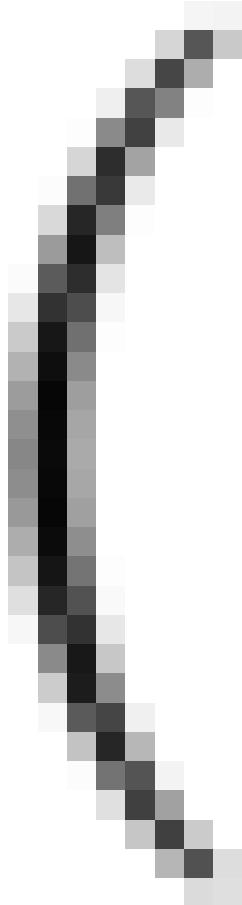


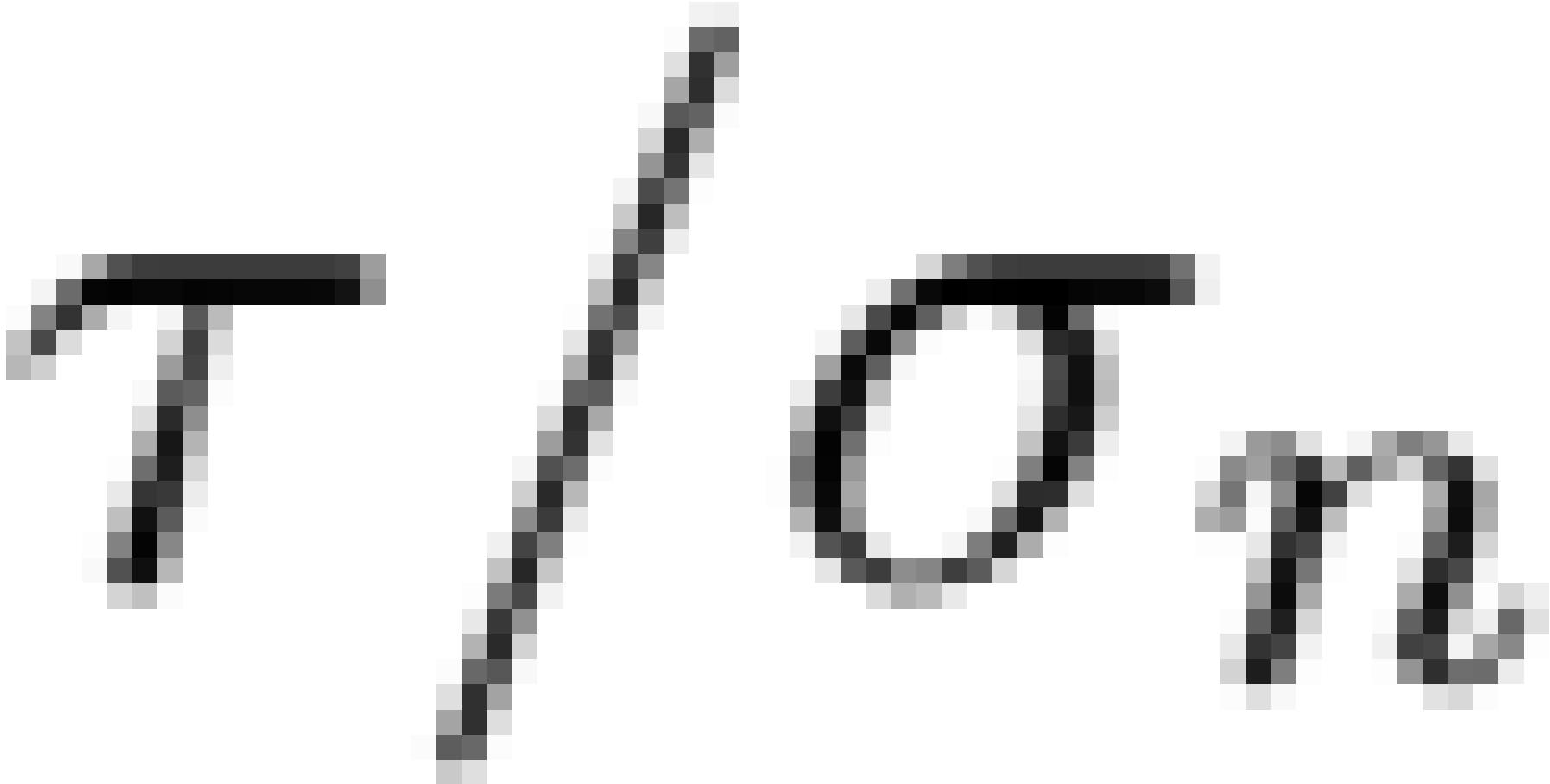
## Interpretation

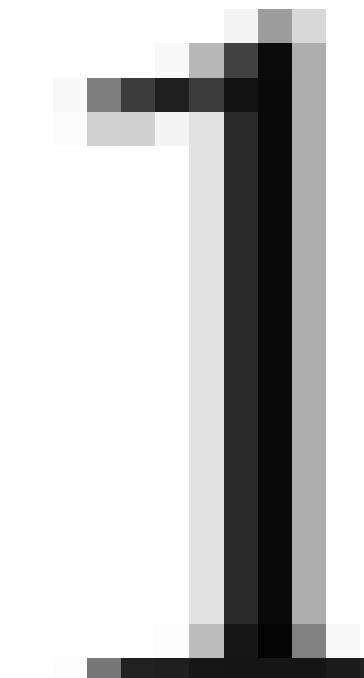
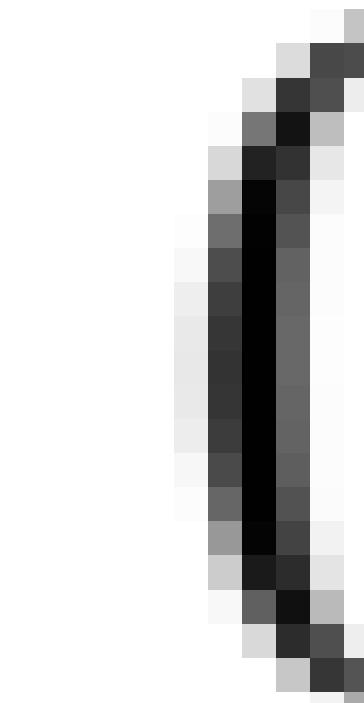
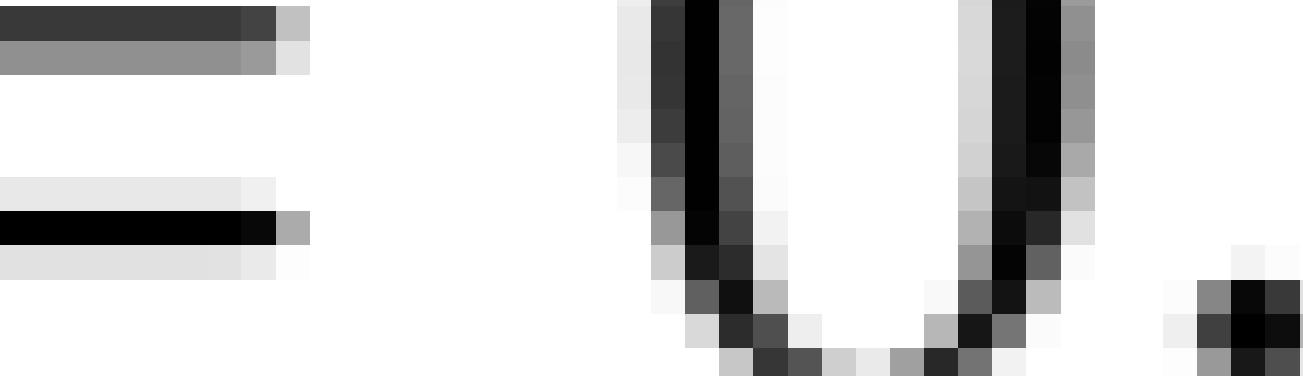
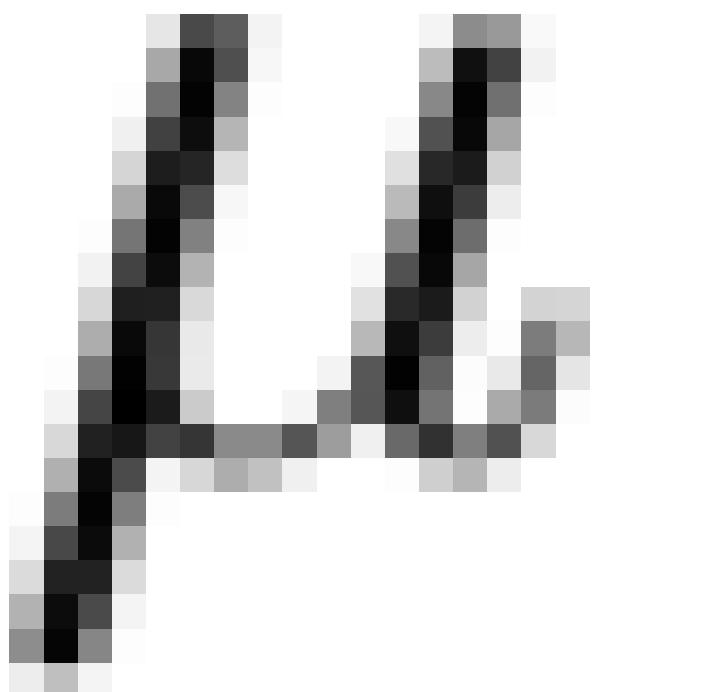


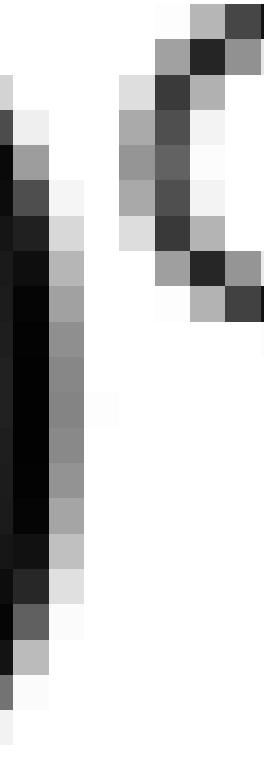
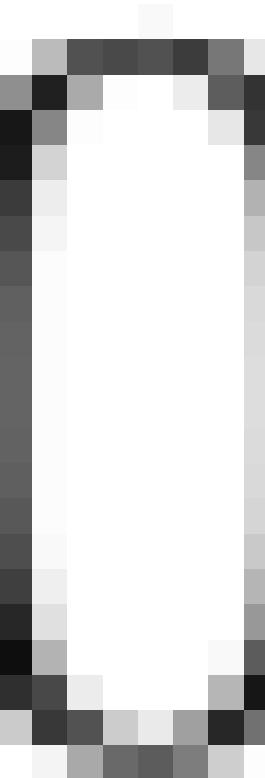
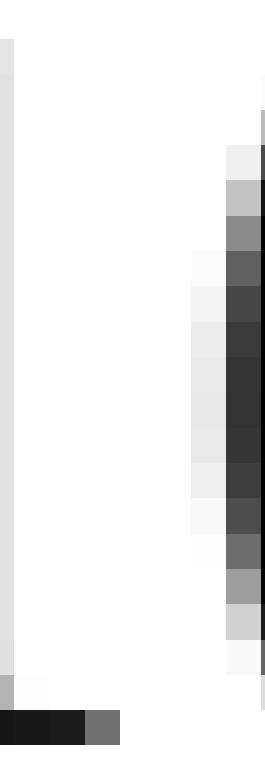
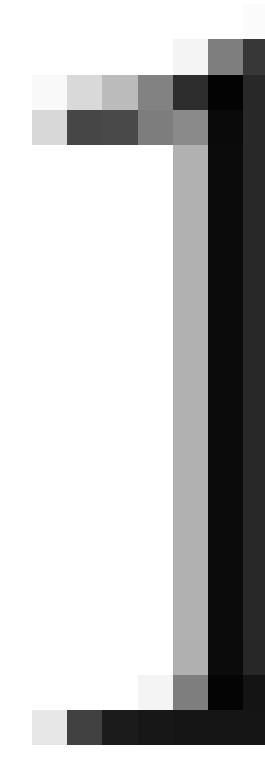
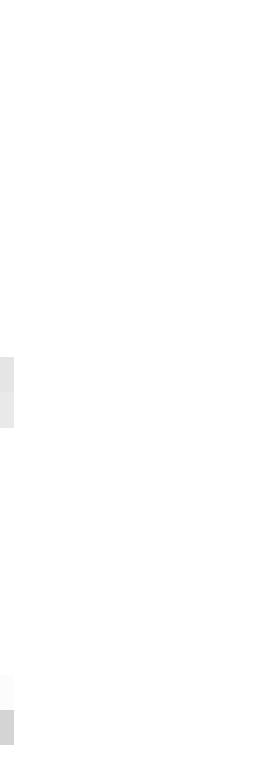
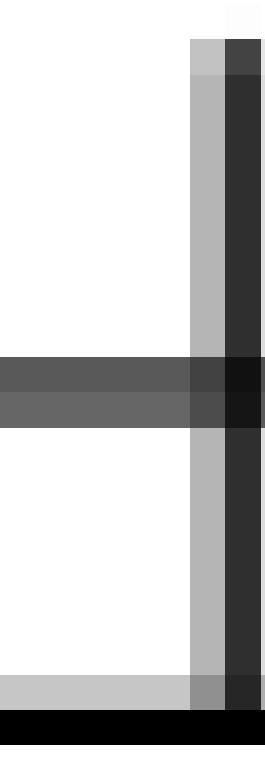
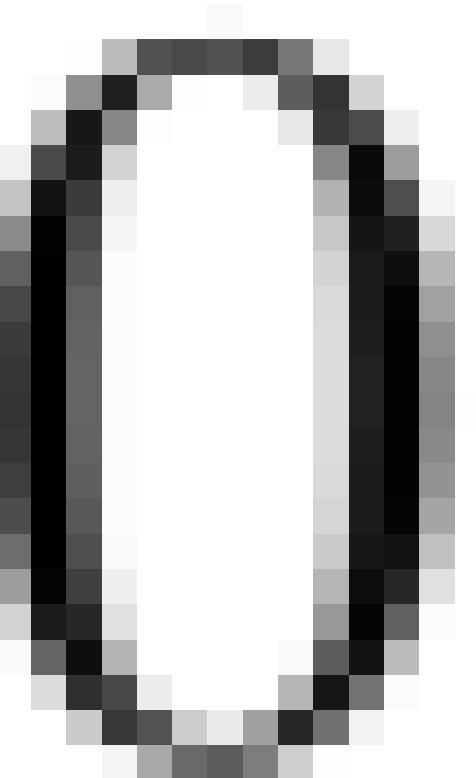
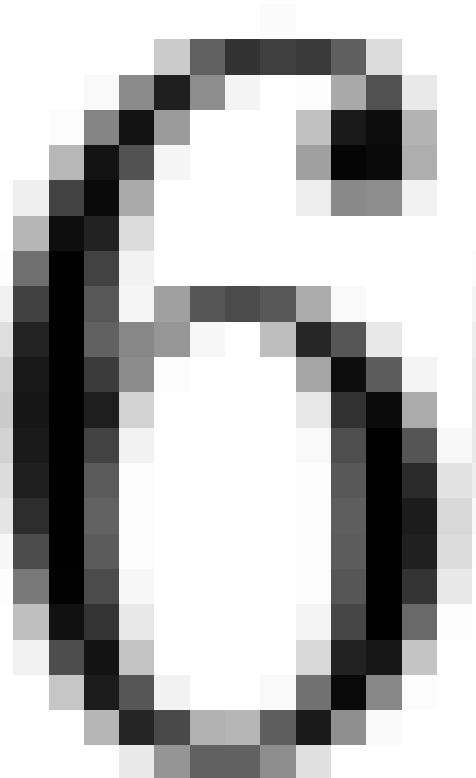
(Zoback 2013, RM, Ch 5.3)

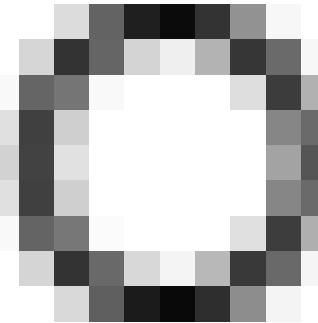
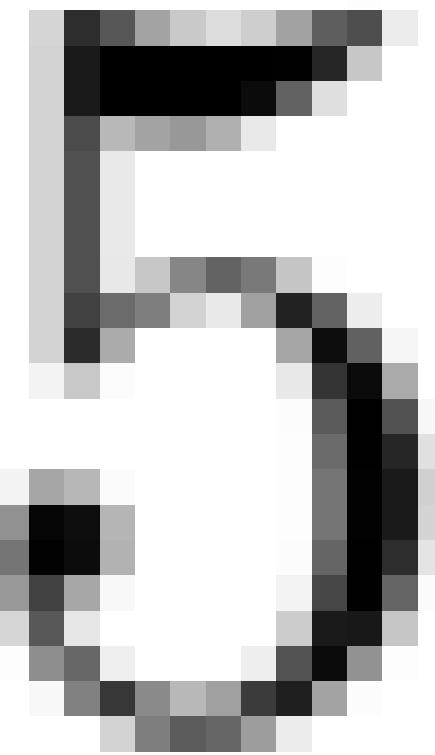
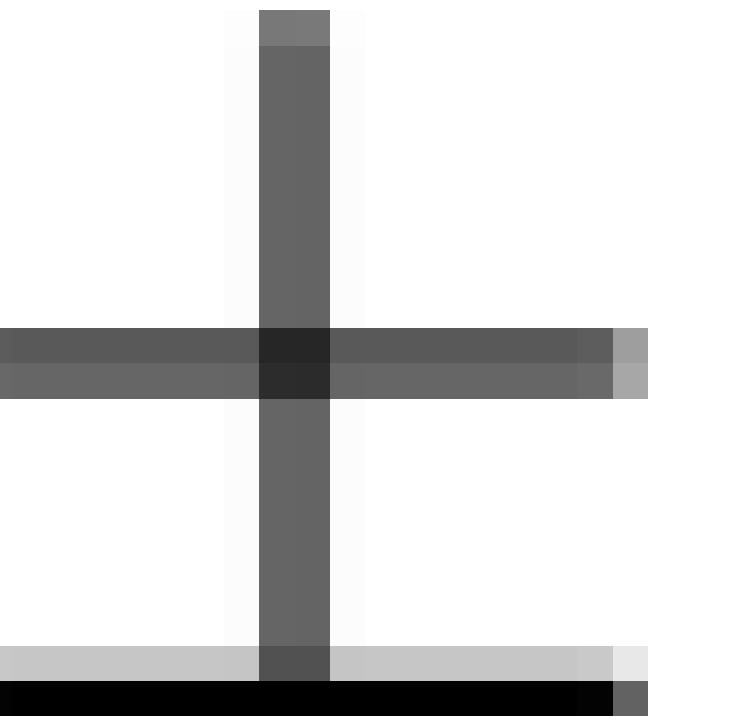
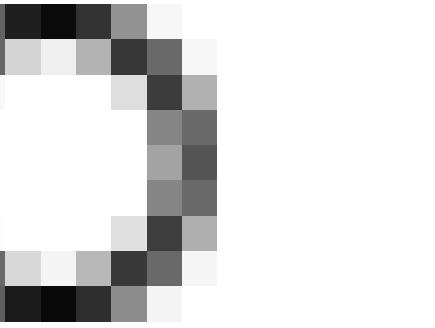
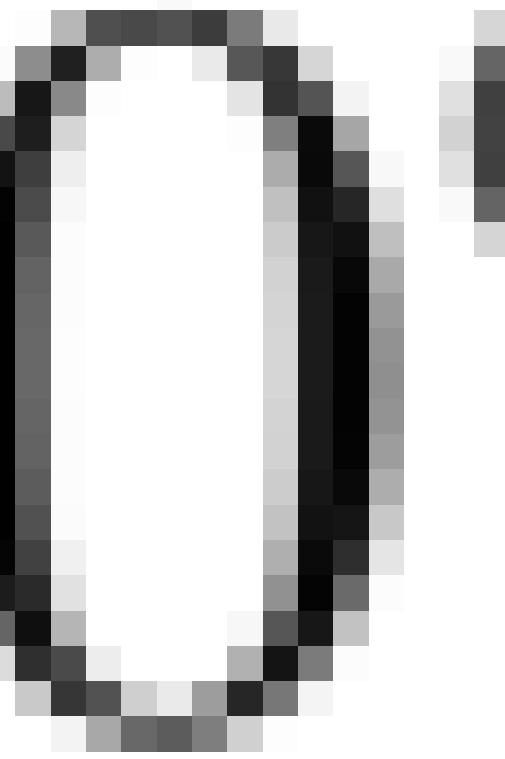
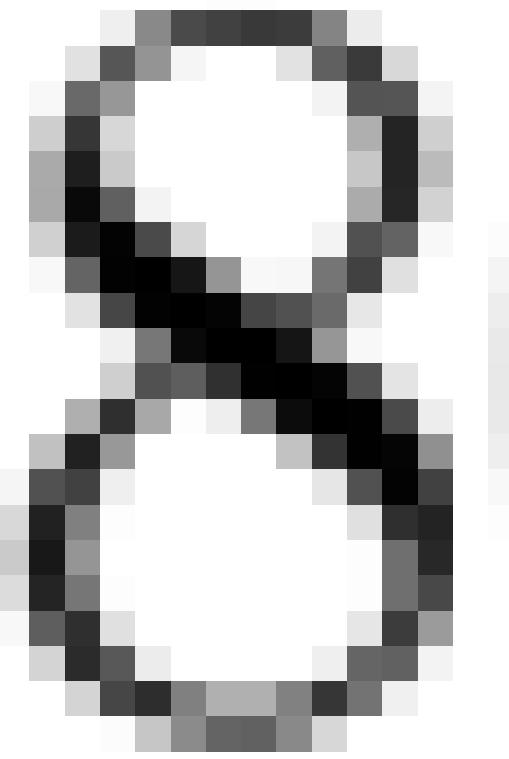


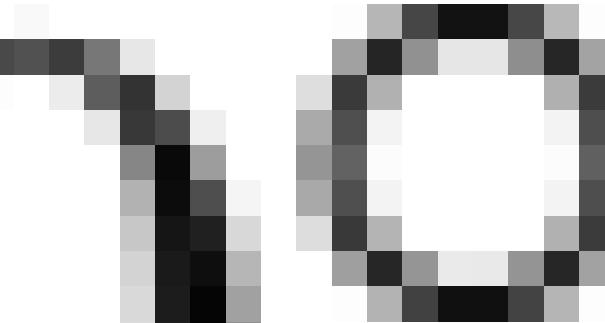
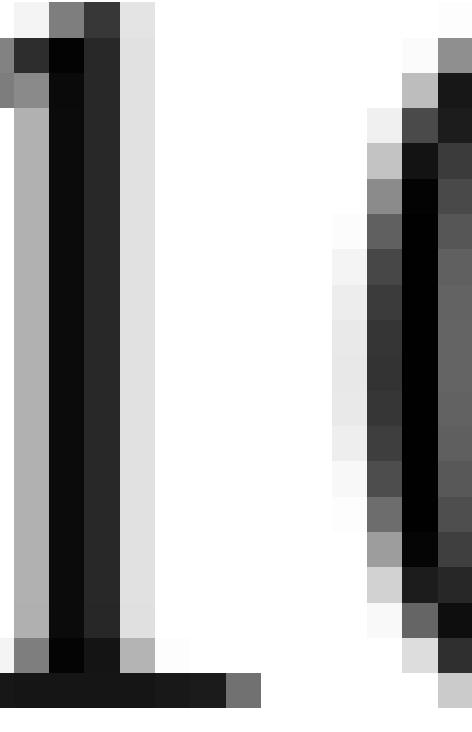
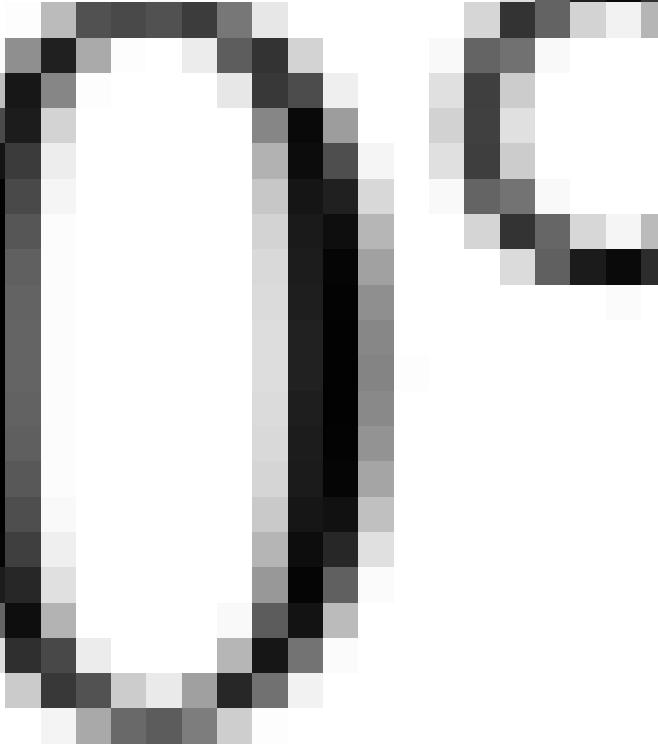
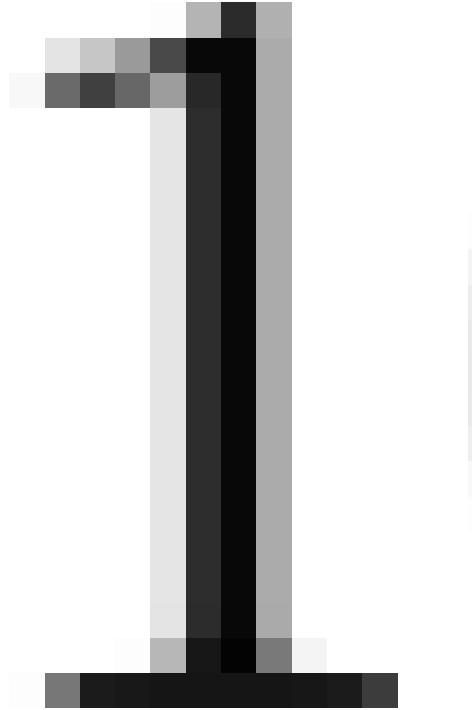


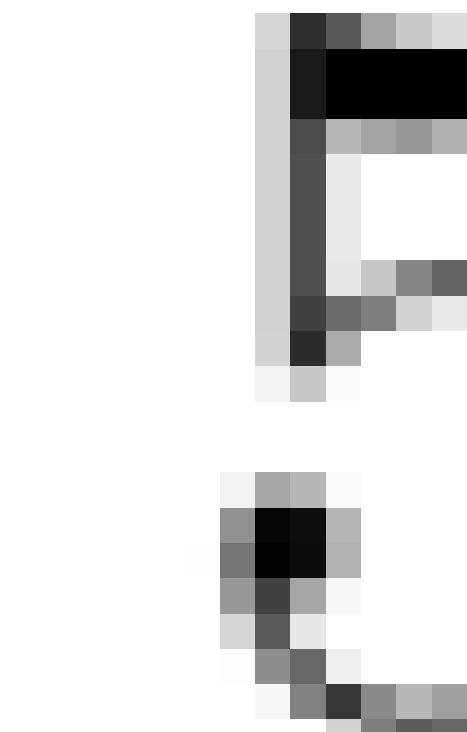
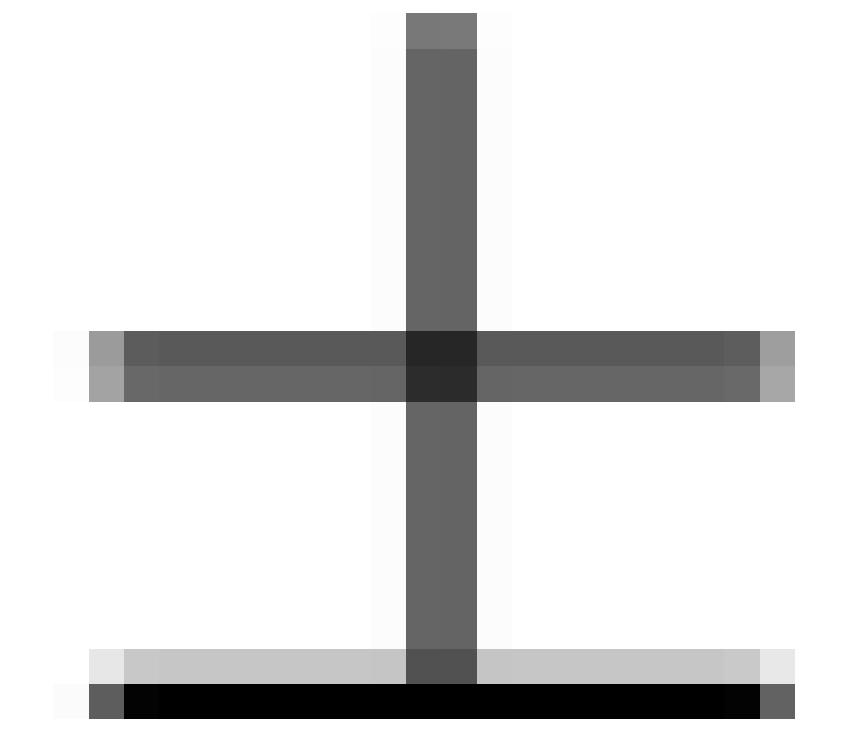
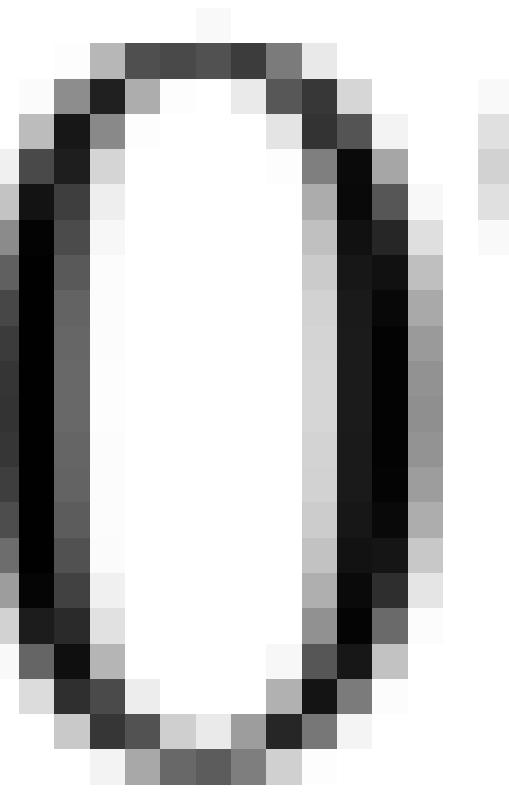
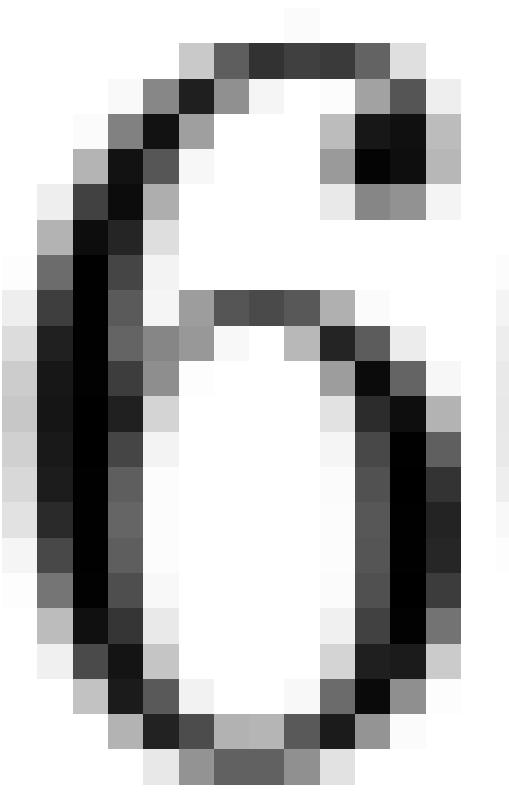


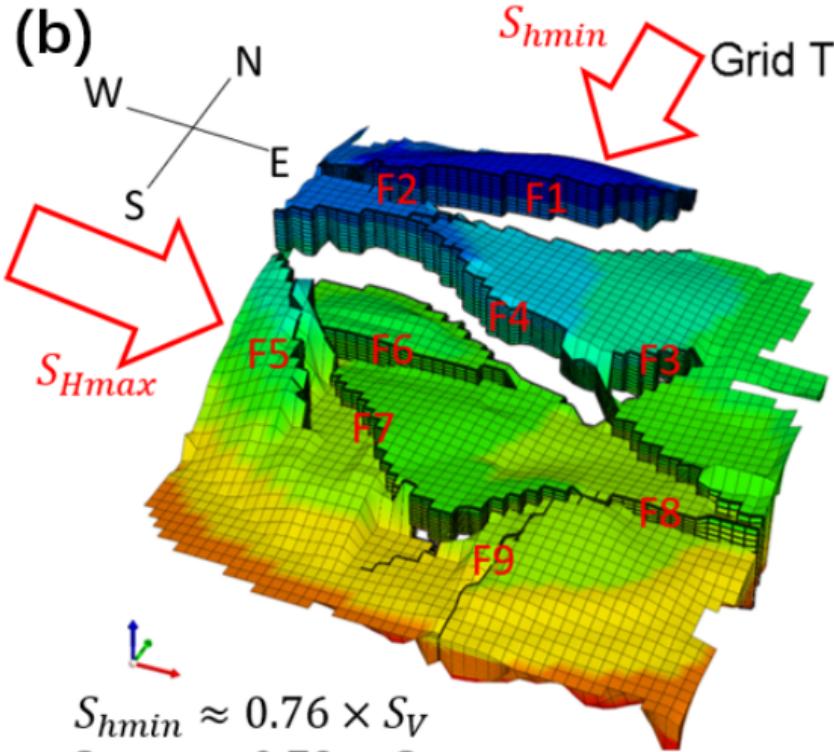
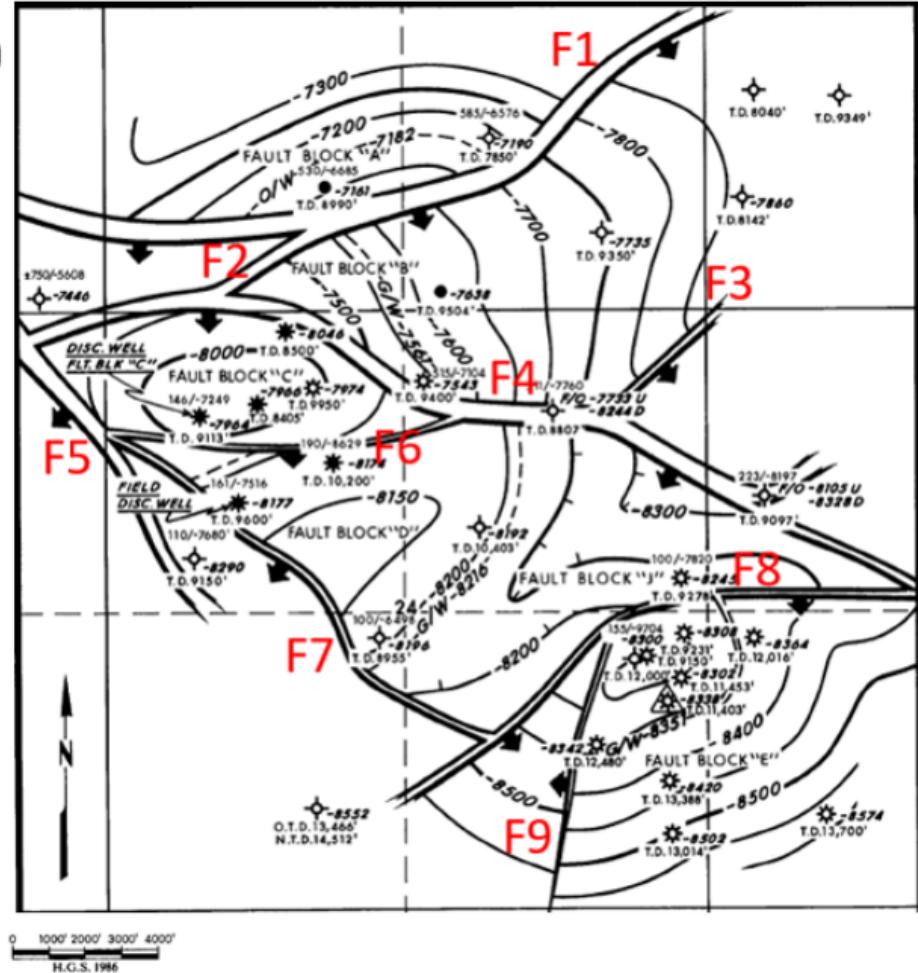












$$S_{hmin} \approx 0.76 \times S_V$$

$$S_{Hmax} \approx 0.79 \times S_V$$

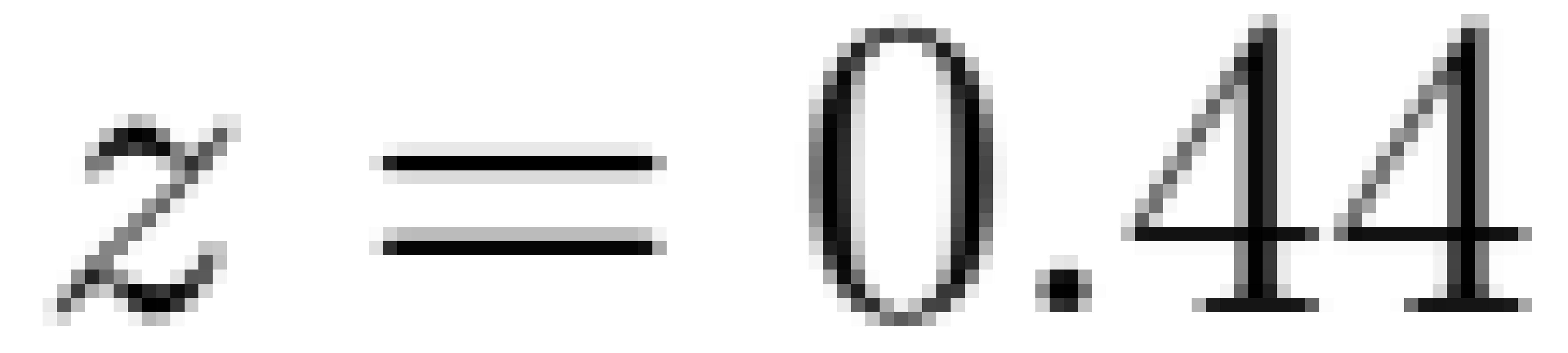
Gradient in  $S_V$ :  
 $\approx 1.00 \text{ psi/ft}$

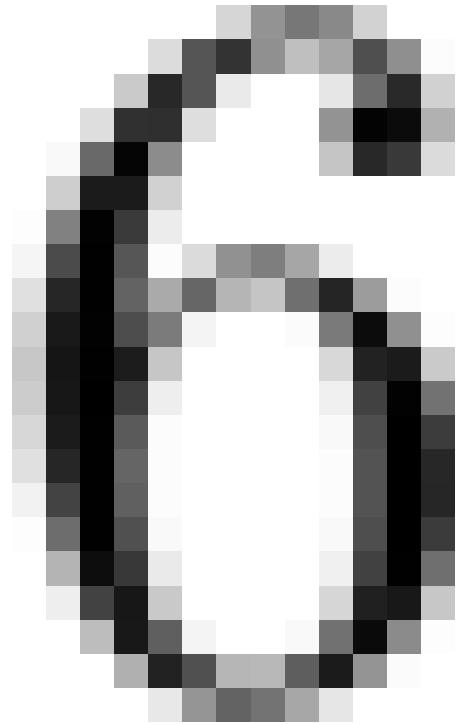
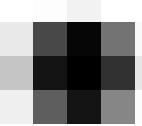
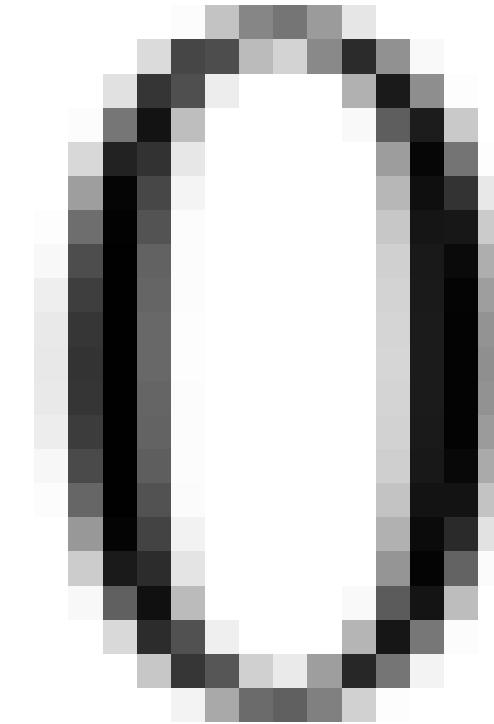
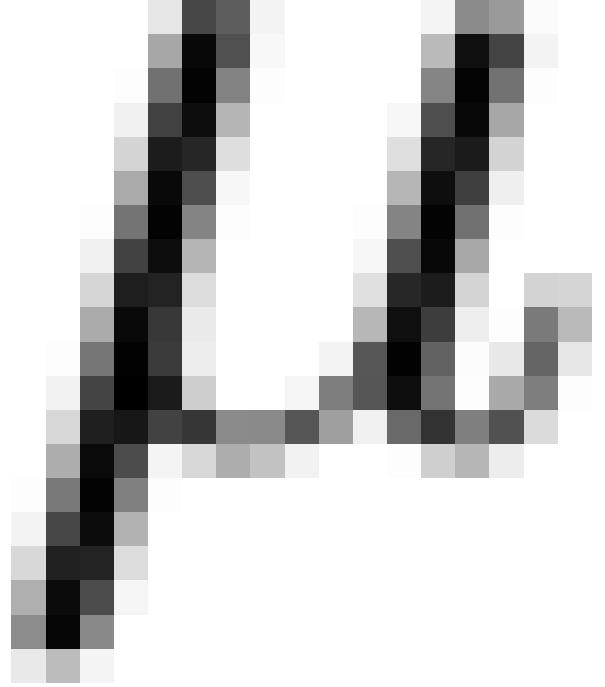




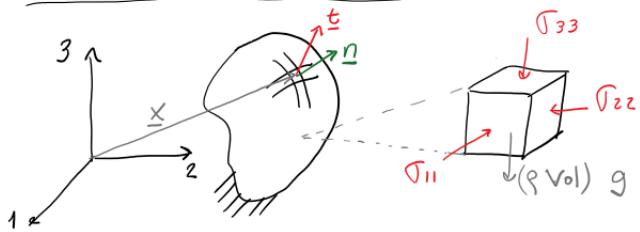








# General solution to a continuum mechanics problem



$$\left\{ \begin{array}{l} \nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \rho \underline{\underline{\alpha}} \\ \underline{\underline{\epsilon}} = F_1(\underline{u}) \\ \underline{\underline{\sigma}} = F_2(\underline{\underline{\epsilon}}) \end{array} \right. \rightarrow \begin{array}{l} \text{Equilibrium (Cauchy's)} \\ \text{Kinematic eq} \\ \text{Constitutive equations} \end{array}$$

→ small strains  
 → large strains

• linear isotropic elastic solid  
 • TVI (VTE)  
 • orthorhombic  
 • visco-elasticity  
 • plasticity

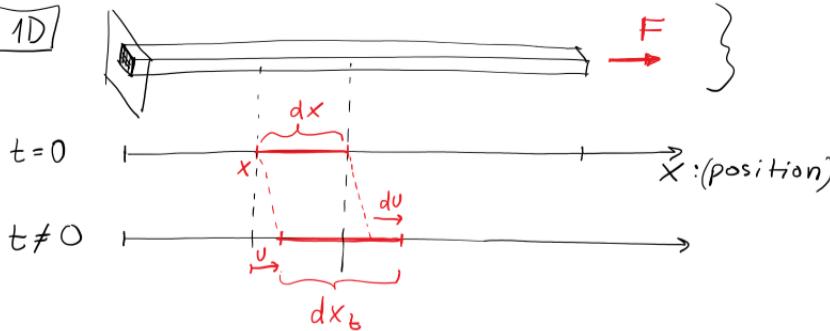
$$\nabla \cdot [F_2(\underline{\underline{\epsilon}})] + \underline{f} = \rho \underline{\underline{\alpha}}$$

$$\boxed{\nabla \cdot [F_2[F_1(\underline{u})]] + \underline{f} = \rho \underline{\underline{\alpha}}}$$

displacement  $\underline{u} \rightarrow \underline{\underline{\epsilon}} \rightarrow \underline{\underline{\sigma}}$

# Kinematic Equations (small strains) $\underline{\epsilon} = F_1(\underline{u})$

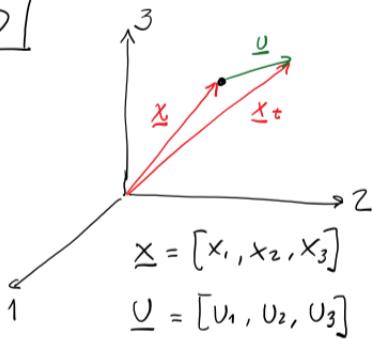
1D



$$\epsilon = \frac{dx_t - dx}{dx} = \frac{[x + u + dx + du - (x + u)] - [x + dx - x]}{[x + dx - x]}$$

$$\boxed{\epsilon = \frac{du}{dx}}$$

3D



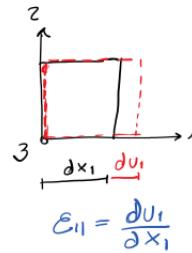
Jacobian

$\uparrow$

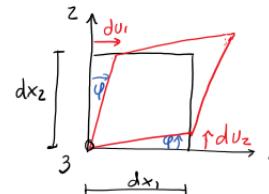
$\downarrow$

$$\boxed{\epsilon = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}}$$

$$\begin{bmatrix} \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} & \frac{\partial U_1}{\partial x_3} \\ \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial x_3} \\ \frac{\partial U_3}{\partial x_1} & \frac{\partial U_3}{\partial x_2} & \frac{\partial U_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial U_1}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial U_2}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial U_3}{\partial x_3} \end{bmatrix}$$



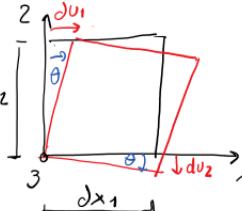
$$+ \begin{bmatrix} 0 & \underbrace{\frac{1}{2} \left( \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right)}_{\tan \varphi} & \frac{1}{2} \left( \frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1} \right) \\ \dots & 0 & \frac{1}{2} \left( \frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2} \right) \\ \dots & \dots & 0 \end{bmatrix}$$



$$\tan \varphi = \frac{\partial U_1}{\partial x_2} = \frac{\partial U_2}{\partial x_1}$$

$$\mathcal{E}_{12} = \frac{1}{2} (\tan \varphi) = \frac{1}{2} \left( \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right)$$

$$+ \begin{bmatrix} 0 & \frac{1}{2} \left( \frac{\partial U_1}{\partial x_2} - \frac{\partial U_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial U_1}{\partial x_3} - \frac{\partial U_3}{\partial x_1} \right) \\ \frac{1}{2} \left( \frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2} \right) & 0 & \frac{1}{2} \left( \frac{\partial U_2}{\partial x_3} - \frac{\partial U_3}{\partial x_2} \right) \\ \dots & \dots & 0 \end{bmatrix}$$



$$\left( \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1} \right) = 0$$

$$\frac{1}{2} \left( \frac{\partial U_1}{\partial x_2} - \frac{\partial U_2}{\partial x_1} \right) = \omega_{12}$$

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\ \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \\ \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) & \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

$\rightarrow$  symmetric } eigen values  $\rightarrow$  principal strains  
 $\rightarrow$  real values } eigen vectors  $\rightarrow$  principal directions  
 own

$$J_1(\underline{\epsilon}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{vol}$$

$$\epsilon_{vol} = \frac{Vol(t) - Vol_0}{Vol_0}$$

$$\underline{\underline{\sigma}} = \underline{\underline{F}}_2 (\underline{\underline{\epsilon}})$$

$\sim$  stress       $\sim$  strain

Superposition  $\begin{cases} \text{space} \\ \text{time} \end{cases}$  } Green's functions

$$\left\{ \begin{array}{l} F_2(A + B) = F_2(A) + F_2(B) \\ F_2(c \cdot A) = c \cdot F_2(A) \end{array} \right.$$

linear relationships

$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\epsilon}}}$$

$$\hookrightarrow \underline{y} = b \cdot \underline{x}$$

$\sim$   
constant

|   |    |   |
|---|----|---|
| 9 | 81 | 9 |
| 6 | 36 | 6 |

Voigt Notation

$$\begin{matrix} \underline{\underline{\sigma}} \\ 3 \times 3 \end{matrix} \rightarrow \begin{matrix} \underline{\sigma} \\ 6 \times 1 \end{matrix}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \vdots & & & & & \vdots \\ C_{61} & C_{62} & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2 \cdot \epsilon_{23} \\ 2 \cdot \epsilon_{13} \\ 2 \cdot \epsilon_{12} \end{bmatrix}$$

$6 \times 1$

$6 \times 6$

$6 \times 1$

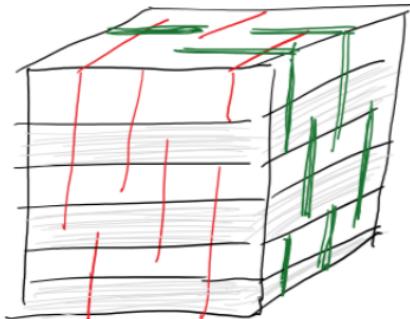
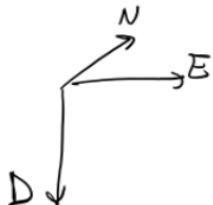
Stiffness Matrix

Shear decoupling < normal  $\leftrightarrow$  shear    shear  $\leftrightarrow$  shear } does not apply  
for plasticity

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{44} & 0 & 0 \\ 0 & C_{55} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2 \cdot \epsilon_{23} \\ 2 \cdot \epsilon_{13} \\ 2 \cdot \epsilon_{12} \end{bmatrix}$$

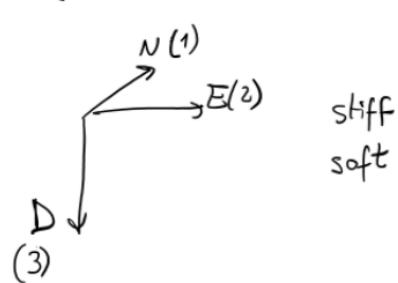


12 coeff  $\rightarrow$  9 coeff  $\rightarrow$  Orthorhombic



$$E_v \neq E_{H(N-S)} \neq E_{H(E-W)}$$

# Vertical Transverse Isotropy



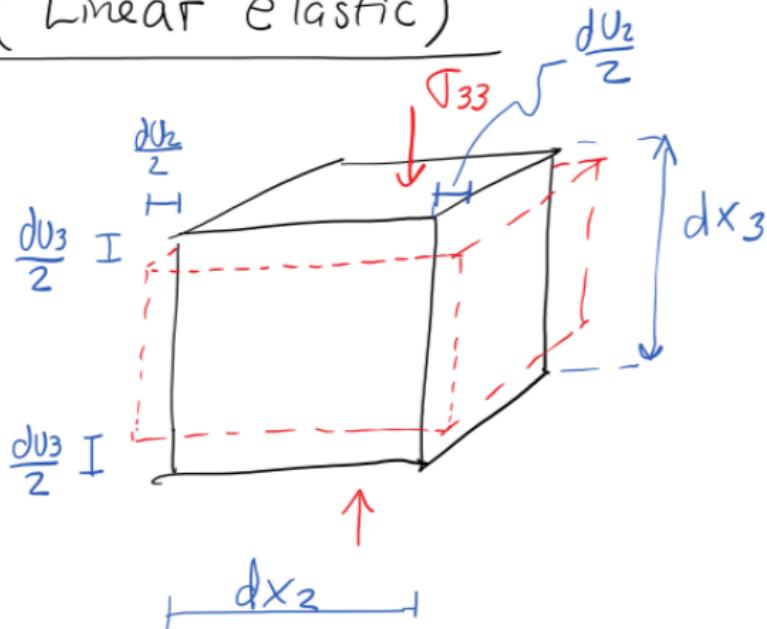
$$E_V < E_H$$

$$\rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} \\ C_{12} & C_{11} & C_{13} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} \\ C_{13} & C_{13} & C_{33} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} \\ \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & C_{44} & \textcolor{green}{\bigcirc} & \textcolor{green}{\bigcirc} \\ \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{green}{\bigcirc} & C_{44} & \textcolor{green}{\bigcirc} \\ \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{red}{\bigcirc} & \textcolor{green}{\bigcirc} & \textcolor{green}{\bigcirc} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} \quad \left. \begin{array}{l} C_{11}, C_{33} \\ C_{12}, C_{13} \\ C_{44}, \cancel{C_{66}} \\ C_{66} = \frac{C_{11} - C_{12}}{z} \end{array} \right\}$$

S independent coefficients

# Isotropy (Linear elastic)

$$\begin{array}{l} N(1) \\ \nearrow E(2) \\ \downarrow D(3) \end{array}$$



$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$E \stackrel{\text{def}}{=} \frac{\sigma_{33}}{\epsilon_{33}}$$

Young's modulus

$$\nu \stackrel{\text{def}}{=} -\frac{\epsilon_{11}}{\epsilon_{33}} = -\frac{\epsilon_{22}}{\epsilon_{33}}$$

Poisson's ratio

$$\underline{\underline{\sigma}} = \begin{vmatrix} 0 \\ 0 \\ \sigma_{33} \\ 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow \underline{\underline{\epsilon}} = \begin{vmatrix} -(\nu/E)\sigma_{33} \\ -(\nu/E)\sigma_{33} \\ \sigma_{33}/E \\ 0 \\ 0 \\ 0 \end{vmatrix} \leftarrow \begin{matrix} \sigma_{33} \\ \sigma_{22} \\ \sigma_{11} \end{matrix}$$

$$\downarrow$$

$$\begin{vmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{vmatrix} = \begin{vmatrix} \frac{1}{E} & -\nu/E & -\nu/E \\ -\nu/E & \frac{1}{E} & -\nu/E \\ -\nu/E & -\nu/E & \frac{1}{E} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{vmatrix} \leftarrow G = \frac{E}{2(1+\nu)}$$

isotropic loading  $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$$\rightarrow \epsilon_4 = \frac{1-2\nu}{E} \sigma_m$$

$$\epsilon_{vol} = \frac{3(1-2\nu)}{E} \sigma_m \Rightarrow K = \frac{E}{3(1-2\nu)}$$

$$\underline{\underline{\epsilon}} = \underline{\underline{D}} \cdot \underline{\underline{\sigma}}$$

$E, \nu$   
compliance matrix (2 indep. coeff)

$$\underline{\underline{D}} \cdot \underline{\underline{\varepsilon}} = \underline{\underline{D}}^{-1} \cdot \underline{\underline{D}} \cdot \underline{\underline{\sigma}}$$

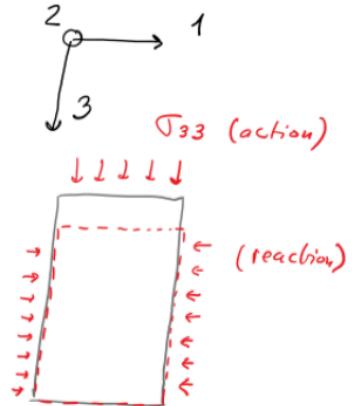
*stiffness matrix*

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}}$$

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \hline \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & | & \varepsilon_1 \\ \nu & 1-\nu & \nu & | & \varepsilon_{22} \\ \nu & \nu & 1-\nu & | & \varepsilon_{33} \\ \hline & & & | & \\ 0 & & & | & 2\varepsilon_{23} \\ 0 & & & | & 2\varepsilon_{13} \\ 0 & & & | & 2\varepsilon_{12} \end{bmatrix}$$

# Uniaxial-strain loading (stress path)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_{23} \\ \epsilon_{13} \\ \epsilon_{12} \end{bmatrix}$$



$$\left\{ \begin{array}{l} \epsilon_{11} = \epsilon_{22} = 0; \epsilon_{33} \neq 0 \\ \epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0 \end{array} \right.$$

$$\sigma_{33} = \frac{(1-v) E}{(1+v)(1-2v)} \cdot \epsilon_{33}$$

M: constrained modulus  
: oedometric modulus  
: P-wave modulus

$$\left. \begin{array}{l} M \geq E \\ \text{for } v \geq 0 \end{array} \right\}$$

$$\sigma_{11} = \frac{v E}{(1+v)(1-2v)} \cdot \epsilon_{33} = \frac{v E}{(1+v)(1-2v)} \cdot \frac{(1+v)(1-2v)}{(1-v) E} \cdot \sigma_{33}$$

$$\sigma_{11} = \frac{v}{1-v} \sigma_{33}$$

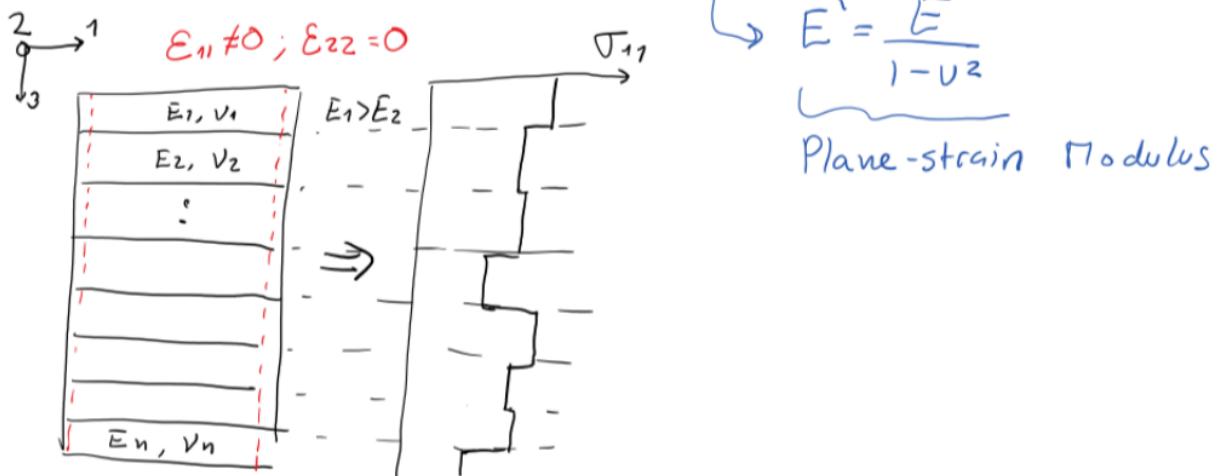
lateral effective stress coefficient  
 $K_o$

# 1D Mechanical Earth Model with tectonic strains

$$\varepsilon_{33} \neq 0; \quad \varepsilon_{11} \neq 0; \quad \varepsilon_{22} \neq 0; \quad \varepsilon_{ij} = 0 \text{ for } i \neq j$$

$$\left\{ \begin{array}{l} \sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33} + \frac{E}{1-\nu^2} \varepsilon_{11} + \frac{\nu E}{1-\nu^2} \varepsilon_{22} \\ \sigma_{22} = \frac{\nu}{1-\nu} \sigma_{33} + \frac{\nu E}{1-\nu^2} \varepsilon_{11} + \frac{E}{1-\nu^2} \varepsilon_{22} \end{array} \right. \rightarrow \begin{array}{l} \text{tectonic} \\ \text{strains} \\ \varepsilon_{11}, \varepsilon_{22} \end{array}$$

$$\Gamma_{33} = S_{33} - P_p$$



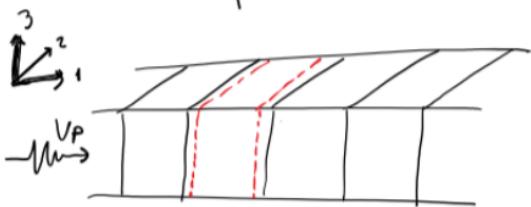
$(E, V)$  from field and laboratory data

lab: static, dynamic

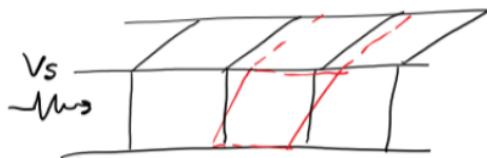
field: dynamic

$(E, V) \rightarrow$

$$V_p = \sqrt{\frac{M}{\rho_{bulk}}} \quad (\text{P-WAVE})$$

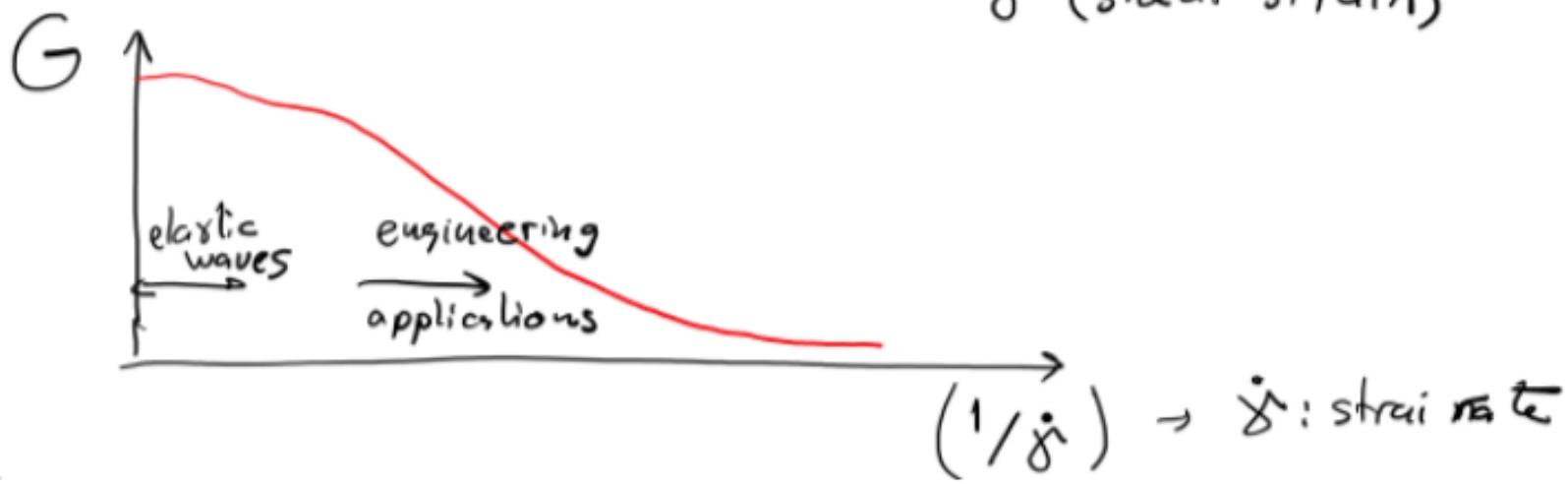
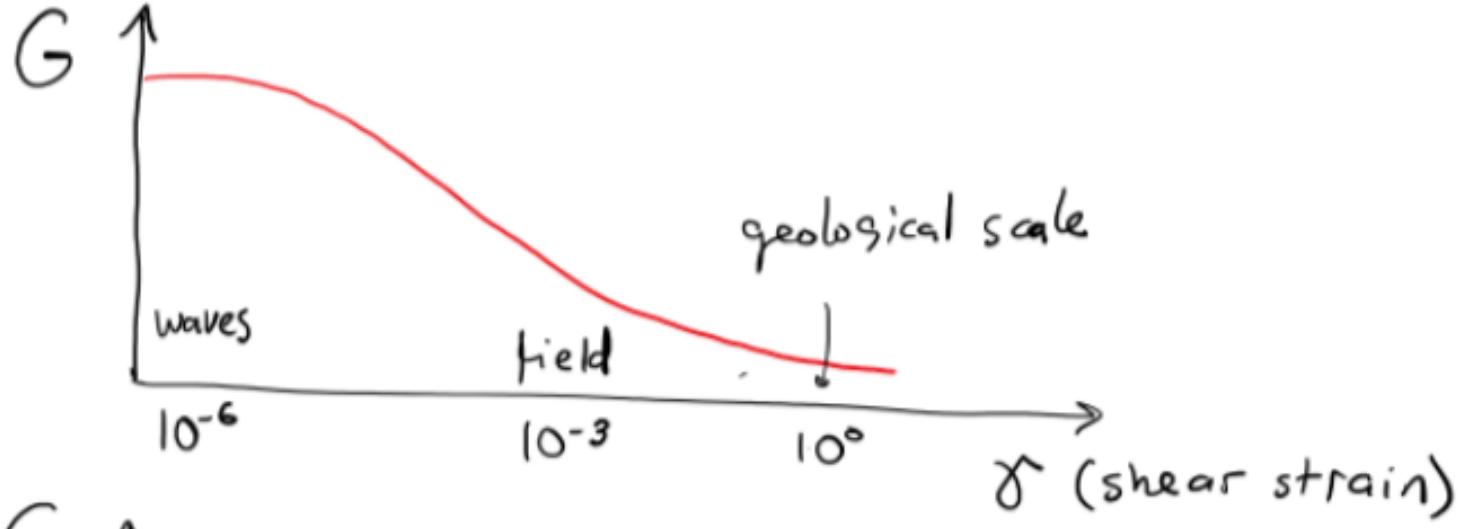


$$V_s = \sqrt{\frac{G}{\rho_{bulk}}} \quad (\text{S-WAVE})$$



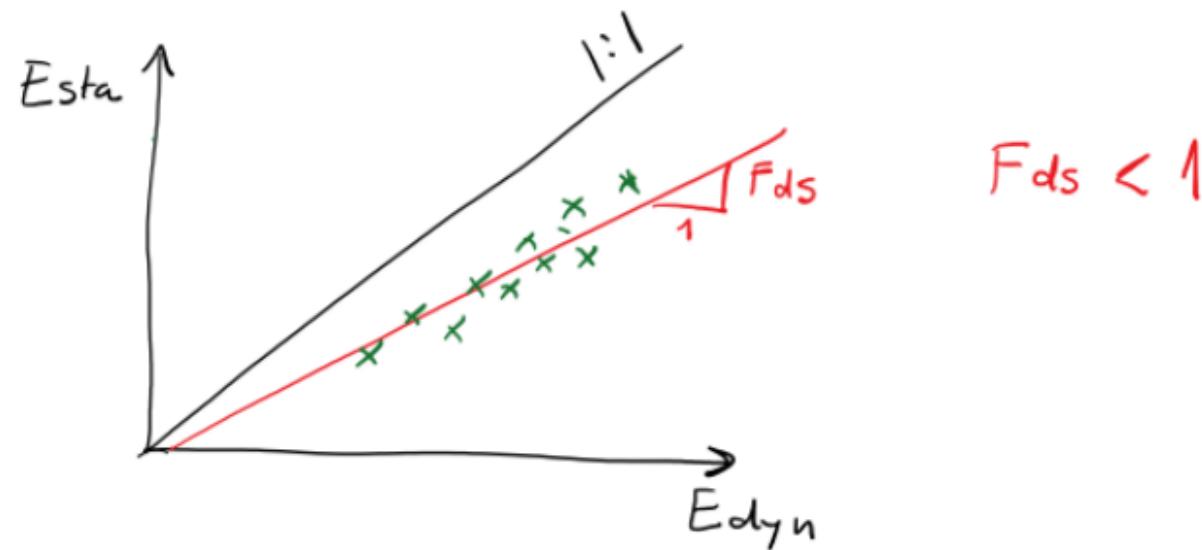
$$E_{dyn} = \rho_{bulk} V_s^2 \left( \frac{3V_p^2 - 4V_s^2}{V_p^2 - V_s^2} \right)$$

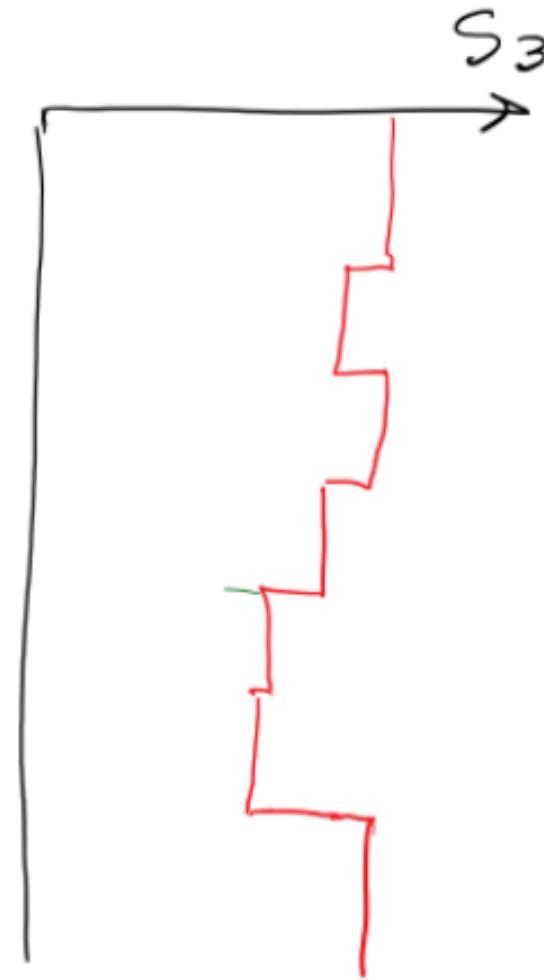
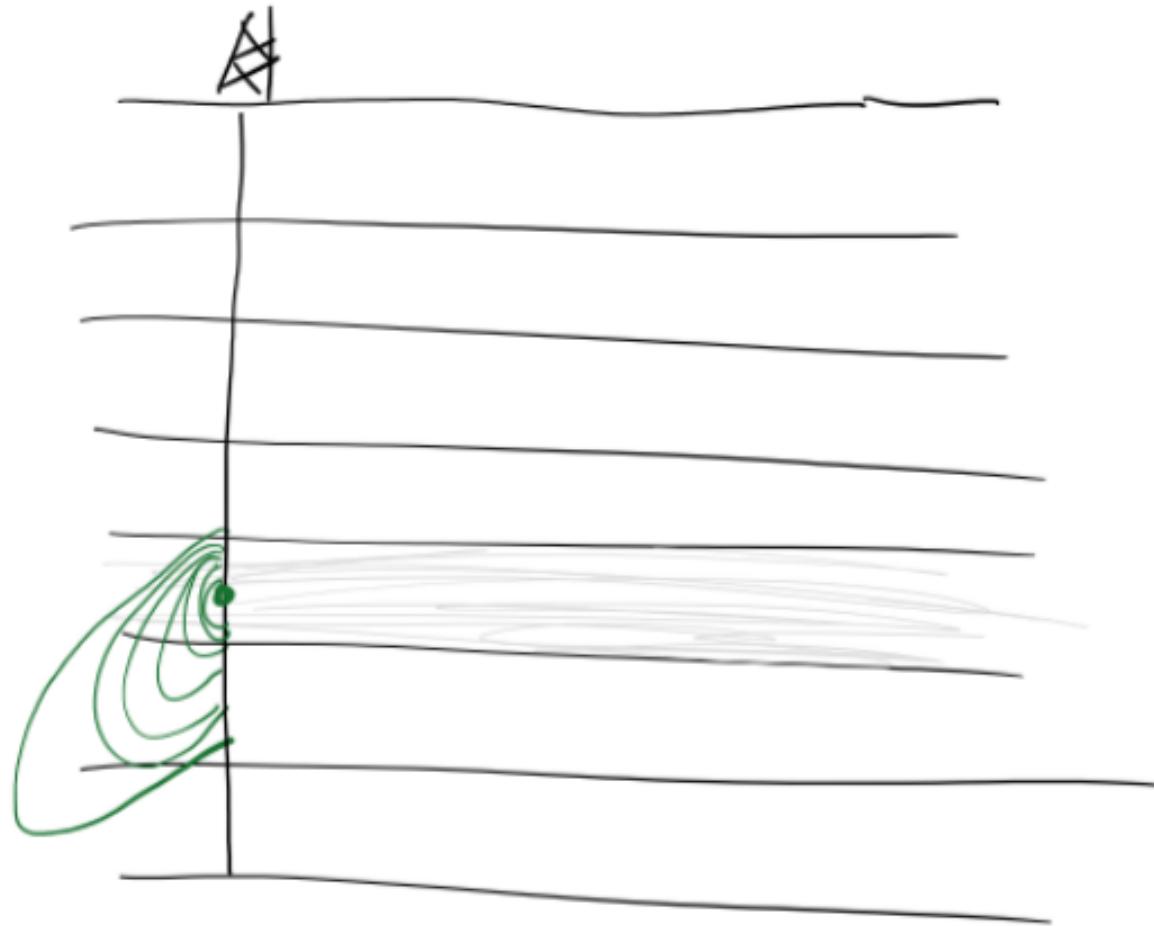
$$V_{dyn} = \frac{V_p^2 - 2V_s^2}{2(V_p^2 - V_s^2)}$$



$$E_{sta} = F_{ds} \cdot E_{dyn}$$

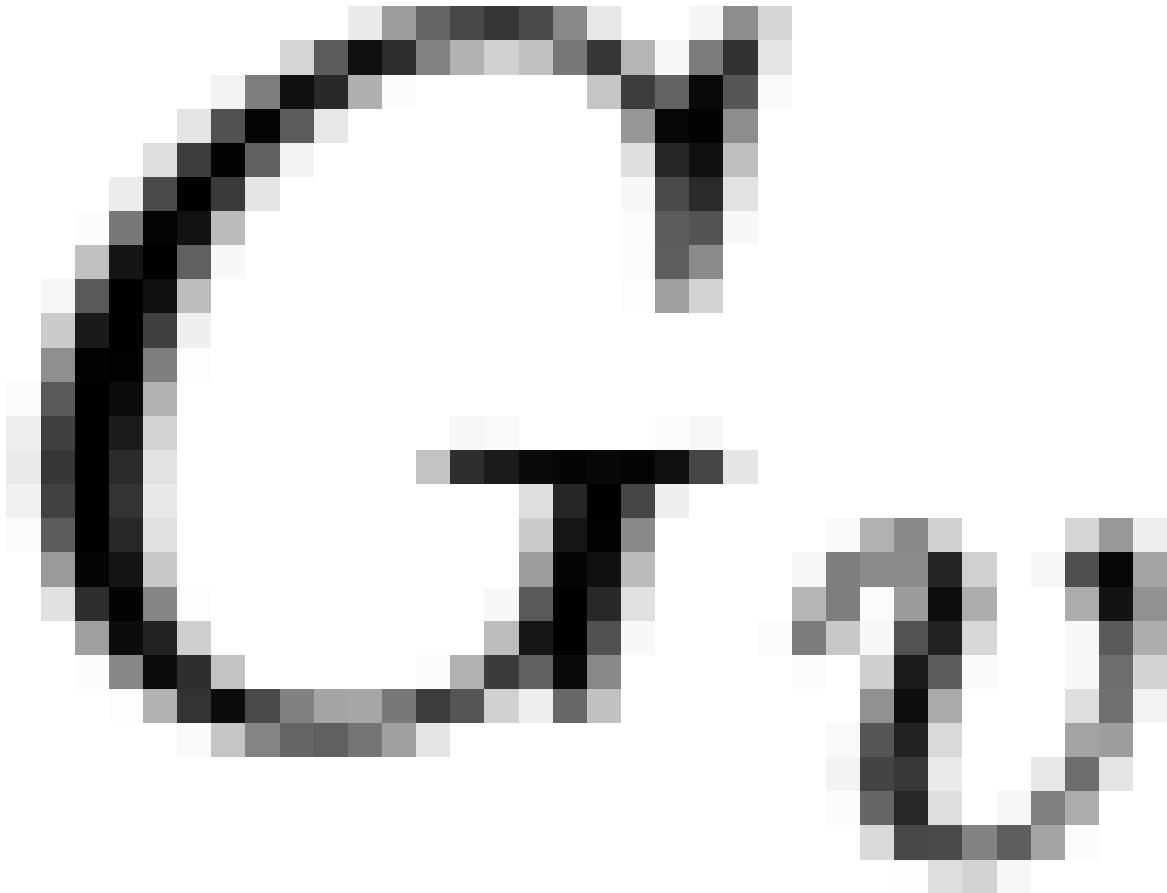
$$; V_{sta} \approx V_{dyn}$$





$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} +\frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_h}{E_h} & +\frac{1}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & +\frac{1}{E_v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_h} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

$$G_h = \frac{E_h}{2(1 + \nu_h)}$$



$$E_h = \frac{(C_{11} - C_{12}) [C_{33}(C_{11} + C_{12}) - 2C_{13}^2]}{C_{11}C_{33} - C_{13}^2}$$

$E_0$  $-$  $C_{33}$  $-$  $C_{11}$  $+$  $C_{12}$  $\frac{2C_2}{C_{13}}$

$v_h$  $=$ 

$$\frac{c_{12}^1 c_{33} - c_{13}^2}{c_{11} c_{33} - c_{13}^2}$$

 $-$

$$v_0 = \frac{c_{13}}{c_{11} + c_{12}}$$



G<sub>b</sub>

=

G<sub>66</sub>

=

G<sub>11</sub>

2

G<sub>12</sub>

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

$$c_{11} = \frac{\left(1 - \nu_h\right)E_v - 2\nu_h^2 E_h}{\left[1 + \nu_h\right] \left(E_h E_v - \frac{\nu_h^2 E_h^2}{1 + \nu_h}\right)}$$

$$C_{33} = \frac{1}{(1 - \nu_h) E_v} \left( \frac{E_v^2 - \nu_h E_v}{E_v - 2\nu_h E_h} \right)$$

$$c_{12} = \frac{\left( \nu_v^2 E_h^2 + \nu_h E_h E_v \right)}{\left( 1 - \nu_h \right) E_v - 2 \nu_v^2 E_h}$$

$$c_{13} = \frac{1}{(1 - \nu_h)E_u - 2\nu_u^2 E_h} (v_u E_h E_u)$$

$$\frac{G_{66}}{2} = \frac{C_{11} - C_{12}}{2} = G_h = \frac{E_h}{2(1 + \nu_b)}$$



# VTI Static Elastic Properties

- Conventional Method

↳ Axisymmetric Triaxial Cell



↳ Deviatoric Loading Stress Path

$$\hookrightarrow \sigma_r = \text{cst}$$

$$\hookrightarrow \Delta(\sigma_a - \sigma_r) = \Delta\sigma_a, \quad \sigma_a \geq \sigma_r$$



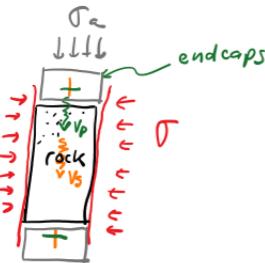
| Sample              | Quasi-static                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|---------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Vertical</b><br> | Vertical Young modulus<br>$E_v = \frac{\Delta\sigma_{33}}{\Delta\varepsilon_{33}} \Big _{\sigma_{11}, \sigma_{22}}$ Vertical Poisson ratio<br>$\nu_v = -\frac{1}{2} \left( \frac{\Delta\varepsilon_{11}}{\Delta\varepsilon_{33}} + \frac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{33}} \right) = -\frac{\underbrace{\Delta\varepsilon_4}_{\Delta\varepsilon_{33}}}{\underbrace{\Delta\varepsilon_{33}}} = -\frac{\Delta\varepsilon_{12}}{\Delta\varepsilon_{33}}$ |

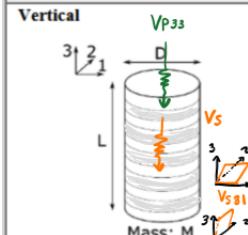
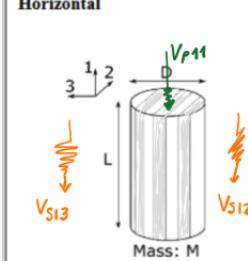
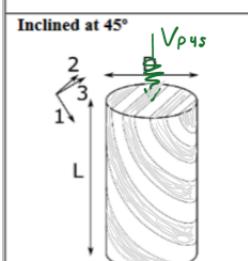
| Horizontal | Quasi-static                                                                                                                                                                                                                                                                                                                                          |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|            | Horizontal Young modulus<br>$E_h = \frac{\Delta\sigma_{11}}{\Delta\varepsilon_{11}}$ Vertical Poisson ratio<br>$\nu_v = -\frac{\Delta\varepsilon_{33}}{\Delta\varepsilon_{11}} = \nu_{31} \quad \leftarrow = \nu_{13}$ Horizontal Poisson ratio<br>$\nu_h = -\frac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{11}} = \nu_{21} \Rightarrow = \nu_{12}$ |

} 4  
 parameters

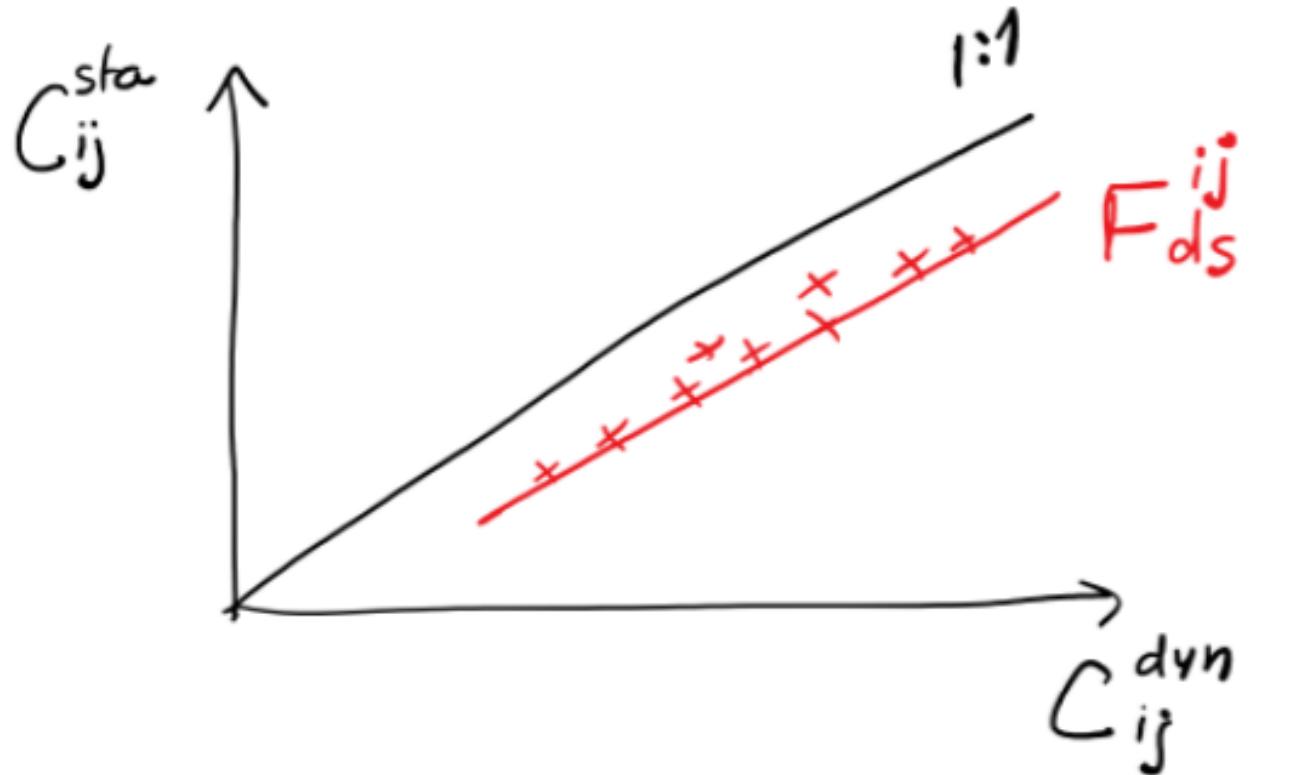
# VTI Dynamic Elastic Properties

- Conventional Method



| Sample                                                                                                      | Dynamic                                                                                                                                                                                                                              |
|-------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Vertical</b><br>         | <p>P-wave stiffness perpendicular to bedding</p> $C_{33} = \rho(V_{p33})^2$ <p>S-wave stiffness perpendicular to bedding</p> $C_{44} = \frac{1}{2} [\rho(V_{s31})^2 + \rho(V_{s32})^2]$ $C_{44} = \rho(V_{s31})^2 = \rho(V_{s32})^2$ |
| <b>Horizontal</b><br>       | <p>P-wave stiffness parallel to bedding</p> $C_{11} = \rho(V_{p11})^2$ <p>S-wave stiffness perpendicular to bedding</p> $C_{44} = \rho(V_{s13})^2$ <p>S-wave stiffness in the plane of bedding</p> $C_{66} = \rho(V_{s12})^2$        |
| <b>Inclined at 45°</b><br> | <p>Off-diagonal stiffness</p> $C_{13} = -C_{44} + [4\rho^2 V_{p45}^4 - 2\rho V_{p45}^2 (C_{11} + C_{33} + 2C_{44}) + (C_{11} + C_{44})(C_{33} + C_{44})]^{1/2}$                                                                      |

## Dynamic to Static conversion



# Quantification of anisotropy

## Static

Young modulus anisotropy

$$\frac{E_h}{E_v} ; E_h > E_v$$

Poisson's ratio anisotropy

$$\frac{v_h}{1-v_h}$$
 } effective lateral stress coefficient

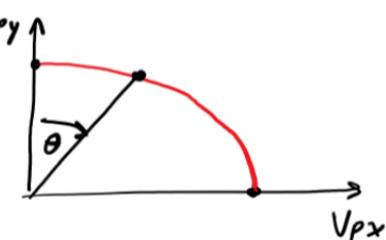
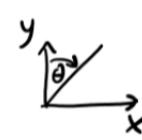
## Dynamic

Thomson Parameters :

$$\epsilon = \frac{V_{p11}^2 - V_{p33}^2}{2 V_{p33}^2} = \frac{C_{11} - C_{33}}{2 C_{33}}$$

$$\gamma = \frac{C_{66} - C_{44}}{2 C_{44}}$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2 C_{33} (C_{33} - C_{44})}$$



Weak anisotropy

$$V_p(\theta) = V_{p33} [1 + \delta \sin^2 \theta \cos^2 \theta + \epsilon \sin^4 \theta]$$

### Iso-stress (Reuss Average)



$$\langle E_v \rangle = \frac{\sigma_v}{\langle \epsilon_v \rangle}$$

$$\hookrightarrow \langle \epsilon_v \rangle = \frac{\sigma_v}{\langle E_v \rangle} \quad (1)$$

$L_{soft}, L_{stiff}$

$$(2) \langle \epsilon_v \rangle = \left( \frac{\sigma_v}{E_{stiff}} \cdot L_{stiff} + \frac{\sigma_v}{E_{soft}} \cdot L_{soft} \right) \cdot \frac{1}{L_T}$$

$$= \frac{\sigma_v}{E_{stiff}} \cdot \underbrace{\frac{L_{stiff}}{L_T}}_{f_{stiff}} + \frac{\sigma_v}{E_{soft}} \cdot \underbrace{\frac{L_{soft}}{L_T}}_{f_{soft}}$$

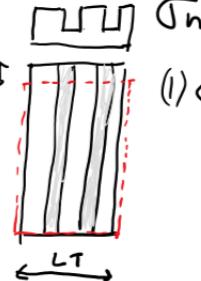
(1,2)  $\rightarrow$

$$\frac{\sigma_v}{\langle E_v \rangle} = \frac{\sigma_v}{E_{stiff}} \cdot f_{stiff} + \frac{\sigma_v}{E_{soft}} \cdot f_{soft}$$

### Iso-strain (Voigt Average)



$\epsilon_h$



$$(2) \langle \sigma_h \rangle = \sigma_n^{soft} \cdot \underbrace{\frac{L_{soft}}{L_T}}_{f_{soft}} + \sigma_n^{stiff} \cdot \underbrace{\frac{L_{stiff}}{L_T}}_{f_{stiff}}$$

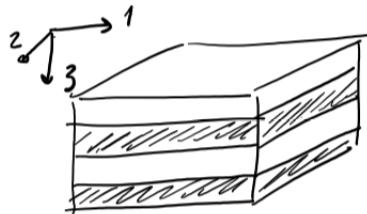
(1,2)

$$\epsilon_h \langle E_h \rangle = \epsilon_h E_{soft} \cdot f_{soft} + \epsilon_h \cdot E_{stiff} \cdot f_{stiff}$$

$$\boxed{\langle E_h \rangle = E_{soft} \cdot f_{soft} + E_{stiff} \cdot f_{stiff}}$$

$$\boxed{\langle E_v \rangle = \left( \frac{f_{stiff}}{E_{stiff}} + \frac{f_{soft}}{E_{soft}} \right)^{-1}}$$

## VTI stiffness matrix



$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_h} & -\frac{v_h}{E_h} & -\frac{v_v}{E_v} \\ -\frac{v_h}{E_h} & \frac{1}{E_h} & -\frac{v_v}{E_v} \\ -\frac{v_v}{E_v} & -\frac{v_v}{E_v} & \frac{1}{E_v} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix}$$

For isotropic loading  $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{nn}$

$$\varepsilon_{11} = \varepsilon_{22} = \left( \frac{1-v_h}{E_h} - \frac{v_v}{E_v} \right) \sigma_{nn}$$

$$\varepsilon_{33} = \left( -\frac{2v_v}{E_v} + \frac{1}{E_v} \right) \sigma_{nn}$$

$$\varepsilon_{vol} = \left[ 2 \left( \frac{1-v_h}{E_h} - \frac{v_v}{E_v} \right) + \left( \frac{1-2v_v}{E_v} \right) \right] \sigma_{nn}$$

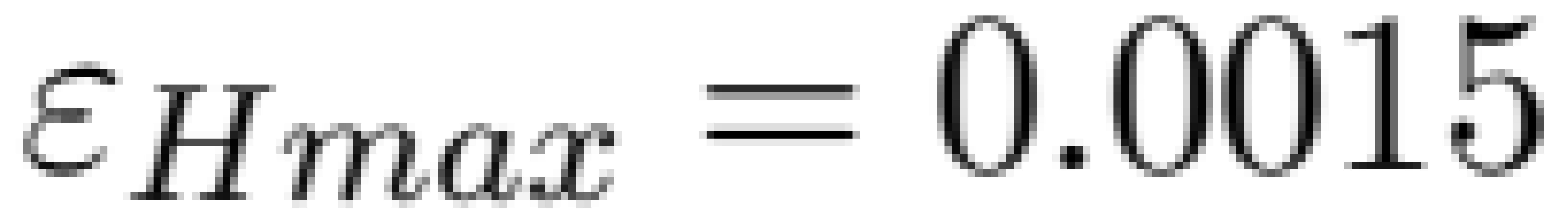
$$\rightarrow K_{VTI} = \left[ \frac{2(1-v_h)}{E_h} + \frac{1-4v_v}{E_v} \right]^{-1}$$

if  $v_h = v_v$ ;  $E_h = E_v$

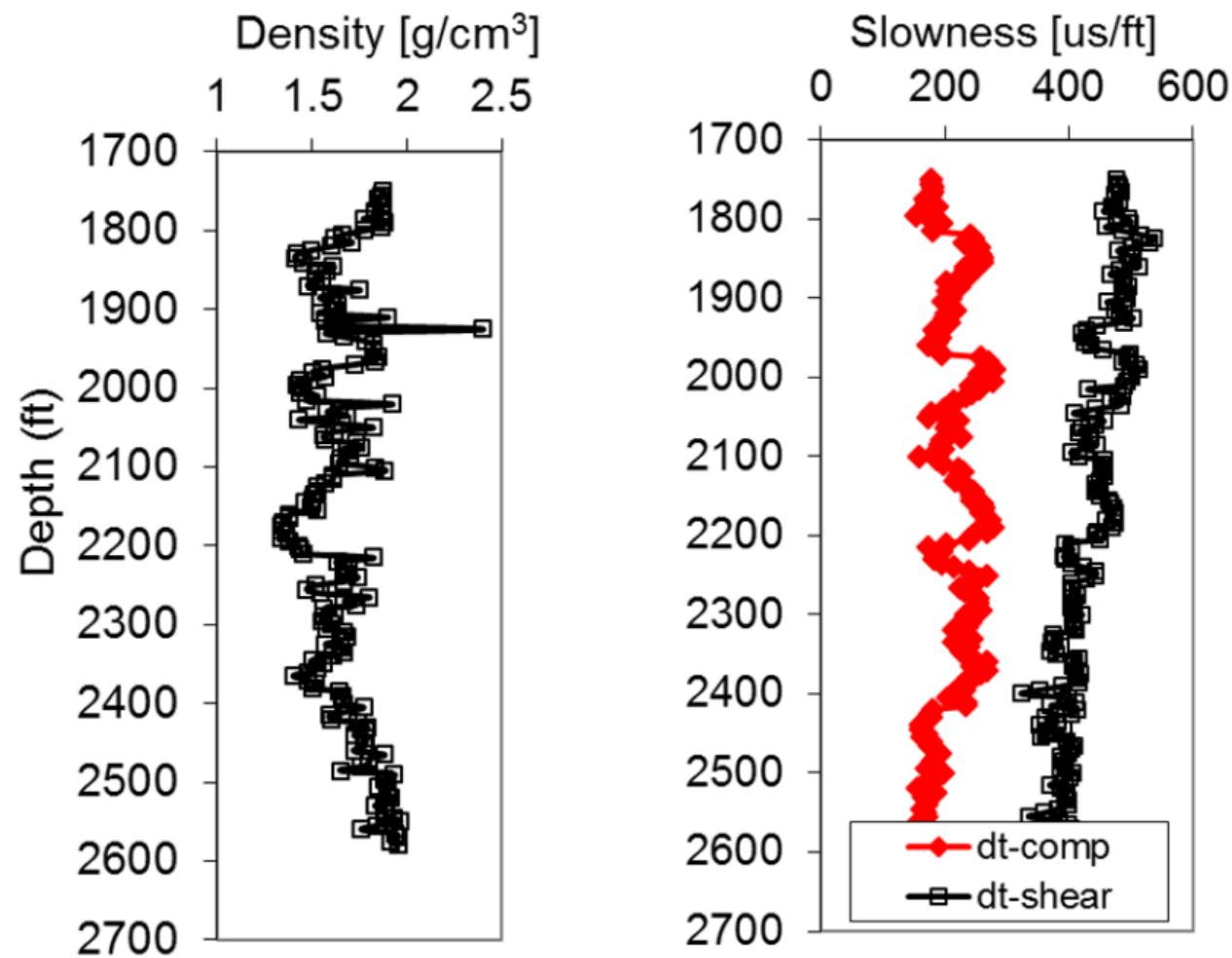
$$K = \left[ \frac{2-2v+1-4v}{E} \right]^{-1} = \frac{E}{3(1-2v)} \quad \checkmark$$







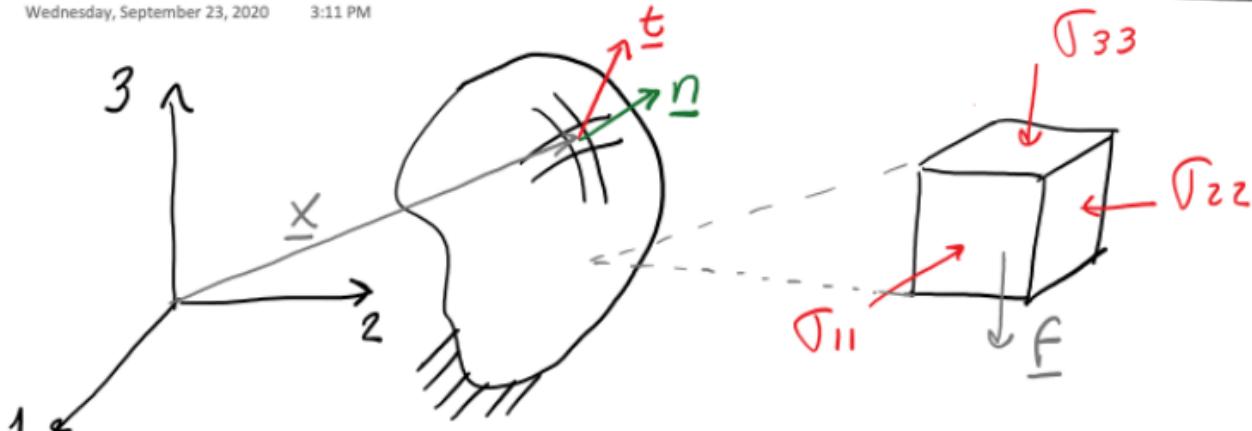




## 2- General solution for a continuum mechanics problem

Wednesday, September 23, 2020

3:11 PM



① Equilibrium :  $\nabla \cdot \underline{\underline{\sigma}} + \underline{F} = 0$

② Kinematic :  $\underline{\underline{\epsilon}} = \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^T) \rightarrow \text{small strains}$

③ Constitutive :  $\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\epsilon}}$   $\rightarrow$  linear elasticity

Plan : ① → ③ → ②

$$①, \text{Coordinate 1: } \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\underline{\sigma} = \underline{C} \cdot \underline{\epsilon} \Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \left[ \begin{array}{ccc|c} \lambda + 2\mu & \lambda & \lambda & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 \\ \hline 0 & 0 & 0 & \nu \\ 0 & 0 & 0 & 2\nu \\ 0 & 0 & \nu & 2\nu \end{array} \right] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}; \quad \begin{aligned} \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \nu &= \frac{E}{2(1+\nu)} = G \end{aligned}$$

$$\frac{\partial}{\partial x_1} (\lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\nu \epsilon_{11}) + \frac{\partial}{\partial x_2} (2\nu \epsilon_{12}) + \frac{\partial}{\partial x_3} (2\nu \epsilon_{13}) + f_1 = 0$$

$$\frac{\partial}{\partial x_1} \left( \lambda \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + 2\nu \frac{\partial u_1}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( 2\nu \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right) + \frac{\partial}{\partial x_3} \left( 2\nu \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \right) + f_1 = 0$$

$$\lambda \left[ \frac{\partial}{\partial x_1} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right] + 2\nu \frac{\partial^2 u_1}{\partial x_1^2} + \nu \frac{\partial^2 u_1}{\partial x_2^2} + \nu \frac{\partial}{\partial x_2} \left( \frac{\partial u_2}{\partial x_1} \right) + \nu \frac{\partial^2 u_1}{\partial x_3^2} + \nu \frac{\partial^2 u_3}{\partial x_3 \partial x_1} + f_1 = 0 \quad \leftarrow \frac{\partial^2 f(x, y)}{\partial x \partial y} = \frac{\partial^2 f(x)}{\partial y \partial x}$$

$$\lambda \left[ \frac{\partial}{\partial x_1} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right] + \nu \underbrace{\left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right)}_{\nabla^2 u_1} + \nu \frac{\partial}{\partial x_1} \underbrace{\left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)}_{\nabla \cdot \underline{u}} + f_1 = 0$$

$$\lambda \frac{\partial}{\partial x_1} (\nabla \cdot \underline{u}) + \nu \frac{\partial}{\partial x_1} (\nabla \cdot \underline{u}) + \nu \nabla^2 \underline{u}_1 + f_1 = 0 \Rightarrow \text{Coord. 1}$$

$$(\lambda + \nu) \nabla (\nabla \cdot \underline{u}) + \nu \nabla^2 \underline{u} + \underline{f} = \underline{0} \quad \underline{u}: \text{unknown}$$

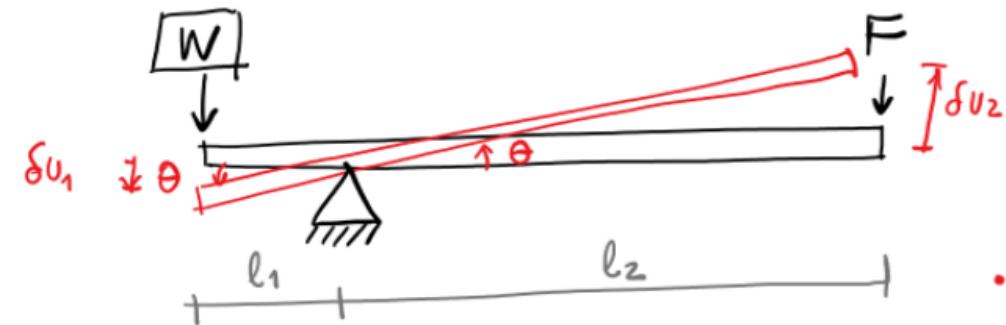
Navier's Equation

## Solution

- Analytical  
↓  
simple boundary conditions
  - Beltrami-Mitchell (strain compatibility)
    - $\underline{U} \rightarrow \underline{\underline{\epsilon}}$  ✓
    - $\underline{\underline{\epsilon}} \rightarrow U$  ✗
  - Airy's functions
    - $\nabla^4 \varphi = 0 ; \left| \sigma_{ii} = \frac{\partial^2 \varphi}{\partial x_i^2} \right.$
  - Green functions (convolution): Boussinesq
  - Examples: Kirsch, Griffith, Sneddon
- Numerical
  - Finite differences; FLAC-Itasca
  - Finite Element Method (FEM): Abaqus  
FreeFEM++  
FEniCS  
Comsol

# Weak formulation of continuum mechanics equations

Analogy with virtual work



• Solution #1: Angular Momentum  
 $E_{\text{ang}}$

$$W \cdot l_1 = F \cdot l_2$$

$$F = \frac{l_1}{l_2} \cdot W$$

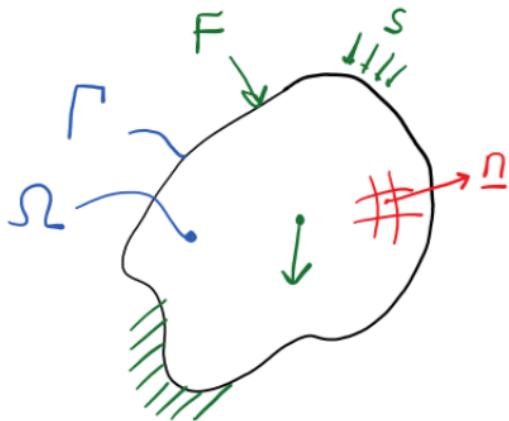
• Solution #2: Energy Conservation  $\leftrightarrow$

Principle  
of  
Virtual  
Work

$$W \cdot \underline{\delta u_1} = F \cdot \underline{\delta u_2}$$

$$W \cdot (l_1 \cdot \tan \theta) = F \cdot (l_2 \tan \theta)$$

$$F = \frac{l_1}{l_2} \cdot W$$



$$\text{Equil: } \nabla \cdot \underline{\underline{\Sigma}} + \underline{F} = \underline{0}$$

$$-\nabla \cdot \underline{\underline{\Sigma}} = \underline{F}$$

$$\int_{\Omega} (\delta \underline{u}) \cdot (-\nabla \cdot \underline{\underline{\Sigma}}) = \int_{\Omega} (\delta \underline{u}) \cdot \underline{F}$$

virtual displacement

Green's Theorem

$$\int_{\Omega} \nabla \delta \underline{u} \cdot \underline{\underline{\Sigma}} - \int_{\Gamma} \delta \underline{u} \cdot (\underline{\underline{\Sigma}} \cdot \underline{n}) = \int_{\Omega} \delta \underline{u} \cdot \underline{F}$$

- Variational form
- Weak form

$$\int_{\Omega} \underline{\epsilon}(\nabla \delta \underline{u}) : \underline{\underline{\Sigma}}(u) = \int_{\Gamma} \delta \underline{u} \cdot (\underline{\underline{\Sigma}}(u) \cdot \underline{n}) + \int_{\Omega} \delta \underline{u} \cdot \underline{F}$$

strain energy
stress boundary condition
body force

Unknowns:  $\underline{u}$  ;  $\delta \underline{u}$   
 actual virtual  
 displacement displacement

$$\mathcal{E}(\nabla \delta \mathbf{u}) : \underline{\underline{\Gamma}}(\underline{\underline{u}}) = \underline{\underline{\xi}}_{11} \cdot \underline{\underline{\Gamma}}_{11} + \underline{\underline{\xi}}_{22} \cdot \underline{\underline{\Gamma}}_{22} + \underline{\underline{\xi}}_{33} \cdot \underline{\underline{\Gamma}}_{33} + \underline{\underline{\xi}}_{12} \cdot \underline{\underline{\Gamma}}_{12} + \dots$$

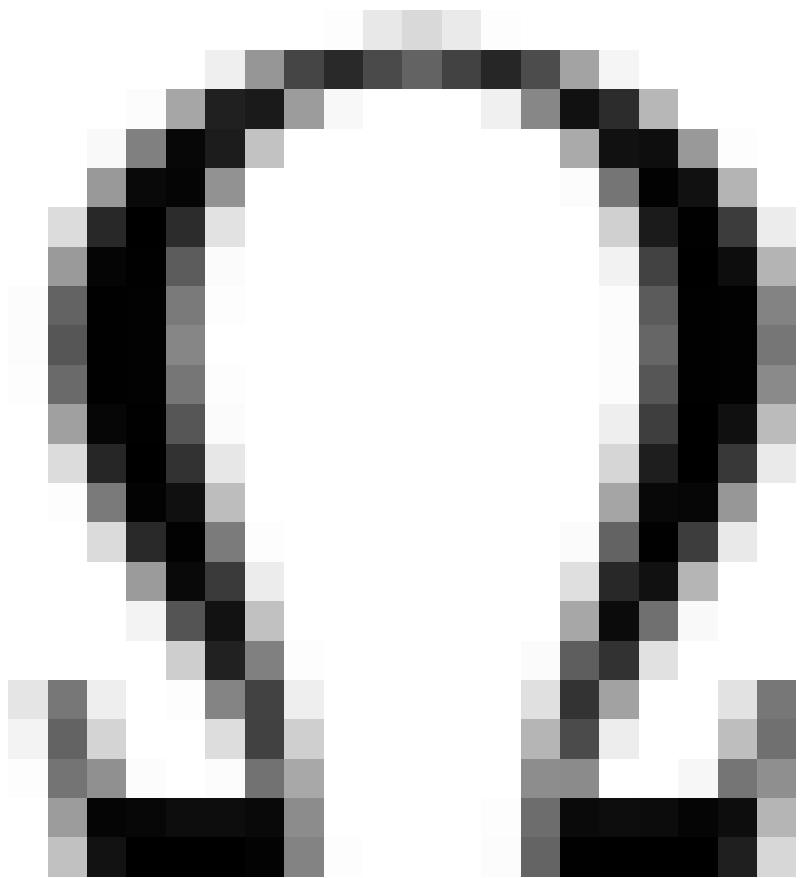
$$\rightarrow E = p \cdot V \quad (\text{Energy})$$

$$E = \Gamma \cdot \underbrace{\frac{dV}{V}}_{\mathcal{E}} \quad (\text{Energy per unit of volume})$$

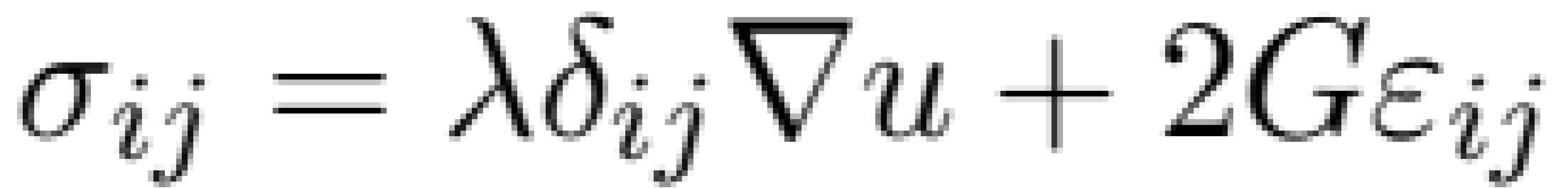
$$\int_{\Omega} \mathcal{E}(\nabla \delta \mathbf{u}) : \underline{\underline{\Gamma}}(\underline{\underline{u}}) = \int_{\Omega} \mathcal{E}(\nabla \delta \mathbf{u}) : [\underline{\underline{C}} \cdot \underline{\underline{e}}(\underline{\underline{u}})]$$

constitutive equation









$$c_{ij} =$$

2

1

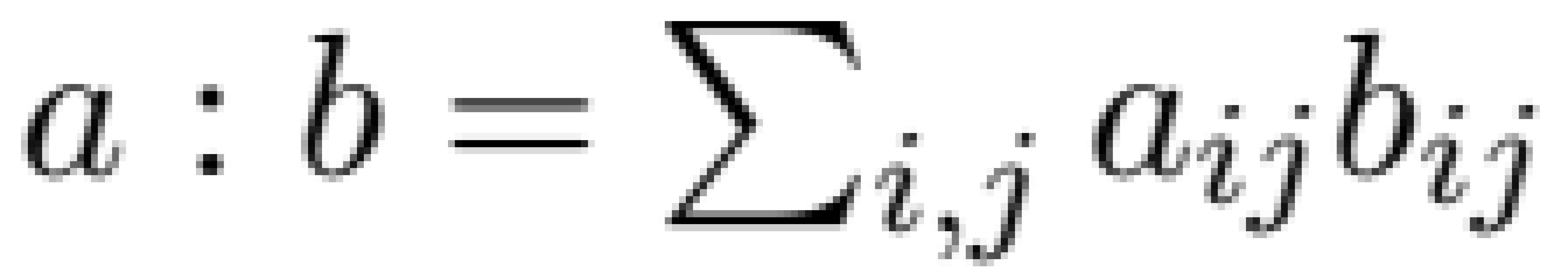
$$\left( \hat{\alpha}_{u_i} - \hat{\alpha}_{c_j} \right)$$

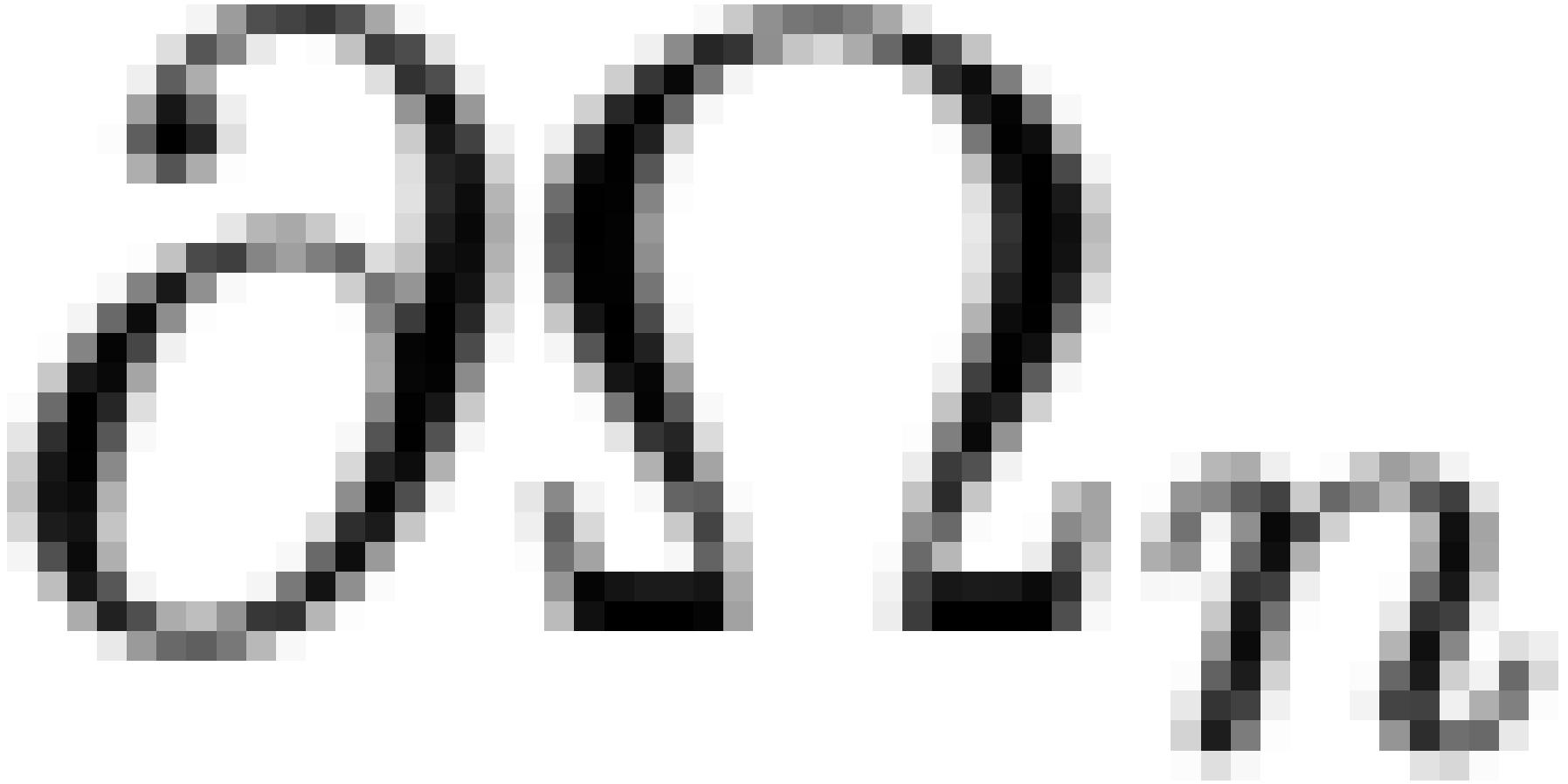
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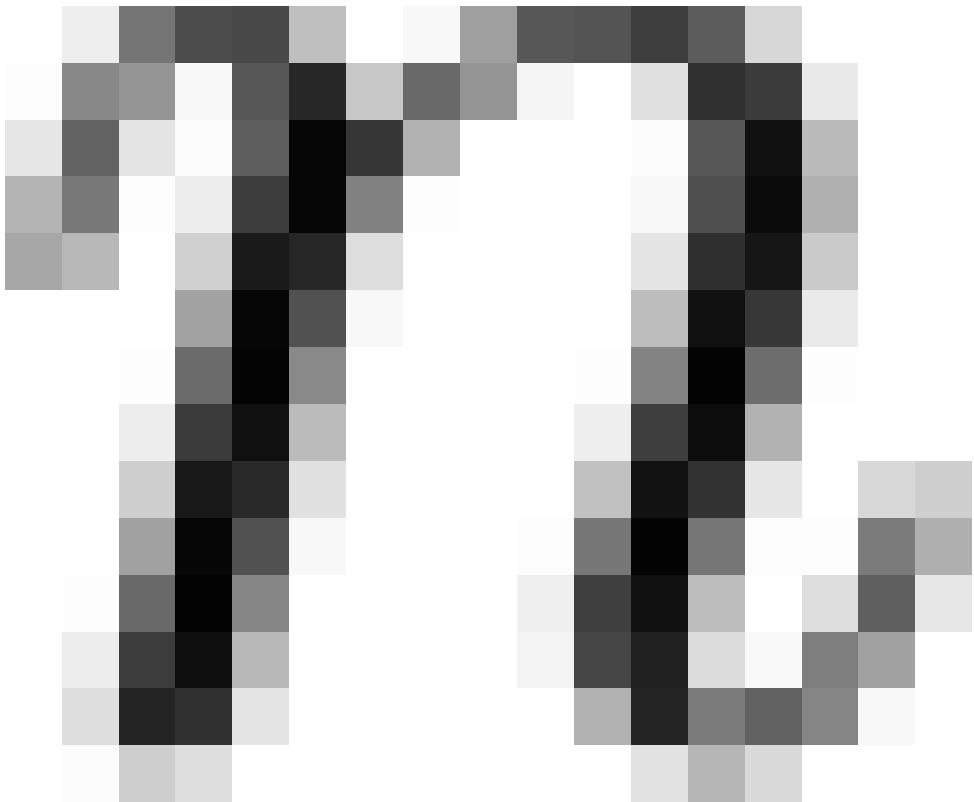
$$\hat{\alpha}_{u_j}$$

$$\hat{\alpha}_{c_i}$$

$$\int_{\Omega} \sigma(u) : \epsilon(v) dV = \int_{\partial\Omega_n} f \cdot v \tau \cdot n dS$$

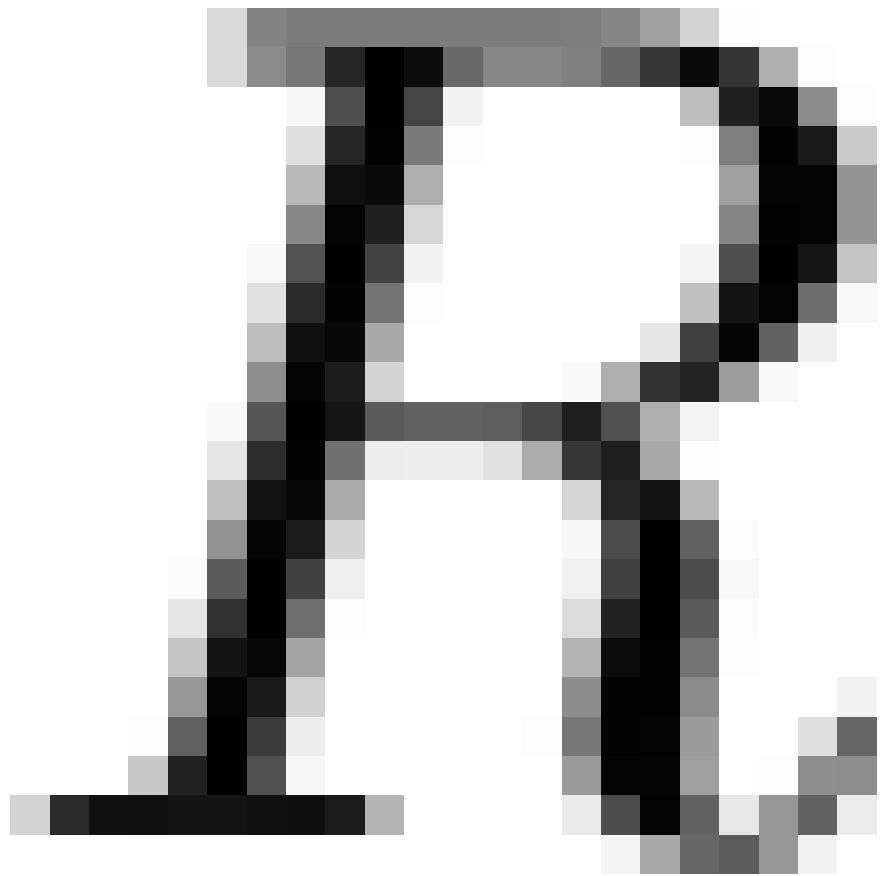








$$\int_{\Omega} [2G \epsilon_{ij}(u) \epsilon_{ij}(v) + \lambda \epsilon_{ii}(u) \epsilon_{jj}(v)] \, dV = \int_{\partial\Omega_n} f \cdot v \, dS$$

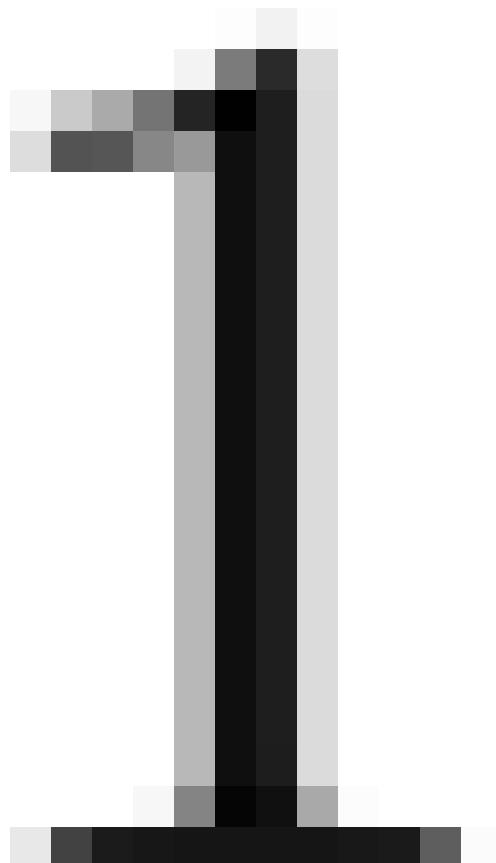
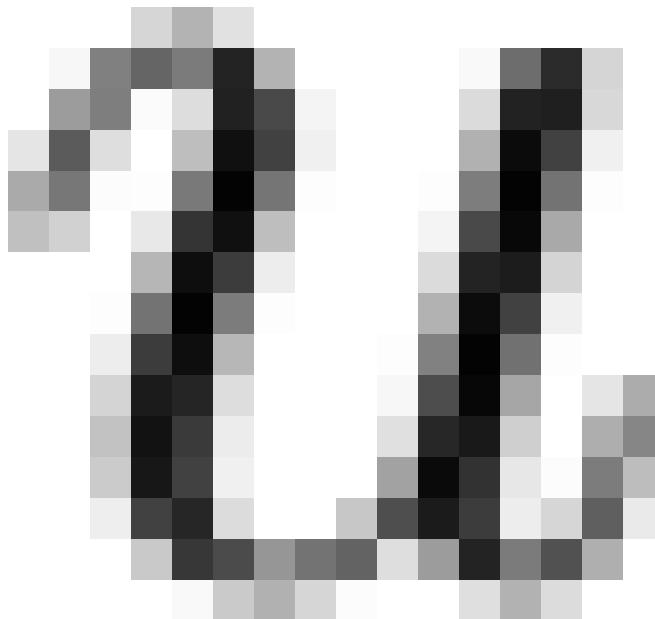


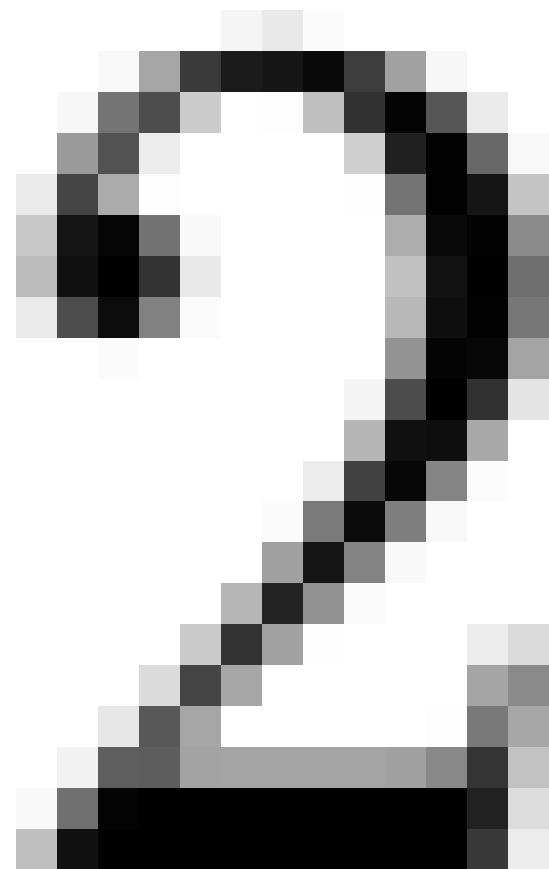
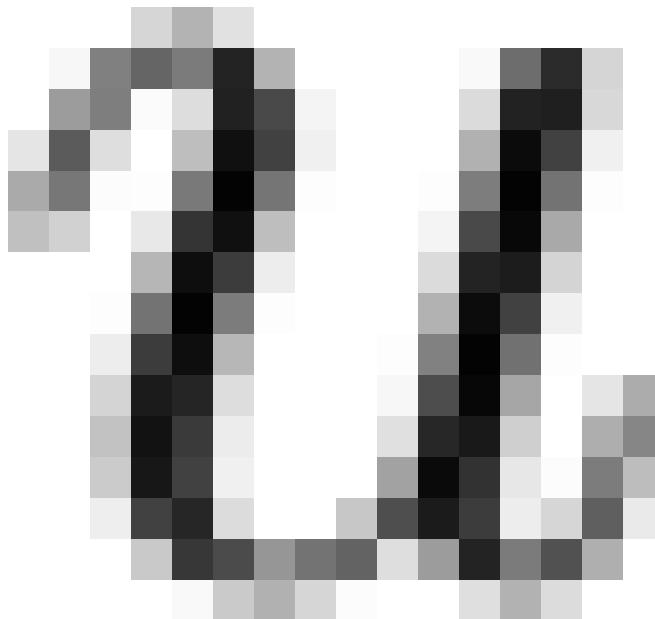
$$\begin{cases} -\nabla \sigma = f \\ \sigma_{xx}(x = \pm\infty, y) = S_1 \\ \sigma_{yy}(x, y = \pm\infty) = S_2 \end{cases}$$

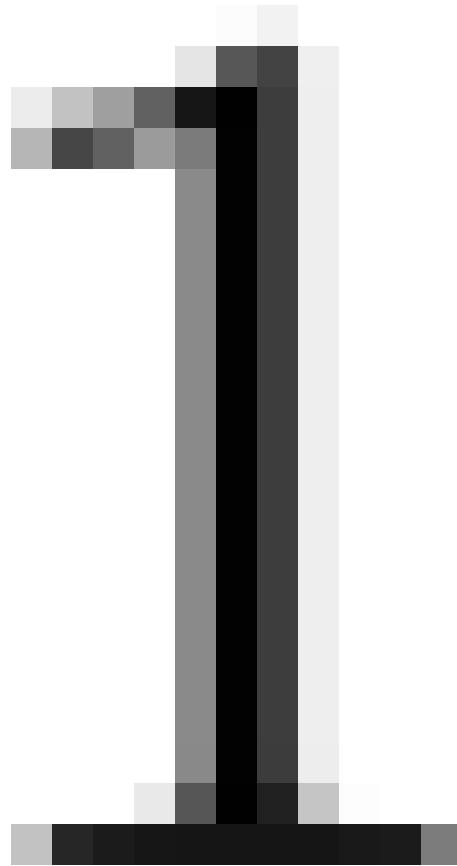
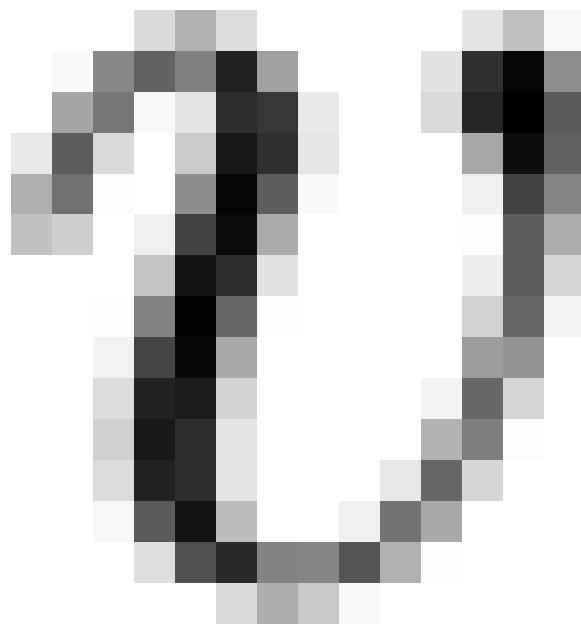
$$\sigma_{rr} = \frac{1}{2}(S_1 + S_2 - 2P_w) \left[ 1 - \left( \frac{R}{r} \right)^2 \right] + \frac{1}{2}(S_1 - S_2) \left[ 1 - \left( \frac{2R}{r} \right)^4 + 3 \left( \frac{R}{r} \right)^4 \right] \cos(2\theta) + P_w \left( \frac{R}{r} \right)^2$$

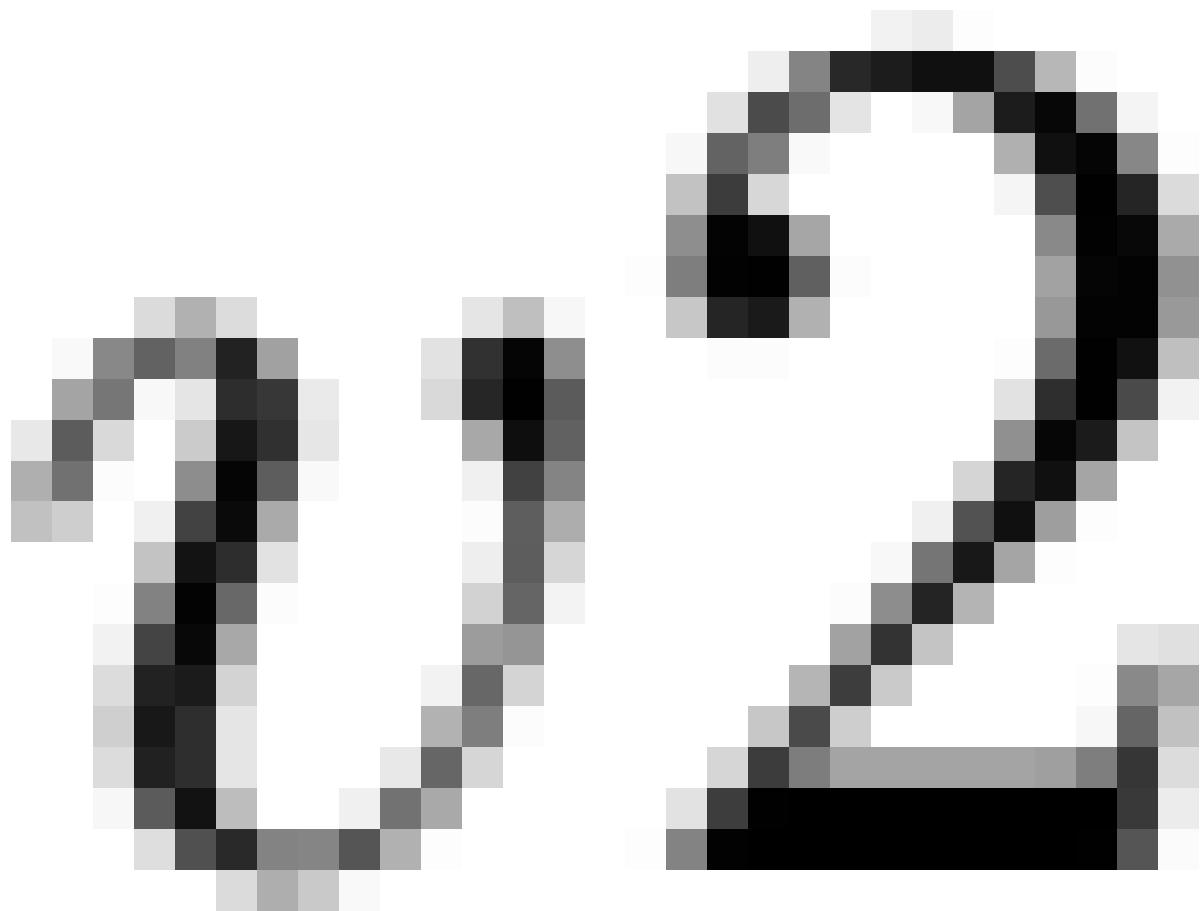
$$\sigma_{\theta\theta} = \frac{1}{2}(S_1 + S_2 - 2P_w) \left[ 1 + \left( \frac{R}{r} \right)^2 \right] - \frac{1}{2}(S_1 - S_2) \left[ 1 + 3 \left( \frac{R}{r} \right)^4 \right] \cos(2\theta) - P_w \left( \frac{R}{r} \right)^2$$

$$\sigma_{r\theta} = \frac{1}{2}(S_1 - S_2) \left[ 1 + \left( \frac{2R}{r} \right)^4 - 3 \left( \frac{R}{r} \right)^4 \right] \sin(2\theta)$$









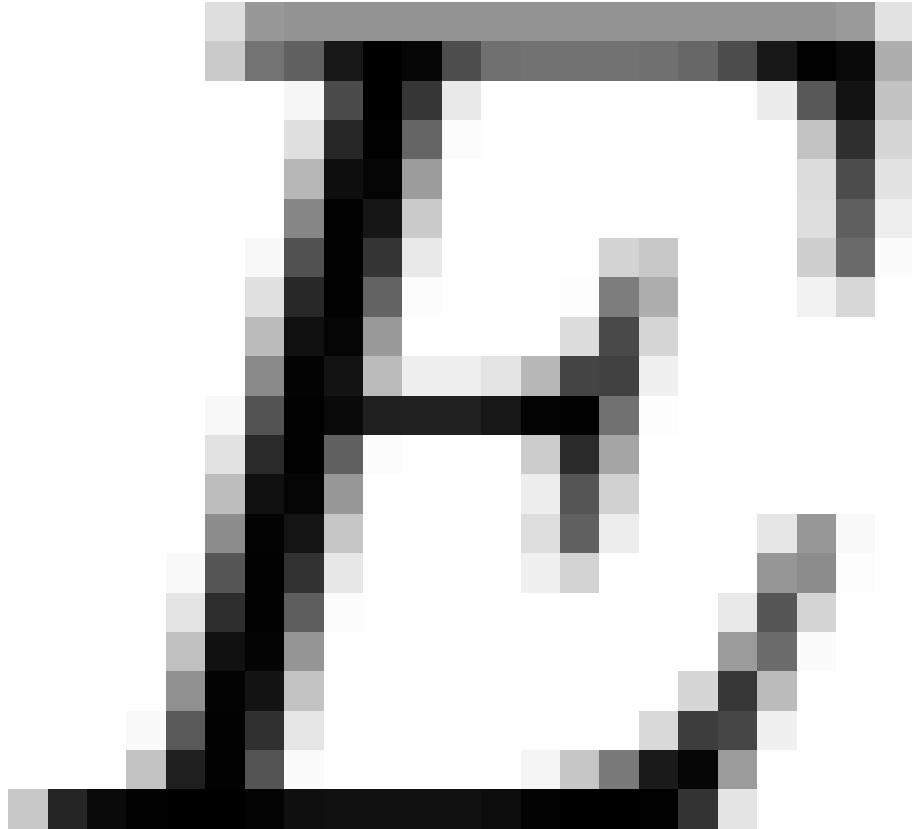


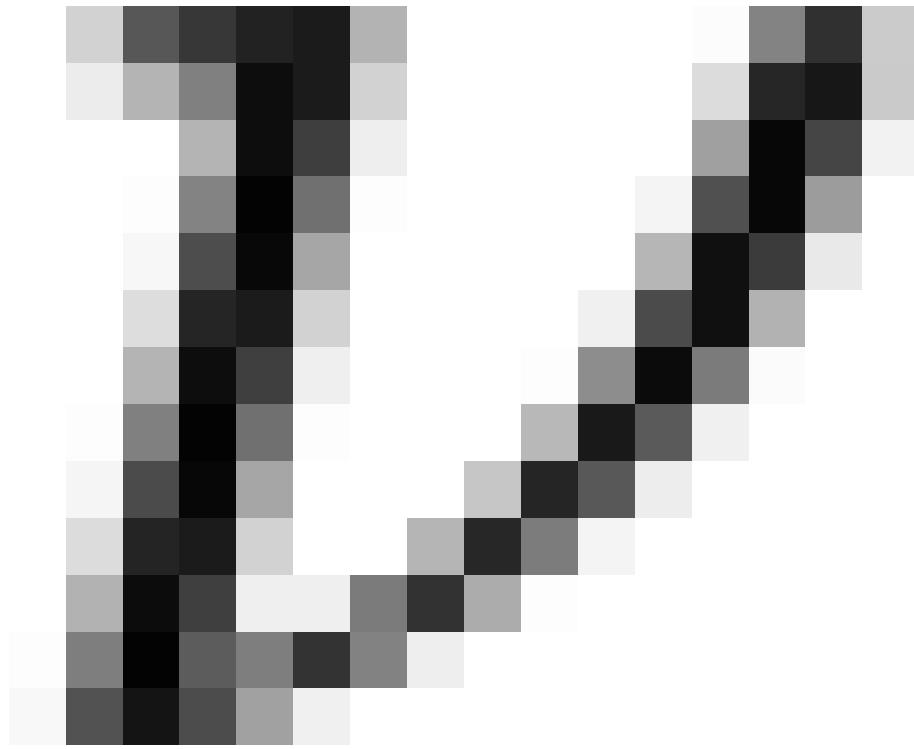


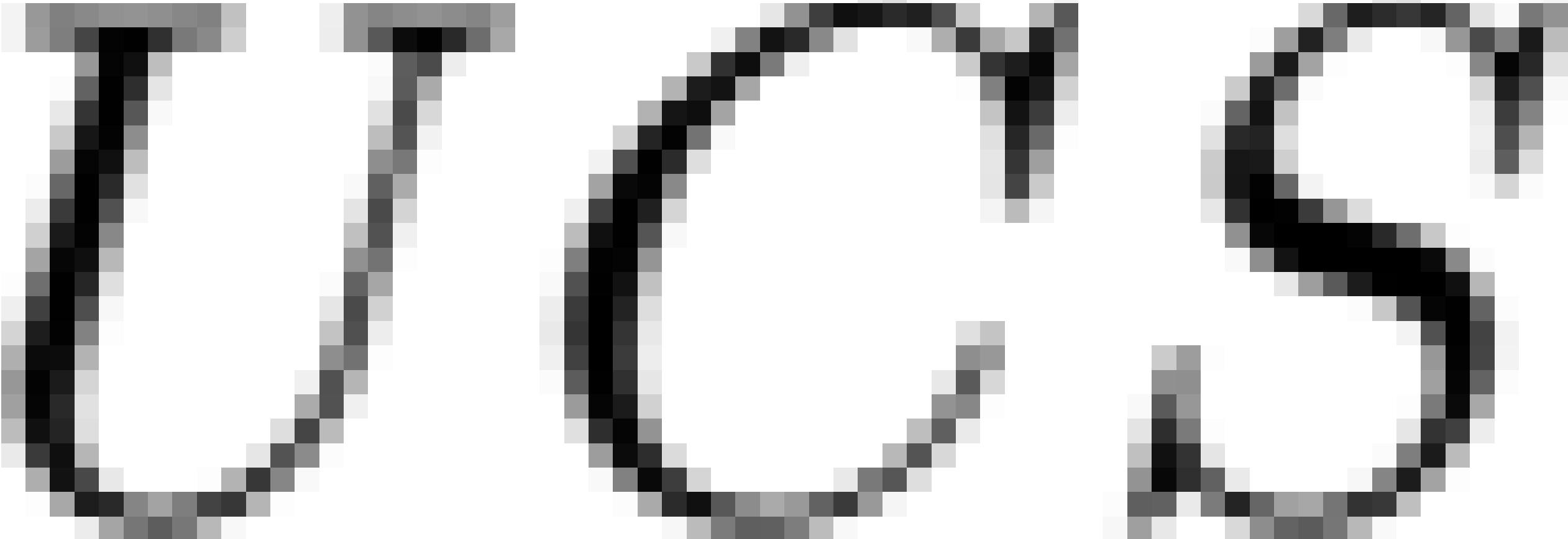


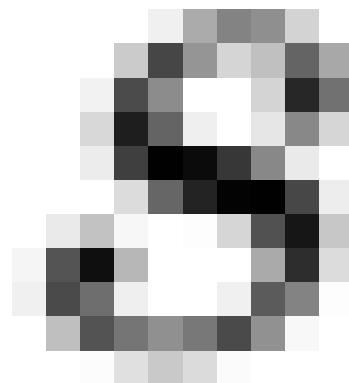
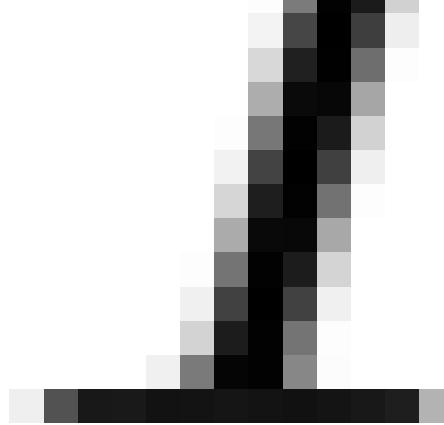
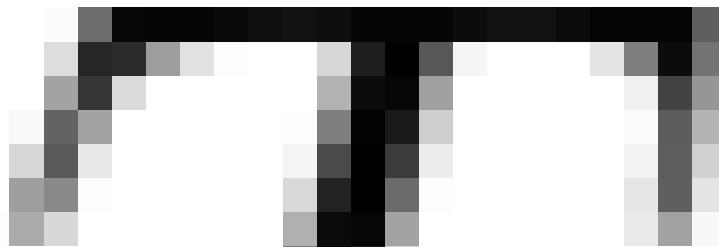


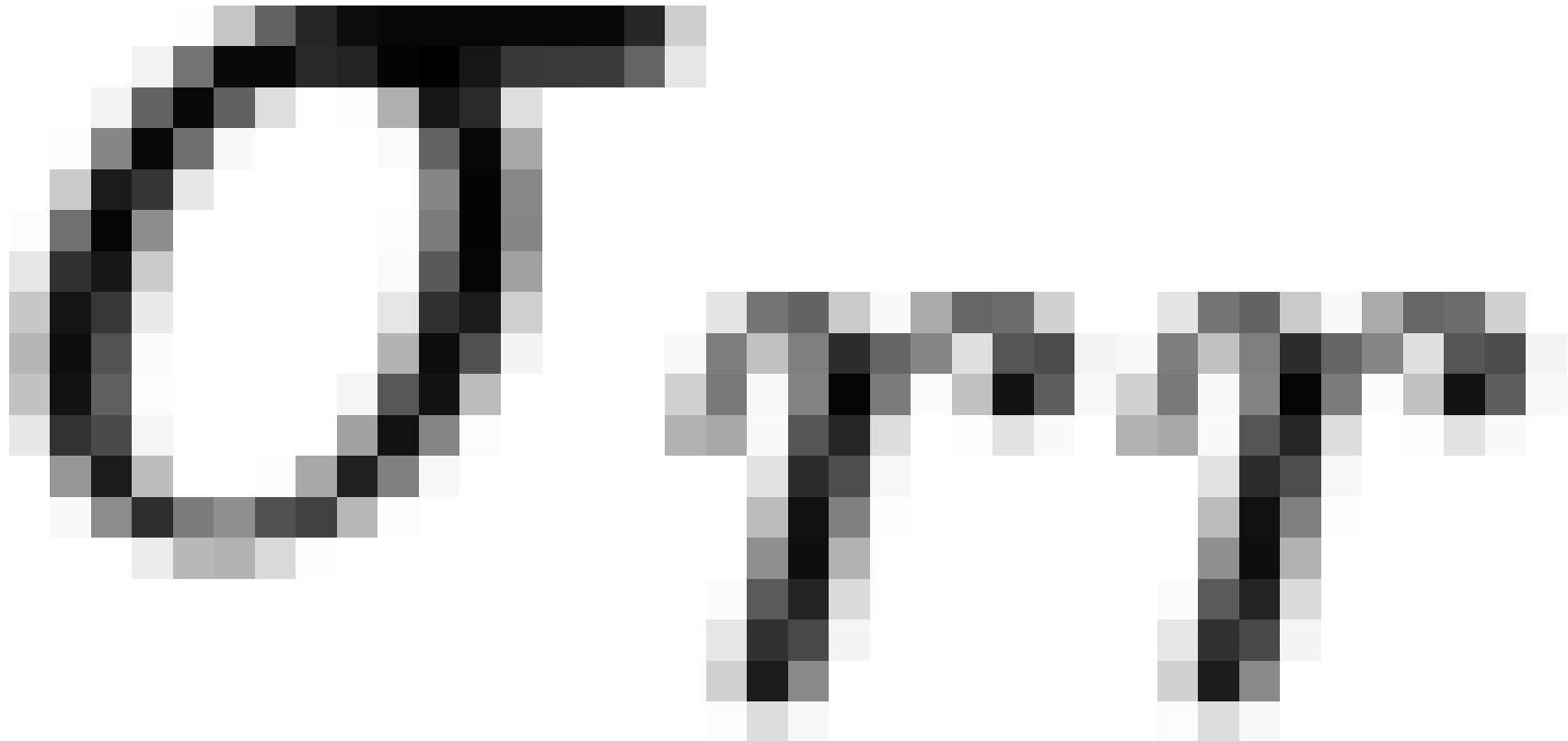


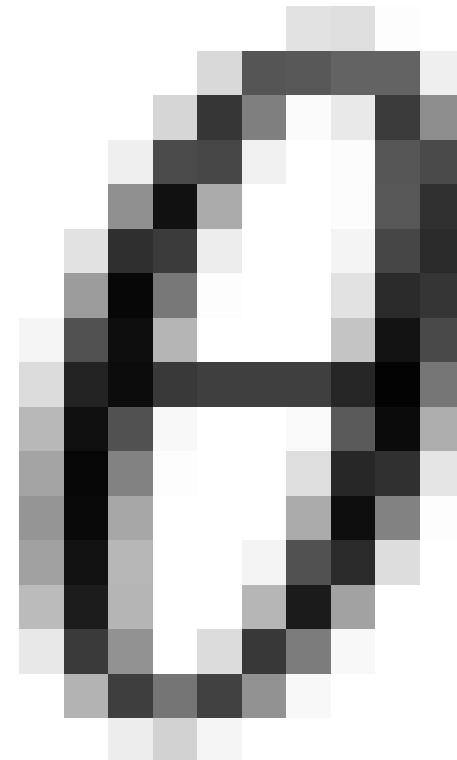
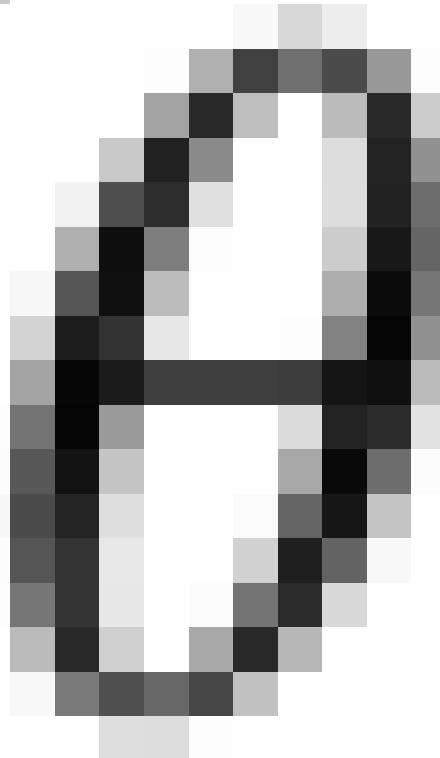
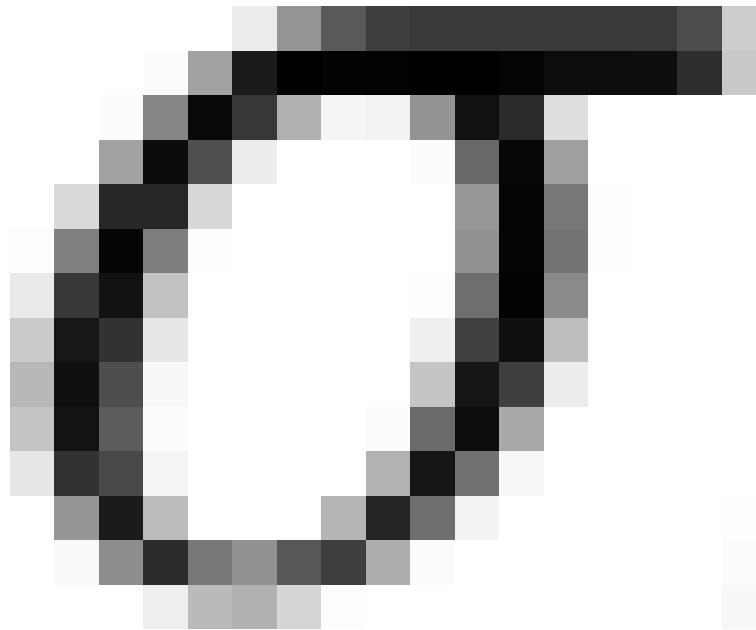


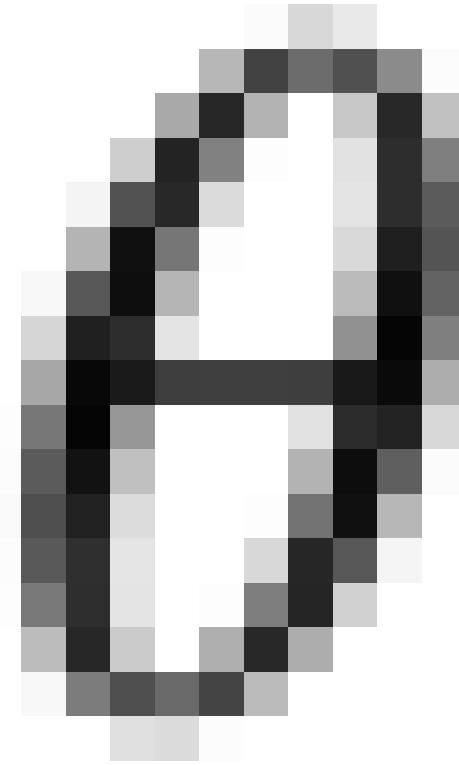
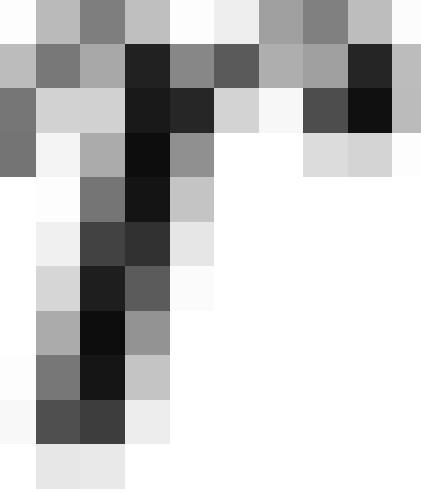
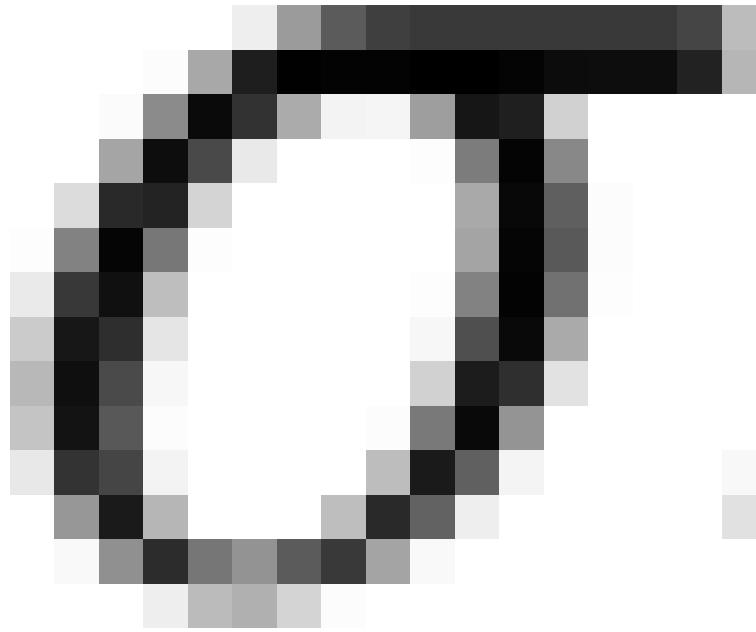


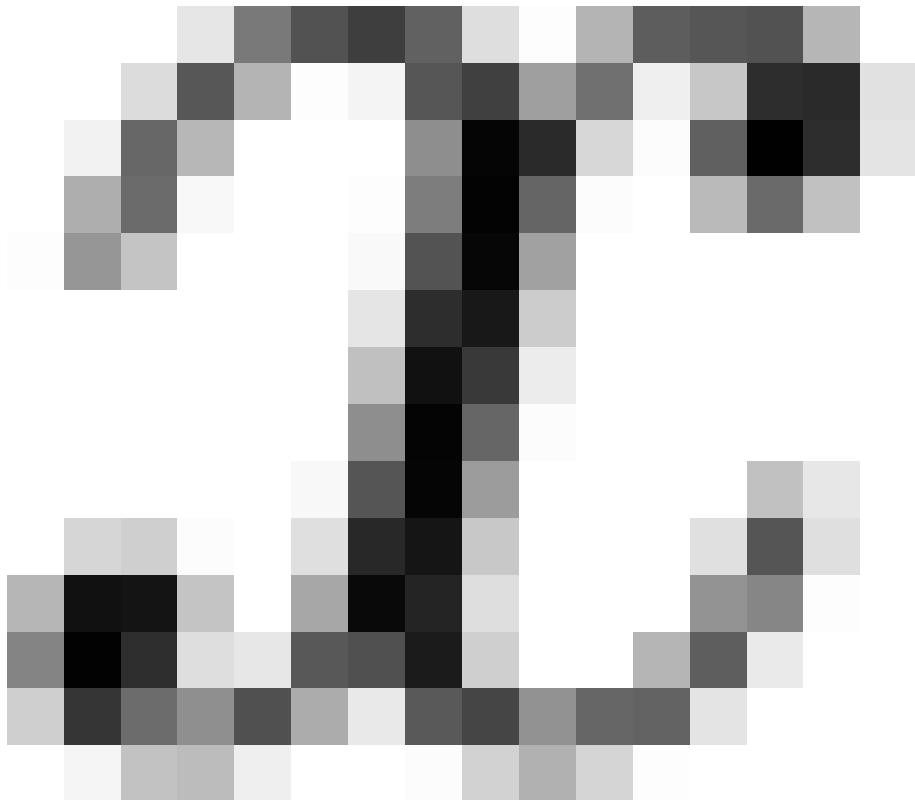


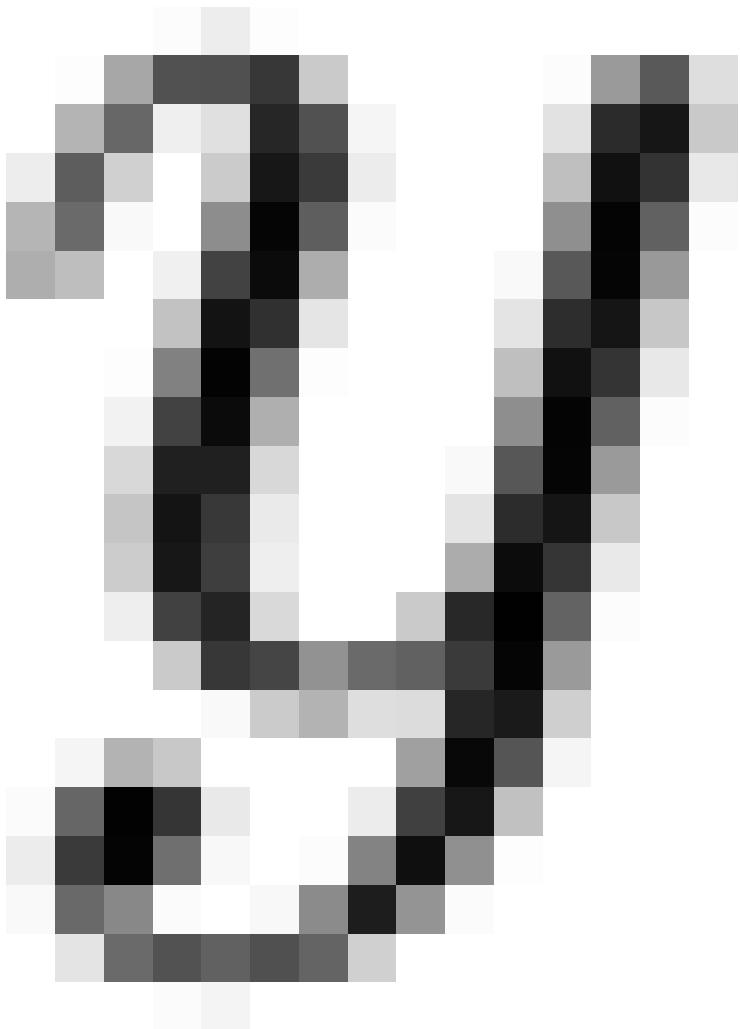


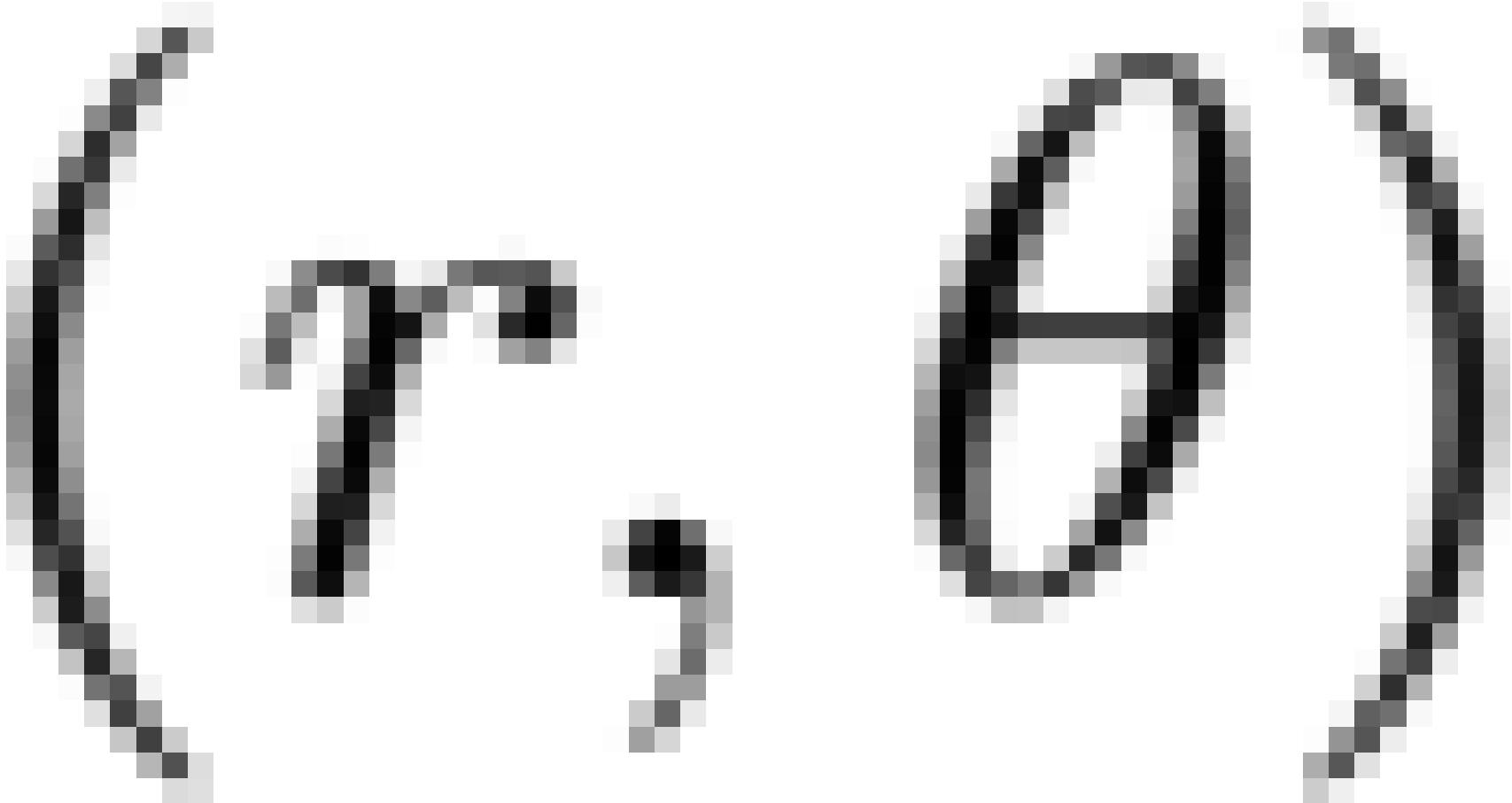






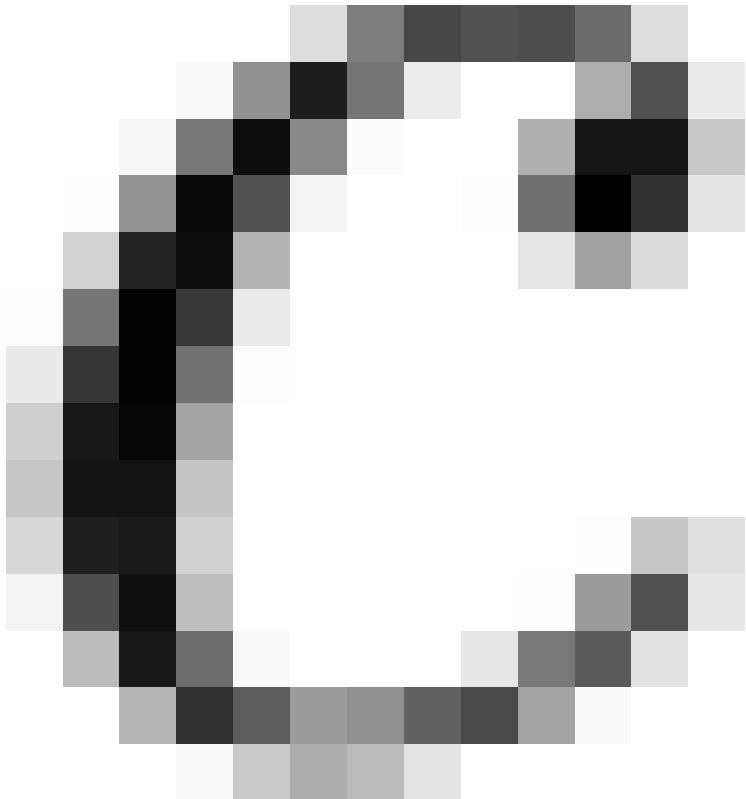


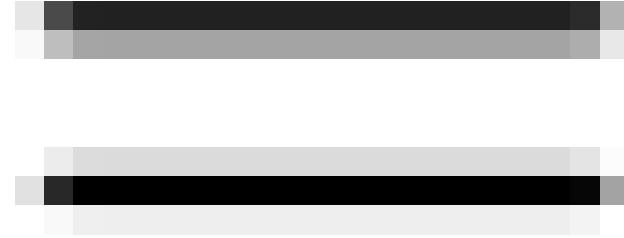
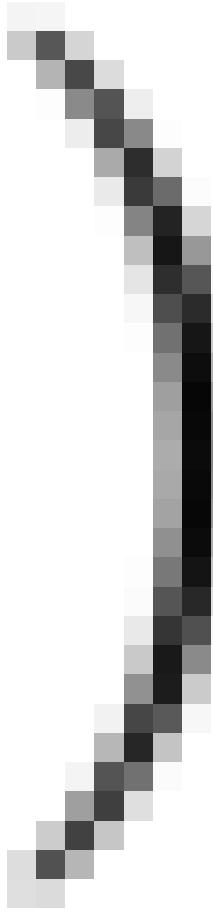
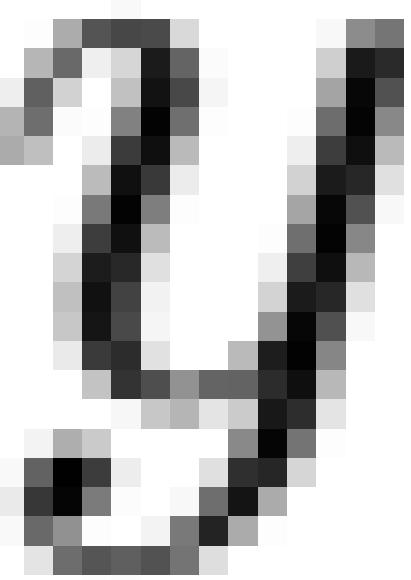
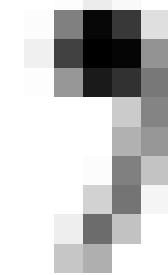
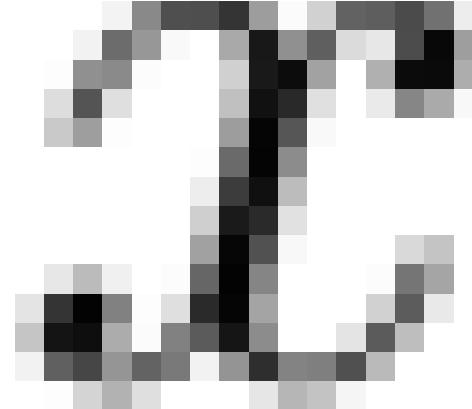
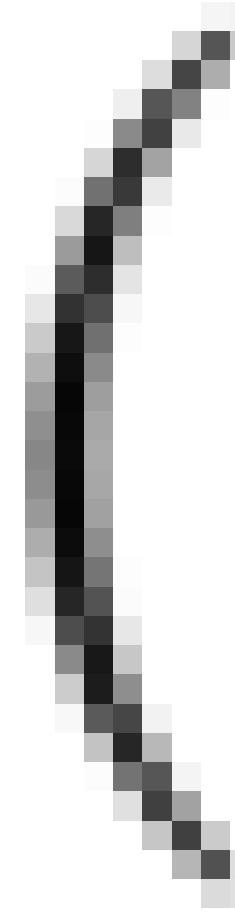




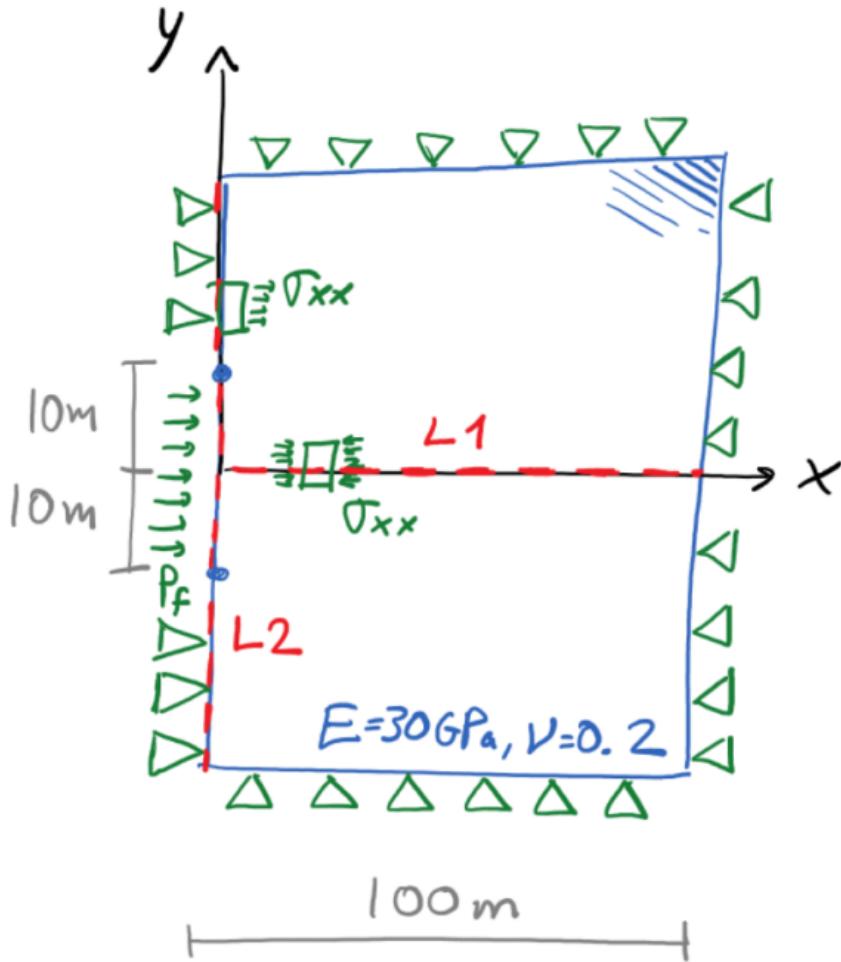
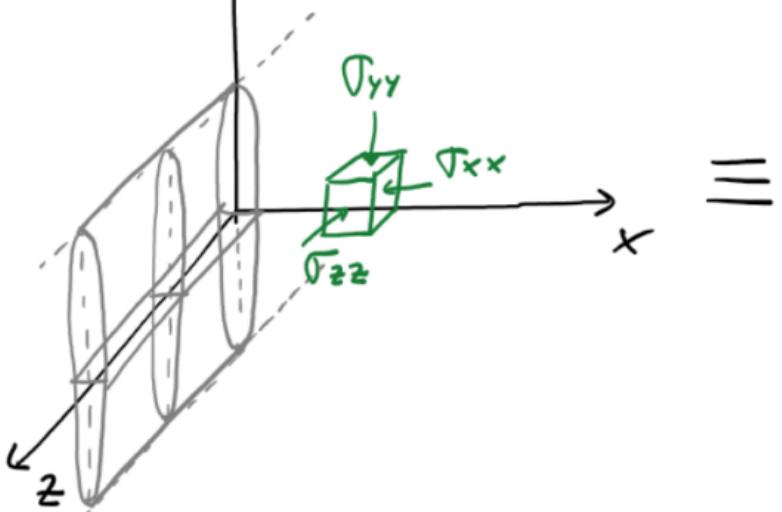


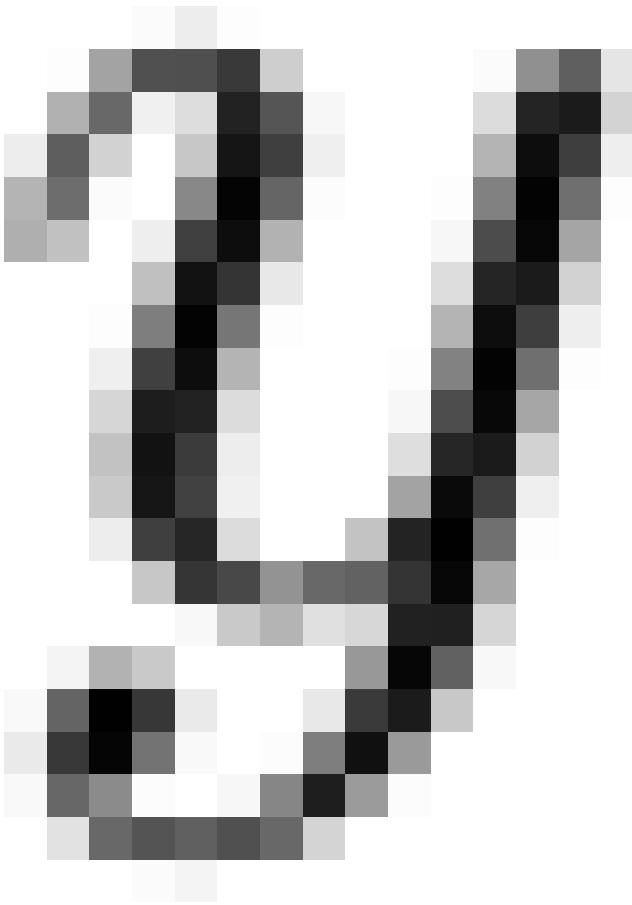






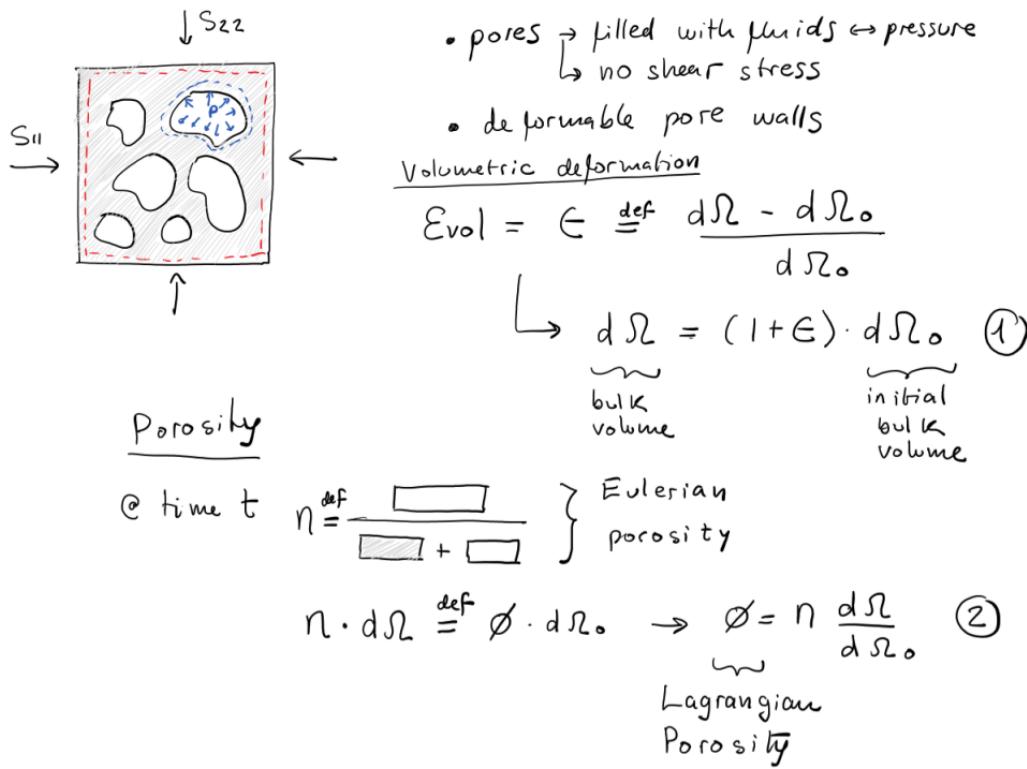
$y$   
 Fracture length in  $z$   
 >> fracture length in  $y$   
 $\Rightarrow$  Plane strain in  $(x, y)$





### 3. Poroelasticity (Cousy, 2010 - Chapters 3 and 4)

Friday, October 2, 2020 5:45 PM



### Solid strain

$$\epsilon_s \stackrel{\text{def}}{=} \frac{d\Omega^s - d\Omega_0^s}{d\Omega_0^s} \Rightarrow d\Omega^s = (1 + \epsilon_s) d\Omega_0^s \quad (3)$$

Combination of ①, ②, ③

$$d\Omega^S = (1 + \epsilon_s) d\Omega_0^S$$

$$(1 - n) d\Omega = (1 + \epsilon_s) (1 - \phi_0) d\Omega_0$$

Bulk Vol. Strain                      Solid Vol. Strain                      Porosity strain

$$\tilde{\epsilon} = (1 - \phi_0) \epsilon_s + (\phi_0) \left( \frac{\phi - \phi_0}{\phi_0} \right)$$

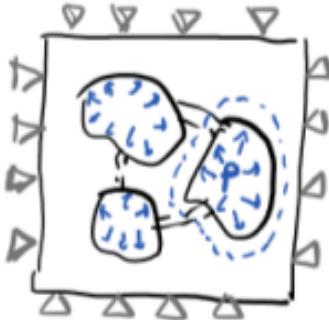
↳ Micromechanical Eq.

↳ Volume average

$$\epsilon = (1 - \phi_0) \epsilon_s + \varphi$$

$$\varphi \stackrel{\text{def}}{=} \phi - \phi_0 : \begin{matrix} \text{change} \\ \text{of} \\ \text{porosity} \end{matrix}$$

## Test 1



$$\epsilon = 0$$

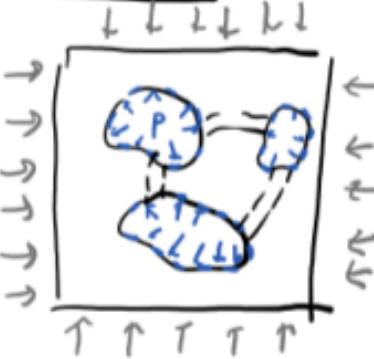
$$\sigma = (1-\phi_0) \epsilon_s + \varphi$$

$$\epsilon_s = -\frac{\varphi}{(1-\phi_0)}$$

if  $\varphi > 0 \Rightarrow \underbrace{\epsilon_s < 0}_{\text{contraction}}$

Mechanics convention

## Test 2

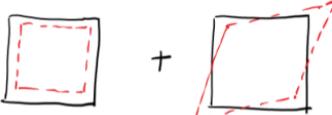


$$\varphi = 0$$

$$\epsilon_s = \frac{\epsilon}{1-\phi_0}$$

# Free energy of the porous solid

Non-porous  
Solid



$$\underbrace{dW}_{=} = \underbrace{S_m \cdot d\epsilon}_{\text{bulk strain}} + \underbrace{S_{ij} \cdot de_{ij}}_{\text{shear strains}}$$

$$\frac{S_{11} + S_{22} + S_{33}}{3}$$

$$S_{12} \leftrightarrow e_{12}$$

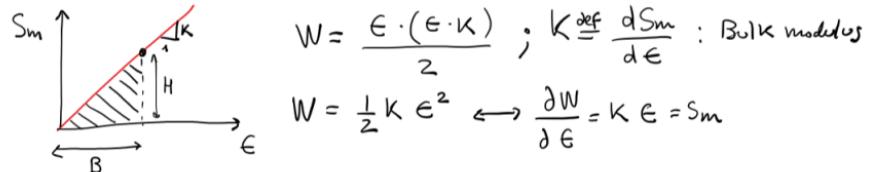
$$S_{23} \leftrightarrow e_{23}$$

$$S_{13} \leftrightarrow e_{13}$$

Porous  
solid



$$\underbrace{dW}_{=} = \underbrace{S_m \cdot d\epsilon}_{\text{bulk strain}} + \underbrace{S_{ij} \cdot de_{ij}}_{\text{shear strains}} + \rho \underbrace{\frac{d\emptyset}{d\varphi}}_{\text{porosity}}$$



$$\underbrace{\chi_s}_{=} = \underbrace{\frac{1}{2} K \epsilon^2}_{\text{bulk vol. strain}} + \underbrace{G e_{ij} \cdot e_{ij}}_{\text{Shear strains}} - \underbrace{d \cdot \epsilon \cdot p}_{\substack{\text{cst 1} \\ \text{pressure}}} - \underbrace{\left(\frac{1}{2} \frac{1}{N}\right) p^2}_{\text{cst 2}}$$

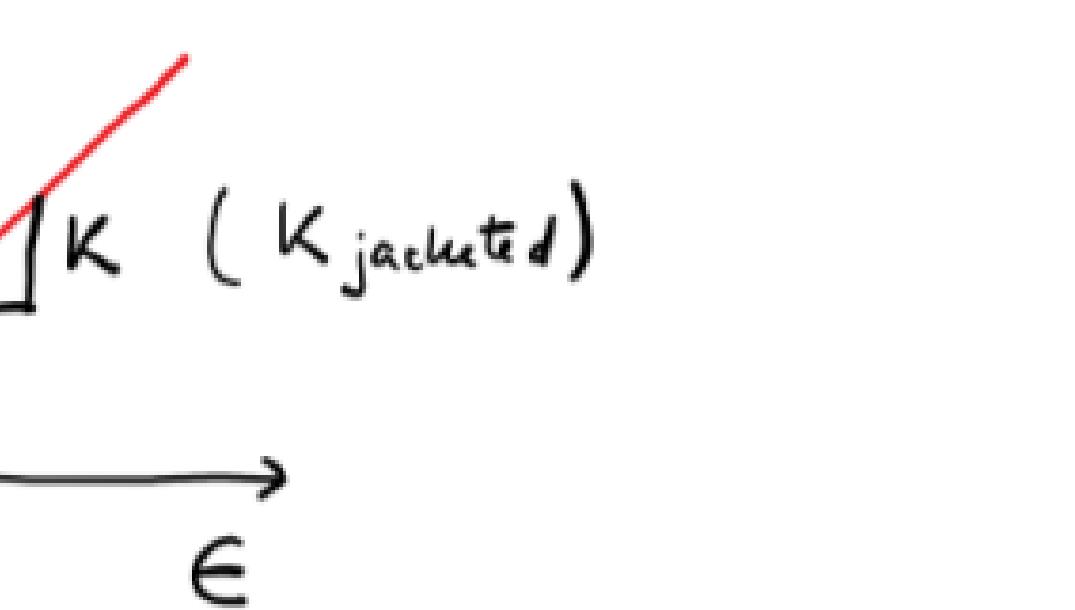
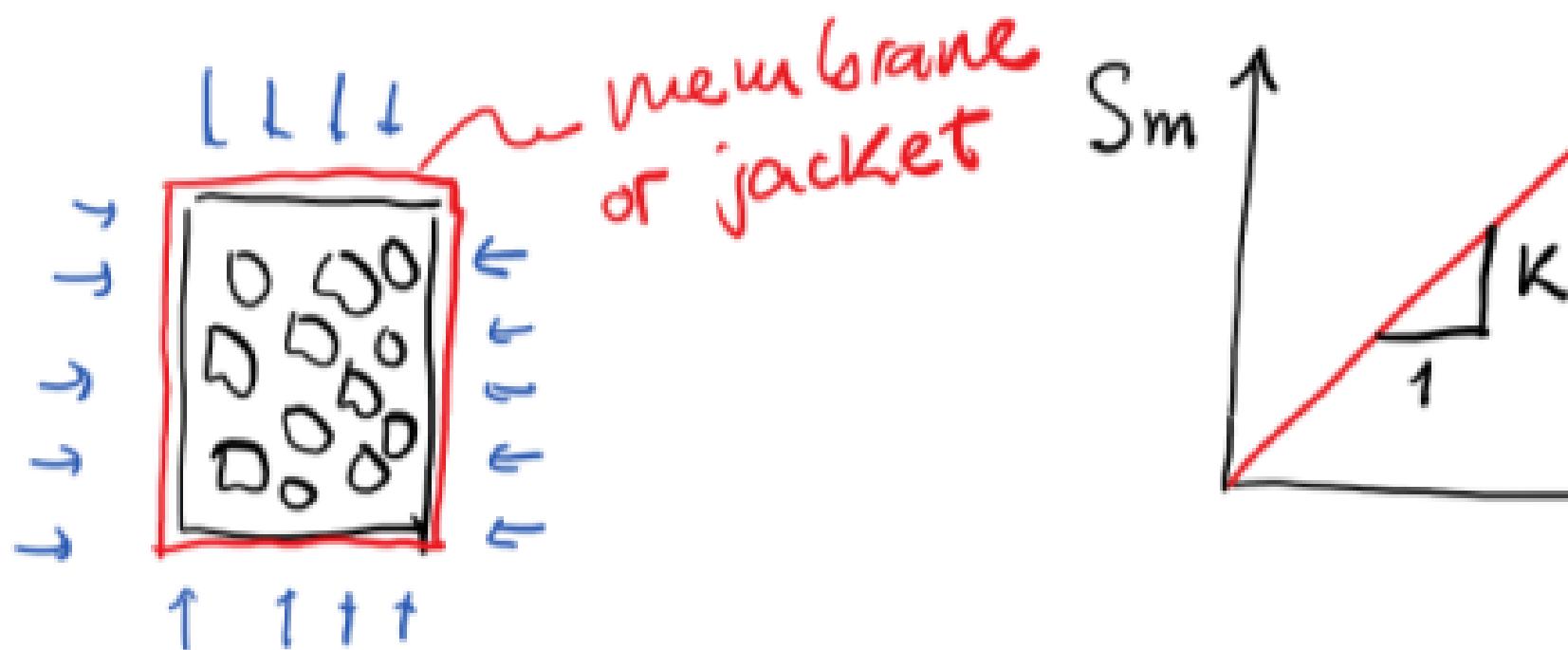
$$\frac{\partial \sigma_s}{\partial \epsilon} = \underbrace{K \epsilon}_{\text{Biot coefficient}} - \underbrace{d p}_{\text{Biot pressure}} = S_m \quad (1) \quad \text{Biot effective stress equation}$$

$$\frac{\partial \Pi_S}{\partial e_{ij}} = 2 G e_{ij} = s_{ij} \quad (2) \quad d = ?$$

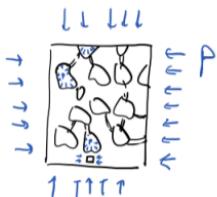
$$\frac{\partial n_s}{\partial p} = \alpha \in + \frac{P}{N} = \varphi \quad (3) \quad N = ?$$

m  
Poroelastic Modulus

## Jacketed loading



## Unjacketed loading



$$\textcircled{1} \quad K\epsilon - \alpha p = S_m = -P$$

$$\epsilon = \frac{P(-1+\alpha)}{K}$$

$$\bullet \quad \epsilon = -\frac{P(1-\alpha)}{K} \quad \bullet \quad \epsilon_s = -\frac{P}{K_s}$$

• isotropic elastic solid

$$\epsilon = \epsilon_s$$

• connected porosity

$$\frac{P(1-\alpha)}{K} = \frac{P}{K_s} \Rightarrow \boxed{\alpha = 1 - \frac{K}{K_s}}$$

$\alpha \leq 1 \text{ for } K_s \geq K$

bulk modulus solid =  
 porous solid =  
 jacketed =

bulk modulus  
 solid skeleton

$$\textcircled{3} \quad \alpha \epsilon + \frac{P}{N} = \varphi$$

$$\left. \begin{array}{l} \alpha \left( -\frac{P}{K_s} \right) + \frac{P}{N} = \varphi \\ \frac{\varphi}{\varphi_0} = -\frac{P}{K_s} \end{array} \right\}$$

$$\cancel{\alpha} \left( -\frac{P}{K_s} \right) + \frac{P}{N} = -\frac{P}{K_s} \cdot \varphi_0$$

$$\boxed{\frac{1}{N} = \frac{\alpha - \varphi_0}{K_s}}$$

- second poromechanical modulus
- $N > 0 \Rightarrow \alpha > \varphi_0$
- $\varphi_0 \leq \alpha \leq 1$

## METHOD 1

- Measure  $K$

- Assume or estimate  $K_s$

mono-mineral \*

multi-mineral

↪ volume average

Reuss

Voigt

Hashin-Shtrikman

$$\alpha = 1 - \frac{K}{K_s}$$

$$N = \frac{K_s}{\alpha - \phi_0}$$

jacketed bulk modulus

dry rock,  $E, V$

$$\hookrightarrow K = \frac{E}{3(1-2v)}$$

$V_p, V_s, \rho \Rightarrow E, V, K$

$$* K_s = K_{\text{mineral}} = K_{SiO_2}$$

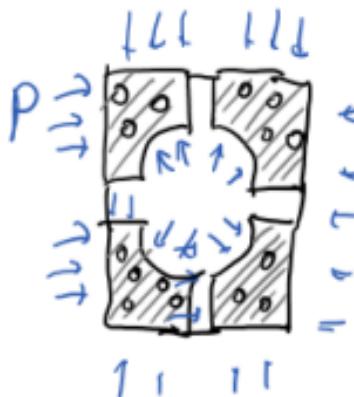
Sandstone 100%  $SiO_2$

$$K_{SiO_2} = 36 \text{ GPa}$$

→ wrong if occluded  $\phi$

## METHOD 2

- Unjacketed test



$$\alpha = 1 - \frac{K}{K_{unj}}$$



- Increase  $P_c$  and  $P_p$  simultaneously

constant Terzaghi's effective stress =  $P_c - P_p \neq 0$

- water < clay sensitive  
short circuit
- oil - high viscosity
- gas
  - small vessel
  - not desirable for large vessels
  - non-sorptive

### METHOD 3

$$S_m = K\epsilon - \alpha P$$

$$K\epsilon = S_m + \alpha P$$

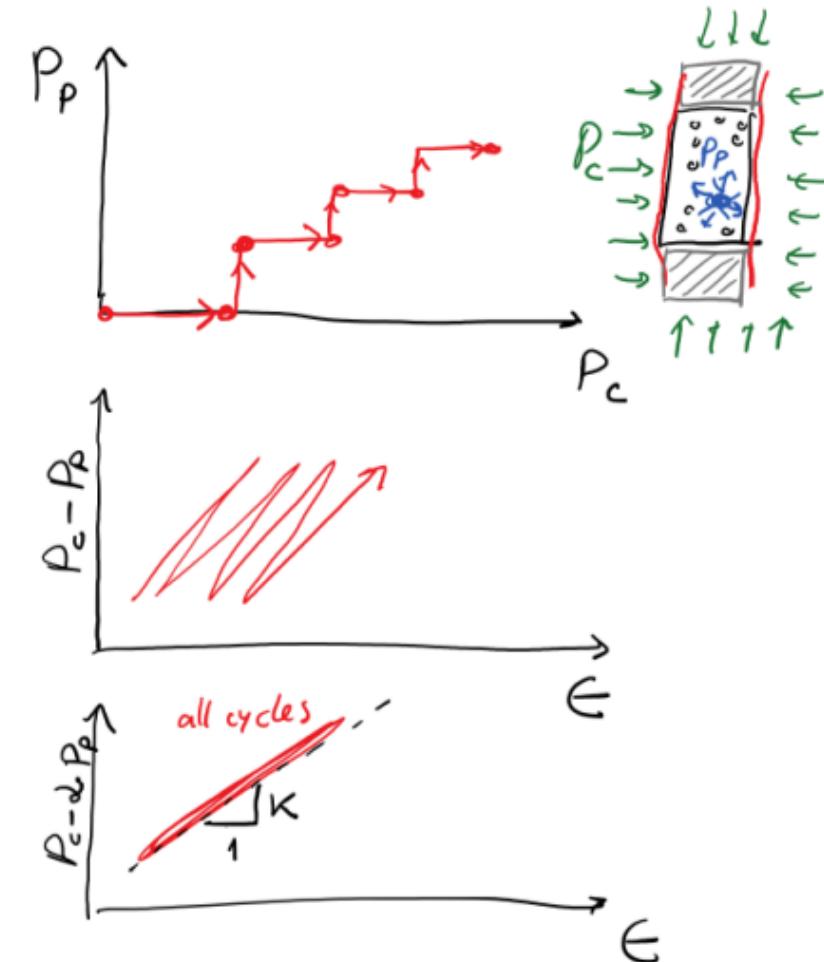
$$\epsilon = \frac{1}{K} (S_m + \alpha P)$$

Biot effective stress

$(K, \alpha) \leftarrow$  error minimization

$$k_m = \frac{K}{1-\alpha}$$

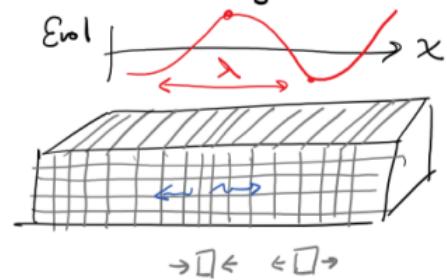
matrix



# Drained and Undrained Problems in Poro mechanics

Terzaghi's effective stress :  $\underline{\sigma} = \underline{\sigma} - p_p \underline{I}$

Poroelasticity



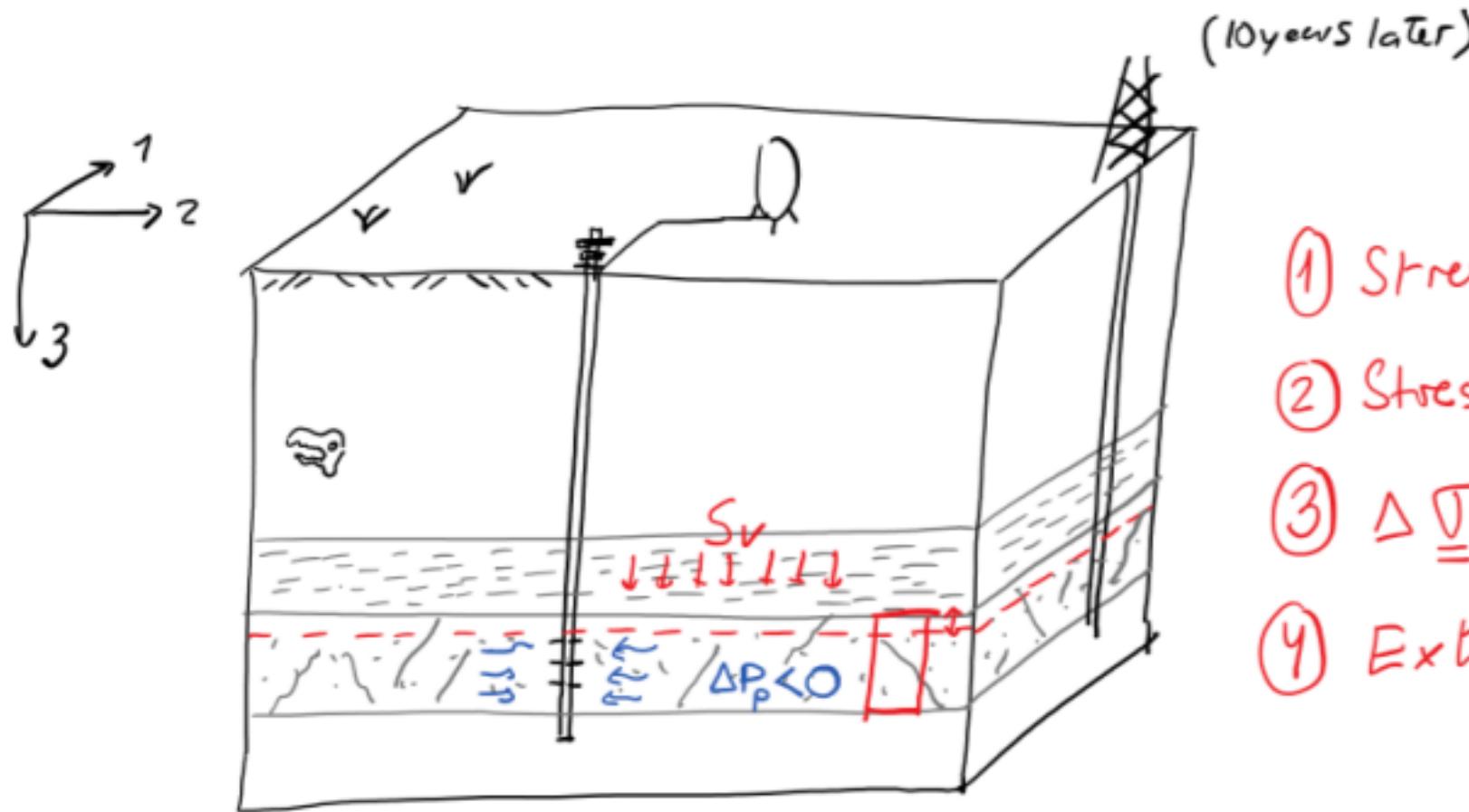
- $\underline{\sigma} = \underline{\sigma} - \alpha p_p \underline{I}$  : Biot's effective stress
- $\varphi \rightarrow$  changes of porosity
- $K_u$ : undrained bulk modulus
- undrained loading  $\leftrightarrow \Delta p_p$
- "squirt flow"  $\leftrightarrow \Delta E_{vol}$ 
  - ↳ attenuation of elastic waves
  - ↳ dispersion of " waves

-

End-members of  
actual solution

drained  
undrained

Diffusivity  $E_g$   
for poroelasticity



(10 years later)

- ① Stresses in-situ ( $\epsilon_{11} = \epsilon_{22} = 0$ )
- ② Stresses in-situ ( $\epsilon_{11} \neq \epsilon_{22} \neq 0$ )
- ③  $\Delta T$  with change in  $P_p$ ?
- ④ Extend to VTI rock

$$\textcircled{1} \quad \underline{\underline{\sigma}} = \underline{\underline{\epsilon}} \cdot \underline{\underline{\epsilon}} \quad ; \quad \underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$$

$$\hookrightarrow \underline{\underline{\epsilon}} = \underline{\underline{D}} \cdot \underline{\underline{S}}$$

$$\begin{bmatrix} \cancel{\epsilon_{11}} \\ \cancel{\epsilon_{22}} \\ \cancel{\epsilon_{33}} \\ \cancel{2\epsilon_{23}} \\ \cancel{2\epsilon_{13}} \\ \cancel{2\epsilon_{12}} \end{bmatrix} = \frac{\begin{bmatrix} 1/E & -v/E & -v/E \\ -v/E & v/E & -v/E \\ -v/E & -v/E & 1/E \end{bmatrix} \circ}{\begin{bmatrix} 1/G & 0 & 0 \\ 0 & 1/G & 0 \\ 0 & 0 & 1/G \end{bmatrix}} \begin{bmatrix} S_{11} - \alpha P_p \\ S_{22} - \alpha P_p \\ S_{33} - \alpha P_p \end{bmatrix}$$

Passive lectoric env  
 $\epsilon_{ii} = \epsilon_{zz} = 0$   
 $\epsilon_{ij} = 0 \text{ for } i \neq j$

$\downarrow$

$$S_{11} = S_{22}$$

$$\epsilon_{11} = 0 = \frac{1}{E}(S_{11} - \alpha P_p) - \frac{v}{E}(S_{22} - \alpha P_p) - \frac{v}{E}(S_{33} - \alpha P_p) \quad ;$$

$$\left(\frac{1-v}{E}\right) \cdot (S_{22} - \alpha P_p) = \frac{v}{E} (S_{33} - \alpha P_p)$$

$$(S_{22} - \alpha P_p) = \underbrace{\frac{v}{1-v}}_{\underline{\sigma}_{22}} \underbrace{(S_{33} - \alpha P_p)}_{\underline{\sigma}_{33}}$$

$$S_{\text{min}} = S_{\text{max}} = S_{22} = \alpha P_p + \underbrace{\frac{v}{1-v} (S_{33} - \alpha P_p)}_{\downarrow S_v}$$

(2)

$$S_{\text{min}} = dP_p + \frac{\nu}{1-\nu} (S_v - \alpha P_p) + \frac{E}{1-\nu} (\varepsilon_{\text{min}} + \nu \varepsilon_{\text{max}})$$

$$S_{\text{max}} = \alpha P_p + \frac{\nu}{1-\nu} (S_v - d P_p) + \frac{E}{1-\nu} (\nu \varepsilon_{\text{min}} + \varepsilon_{\text{max}})$$

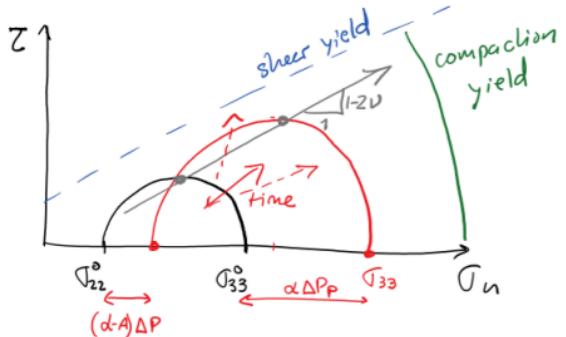
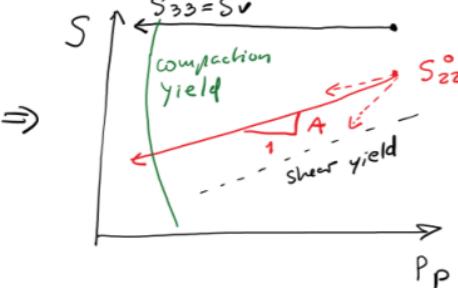
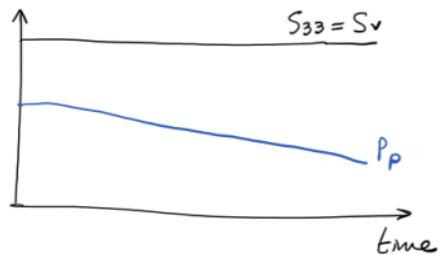
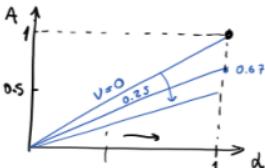
$$S_v = \int_0^t \rho_{\text{bulk}} g \, dz$$



$$③ S_{22} = \frac{v}{1-v} S_{33} + d \left( \frac{1-2v}{1-v} \right) P_p$$

~~~~~ Total He stress  
 ~~~~~ Contrib Total Vert stress  
 A Pore pressure

$$\frac{\partial S_{22}}{\partial P_p} = d \left( \frac{1-2v}{1-v} \right) = A \rightarrow$$



$$\sigma_{33} = S_{33} - d P_p$$

$$\frac{\partial \sigma_{33}}{\partial P_p} = -d$$

$$\frac{\partial \sigma_{22}}{\partial P_p} = - (d - A) = -d \left( \frac{v}{1-v} \right)$$

$$\varepsilon_{33} = -\frac{2\nu}{E}(s_{22} - \alpha p_p) + \frac{1}{E}(s_{33} - \alpha p_p)$$

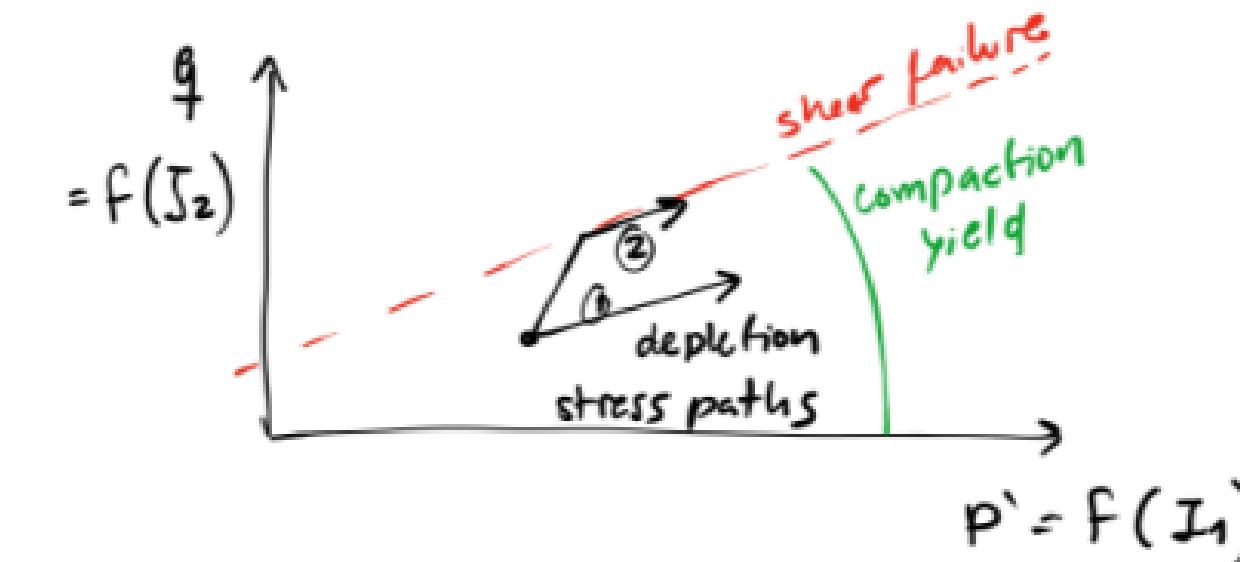
$$\varepsilon_{33} = \frac{(1-2\nu)(1+\nu)}{E(1-\nu)}(s_{33} - \alpha p_p)$$

$\frac{\partial \varepsilon_{33}}{\partial p_p} = -\frac{\alpha}{M}$  constrained modulus



$$\Delta \varepsilon_{33} = -\frac{\alpha}{M} \Delta p_p$$

$$\Delta H = \Delta \varepsilon_{33} \cdot H = \left(-\frac{\alpha}{M} \Delta p_p\right) \cdot H$$



4

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} +\frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_h}{E_h} & +\frac{1}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & +\frac{1}{E_v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_h} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

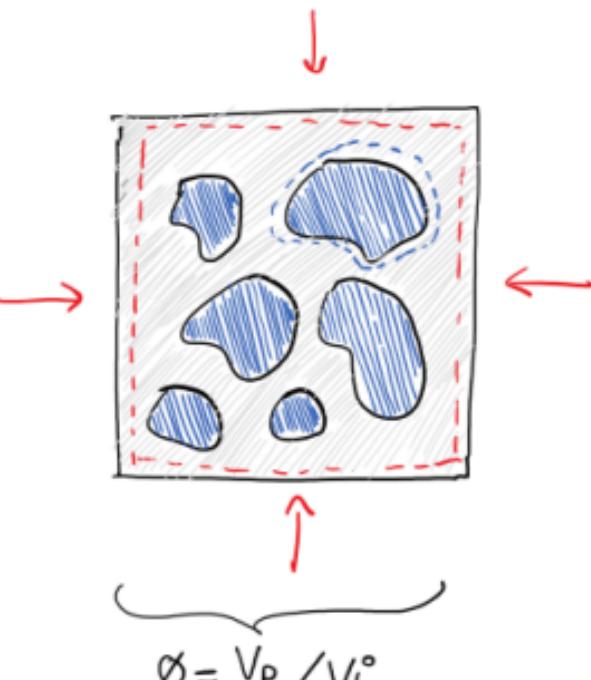
$$\left\{ \begin{array}{l} S_{hmin} = \alpha_h P_p + \frac{V_h}{1-V_h} \frac{E_h}{E_v} (S_v - \alpha_v P_p) + \frac{E_h}{1-V_h^2} (\varepsilon_{hmin} + V_h \varepsilon_{hmax}) \\ S_{hmax} = \alpha_h P_p + \frac{V_h}{1-V_h} \frac{E_h}{E_v} (S_v - \alpha_v P_p) + \frac{E_h}{1-V_h^2} (V_h \varepsilon_{hmin} + \varepsilon_{hmax}) \\ S_v = \int_0^Z p_{bulk} g \, dz \end{array} \right.$$

$$\left\{ \begin{array}{l} \Omega_{hmin} = S_{hmin} - \alpha_h P_p \\ \Omega_{hmax} = S_{hmax} - \alpha_h P_p \\ \Gamma_v = S_v - \alpha_v P_p \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha_h = 1 - \frac{C_{11} + C_{12} + C_{13}}{3 \, km} \\ \alpha_v = 1 - \frac{2C_{13} + C_{33}}{3 \, km} \\ \frac{1}{N} = \frac{(2\alpha_h + \alpha_v)/3 - \emptyset_o}{km} \end{array} \right.$$

3- Pore pressure diffusivity equation coupled with  
poroelasticity

Wednesday, September 28, 2022 11:28 AM



$$\frac{\text{Fluid mass}}{V_b^\circ} = \frac{V_p}{V_b^\circ} \rho_F = \phi \rho_F$$

$$d(\phi \rho_F) = \rho_F d\phi + \phi d\rho_F \quad \left\{ \begin{array}{l} = 0 \rightarrow \text{undrained loading} \\ \neq 0 \end{array} \right.$$

$$\frac{d(\phi \rho_F)}{\rho_F} = d\phi + \phi \frac{d\rho_F}{\rho_F} \quad \left\{ \begin{array}{l} \varphi = d\epsilon + \frac{P}{N} \\ \sim C_F = \frac{1}{\rho_F} \frac{d\rho_F}{dP}; K_F = C_F^{-1} \end{array} \right.$$

$$\frac{d(\phi \rho_F)}{\rho_F} = \left( \alpha d\epsilon + \frac{dp}{N} \right) + \phi \frac{dP}{K_F}$$

$$\frac{d(\phi \rho_F)}{\rho_F} = \alpha d\epsilon + \left( \frac{1}{N} + \frac{\phi_0}{K_F} \right) dP$$

$\downarrow$   
 $d\phi \ll \phi_0$  (small strains)

Biot Modulus

$$\boxed{\frac{1}{M^*} \stackrel{\text{def}}{=} \left( \frac{\alpha - \phi_0}{K_s} + \frac{\phi_0}{K_F} \right)}$$

$; \frac{1}{N} = \frac{\alpha - \phi_0}{K_s}$

Continuity equation  
(Mass conservation)

$$\frac{d(\rho_f \phi)}{dt} + \nabla \cdot (\rho_f \underline{\underline{\sigma}}) = 0$$

↓ isotropic Darcy's

$$\frac{1}{dt} \left( \cancel{\rho_f} \left[ \alpha d\epsilon + \frac{1}{M^*} dP \right] \right) + \nabla \cdot \left[ \cancel{\rho_f} \left( -\frac{\kappa}{N} \nabla P \right) \right] = 0$$

✓ homogeneous porous solid

$$\alpha \frac{d\epsilon}{dt} + \frac{1}{M^*} \frac{dP}{dt} + \left( -\frac{\kappa}{N} \nabla^2 P \right) = 0$$

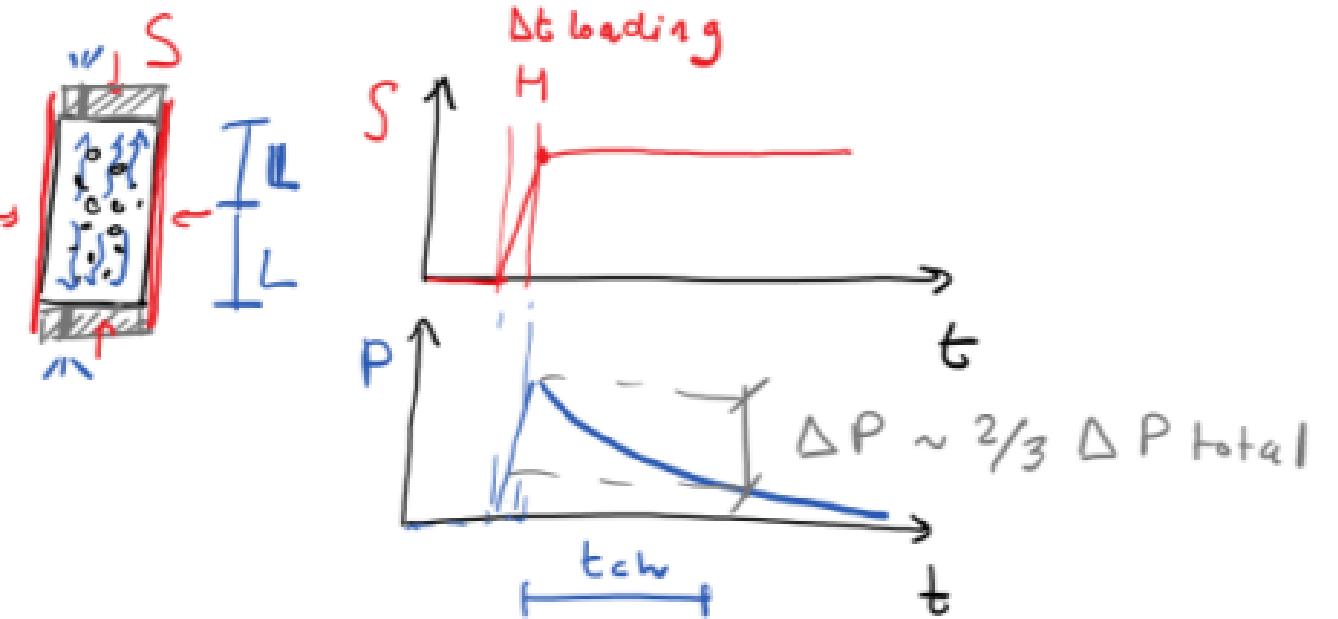
$$\frac{dP}{dt} = \frac{\kappa M^*}{N} \nabla^2 P - \alpha M^* \frac{d\epsilon}{dt}$$

Diff. Eq.  
coupled with  
poroelasticity

$$\frac{\partial P}{\partial t} \sim D \nabla^2 P$$

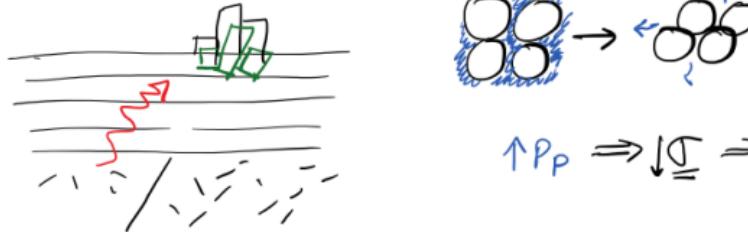
$$\hookrightarrow t_{ch} = \frac{L^2}{D}$$

$$D \sim \frac{K n^*}{N}$$



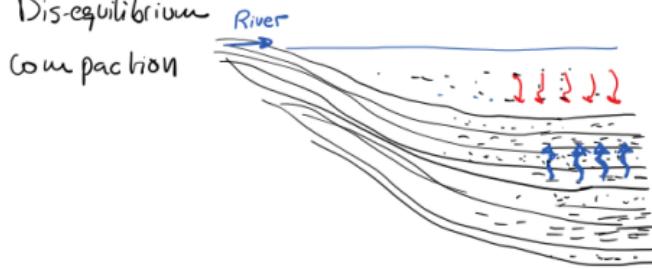
{  $\Delta t_{loading} \ll t_{ch} \Rightarrow$  undrained loading  
 $\Delta t_{loading} \gg t_{ch} \Rightarrow$  drained loading

### ① Liquefaction



$$\uparrow P_p \Rightarrow \downarrow \underline{\sigma} \Rightarrow \downarrow \text{shear strength}$$

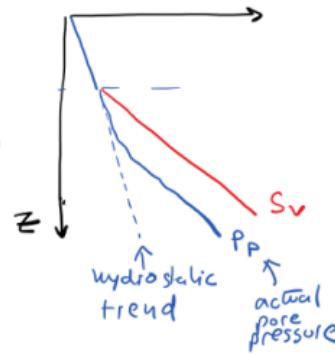
### ② Dis-equilibrium compaction



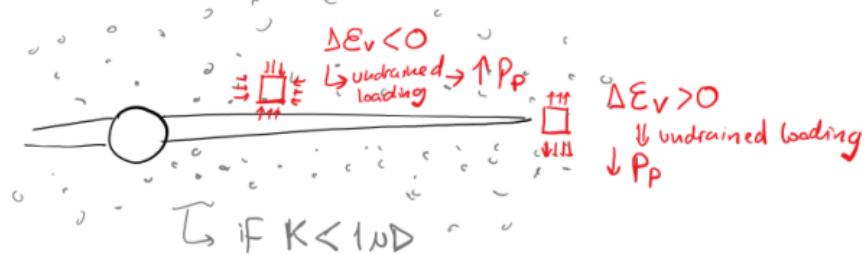
rate of sedimentation

$$\frac{dm}{dt} >> \frac{dp}{dt}$$

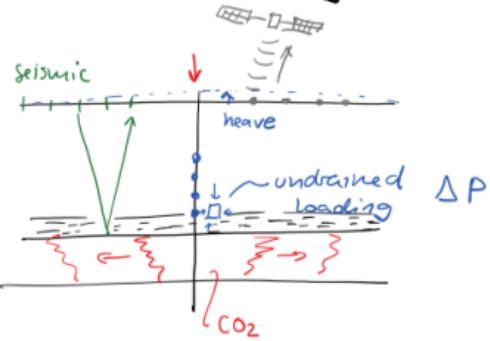
} overpressure



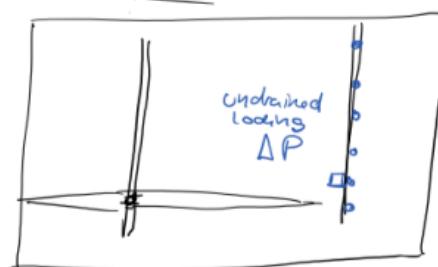
### ③ Hydraulic fracture



#### ④ Poroelastic monitoring

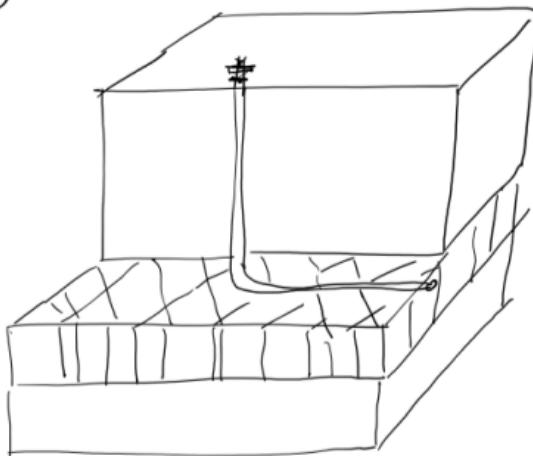


top view

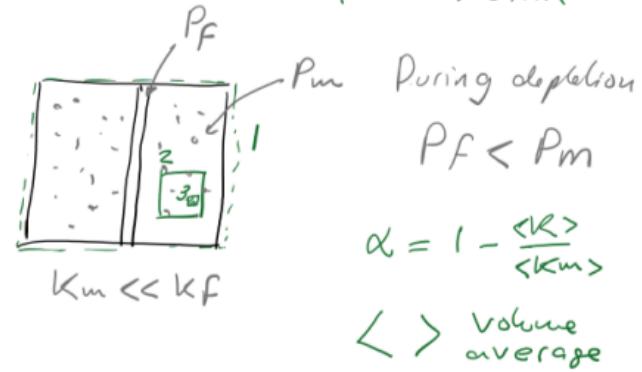


Rouse II  
Conus o  
Phil. ps

#### ⑤



undrained loading  
of the matrix



# ① Pore pressure change due to volumetric strain

Diff. Eq.  
coupled with  
poroelasticity

$$\frac{dp}{dt} = \frac{\kappa n^*}{\nu} \nabla^2 p - \alpha M^* \frac{d\epsilon}{dt}$$

$$\left. \frac{dp}{d\epsilon} \right|_{d(\rho_F \phi) = 0} = -\alpha M^* = -d \left( \frac{d - \phi_0}{K_m} + \frac{\phi_0}{K_F} \right)^{-1}$$

$$\Delta p = (-\alpha n^*) \Delta \epsilon$$

## ② Undrained Bulk Modulus

$$\frac{\partial S_m}{\partial \epsilon} \Big|_{\text{drained, dry}} = K \quad \left. \begin{array}{l} \text{Bulk modulus} \\ \cdot \text{drained} \\ \cdot \text{dry} \end{array} \right\}$$

$$\frac{\partial S_m}{\partial \epsilon} \Big|_{\partial(\rho_f \phi) = 0} \stackrel{\text{def}}{=} K_u \quad \left. \begin{array}{l} \text{Undrained Bulk} \\ \text{modulus} \end{array} \right\}$$

$$S_m = K \epsilon - \alpha P$$

$$\frac{\partial S_m}{\partial \epsilon} = K - \alpha \left( \frac{\partial P}{\partial \epsilon} \right) \Rightarrow \frac{\partial S_m}{\partial \epsilon} \Big|_{\partial(\rho_f \phi) = 0} = K - \alpha (-\alpha M^*)$$

$$K_u = K + \alpha^2 M^*$$

### ③ Skempton's parameter

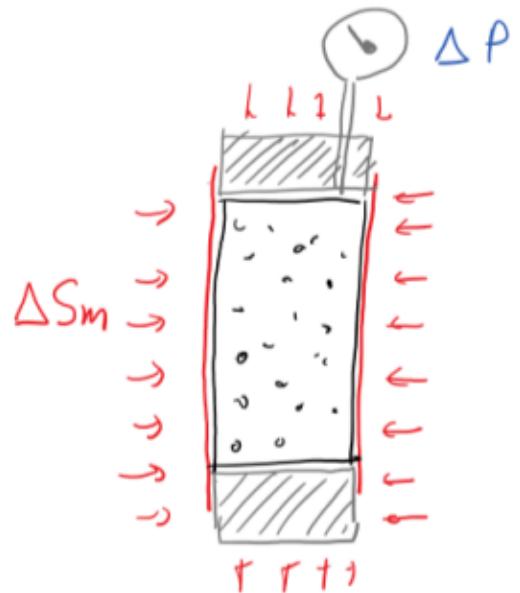
$$B \stackrel{\text{def}}{=} - \left. \frac{\partial P}{\partial S_m} \right|_{d(\rho, \theta) = 0}$$

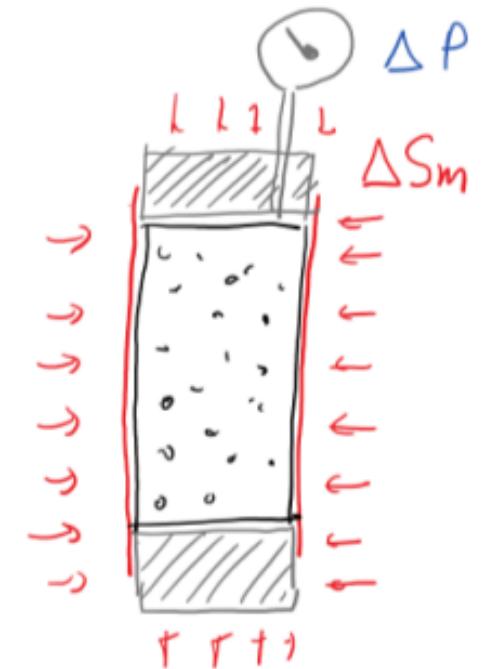
$$S_m = K e^{-\alpha P}$$

$$\frac{\partial S_m}{\partial P} = K \frac{\partial e}{\partial P} - \alpha$$

$$\frac{\partial S_m}{\partial P} = K \left( -\frac{1}{\alpha \eta^*} \right) - \alpha$$

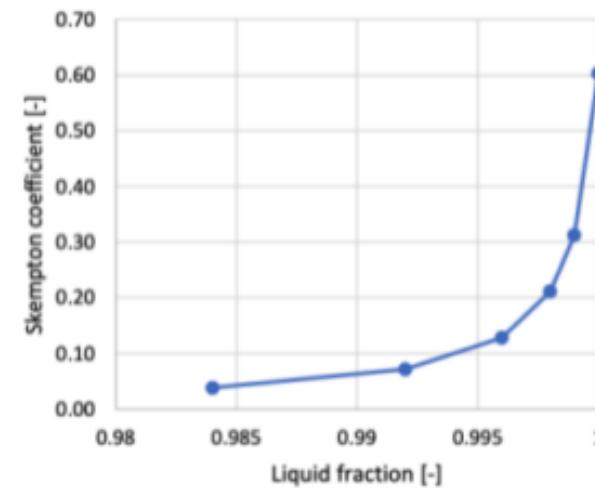
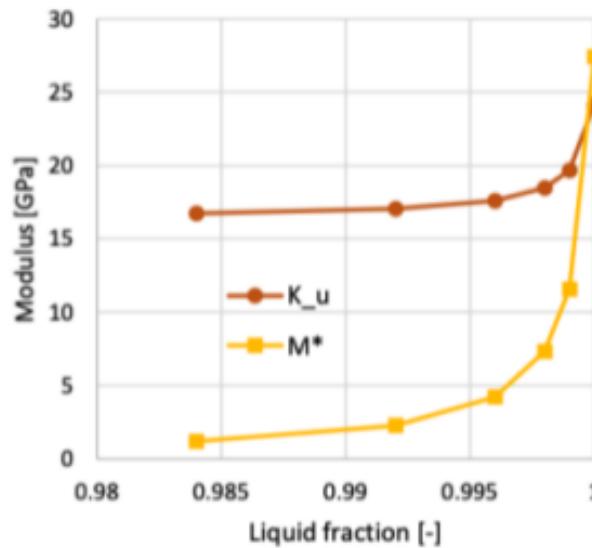
$$\boxed{B = \frac{1}{\alpha} \left( \frac{K}{\eta^*} + \alpha^2 \right)}$$





$$K_u = K + \alpha^2 M^*$$

$$\bar{B}^{-1} = \frac{1}{\alpha} \left( \frac{K}{M^*} + \alpha^2 \right)$$

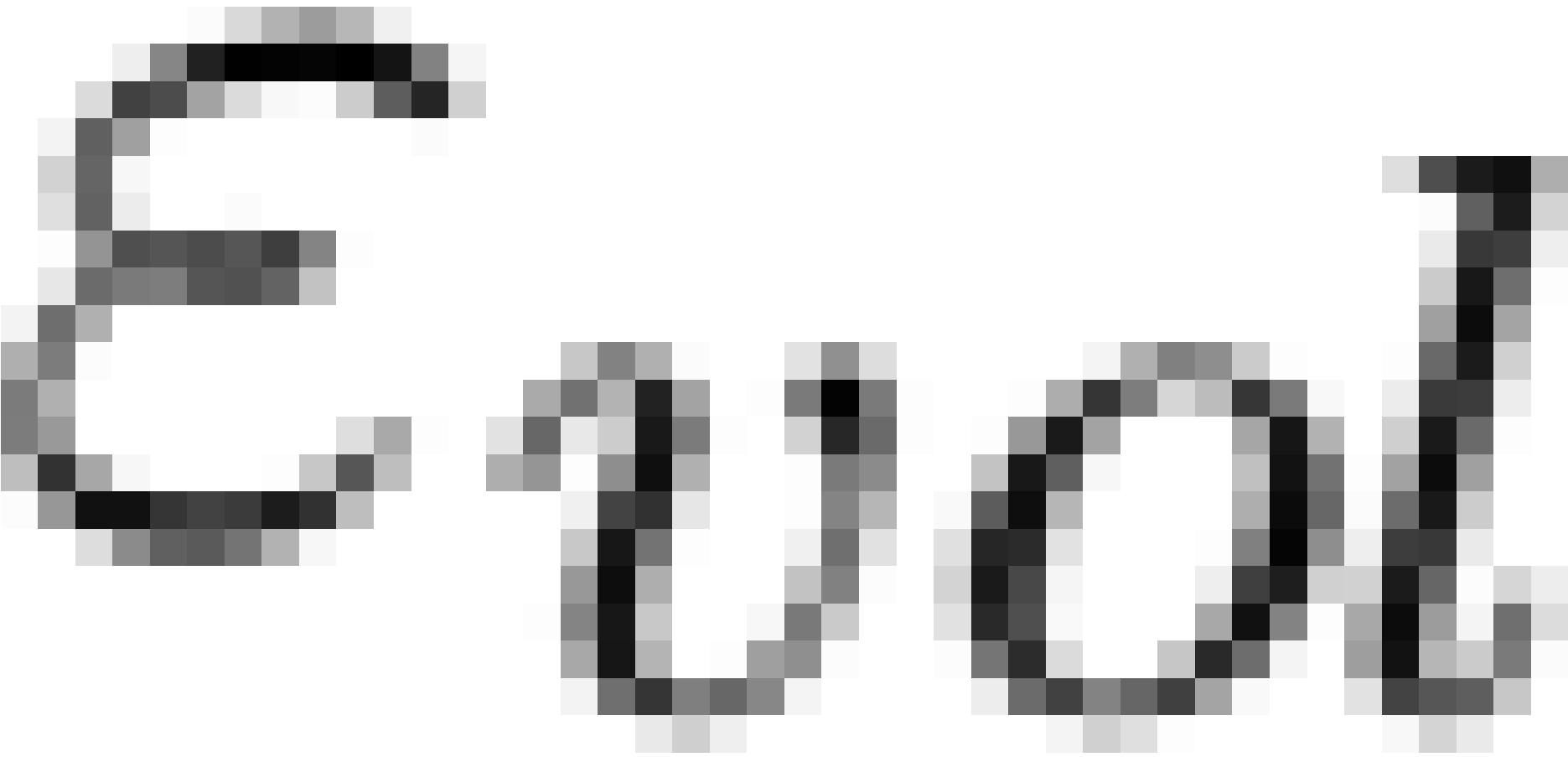


$$V_p = \sqrt{\frac{M}{\rho}}$$

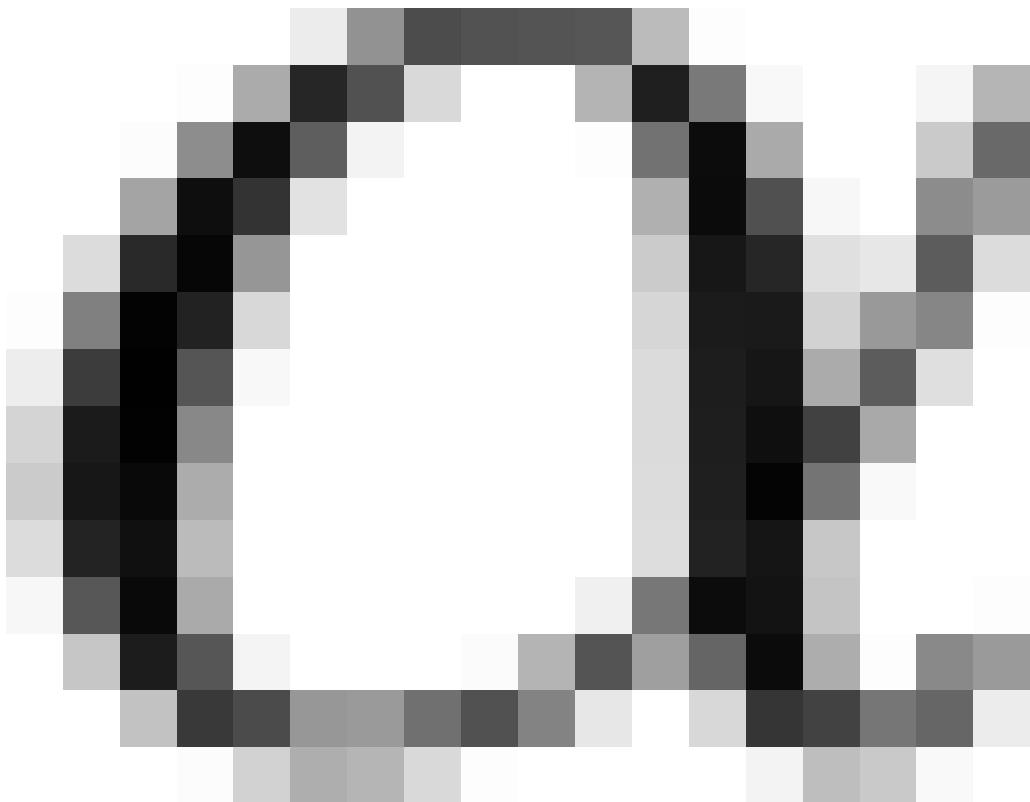
$$\rightarrow K_F^{-1} = \left( \frac{1-S_w}{K_g} + \frac{S_w}{K_w} \right)$$



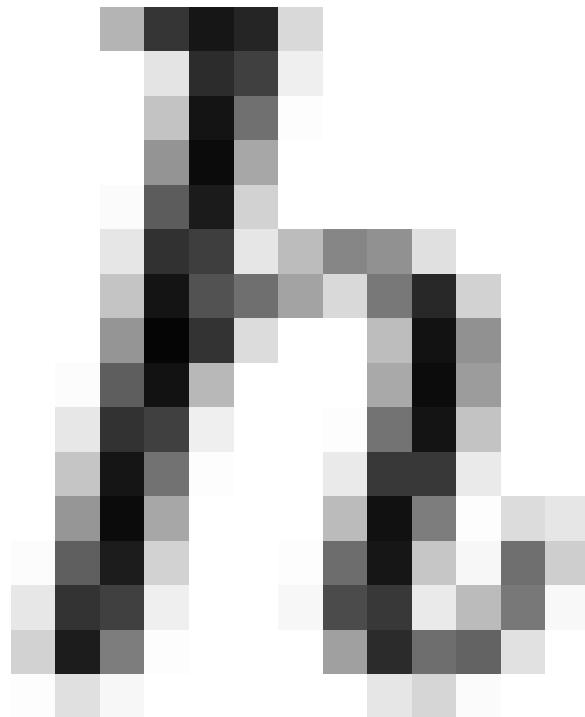
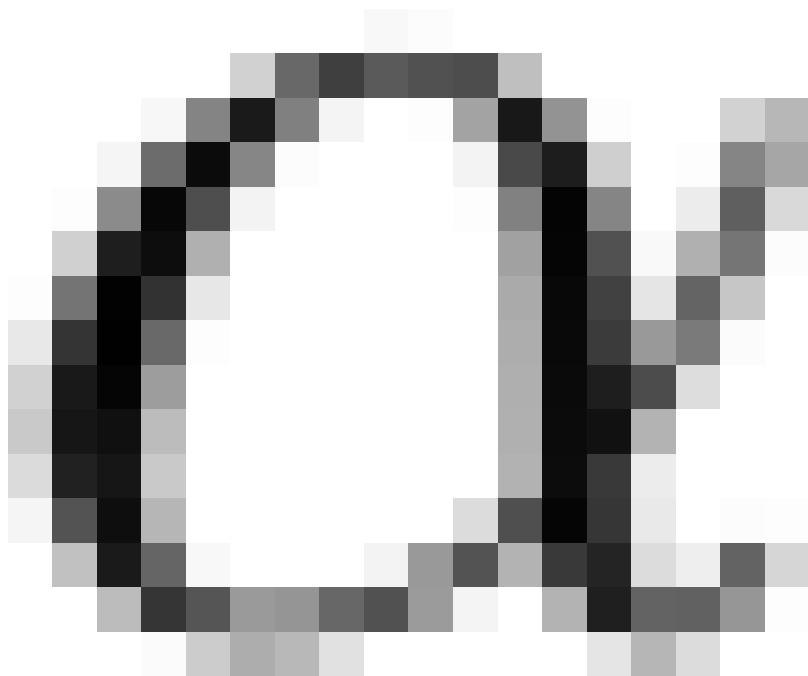


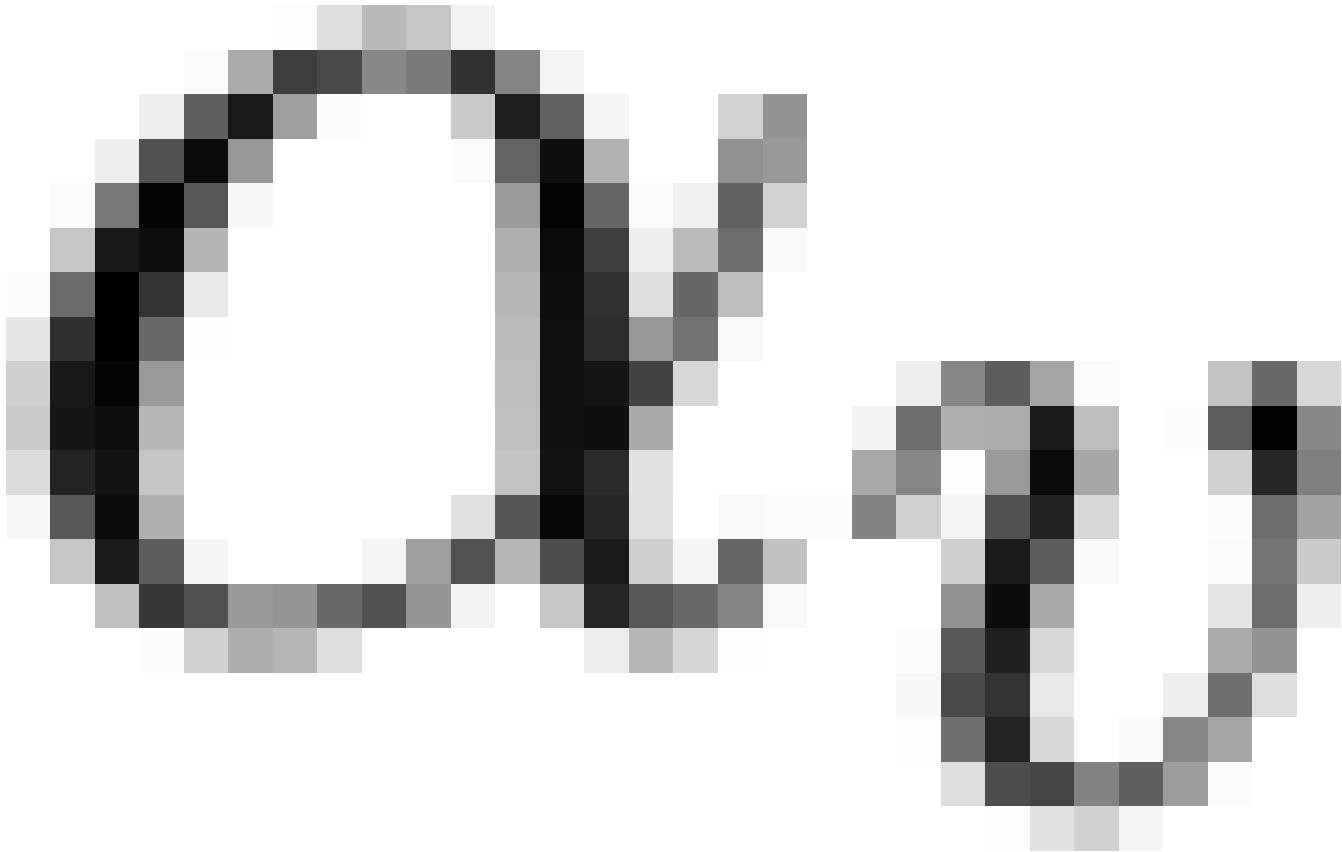


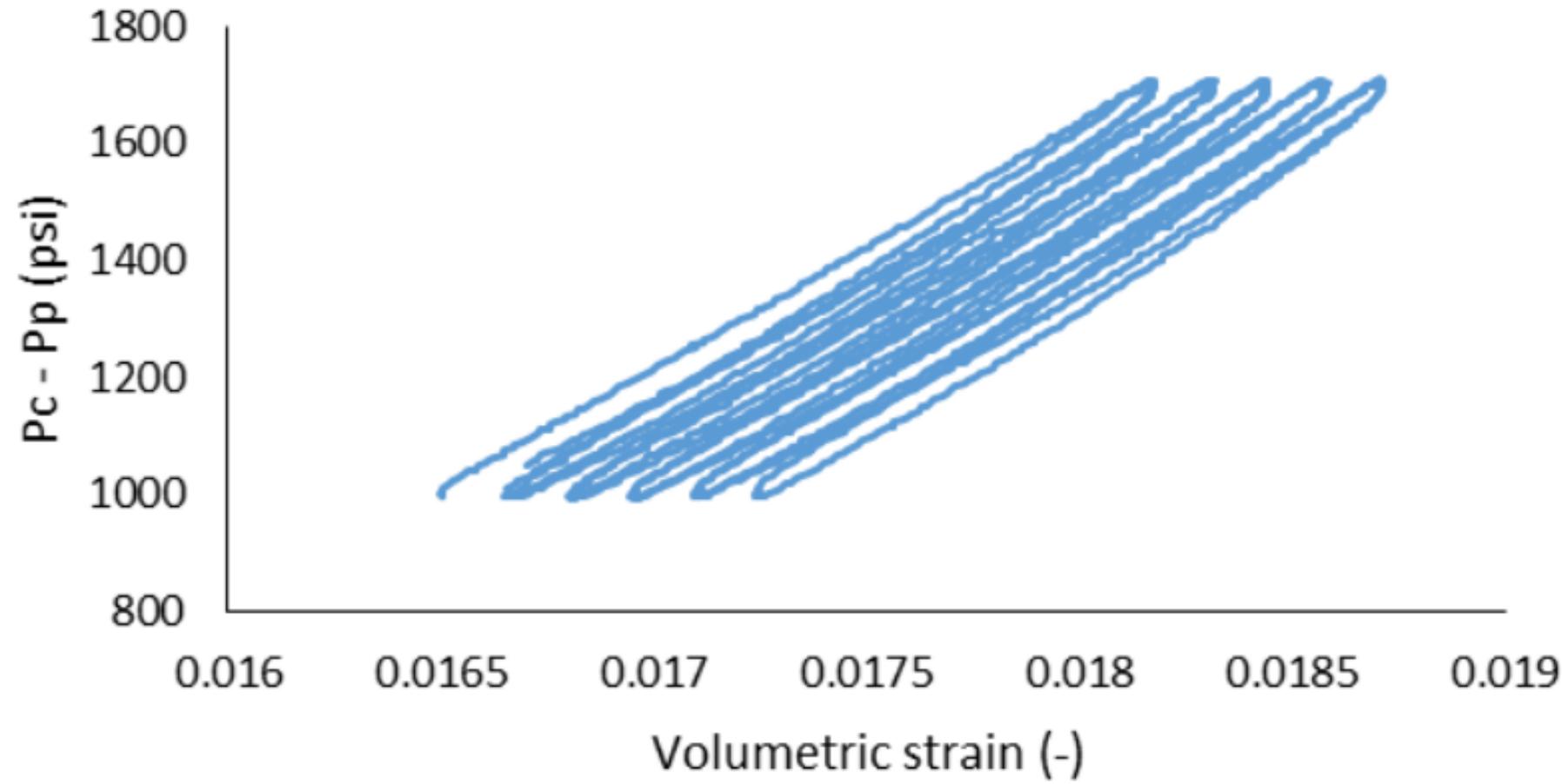












Production : 120 bbl/day  
Min. BHP : 240 psi

1000 ft

$$S_v = 1200 \text{ psi}$$

Zero displacements  
except for vertical  
direction

Permeability : 50 mD  
Porosity : 0.3  
 $P_{init}$ : 460 psi

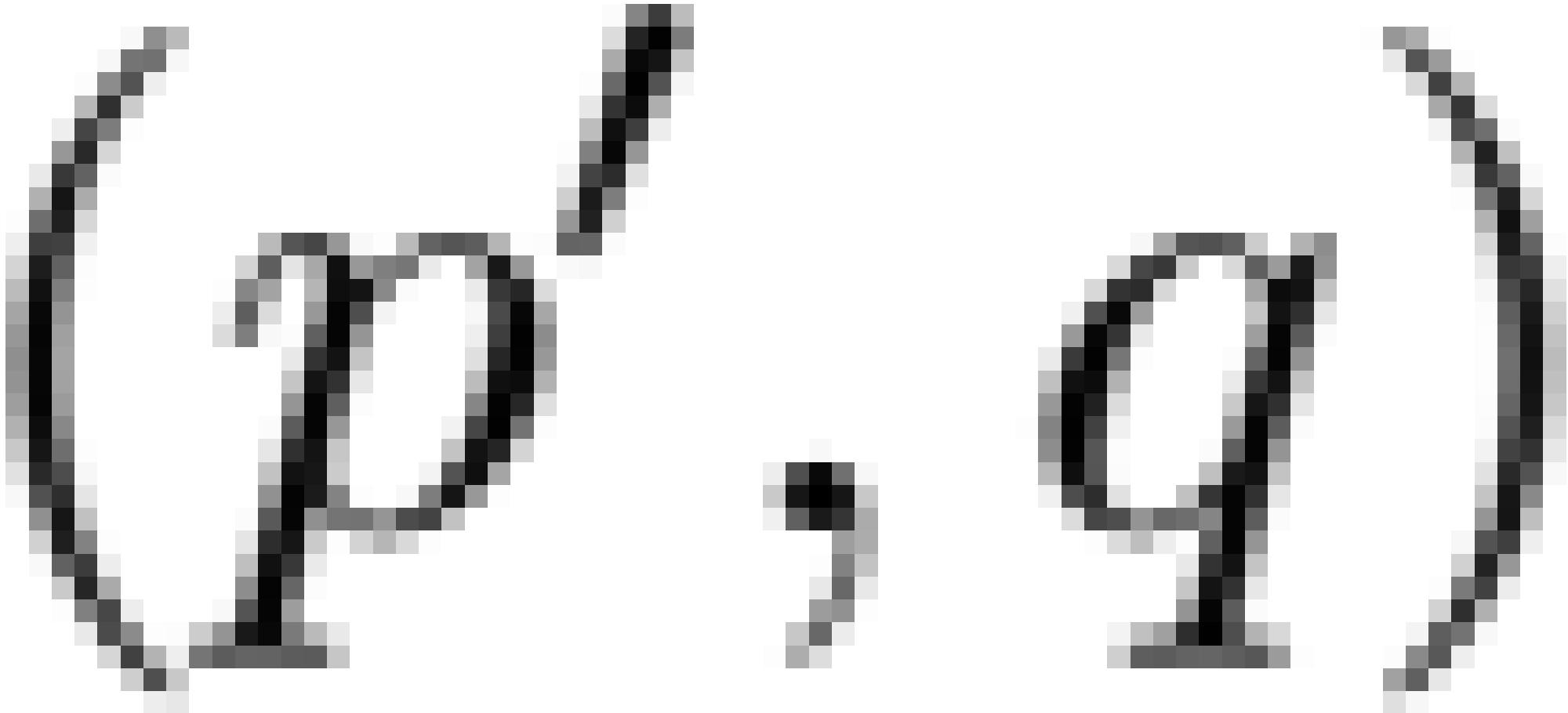
100 ft

1000 ft

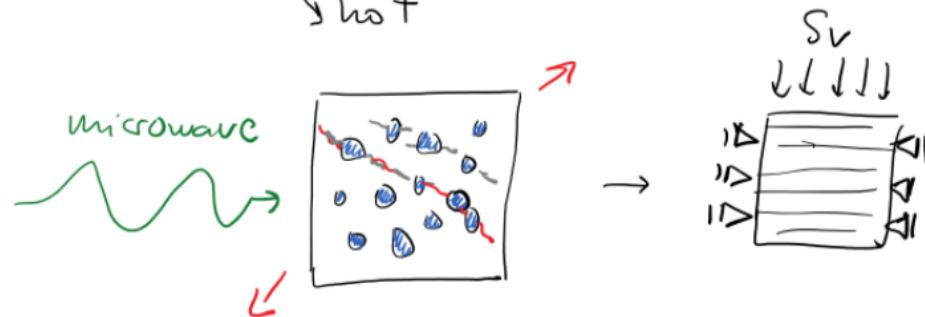
Q

(1) 22

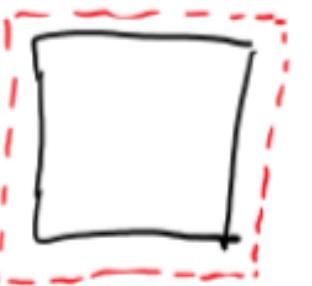
1 - V



- Applications {
- geothermal energy
    - Shallow
    - deep { HF,  $\tau/\sigma_n \uparrow$ ,  $\Delta\sigma$
  - drilling —  $T_{mud} < T_{form} \Rightarrow \downarrow \sigma_{\theta\theta} \rightarrow$  tensile fractures
  - subsurface fluid injection/disposal
    - EOR
      - water
      - CO<sub>2</sub>
      - chemicals
      - steam
    - disposal
      - produced water, HF water
      - CO<sub>2</sub>
  - hydraulic fracturing
    - cold  $\rightarrow$  cryogenic fracturing
    - hot +



$$\Delta T > 0 \Rightarrow$$



$\text{Evol} < 0$  (Dilation)

$$\Delta T < 0 \Rightarrow$$

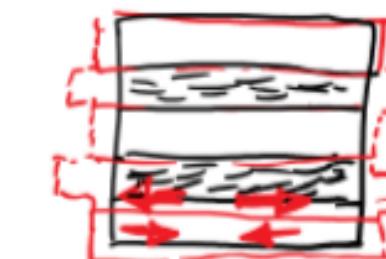


$\text{Evol} > 0$  (Contraction)

$$\Delta T \leq 0 \Rightarrow$$

no shear strains  
(isotropic homogeneous)

$\neq$



} layers  
 $\neq \beta$

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\epsilon}} + 3\beta K (\Delta T) \underline{\underline{I}}$$

Linear thermal dilation coefficient      Bulk modulus

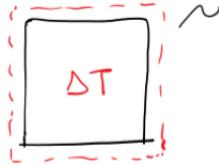
Change of temperature

Identity matrix

$$\Delta T = T - T_0$$

$$\Delta T = \theta$$

## Dilation



- unconstrained
- free dilation

$$\frac{\circ}{\square} = \frac{C}{\square} \cdot \underline{\underline{\varepsilon}} + 3\beta K (\Delta T) \underline{\underline{\varepsilon}}$$

$$\underline{\underline{\varepsilon}} = - \frac{D}{\square} (3\beta K \theta \underline{\underline{\varepsilon}})$$

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} 1/E & -v/E & -v/E \\ -v/E & 1/E & -v/E \\ -v/E & -v/E & 1/E \end{vmatrix} \begin{matrix} -3\beta K \theta \\ -3\beta K \theta \\ -3\beta K \theta \end{matrix}$$

$$\varepsilon_{11} = - \frac{(1-2v)}{E} [ + 3\beta K \theta ]$$

$1/K$

$$-\frac{\partial \varepsilon_{11}}{\partial \theta} \stackrel{\text{def}}{=} \beta$$

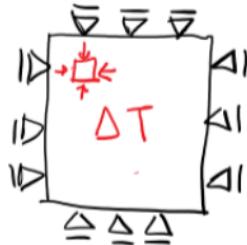
$$\varepsilon_{11} = -\beta \theta$$

$$\left. \begin{array}{l} \beta = 10^{-6} \text{ to } 10^{-5} \frac{1}{^{\circ}\text{K}} \\ \Delta T \sim 100 \text{ }^{\circ}\text{C} \end{array} \right\} \varepsilon \sim 10^{-4} \text{ to } 10^{-3}$$

$\theta = \Delta T > 0 \Rightarrow \Delta \varepsilon < 0$  (Dilation)

$\theta = \Delta T < 0 \Rightarrow \Delta \varepsilon > 0$  (Contraction)

## Constrained dilation



$$\underline{\underline{\sigma}} = \underline{\underline{C}} + 3\beta K \Delta T \underline{\underline{I}}$$

$$\theta = \Delta T > 0$$

↓

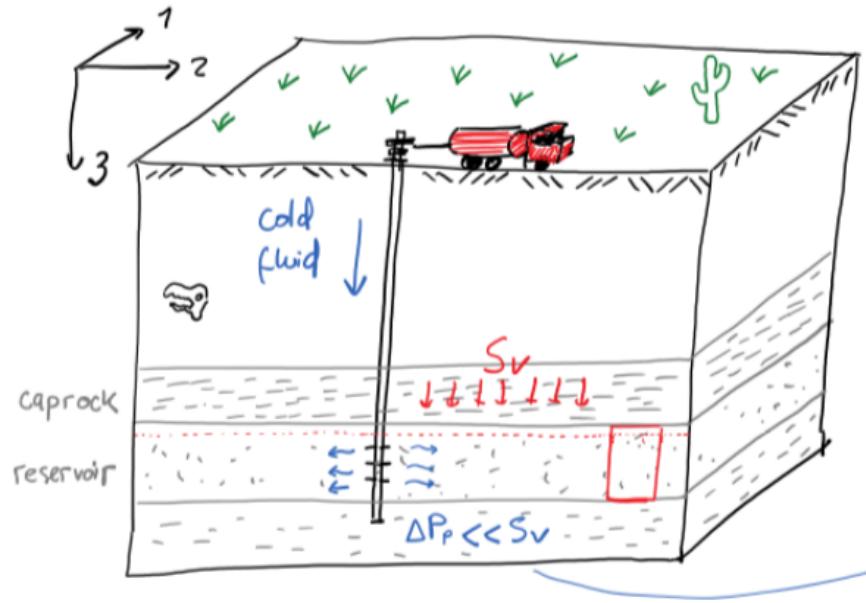
$$\Delta T > 0$$

$$\begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{vmatrix} = \begin{vmatrix} +3\beta K \theta \\ +3\beta K \theta \\ +3\beta K \theta \end{vmatrix}$$

$$\sigma_{11} = +3\beta K \theta$$

$$\left. \begin{array}{l} \beta = 10^{-5} \frac{1}{^{\circ}K} \\ \theta = 100^{\circ}K \\ K = 10 \text{ GPa} \end{array} \right\} \sigma_{11} = 3 \cdot 10^{-5} \frac{1}{^{\circ}K} \cdot 10^{10} \text{ Pa} \cdot 10^2 \cancel{^{\circ}K}$$

$$= 3 \cdot 10^7 \text{ Pa} = \underbrace{30 \text{ MPa}}_{\text{compression}} \approx 4400 \text{ psi}$$



$$\underline{\sigma} = \underline{\epsilon} + 3\beta K \theta \underline{I}$$

$$= \Delta T = T - T_0$$

No horizontal strains

$$\epsilon_{11} = \epsilon_{22} = 0$$

$$\epsilon_{33} \sim \text{constant}$$

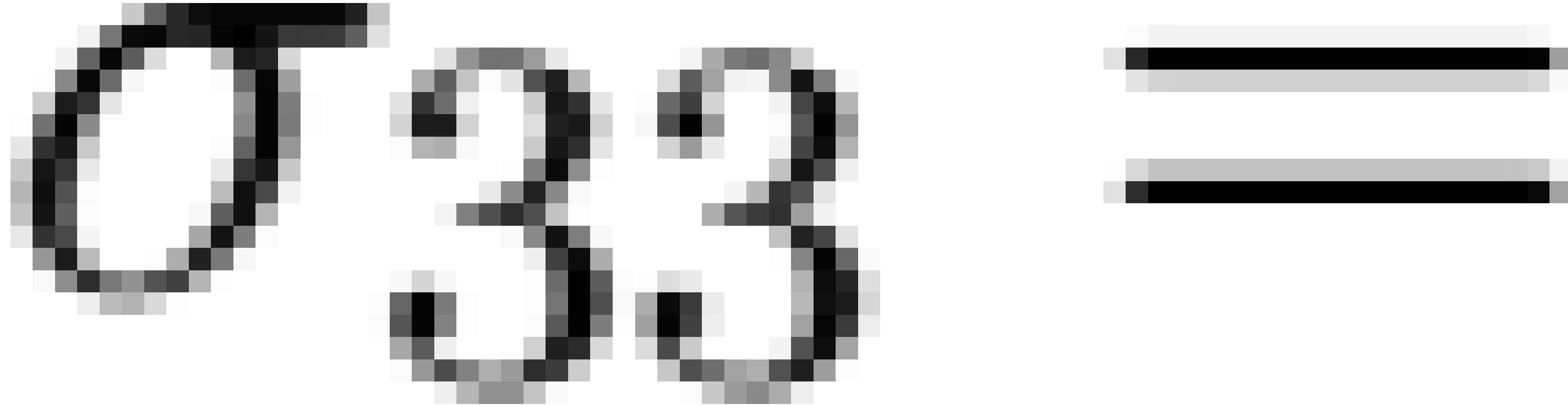


Uniform  
cooling in  
reservoir

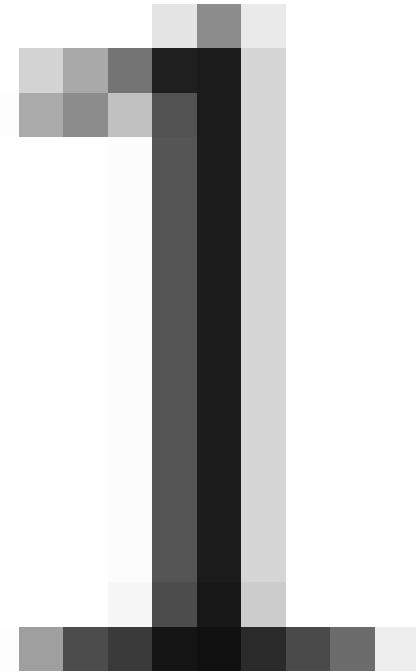
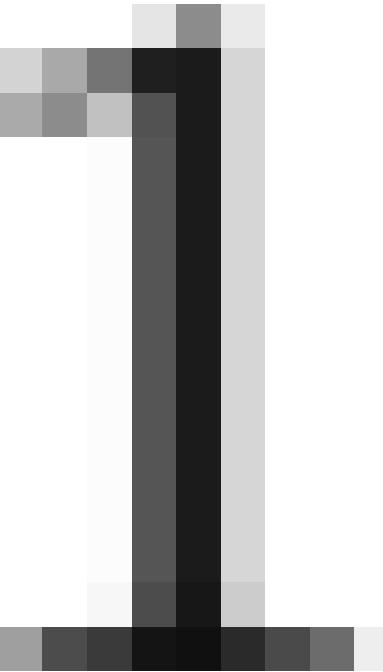
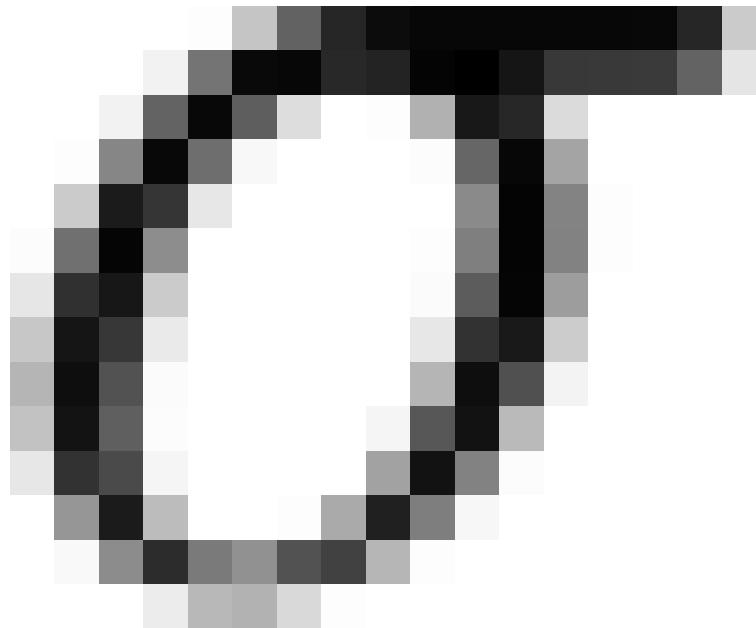
$$\begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{vmatrix} = \frac{E}{(1+v)(1-2v)} \begin{vmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \end{vmatrix} \begin{vmatrix} \cancel{\epsilon_{11}} \\ \cancel{\epsilon_{22}} \\ \epsilon_{33} \end{vmatrix} + \begin{vmatrix} 3\beta K \theta \\ 3\beta K \theta \\ 3\beta K \theta \end{vmatrix}$$

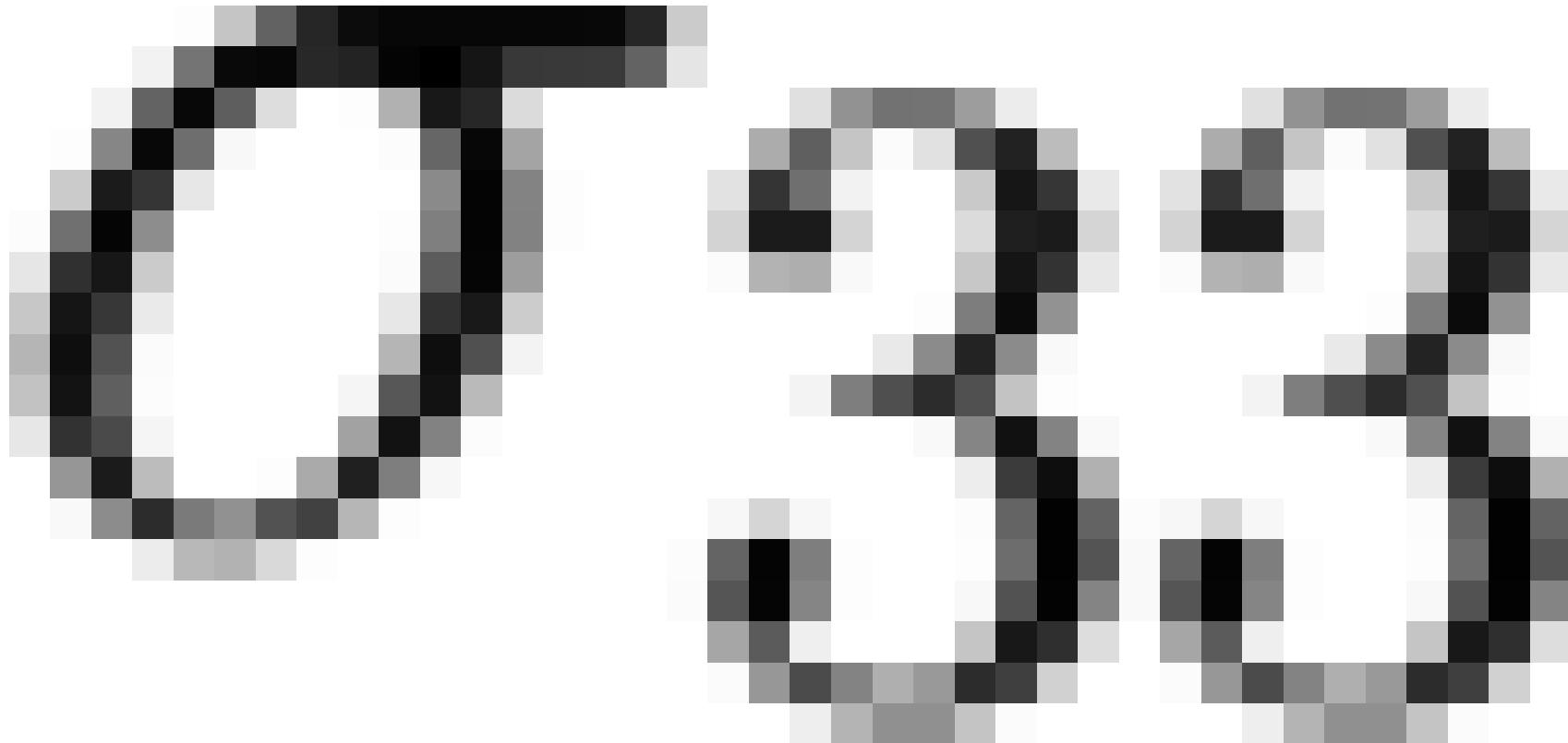




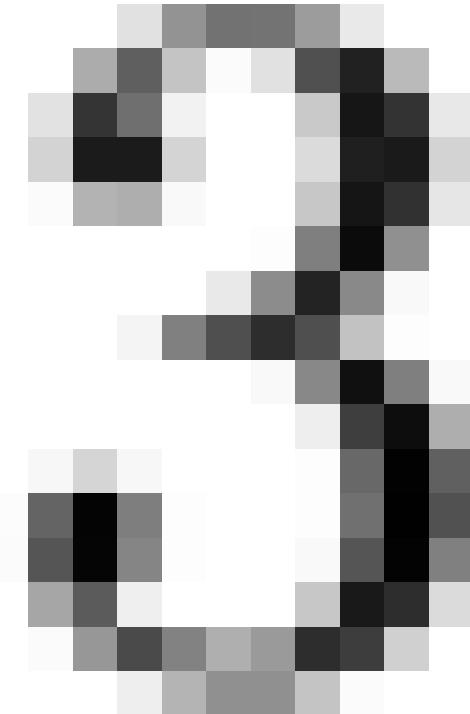
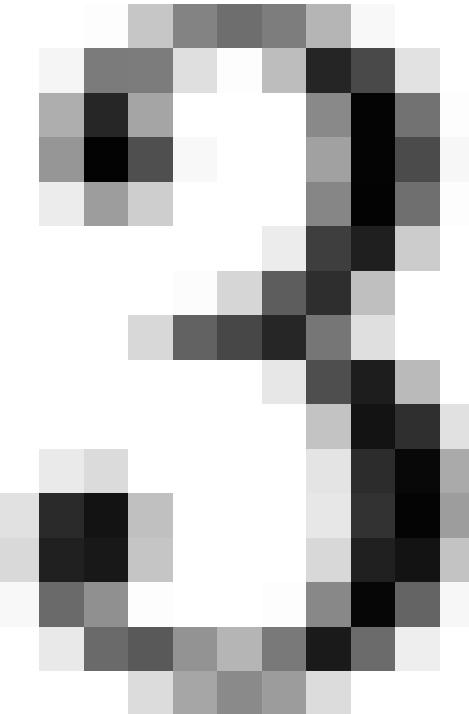
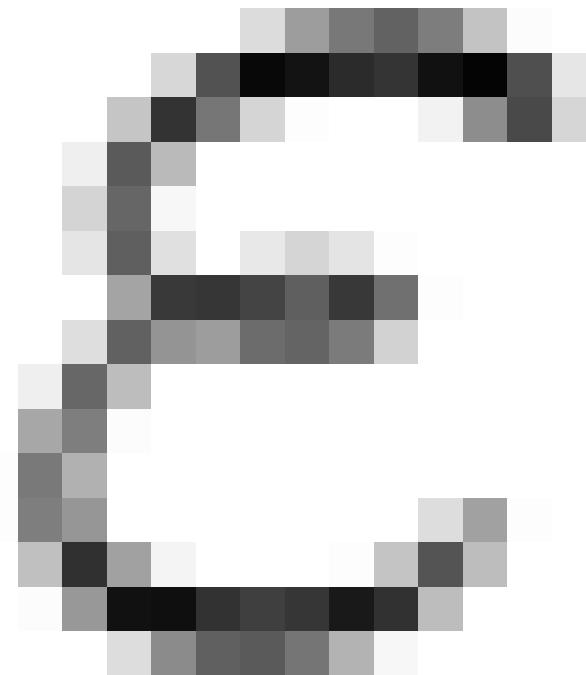


$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{33} \end{bmatrix} + \begin{bmatrix} 3\alpha_L K \theta \\ 3\alpha_L K \theta \\ 3\alpha_L K \theta \end{bmatrix}$$

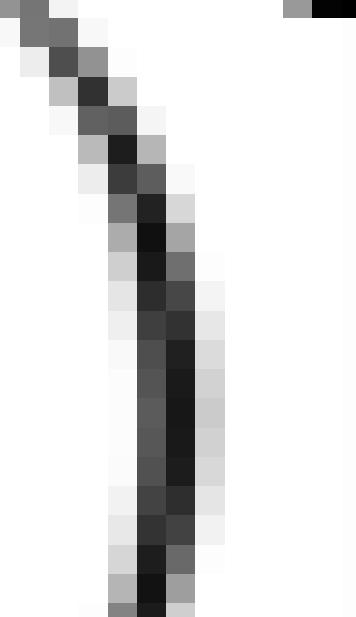
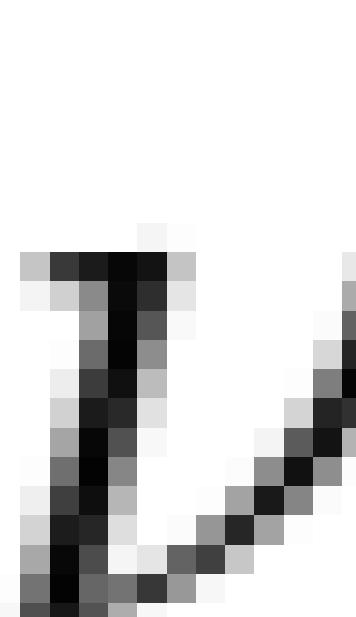
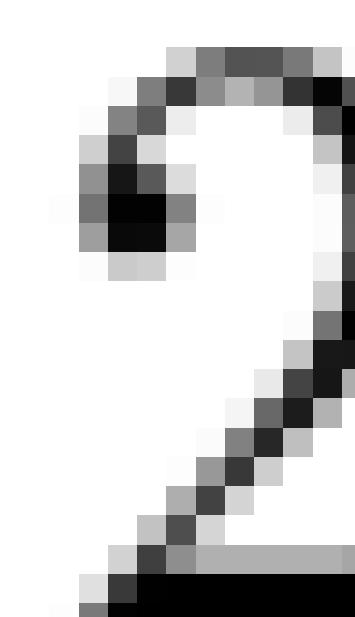
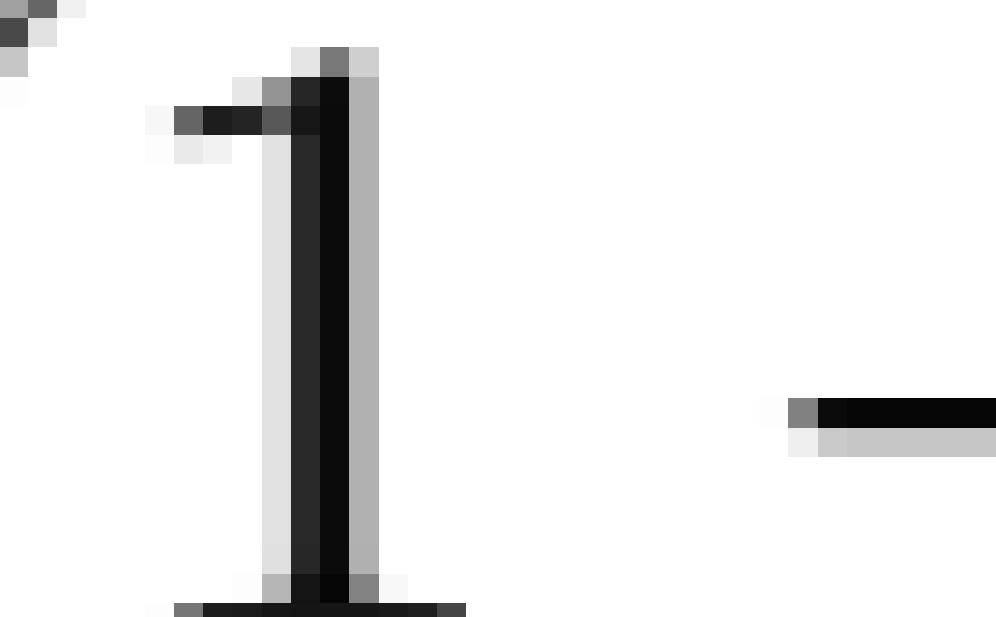
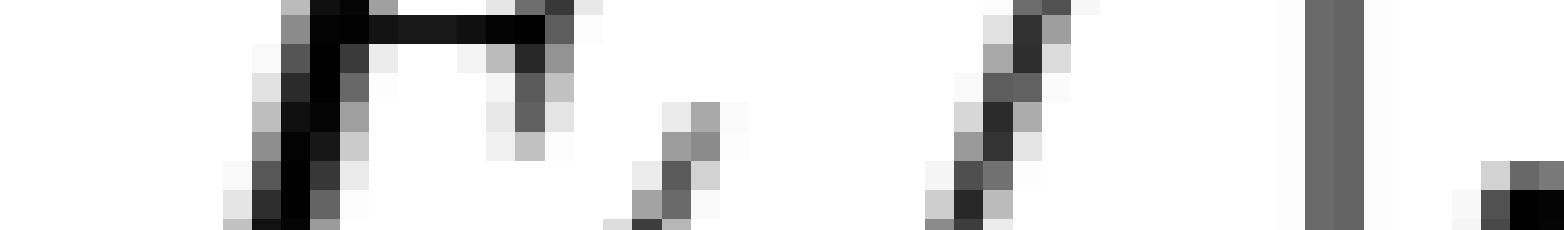
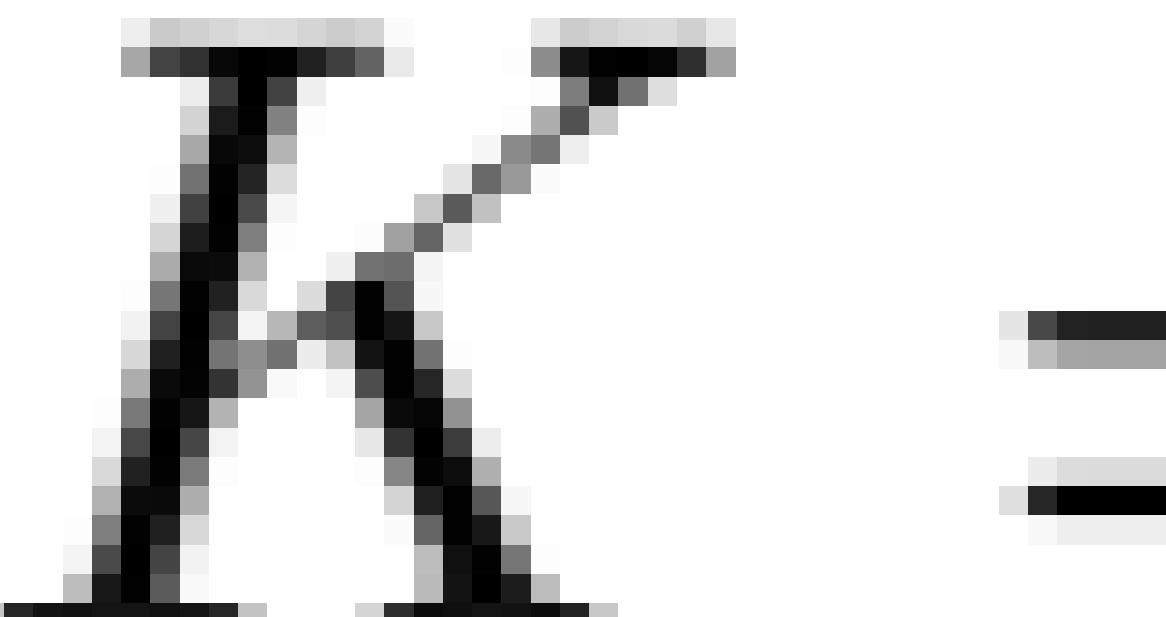




$$\left\{ \begin{array}{l} \sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)} \epsilon_{33} + 3\alpha_L K \theta \\ \sigma_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \epsilon_{33} + 3\alpha_L K \theta \end{array} \right.$$



$$\sigma_{11} = \left( \frac{v}{1-v} \right) \sigma_{33} + \left( \frac{1-v}{1+v} \right) 3\alpha_L K_0$$



$$\sigma_{11} = \left( \frac{1 - \nu}{1 + \nu} \right) \sigma_{33} + \frac{\alpha E}{1 - \nu}$$

$\hat{\sigma}_{11}$

$\hat{\sigma}_{11}$

$\hat{\sigma}_0$

$\hat{\sigma}_0$

$\hat{\sigma}_0$

$\hat{\sigma}_1$

$\hat{\sigma}_1$

$\hat{\sigma}_2$

$\hat{\sigma}_{DE}$

$\hat{\sigma}_{11}$

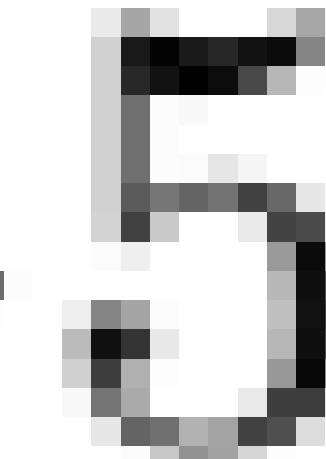
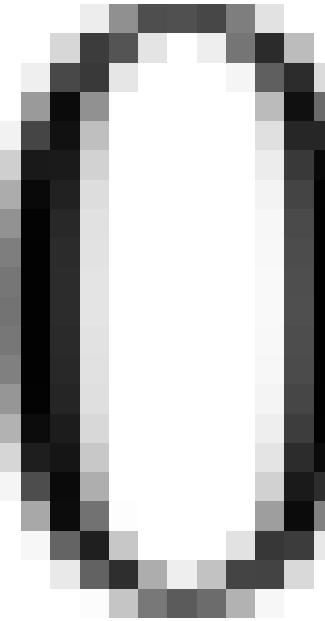
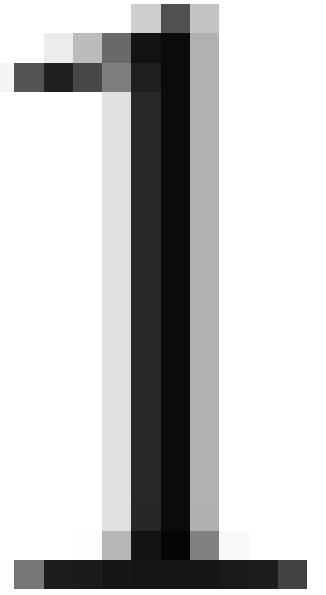
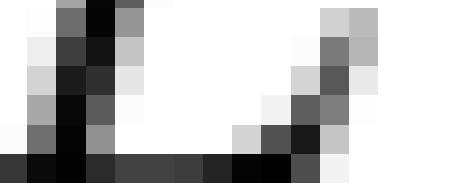
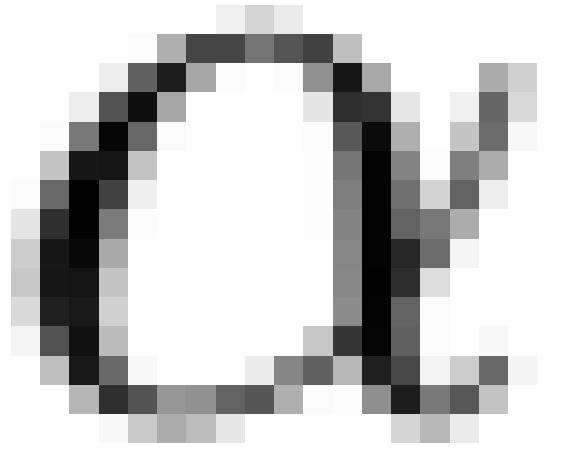
$\hat{\theta}$



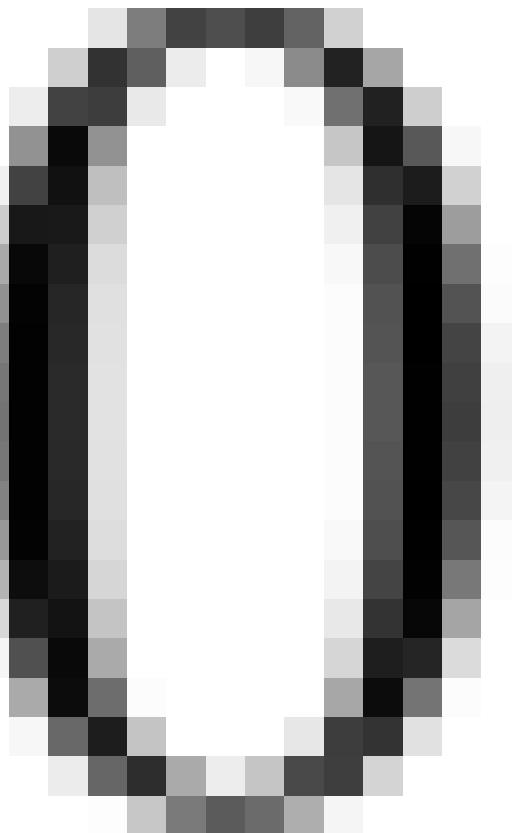
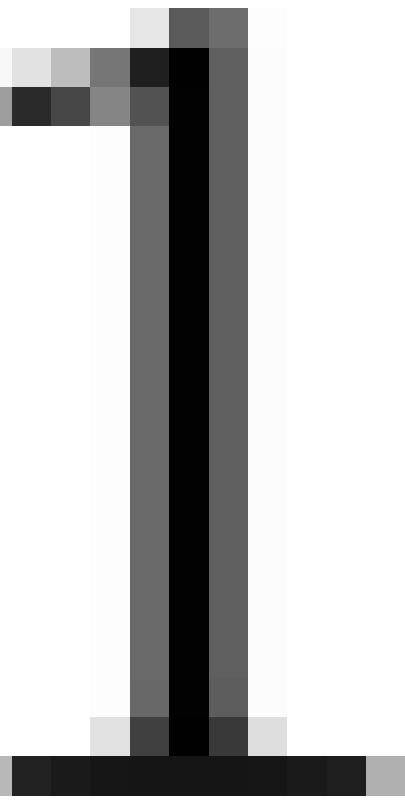
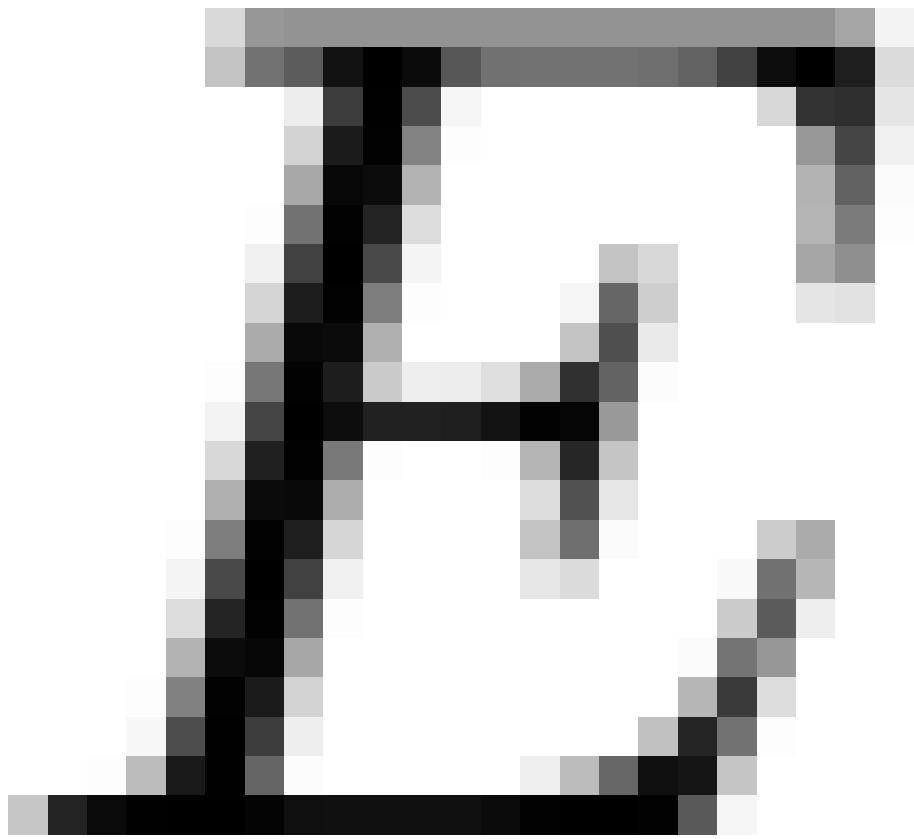
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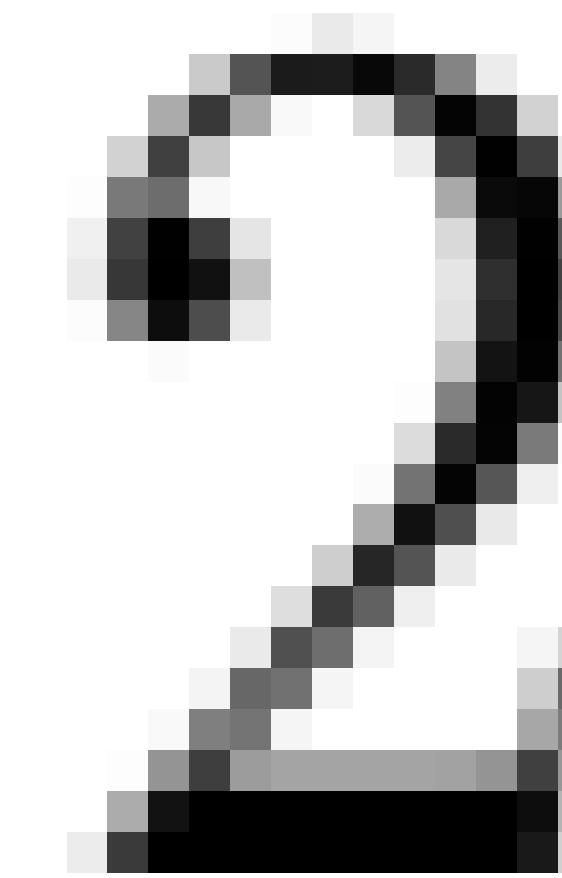
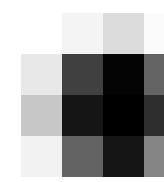
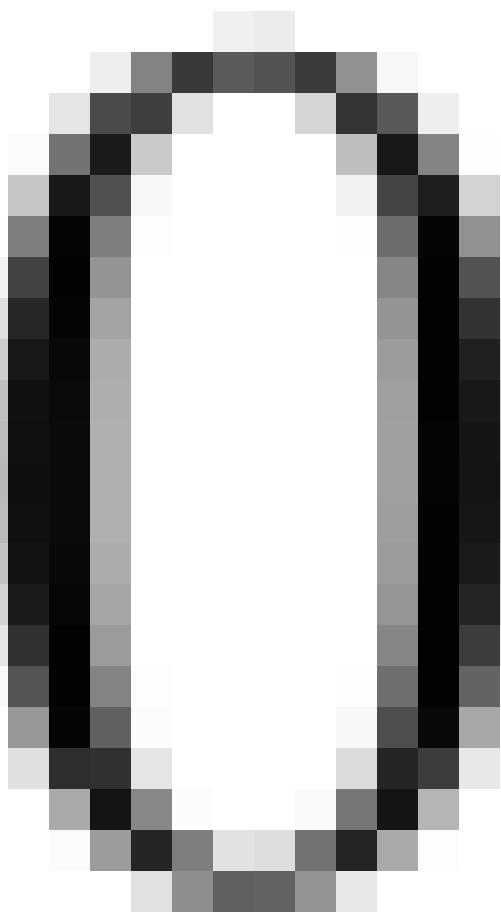
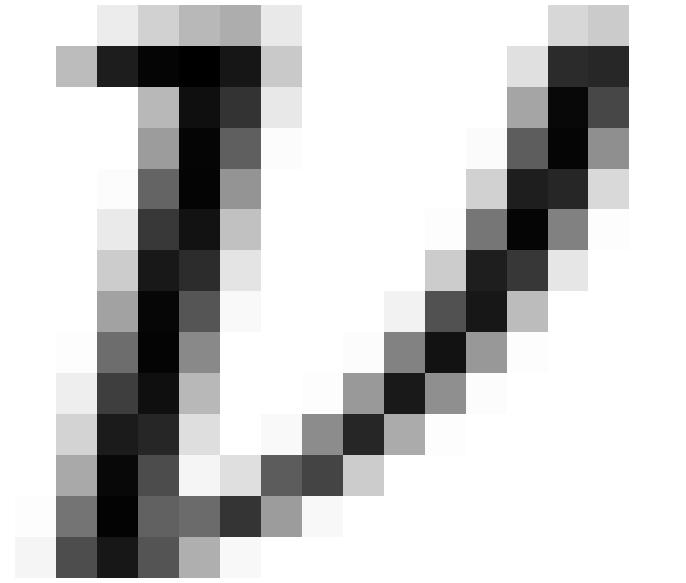
C

MPa









$$\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)} \epsilon_{33} + 3\beta K \theta$$

$$\sigma_{33} = \frac{(1-\nu) E}{(1+\nu)(1-2\nu)} \epsilon_{33} + 3\beta K \theta$$

$$\sigma_{11} = \frac{\nu E}{\cancel{(1+\nu)(1-2\nu)}} \left[ \frac{\cancel{(1+\nu)(1-2\nu)}}{1-\nu} (\sigma_{33} - 3\beta K \theta) \right] + 3\beta K \theta$$

$$\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33} + \left(1 - \frac{\nu}{1-\nu}\right) 3\beta K \theta$$

$$\boxed{\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33} + \left(\frac{1-2\nu}{1-\nu}\right) 3\beta K \theta}$$

$$\frac{\partial \sigma_{11}}{\partial \theta} \Big|_{\sigma_{33}} = \left(\frac{1-2\nu}{1-\nu}\right) 3\beta K = \frac{1-2\nu}{1-\nu} \cdot \frac{E}{\cancel{3(1-2\nu)}} \cdot \cancel{3\beta}$$

$$\boxed{\frac{\partial \sigma_{11}}{\partial \theta} \Big|_{\sigma_{33}} = + \frac{\beta E}{1-\nu}}$$

$$\left. \begin{array}{l} \downarrow \theta \Rightarrow \downarrow \sigma_{11} \\ \uparrow \theta \Rightarrow \uparrow \sigma_{11} \end{array} \right.$$

just mechanics

$$\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33}$$

$\Delta P_p$ : pore pressure

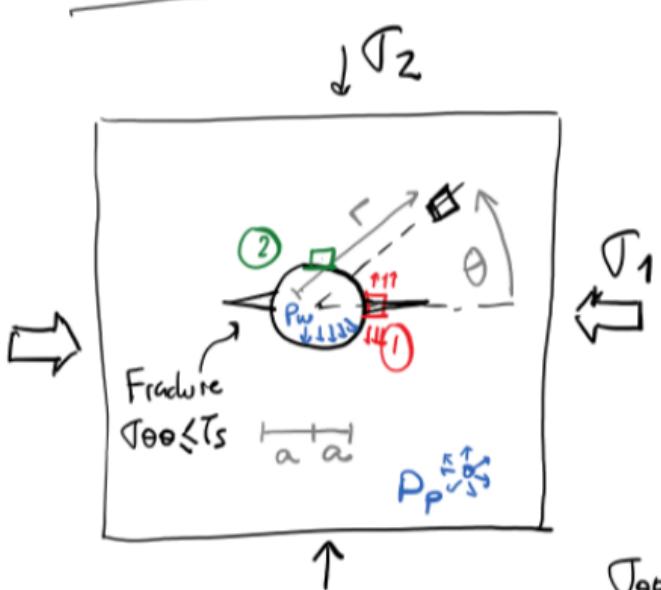
$$\frac{\partial \sigma_{11}}{\partial P_p} = \nu \left(\frac{1-2\nu}{1-\nu}\right)$$

$\Delta \text{Temp}$

$$\frac{\partial \sigma_{11}}{\partial \theta} = \frac{\beta E}{1-\nu}$$

$$\left. \begin{array}{l} \beta = 10^{-5} \text{ /K} \\ E = 10 \text{ GPa} \\ \nu = 0.2 \end{array} \right. \Rightarrow \frac{\partial \sigma_{11}}{\partial \theta} = 0.1 \frac{\text{MPa}}{\text{°K}}$$

# Changes of stresses due to temperature around a wellbore

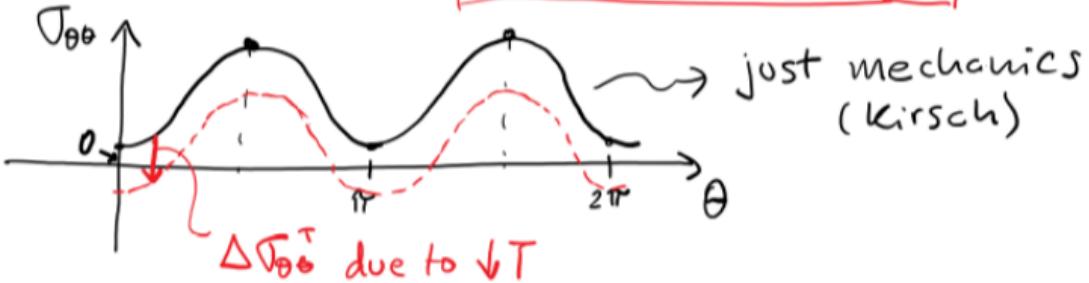


$$\text{Kirsch} \quad \Delta \sigma_{\theta\theta}^T = -(\rho_w - \rho_p) - \sigma_1 + 3\sigma_2 + \Delta \sigma_{\theta\theta}^T$$

$$② \sigma_{\theta\theta}(r=a, \theta=90^\circ) = -(\rho_w - \rho_p) + 3\sigma_1 - \sigma_2 + \Delta \sigma_{\theta\theta}^T$$

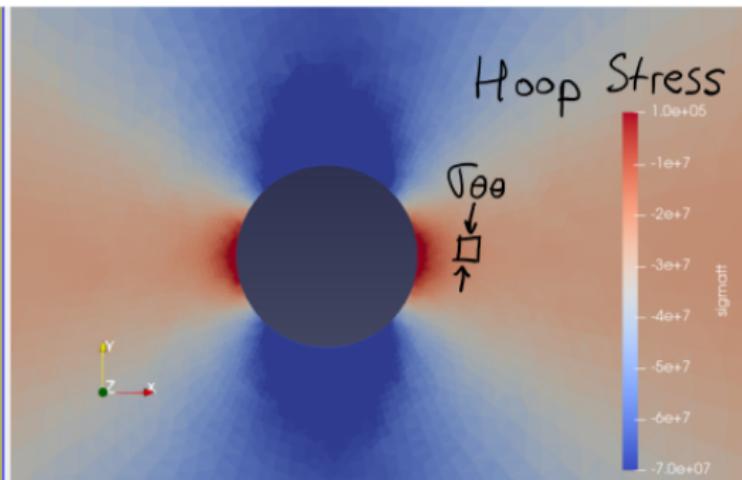
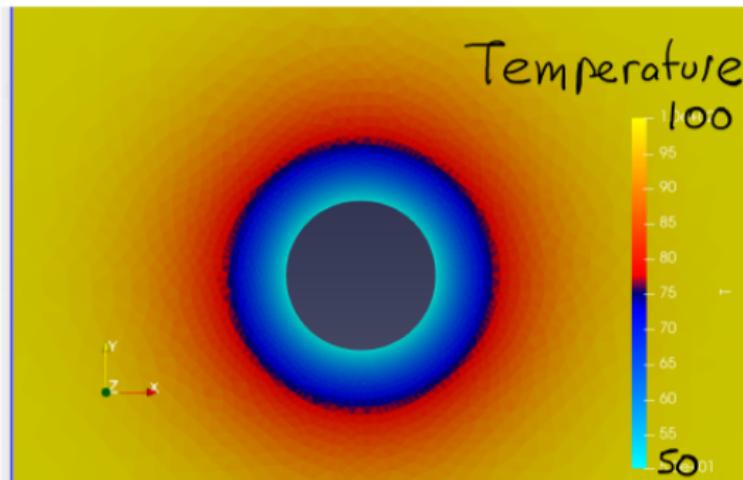
steady state  
solution

$$\Delta \sigma_{\theta\theta}^T = \frac{\beta E}{1-\nu} \cdot \Delta T$$



$$\left\{ \begin{array}{l} \nabla \underline{\underline{\sigma}} + \underline{\underline{f}} = \underline{\underline{0}} \\ \\ \underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) \\ \underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}} + 3\alpha_L K \theta \underline{\underline{I}}; \quad \theta = T - T_0 \\ \\ \frac{\partial \theta}{\partial t} = \frac{k_T}{\rho c_v} \nabla^2 \theta + \frac{3\beta K T_0 \partial \varepsilon_{vol}}{\rho c_v - \partial t} \end{array} \right. \begin{array}{l} \text{Equilibrium} \\ \\ \text{Kinematic} \\ \\ \text{Constitutive} \\ \\ \text{Diffusivity} \end{array}$$

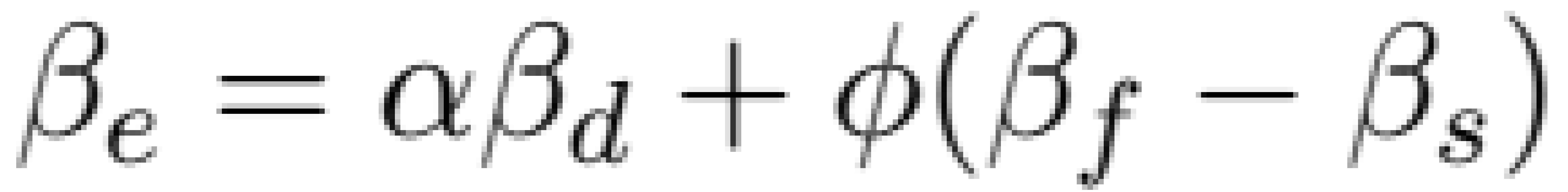
$$\left\{ \begin{array}{l} \nabla \underline{\underline{\sigma}} + \underline{\underline{F}} = \underline{\underline{0}} \quad \longrightarrow \text{Equilibrium} \\ \underline{\underline{\varepsilon}} = \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^T) \quad \longrightarrow \text{Kinematic} \\ \underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}} + 3\beta K \theta \underline{\underline{I}} \rightarrow \text{Constitutive} \quad (\theta = \Delta T = T - T_0) \\ \frac{\partial \theta}{\partial t} = \frac{K_T}{\rho C_v} \nabla^2 \theta + \frac{3\beta K T_0}{\rho C_v} \cdot \frac{\partial \varepsilon_v}{\partial t} \rightarrow \text{Diffusivity} \end{array} \right.$$

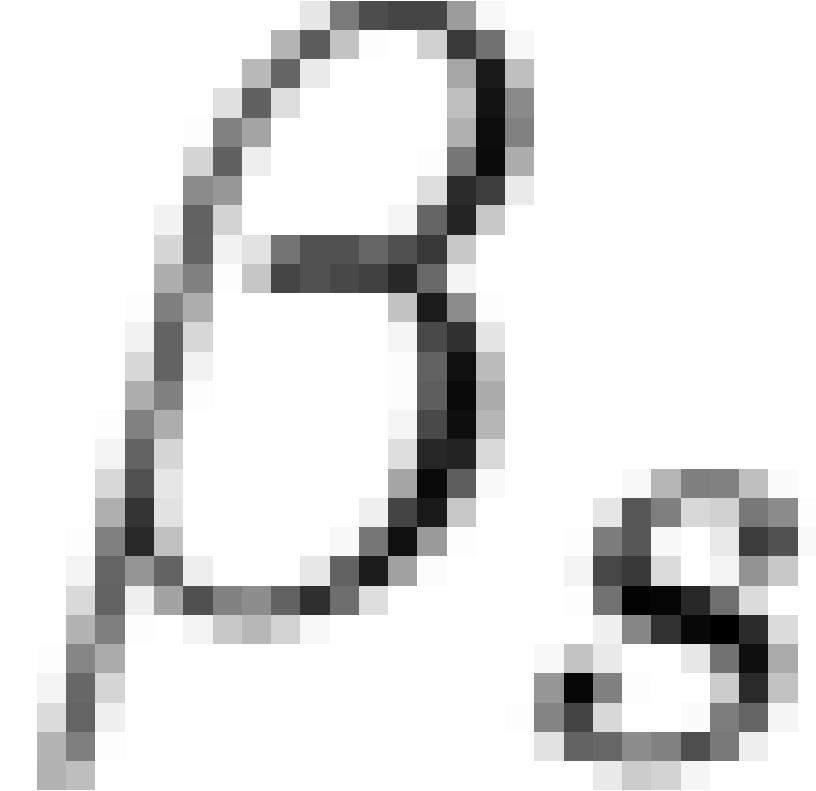
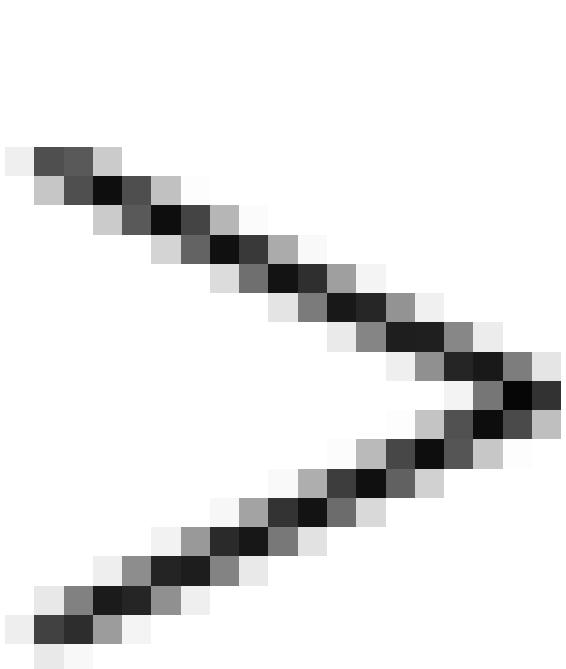


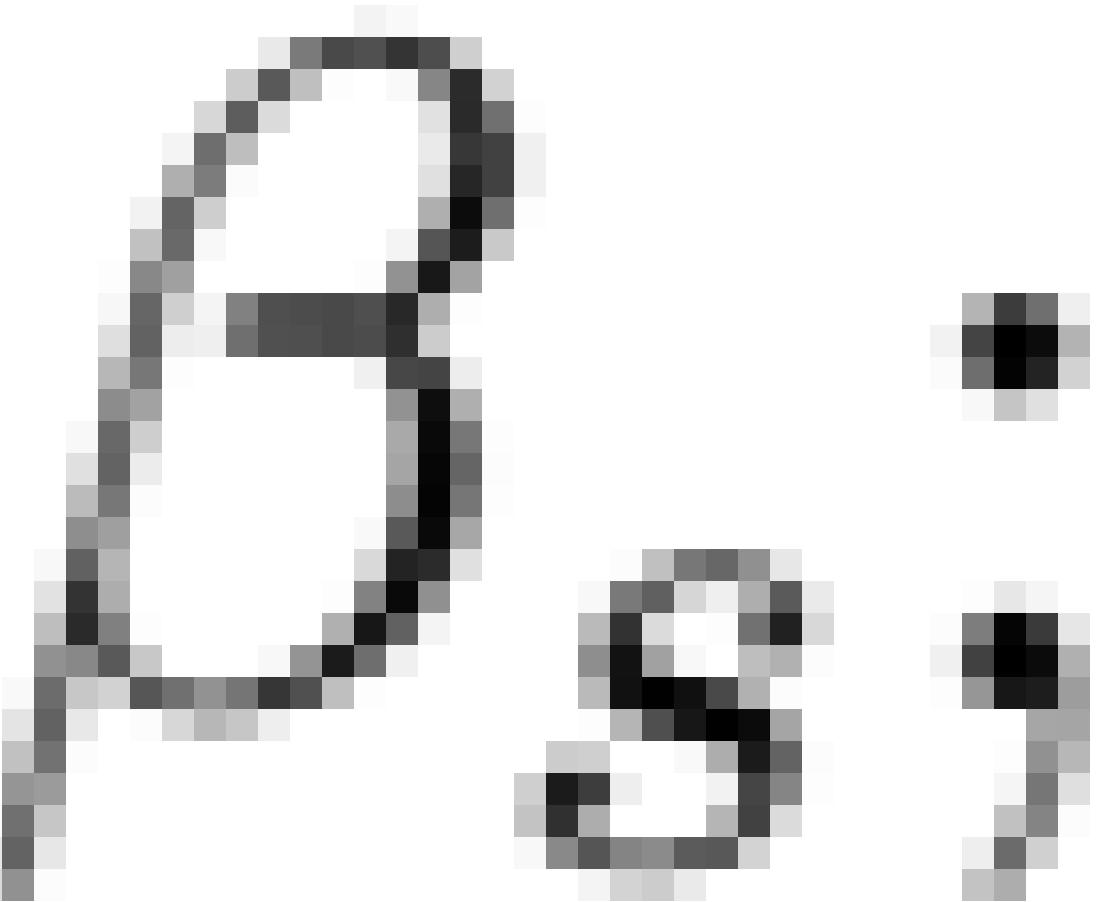
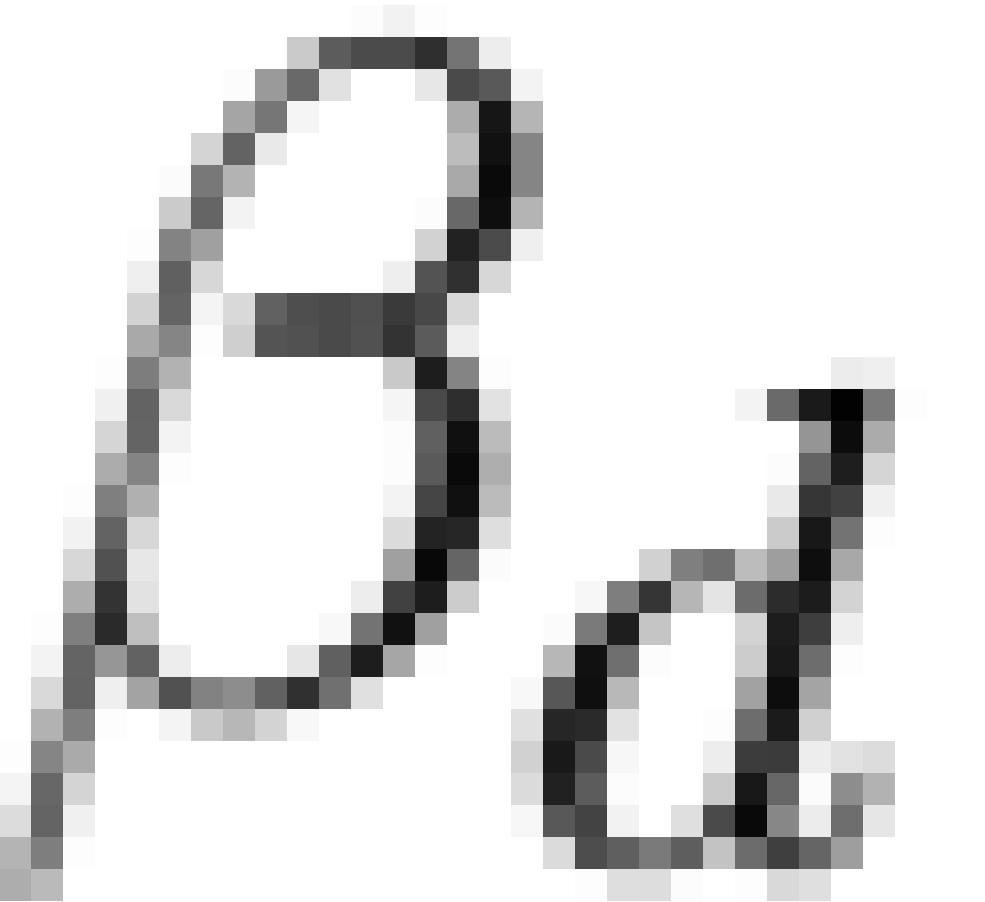
$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} - \frac{\kappa}{\mu} M \nabla^2 p = -\alpha M \frac{\partial \epsilon_{vol}}{\partial t} + \beta_e M \frac{\partial T}{\partial t} \\ \frac{\partial T}{\partial t} - \kappa_T \nabla^2 T = -\frac{\alpha_d}{m_d} \frac{\partial \epsilon_{vol}}{\partial t} + \frac{\beta_e}{m_d} \frac{\partial p}{\partial t} \end{array} \right.$$

Pore pressure diffusivity

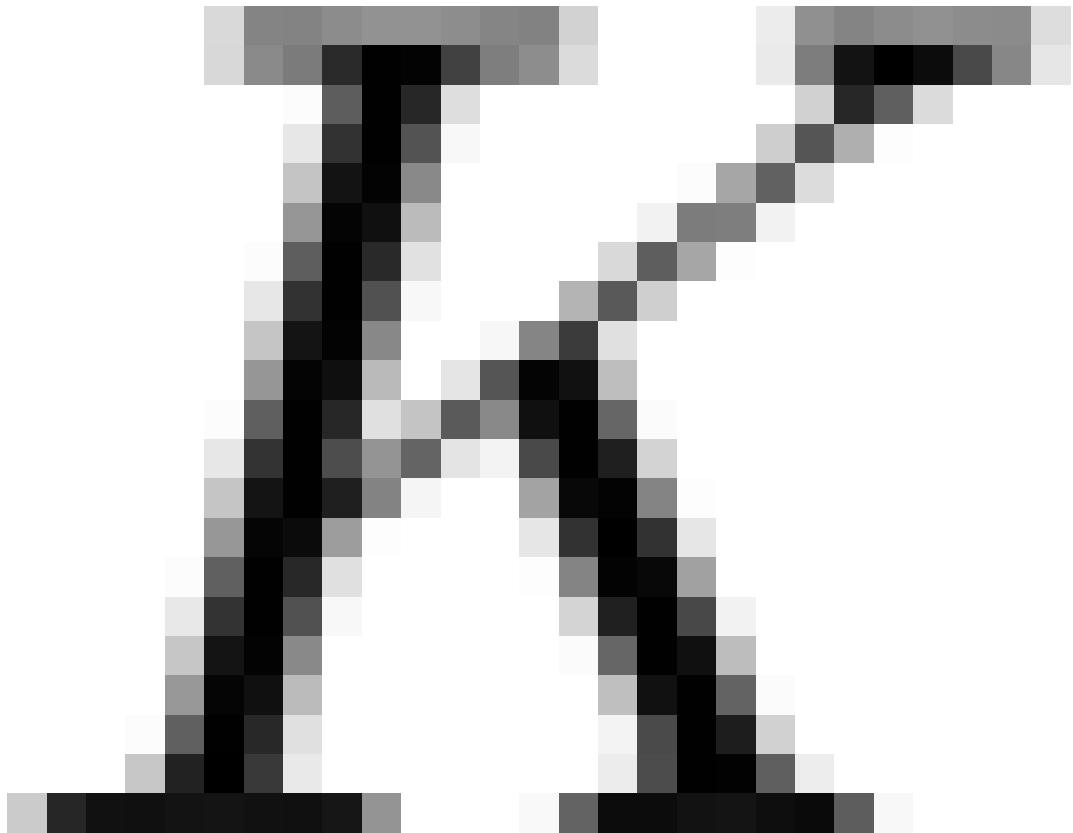
Temperature diffusivity

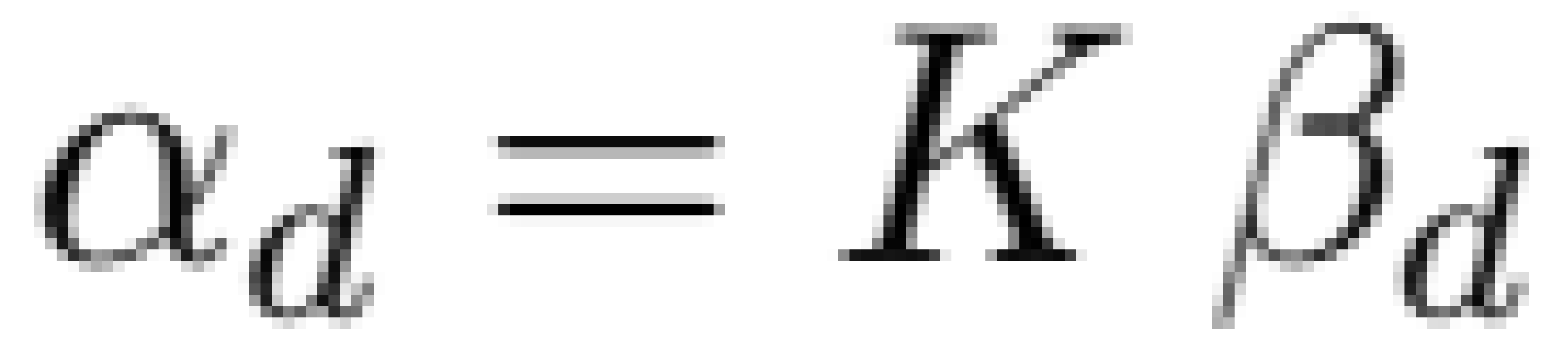






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## Chemo-mechanical coupled processes

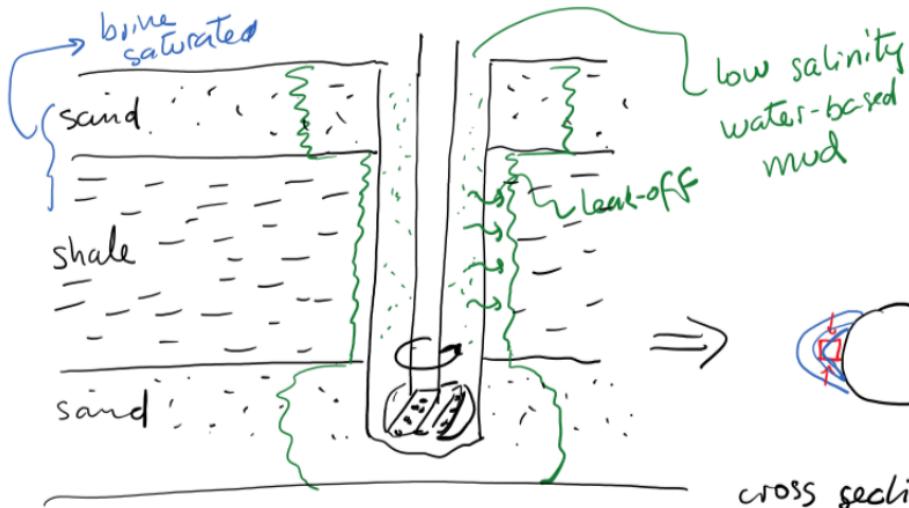
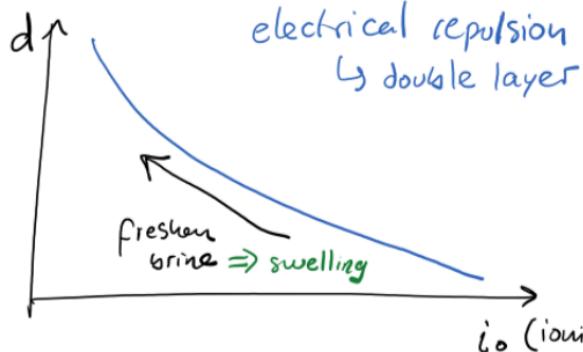
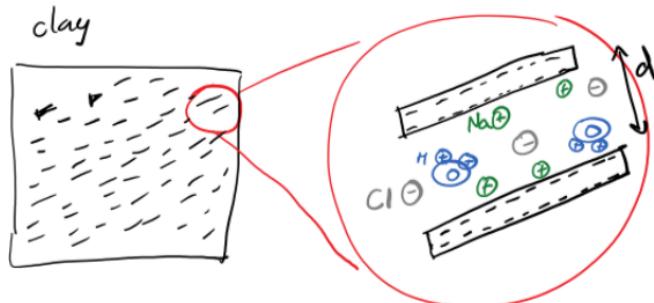
- chemo-plasticity : inelastic deform assisted by chem processes
- chemo-elasticity

$$\hookrightarrow \underline{\underline{S}} = \underline{\underline{\epsilon}} + \alpha P \underline{\underline{I}} + 3\beta K \theta \underline{\underline{I}} + \underbrace{\gamma}_{\text{coeff}} \overset{\text{chem potential}}{\overbrace{N}} K \underline{\underline{I}}$$

Neck      poro hydro      thermal      chemical

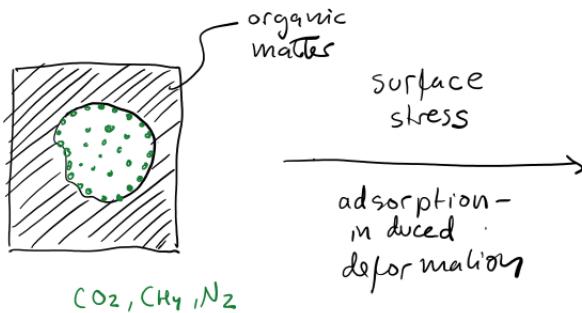
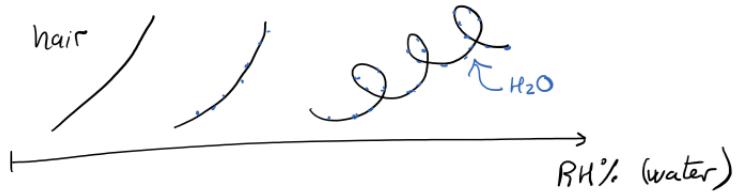
$\underbrace{\text{THCM}}$       coupled processes  $\Rightarrow$  emergent phenomena

# 1) Chemical sensitivity of shales

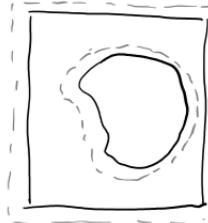


- drilling mud
  - water-based  
↳  $\uparrow$  salinity
  - oil-based mud  
↳ non-polar fluid
- under balance drilling  
 $P_{mud} < P_p$

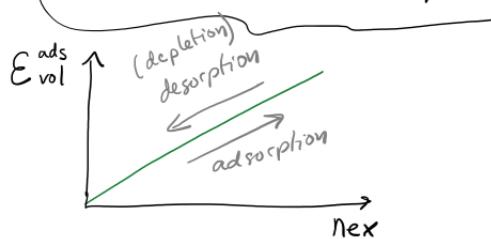
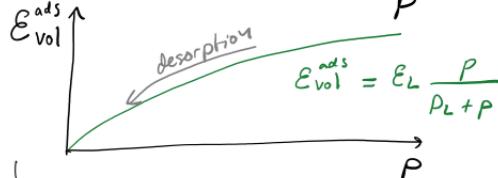
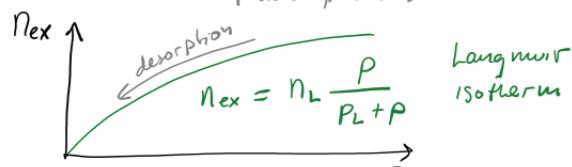
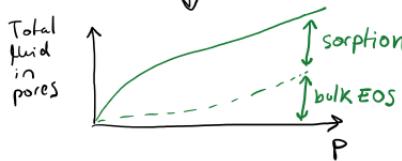
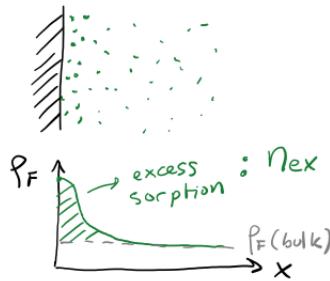
## 2) Adsorption-induced deformation



adsorption-induced deformation

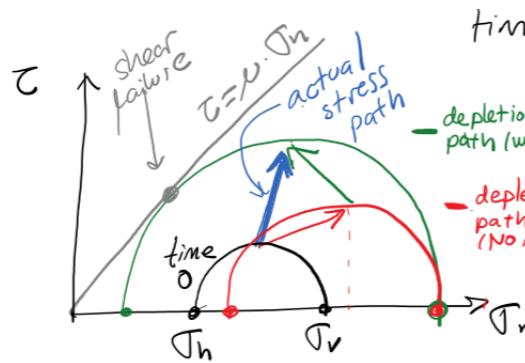
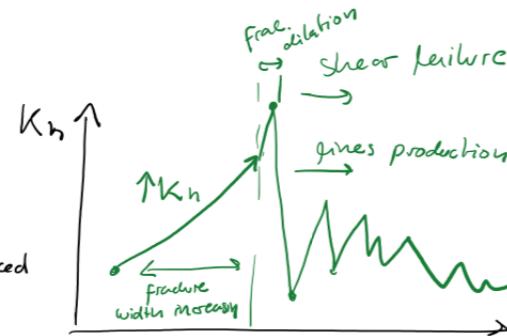
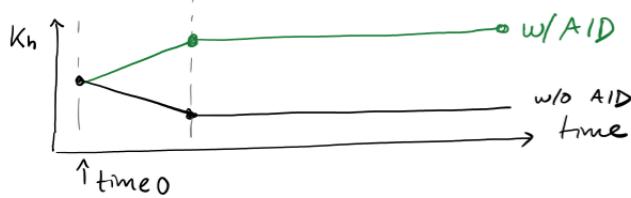
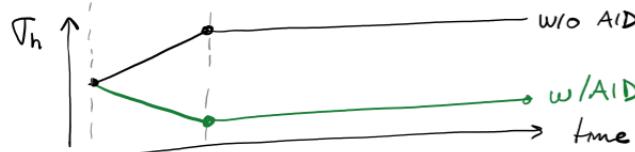
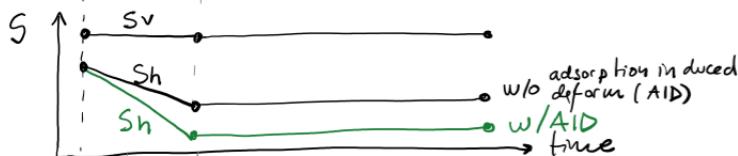
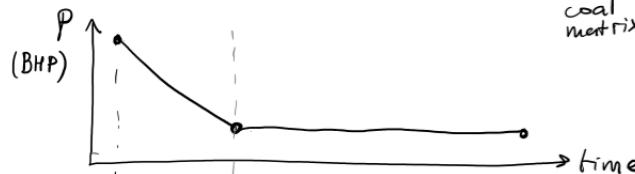
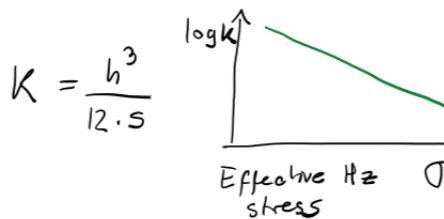
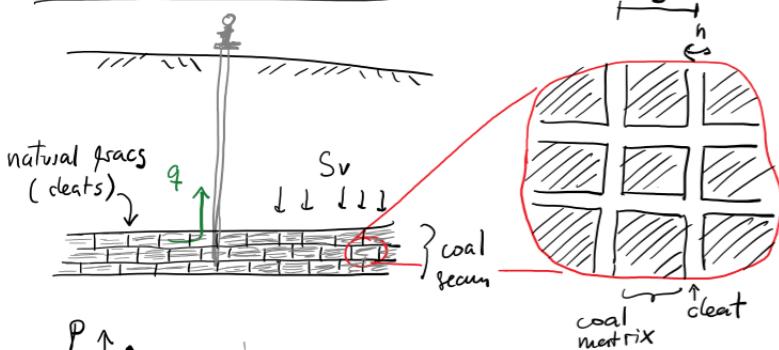


$\uparrow$  adsorption  $\Rightarrow$   $\uparrow$  volume



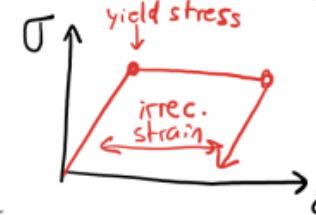
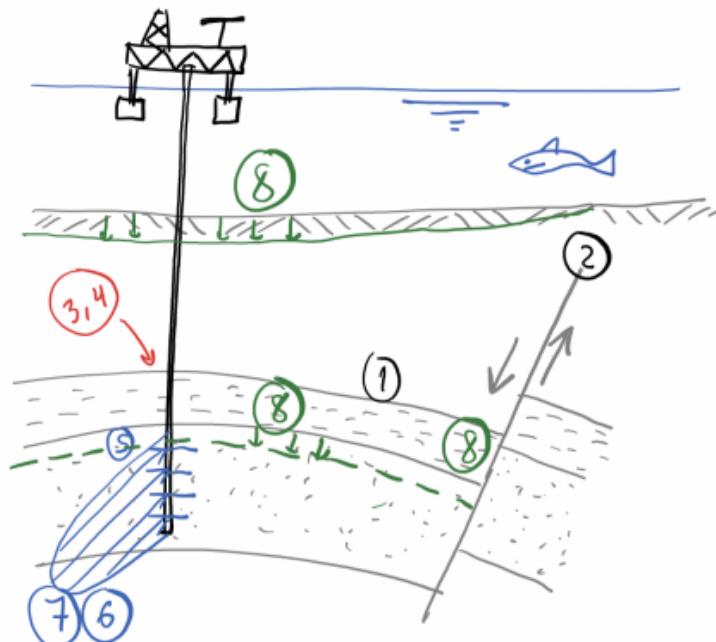
$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_0 \cdot \underline{\underline{\epsilon}} + \alpha P \underline{\underline{\epsilon}} + (1-\alpha) S_a(P) \underline{\underline{\epsilon}} \rightarrow \text{Adsorption stress: } S_a(P)$$

# Coal bed methane (desorption-induced shrinkage and stress relaxation)



# 4-Inelasticity in subsurface engineering applications

Monday, October 26, 2020 5:17 PM

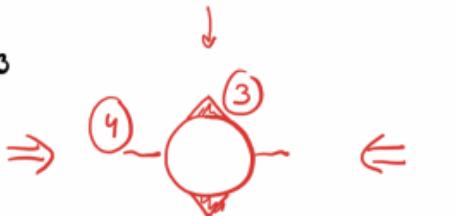


Inelasticity

- limits max stresses
- strength passed yield stress
- permanent strain

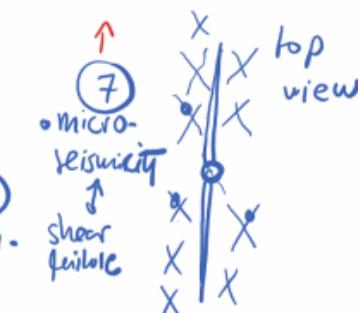
- Exploration
  - folding (1)
  - faulting (2)
$$\sigma_1 = \frac{1 + \sin\phi}{1 - \sin\phi} \sigma_3$$

- Drilling
  - well bore cuttings
  - break outs (3) and Tensile fractures (4)

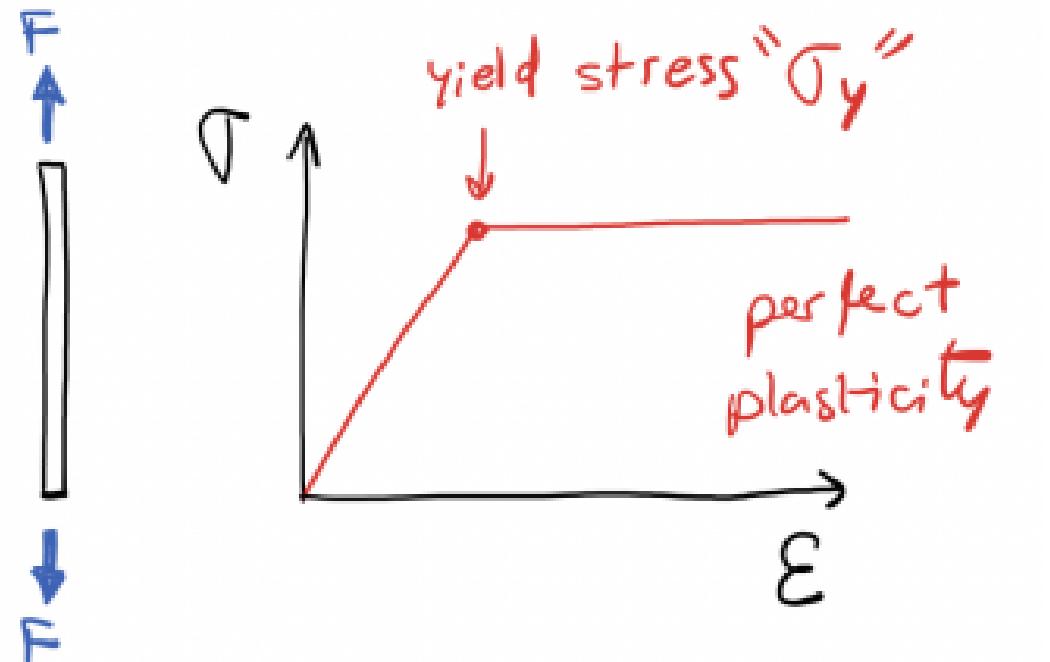


- Production
  - reservoir compaction  
→ fault reactivation
  - injection
  - sand production

- Completion
  - perforation (5)
  - hydraulic fracture (6)
  - fracture reacts → microseism. (7)



## Yield criteria insensitive to mean (effective) stress



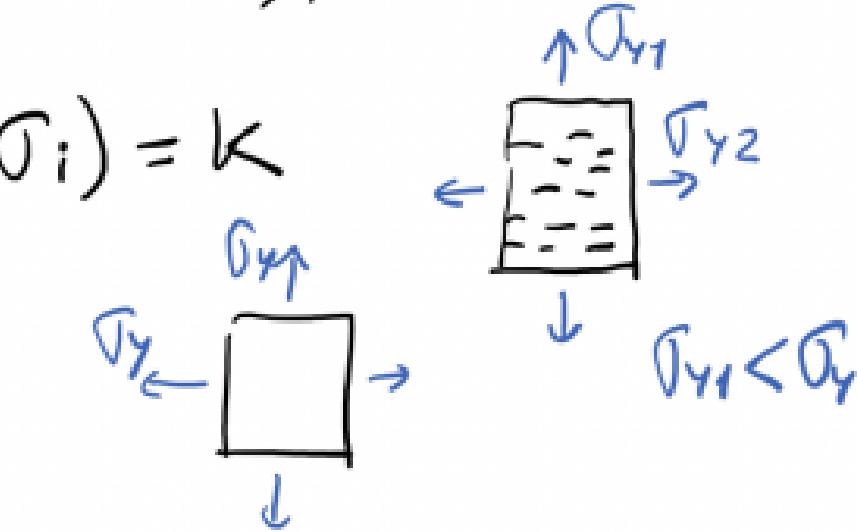
$$f(\underline{\sigma}) = K \rightarrow \text{yield criterion, failure criterion}$$

$$f(\sigma_1, \sigma_2, \sigma_3, \text{direction } \sigma_i) = K$$

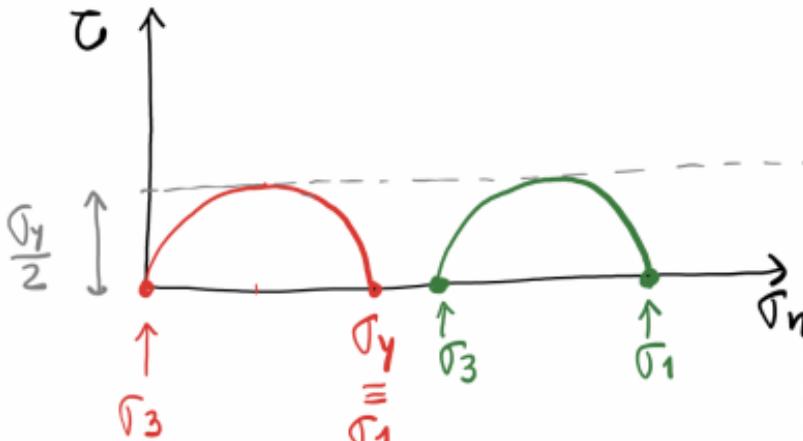
$\downarrow$  isotropy

$$f(\sigma_1, \sigma_2, \sigma_3) = K$$

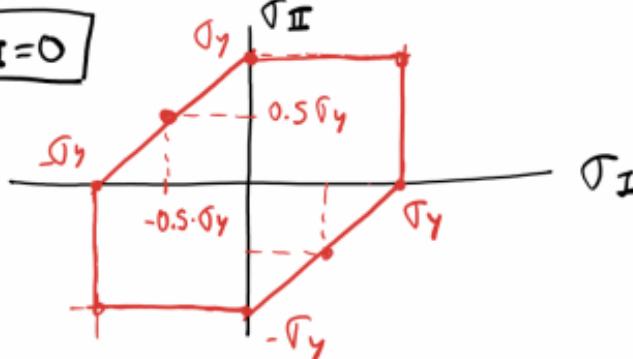
$$f(I_1, I_2, I_3) = K^*$$



## Tresca



$$\sigma_{\text{III}} = 0$$



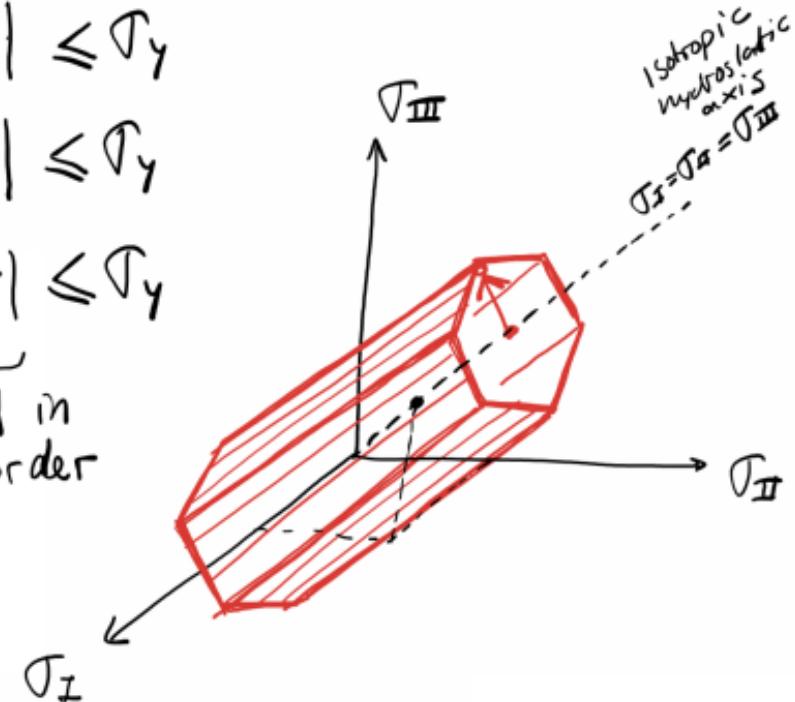
$$\sigma_1 - \sigma_3 \leq \sigma_y$$

$$|\sigma_1 - \sigma_{\text{II}}| \leq \sigma_y$$

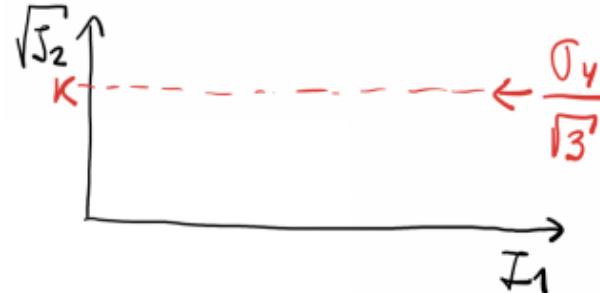
$$|\sigma_1 - \sigma_{\text{III}}| \leq \sigma_y$$

$$|\sigma_{\text{II}} - \sigma_{\text{III}}| \leq \sigma_y$$

not ordered in particular order



## Von Mises



$$J_2 = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

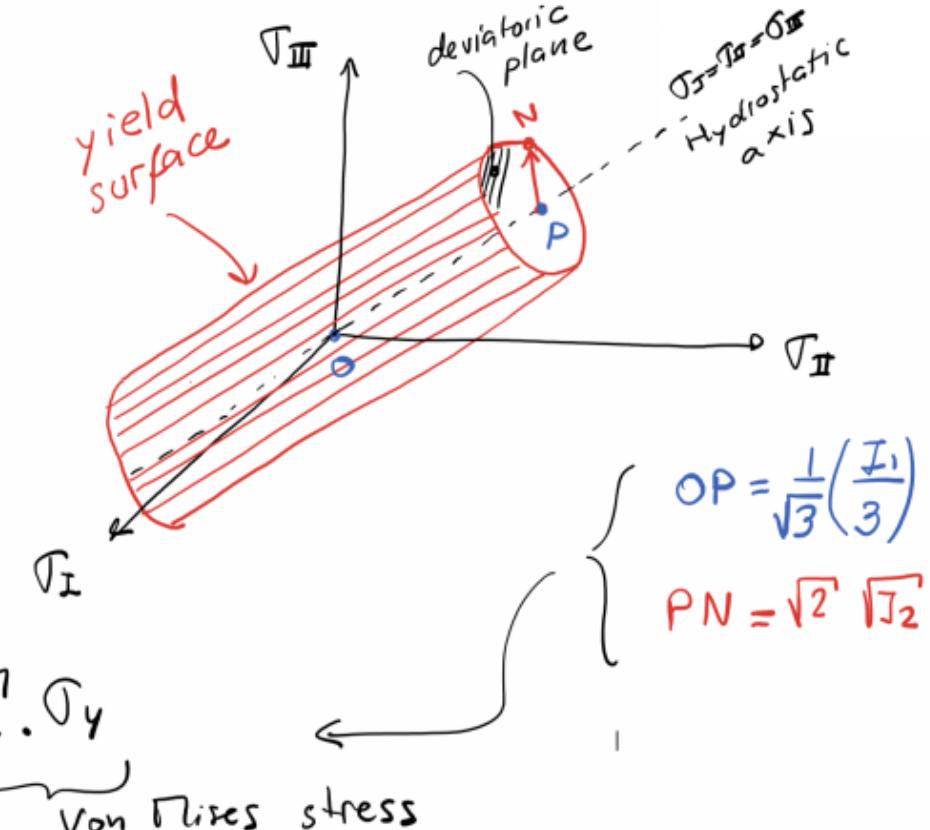
$$\sigma_1 = \sigma_y ; \quad \sigma_2 = \sigma_3 = 0$$

$$J_2 = \frac{1}{6} [2\sigma_y^2]$$

$$\sqrt{J_2} = \frac{\sigma_y}{\sqrt{3}}$$

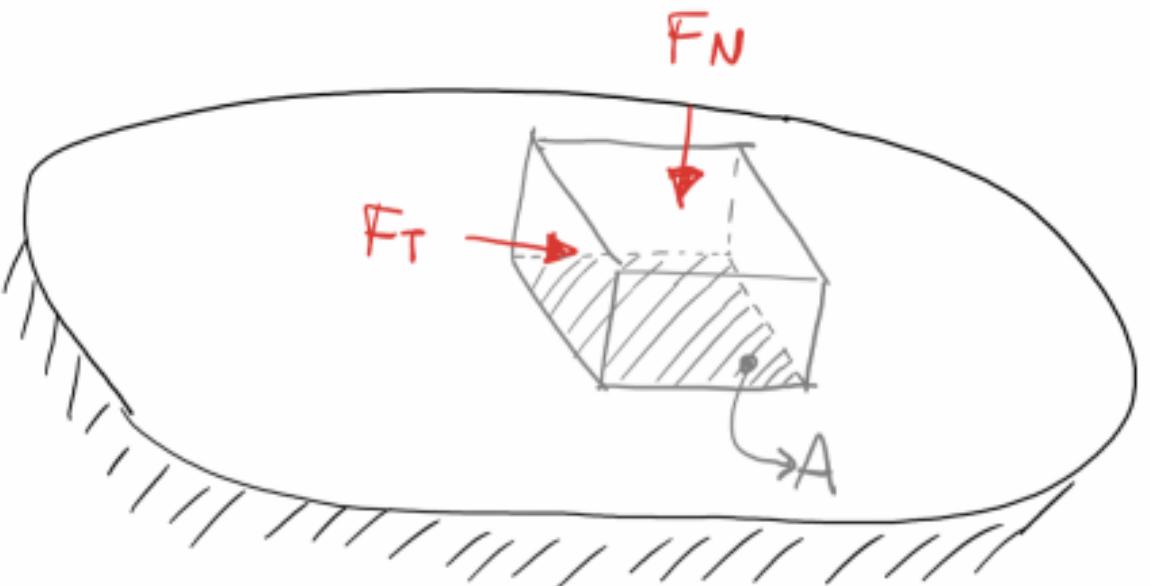
$\rightarrow$  Von Mises stress

$$\sigma_N = \underbrace{\sqrt{\frac{2}{3}} \cdot \sigma_y}_{\text{Von Mises stress}}$$



## Stress sensitive yield criteria

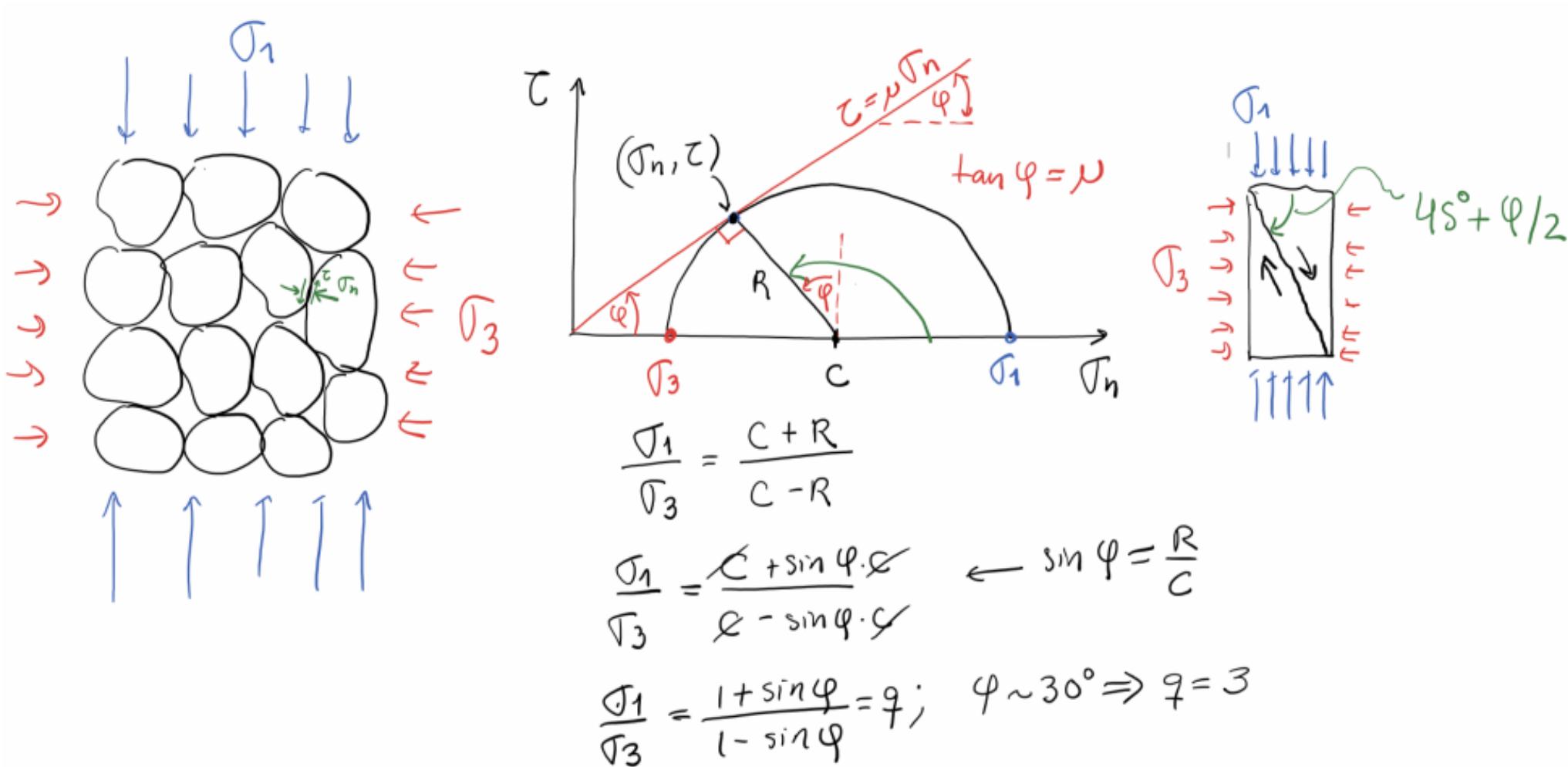
↳ Frictional strength

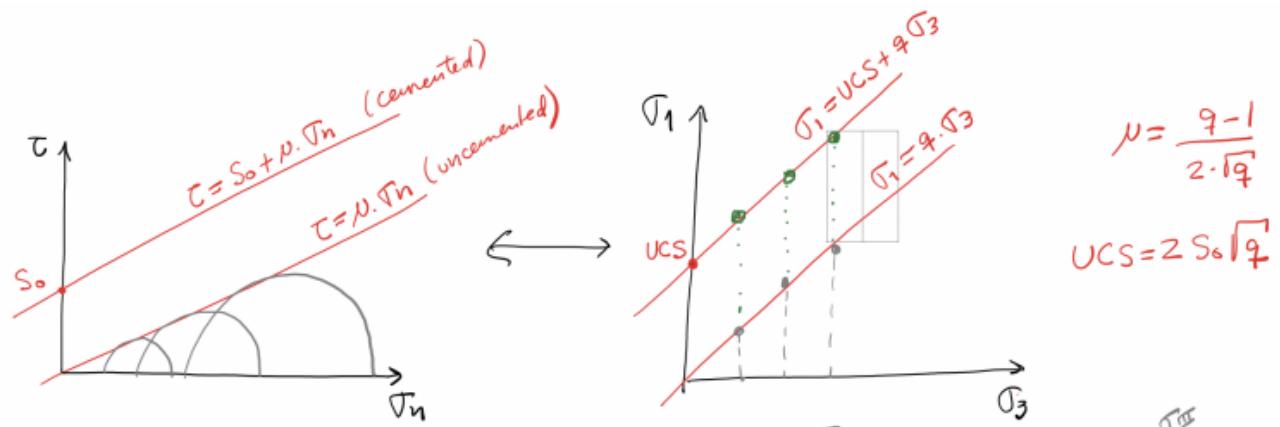


Friction coefficient ( $\sim 0.3$  to  $1.0$ )

$$\tau = \mu \cdot \sigma_n$$

$$\frac{F_T}{A} = \mu \frac{F_N}{A}$$

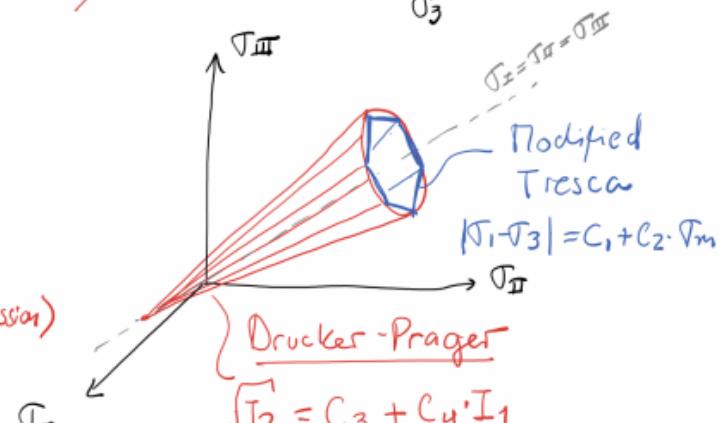
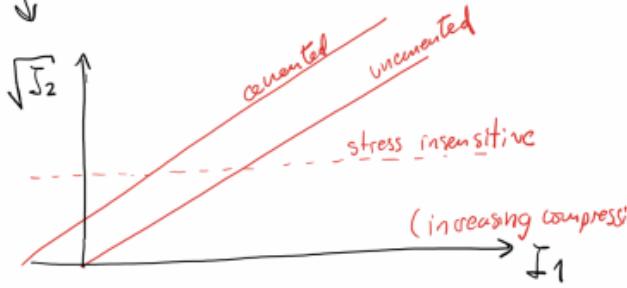




$$\mu = \frac{q-1}{2 \cdot \sqrt{q}}$$

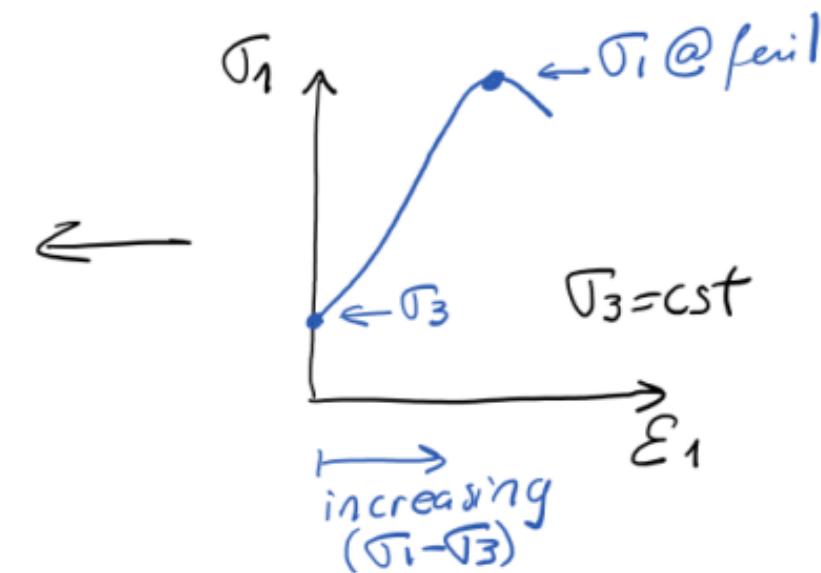
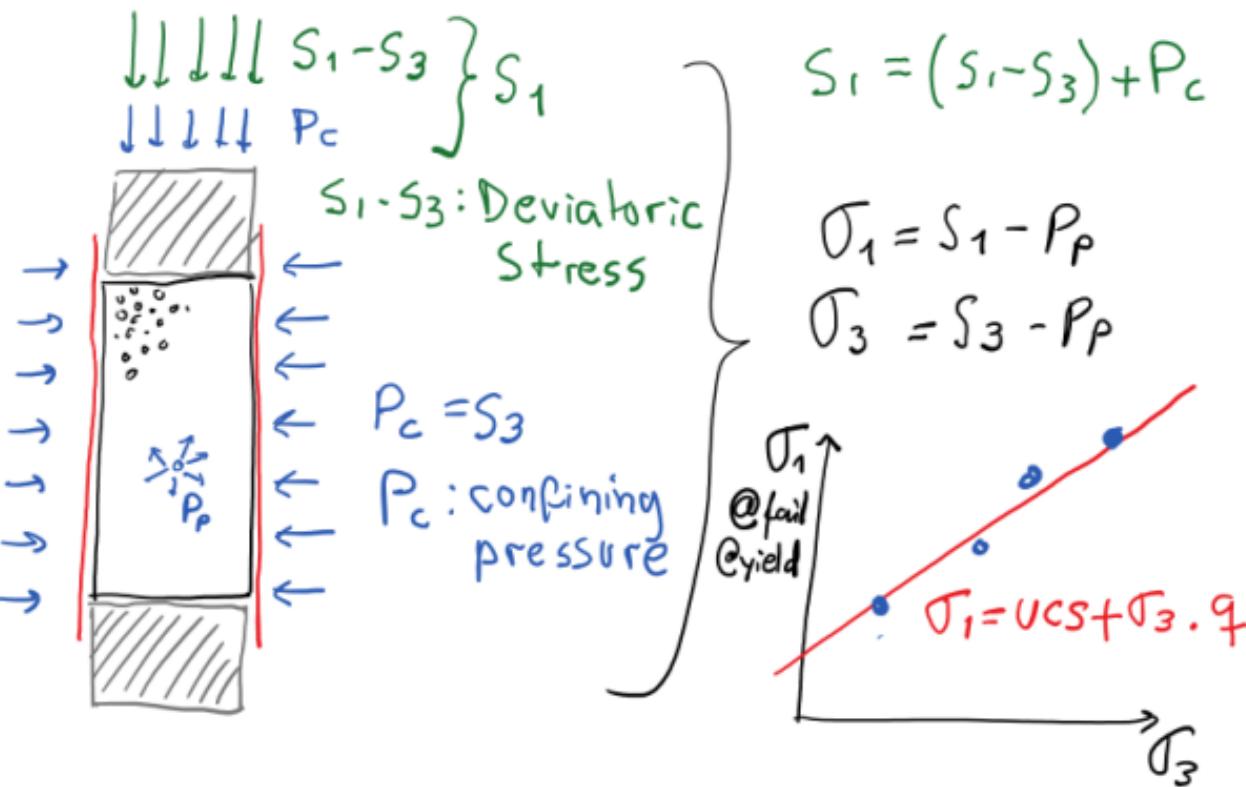
$$UCS = 2 S_0 \sqrt{q}$$

↙ Extension to 3D

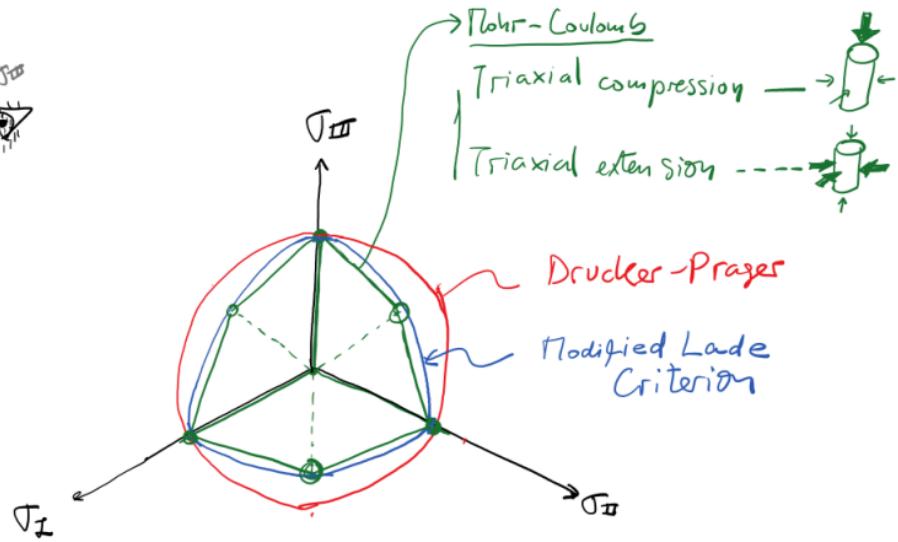
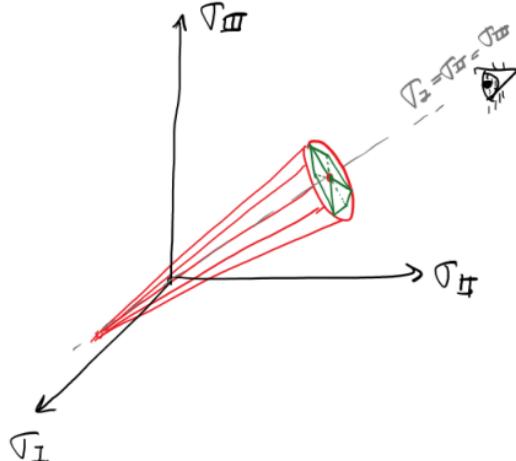


Mohr Coulomb  $\Rightarrow \left\{ \begin{array}{l} C_3 = \frac{6 \cdot S_0 \cdot \cos \varphi}{\sqrt{3} (3 - \sin \varphi)} \\ C_4 = \frac{2}{\sqrt{3}} \frac{\sin \varphi}{(3 - \sin \varphi)} \end{array} \right.$

# Triaxial tests (axisymmetric)



## Modified Lade criterion



$$f(I_1, I_3) = K$$

$$\left[ \frac{(I_1^*)^3}{I_3^*} = 27 + n \right]$$

$$I_1^* = \sigma_1^* + \sigma_2^* + \sigma_3^*$$

$$I_3^* = \sigma_1^* \cdot \sigma_2^* \cdot \sigma_3^*$$

$$\sigma_i^* = \sigma_i + S$$

$$S = S_0 / \tan \varphi$$

$$n = \frac{4(\tan \varphi)^2(9 - 7 \sin \varphi)}{1 - \sin \varphi}$$

# Beyond the yield point: determination of plastic strains

→ determination of strains

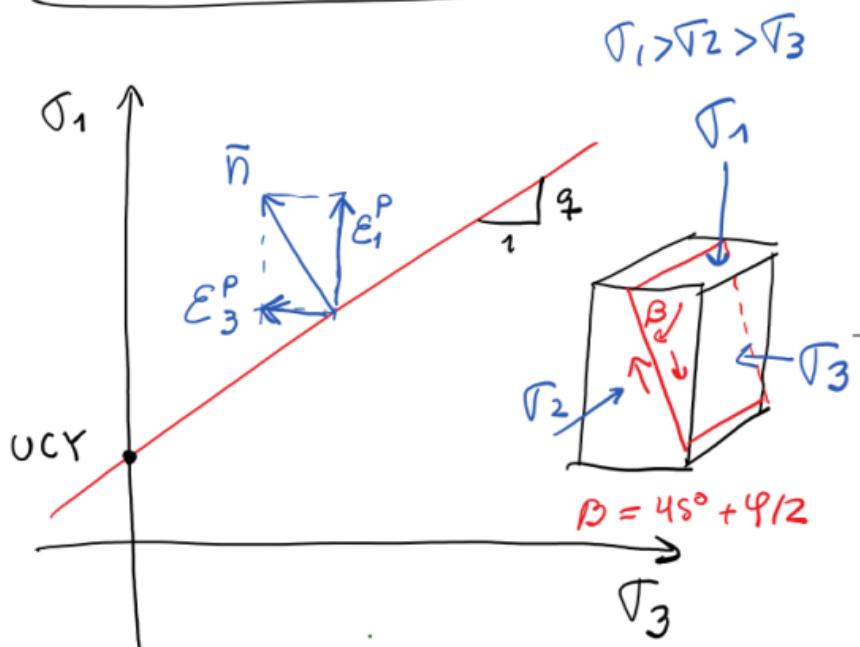
elastic       $\begin{cases} \underline{\underline{\varepsilon}} = \underline{\underline{\Delta}} \cdot \underline{\underline{\sigma}} \\ \varepsilon_{ij} = D_{ijmn} \sigma_{mn} \\ d\varepsilon_{ij} = D_{ijmn} d\sigma_{mn} \end{cases}$

→ plastic -  $d\varepsilon_{ij}^p \leftrightarrow d\sigma_{ij}$

- Small strain
- continuous strain field
- strain-rate independent

- ① Yield criterion  
 $f(\sigma_{ij}) = Y$
- ② Strain-hardening rule  
 $Y = F^*(\delta\varepsilon^p)$
- ③ Strain decomposition  
 $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$
- ④ Plastic flow rule  
 $d\varepsilon_{ij}^p \leftrightarrow d\sigma_{ij}$
- ⑤ Elastic unloading criterion

## Example: Mohr - Coulumb



$$\sigma_1 > \sigma_2 > \sigma_3$$

$$C = S_0 + n \cdot \sigma_n \quad ; \quad n = \tan \varphi$$

$$\boxed{\sigma_1 = UCY + q \sqrt{3}} \quad ; \quad q = \frac{1 + \sin \varphi}{1 - \sin \varphi}$$

$$\varphi = 30^\circ \Rightarrow q =$$

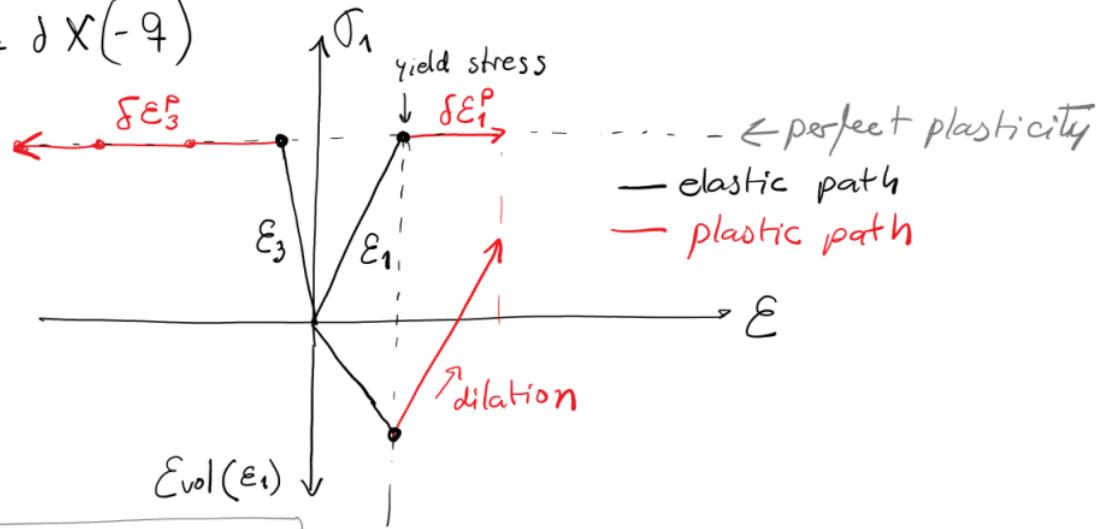
$$f = \sigma_1 - UCY - q \sqrt{3}$$

$$\bar{n} = \left( \frac{\partial F}{\partial \sigma_1}, \frac{\partial F}{\partial \sigma_2}, \frac{\partial F}{\partial \sigma_3} \right)$$

$$\bar{n} = (1, 0, -q)$$

$$\text{Flow rule: } \delta \epsilon_{ij}^P = \underbrace{\delta \lambda}_{\text{cst}} \cdot \frac{\partial F}{\partial \sigma_{ij}}$$

$$\left\{ \begin{array}{l} \delta \varepsilon_1^P = \delta X \cdot 1 \\ \delta \varepsilon_2^P = \delta X \cdot 0 \\ \delta \varepsilon_3^P = \delta X (-q) \end{array} \right. \Rightarrow \delta \varepsilon_{\text{vol}}^P = \delta X (1 - q)$$



$$\boxed{\delta \varepsilon_{ij}^P = \delta X \frac{\partial F}{\partial \sigma_{ij}}} \Rightarrow \text{Associated Flow Rule}$$

↑ yield surface

$$\boxed{\delta \varepsilon_{ij}^P = \delta X \frac{\partial g}{\partial \sigma_{ij}}} \Rightarrow \text{Non-associated flow rule}$$

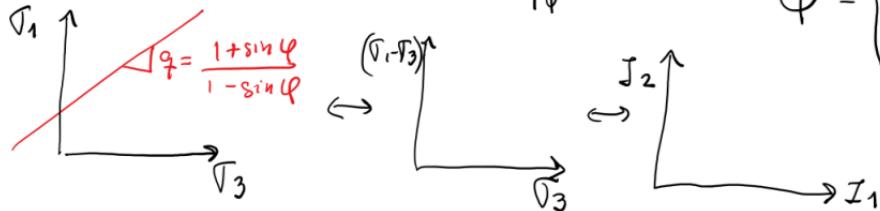
Plastic potential function,  $g \neq F$

$$g = \sigma_1 - \sigma_3 \cos \psi - \frac{1 + \sin \psi}{1 - \sin \psi} \sigma_3$$

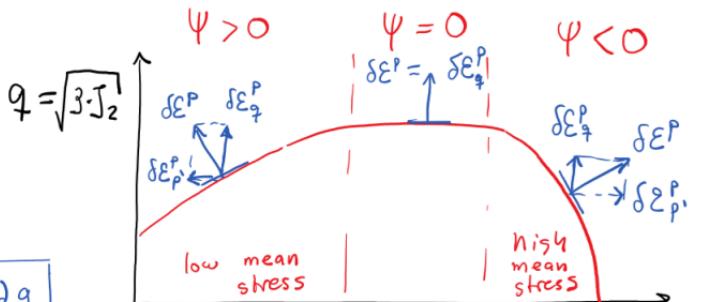
dilation angle

$$\psi < \varphi$$

$$\psi = \begin{cases} \psi > 0 \Rightarrow \text{dilation} \\ \psi = 0 \Rightarrow \text{iso-chronic} \\ \psi < 0 \Rightarrow \text{contraction} \end{cases}$$



$P' - q$   
space



$$\delta \varepsilon_{ij}^P = \delta \lambda \frac{\partial g}{\partial \sigma_{ij}}$$



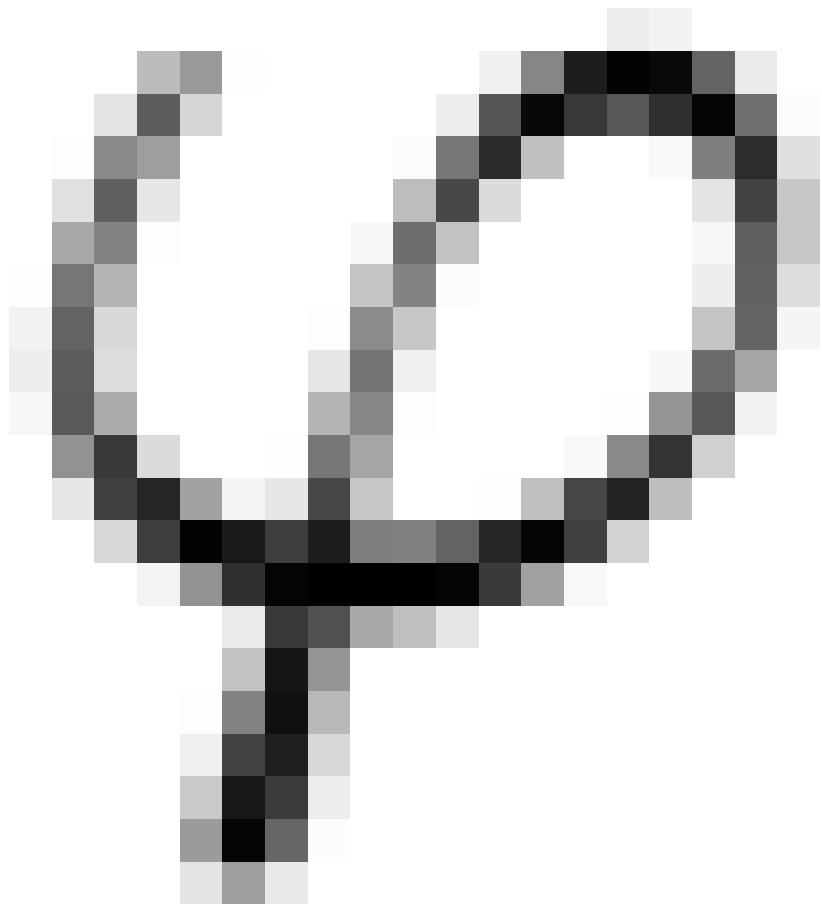


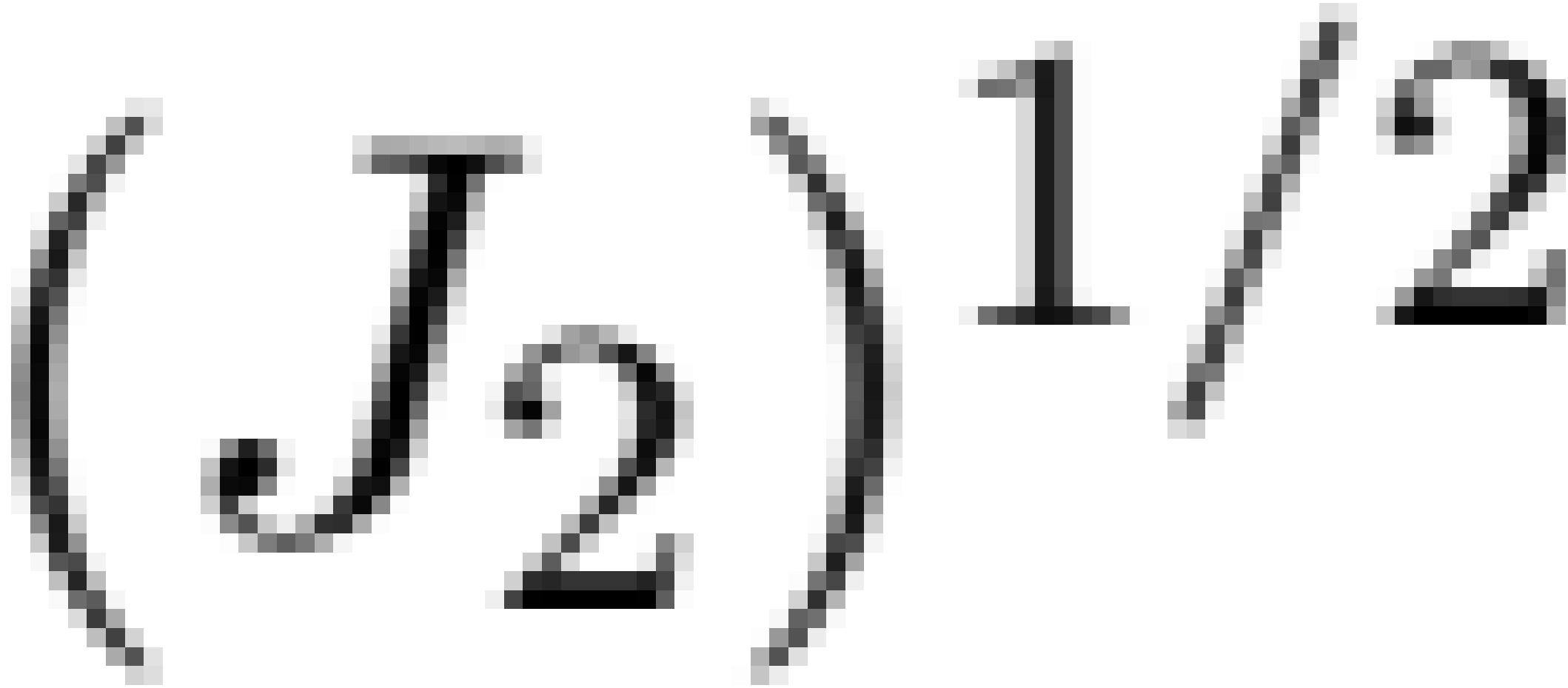




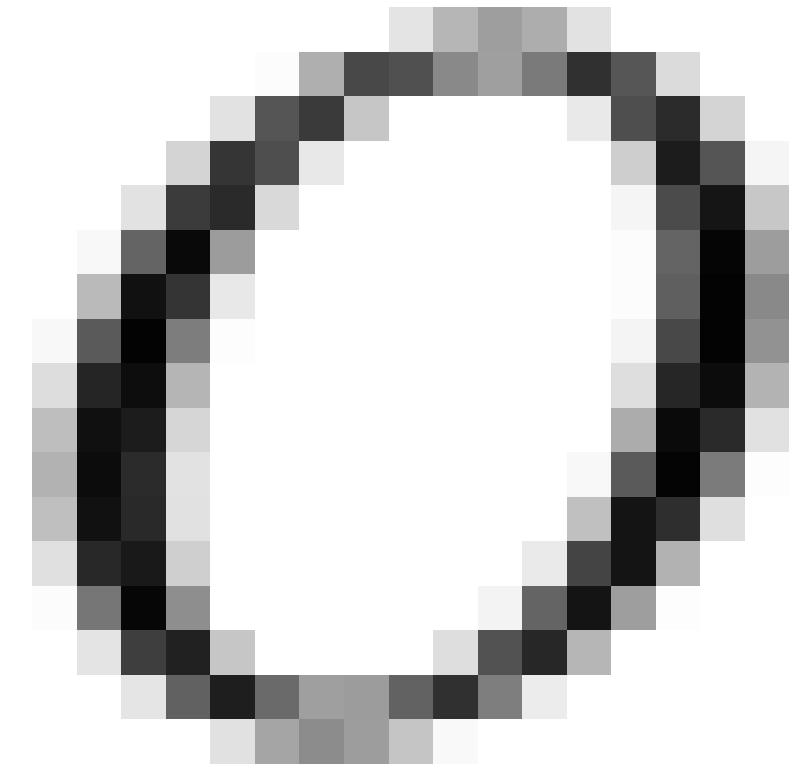
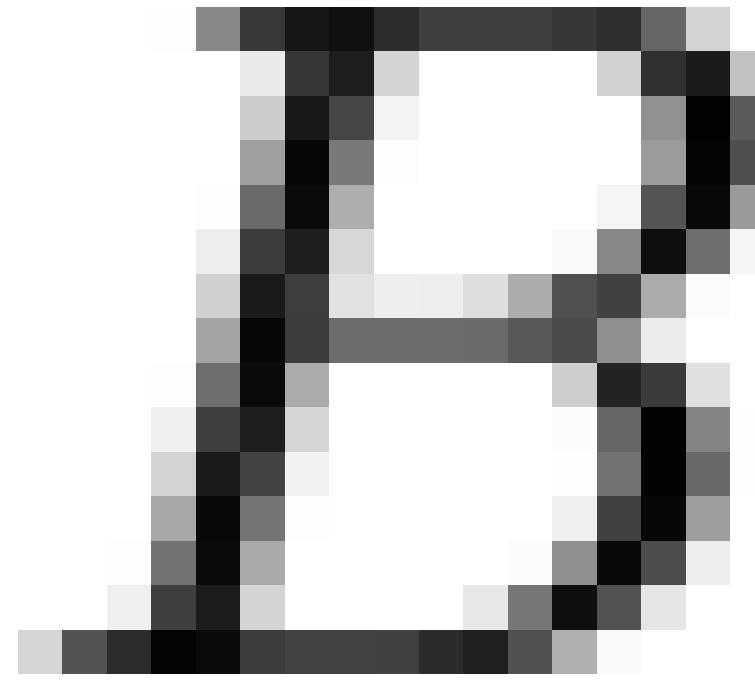
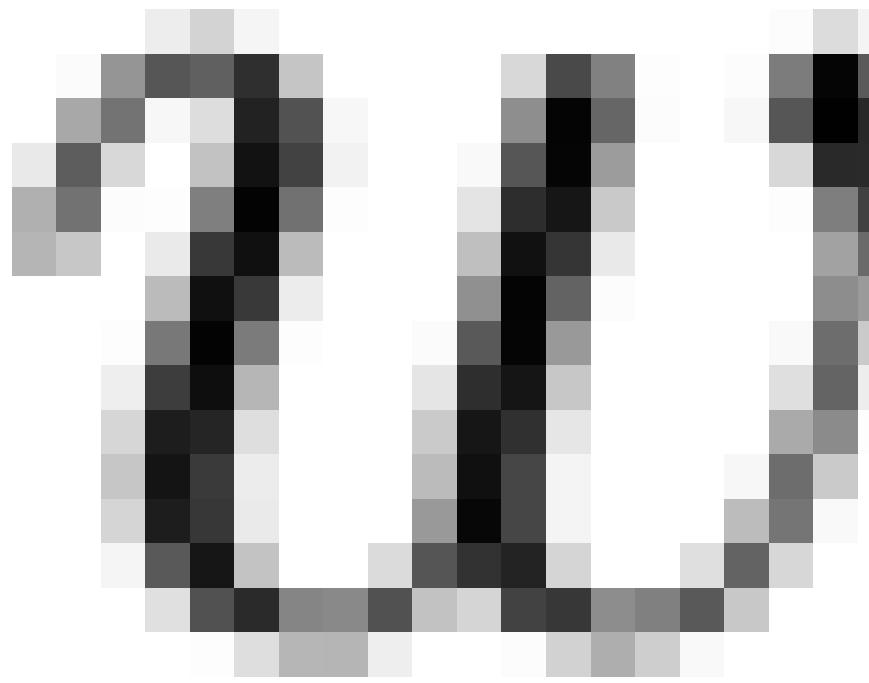


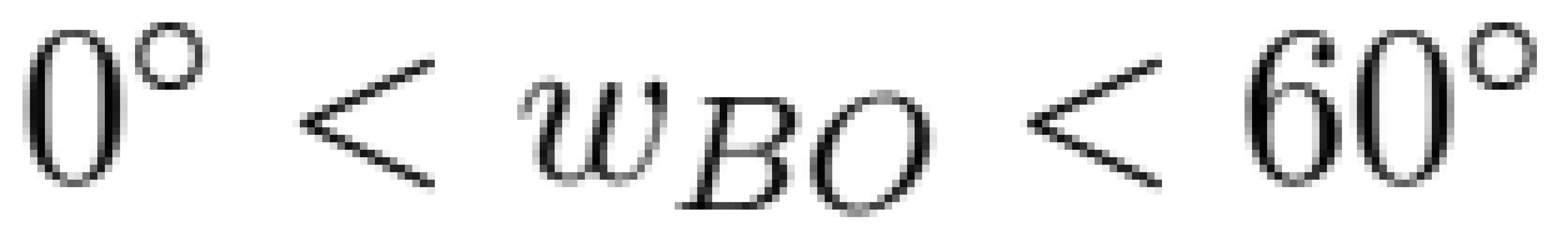


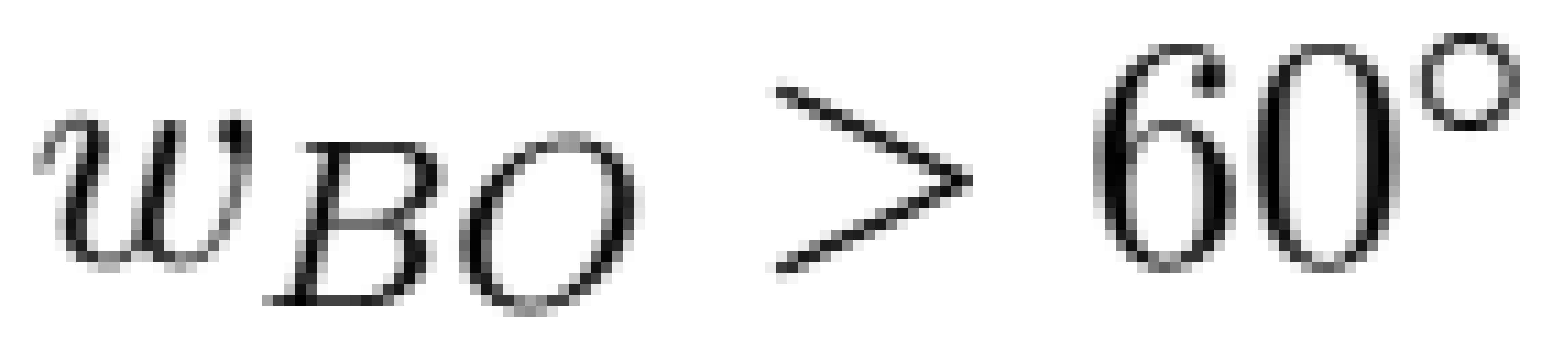


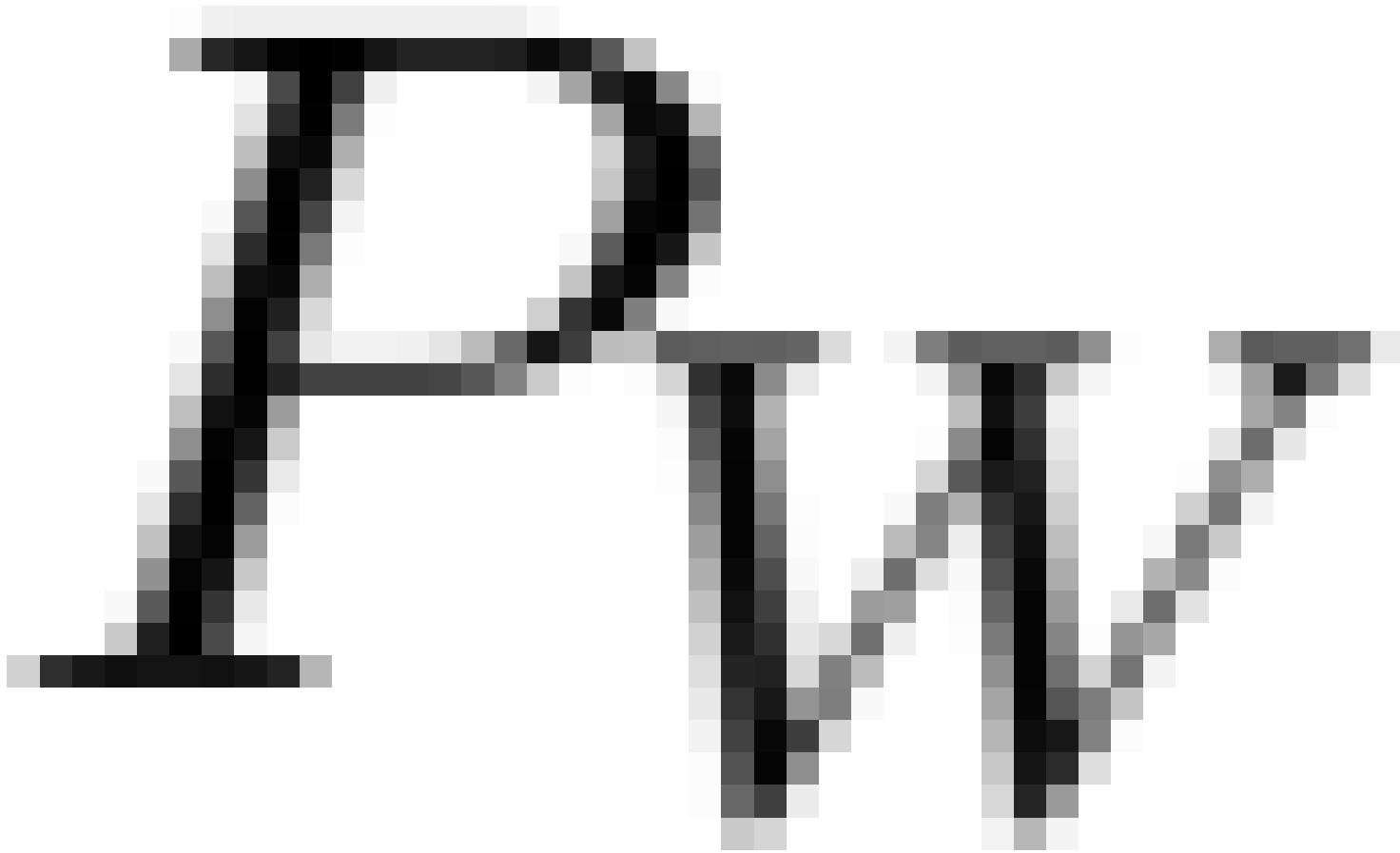


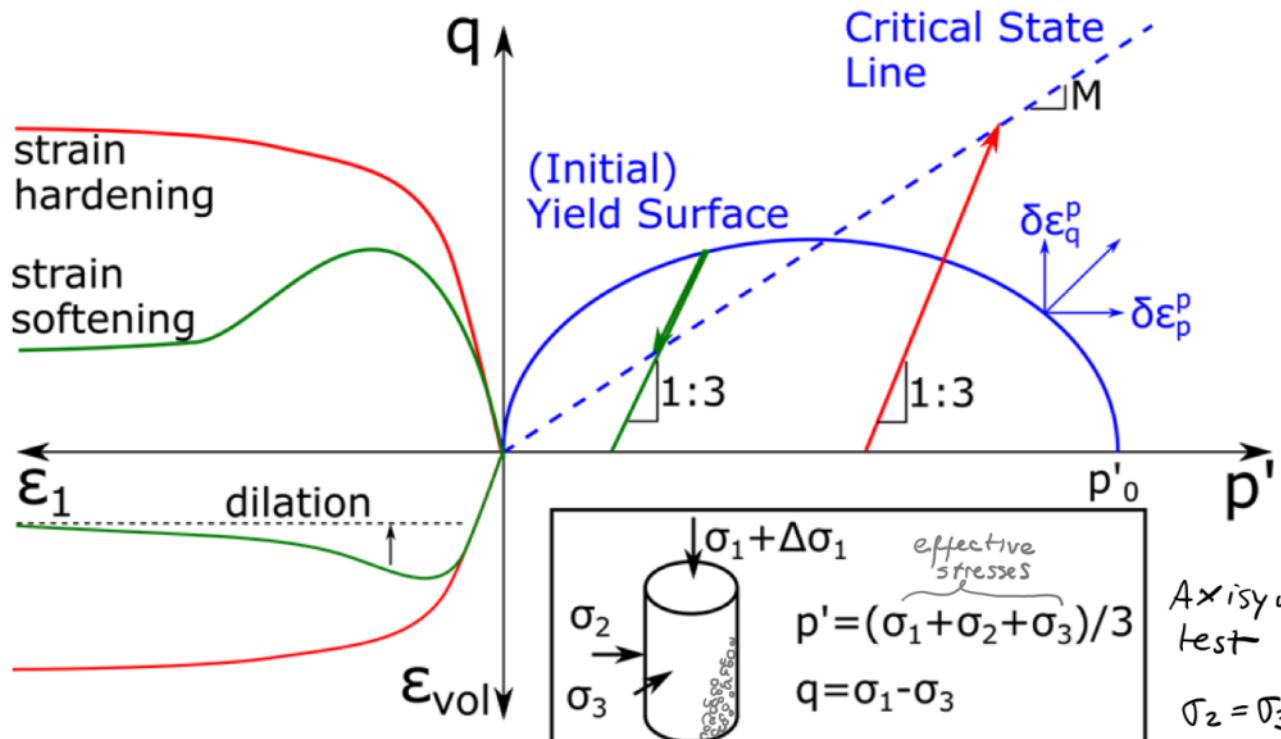












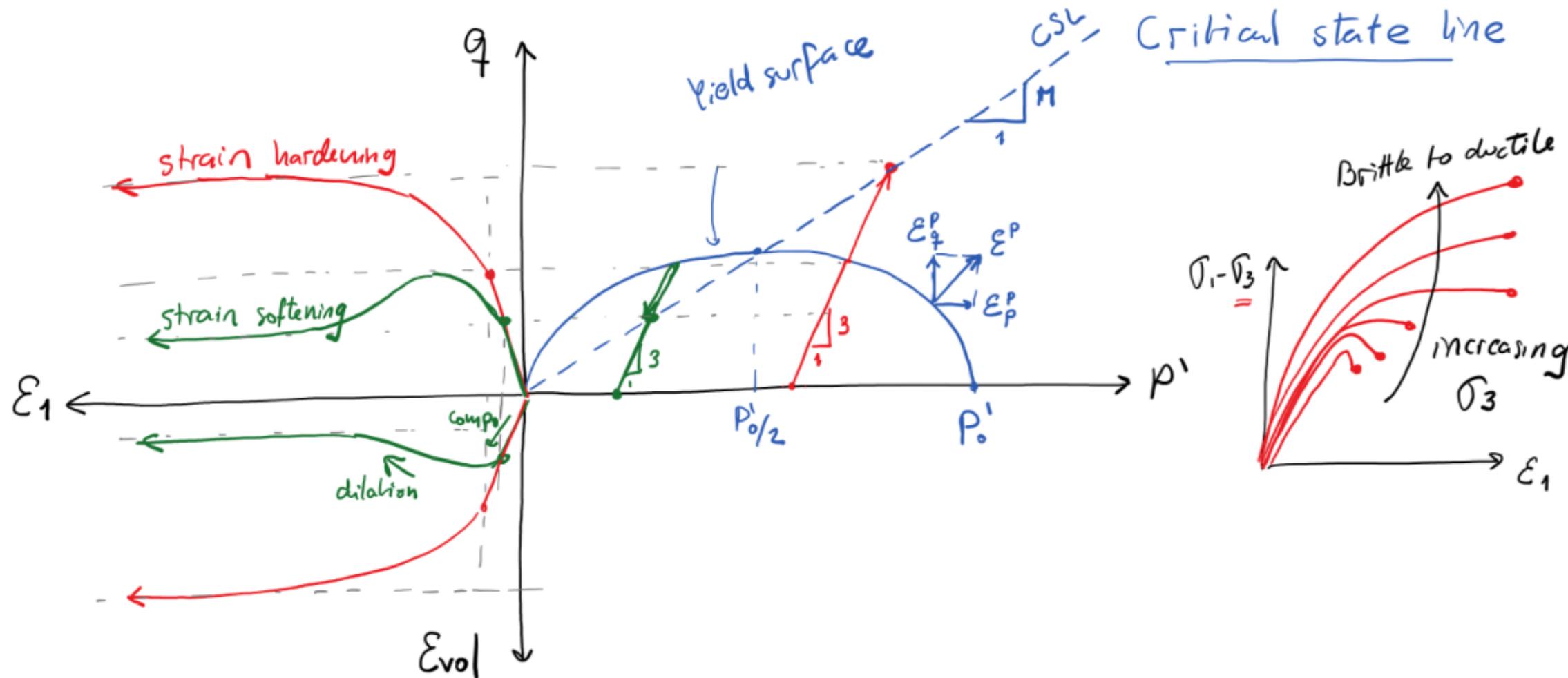
Deviatoric loading

$$\left\{ \begin{array}{l} \Delta p' = \frac{\Delta \sigma_1}{3} \\ \Delta q = \Delta \sigma_1 \end{array} \right\} \frac{\Delta q}{\Delta p'} = 3$$

$\Delta \sigma_1 \rightarrow$  until yield

Axisymmetric triaxial test (drained)

$\sigma_2 = \sigma_3 = cst$  for a given test



$$q \stackrel{\text{def}}{=} \Pi \cdot p^1 (@ \text{ CSL})$$

$$\Pi = \frac{q}{p^1} = \frac{\sigma_1 - \sigma_3}{\frac{\sigma_1 + 2\sigma_3}{3}}$$

$$\Rightarrow \Pi (\varphi_{cs} = 30^\circ) = \frac{\cancel{2\sigma_3}}{\frac{\cancel{5\sigma_3}}{3}} = \frac{6}{5} = 1.2$$

$$\sigma_1 = 3\sigma_3$$

$$\boxed{\Pi = \frac{3 \left( \frac{1 + \sin \varphi_{cs}}{1 - \sin \varphi_{cs}} - 1 \right)}{2 + \left( \frac{1 + \sin \varphi_{cs}}{1 - \sin \varphi_{cs}} \right)}}$$

Yield surface

$$f(\underbrace{q, p^*, p_0^*}_{\sim}) = q^2 - M^2 p^*(p_0^* - p^*) = 0$$

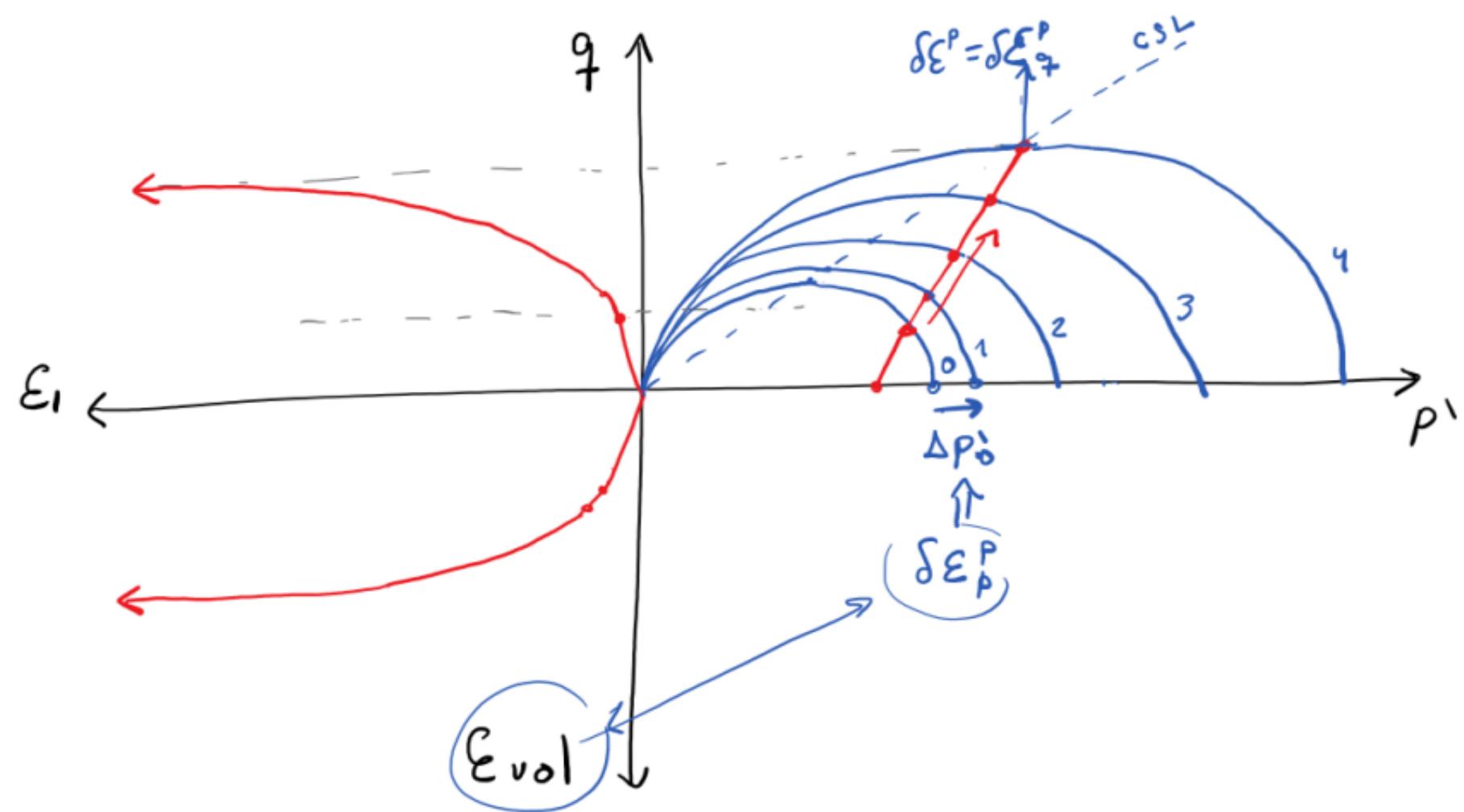
$$\begin{cases} q = 0 \Rightarrow \\ p^* = 0 \\ p^* = p_0^* \end{cases}$$
$$\rightarrow q = Mp^* \Rightarrow p^* = \frac{p_0^*}{2}$$

→ pre-consolidation pressure (stress)

↳ variable

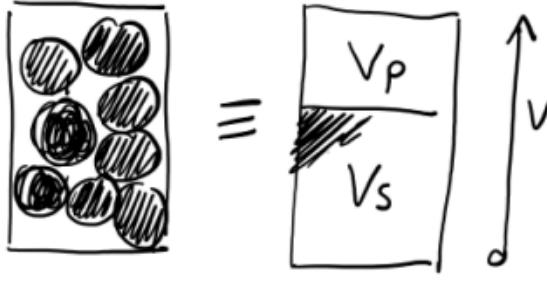
↳ size of the yield surface

↳ hardening parameter  $\leftrightarrow f(\underline{\delta\varepsilon_p^P}) \sim f(\emptyset)$



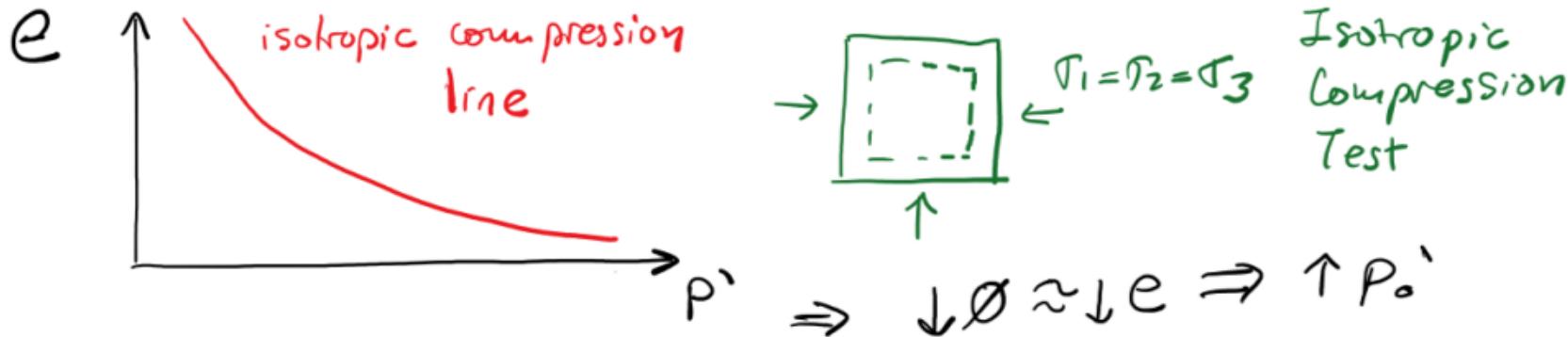
$$P_0' = f(\delta \varepsilon_p^p) = f^*(e = \frac{\phi}{1-\phi})$$

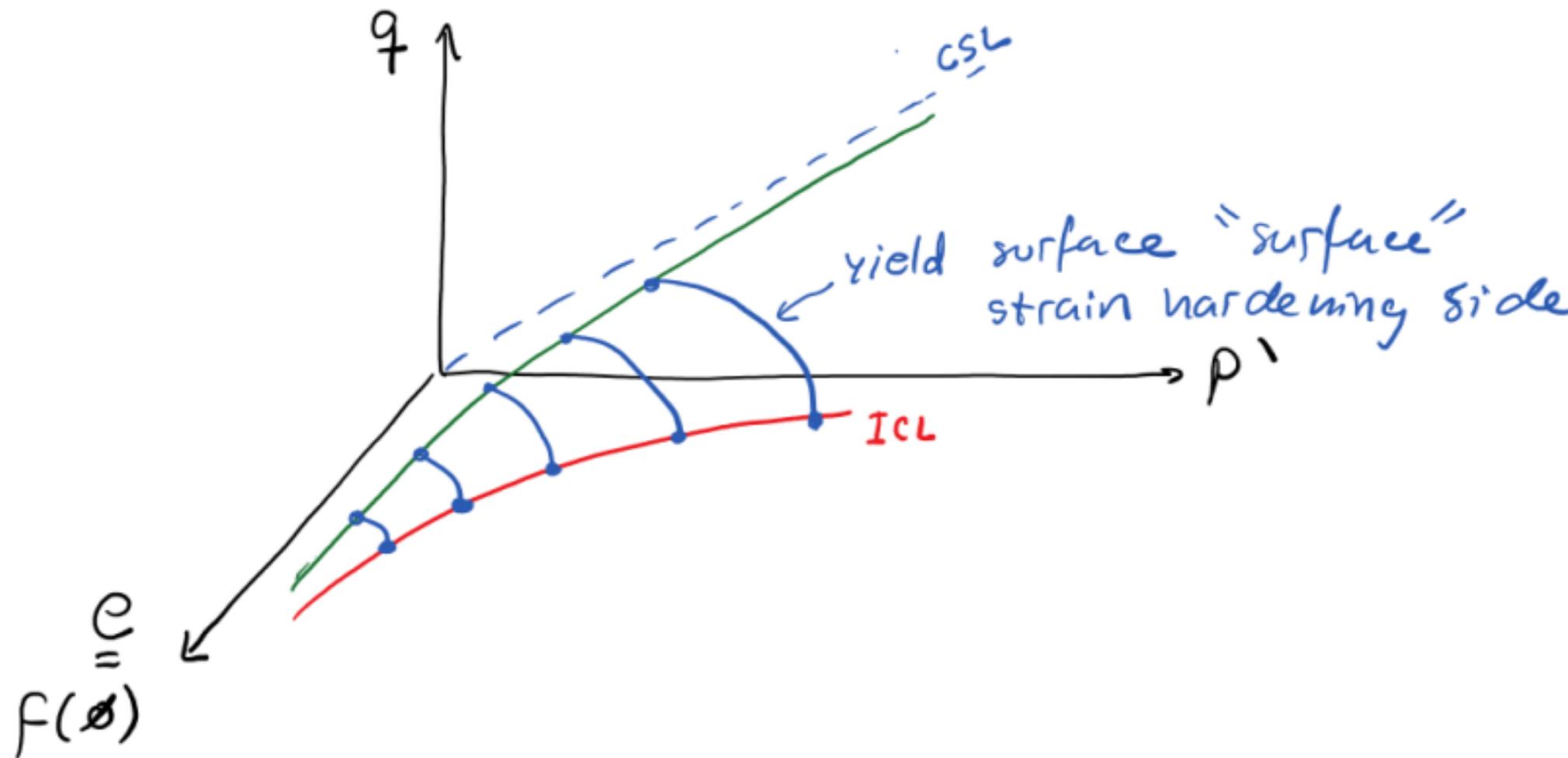
void ratio



$$\left. \begin{array}{l} \phi \stackrel{\text{def}}{=} \frac{V_p}{V_T} \\ e \stackrel{\text{def}}{=} \frac{V_p}{1-V_p} \end{array} \right\} \Rightarrow e = \frac{V_p}{1-V_p} = \frac{\phi}{1-\phi}$$

$$V_T = 1$$





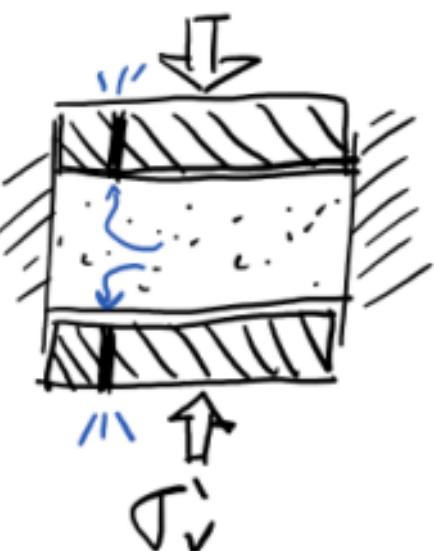
## Calculation of elastic and plastic strains

$$\underline{\underline{\epsilon}} = \underline{\underline{\epsilon}}^e + \underline{\underline{\epsilon}}^p \leftarrow \text{strain de composition}$$

$$d\underline{\underline{\epsilon}} = d\underline{\underline{\epsilon}}^e + d\underline{\underline{\epsilon}}^p \quad (\text{incremental version})$$

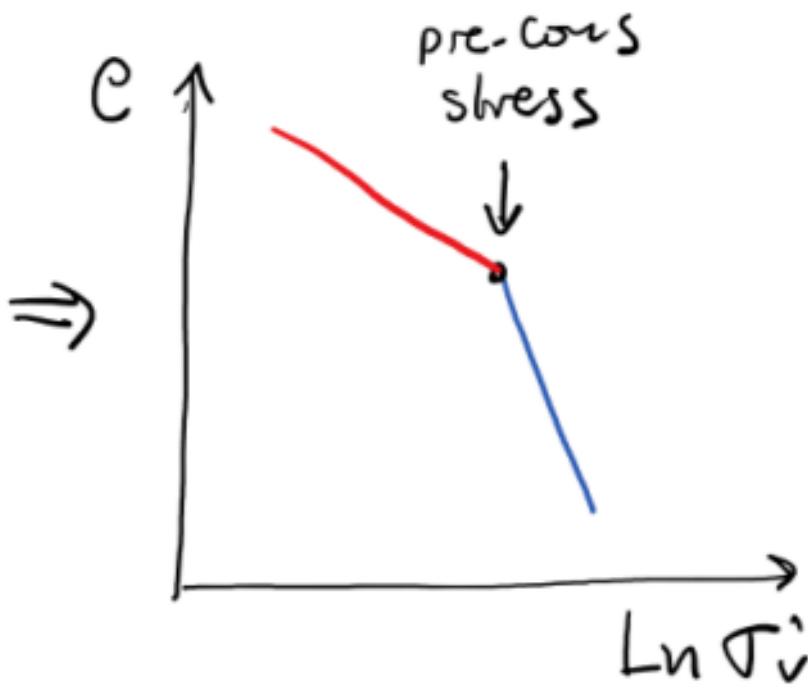
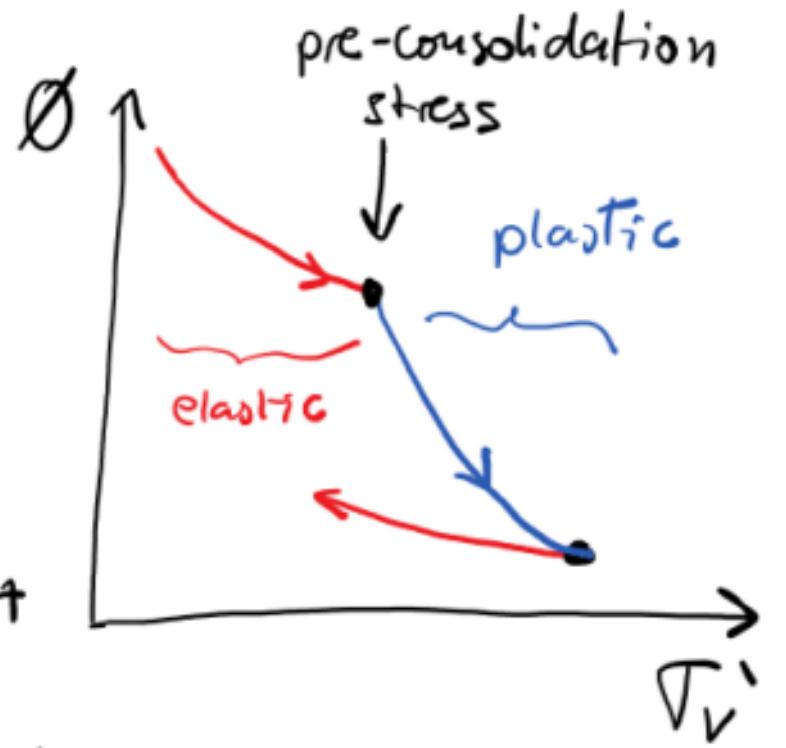
$$d\underline{\underline{\epsilon}} = \underbrace{\underline{\underline{C}}^e d\underline{\underline{\sigma}}}_{\text{elastic}} + \underbrace{\underline{\underline{C}}^p d\underline{\underline{\sigma}}}_{\text{plastic}}$$

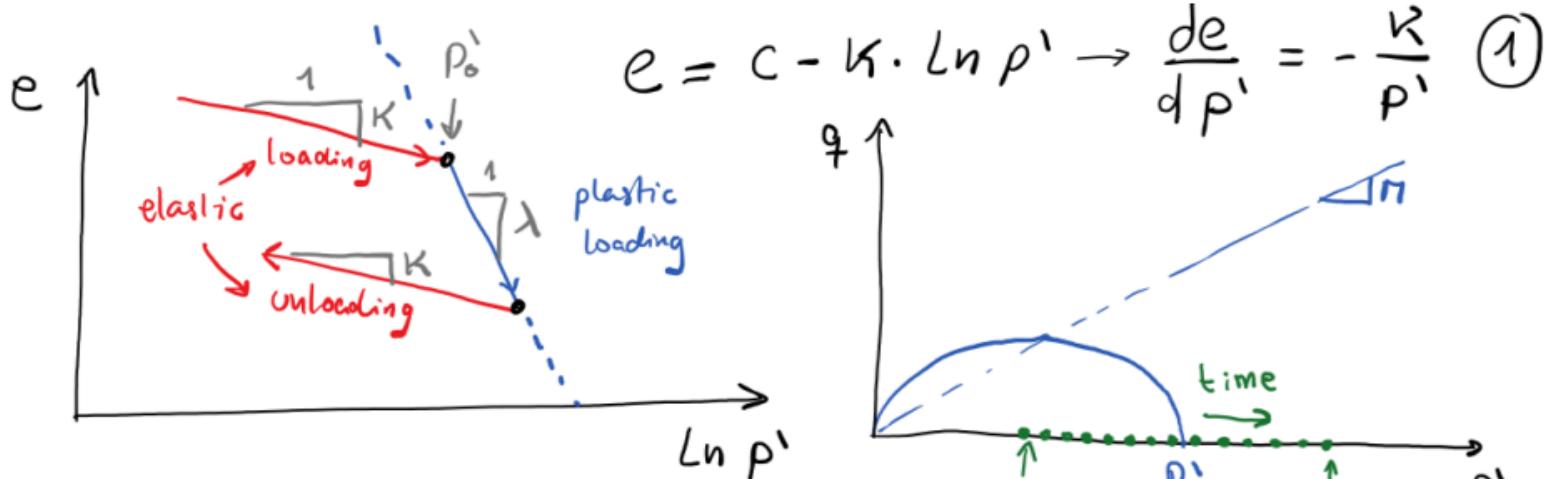
# ① Elastic strains



Consolidation Test  
(Uniaxial strain)

Oedometric condition



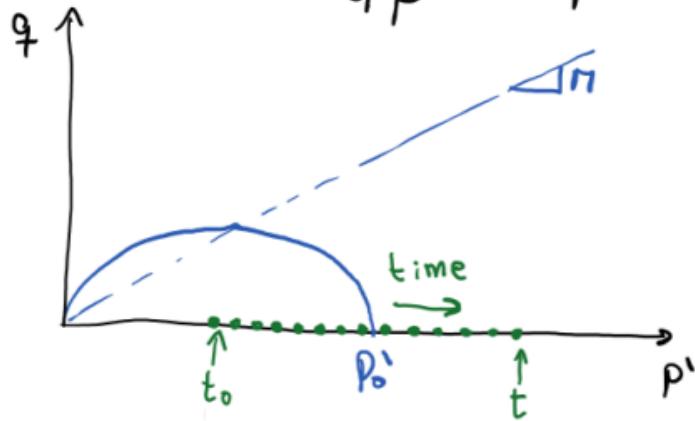


$$e \stackrel{\text{def}}{=} \frac{V_p}{V_b - V_p} \rightarrow de = \dots dV_p + \dots dV_b$$

↓ assumption  $dV_b = dV_p \quad (d \approx 1)$

$$\delta \varepsilon_p = \frac{dV_b}{V_b} = \frac{dV_p}{V_b}$$

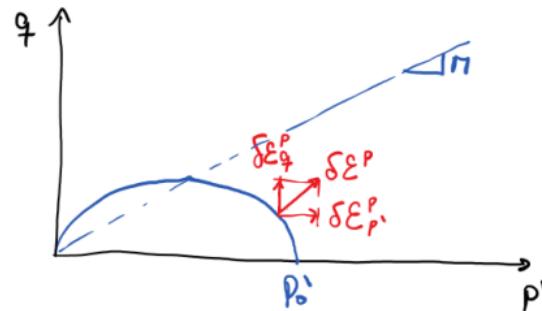
$$de = -(1+e) \delta \varepsilon_p \quad (2)$$



$$\textcircled{1} + \textcircled{2} \quad \left\{ \begin{array}{l} \delta \varepsilon_{p'}^e = \frac{\kappa}{1+e} \cdot \frac{dp'}{p'} \\ \delta \varepsilon_q^e = \frac{1}{3G} \cdot dq ; \quad \varepsilon_q \stackrel{\text{def}}{=} \frac{2}{3}(\varepsilon_1 - \varepsilon_3) \end{array} \right.$$

$$\begin{bmatrix} \delta \varepsilon_{p'}^e \\ \delta \varepsilon_q^e \end{bmatrix} = \begin{bmatrix} \frac{\kappa}{1+e} \cdot \frac{1}{p'} & 0 \\ 0 & \frac{1}{3G} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix} \quad \begin{array}{l} \text{Elastic strains} \\ (\text{before yield}) \end{array}$$

## ② Plastic strains



$$f = q^2 - M^2 p' (P_0' - p')$$

$$f^* = \frac{p'}{P_0'} - \frac{M^2}{\eta^2 + M^2} ; \quad \eta = \frac{q}{p'}$$

derivatives

$$\left\{ \begin{array}{l} \frac{\partial f^*}{\partial p'} = P_0' M^2 \left( \frac{\eta^2 - M^2}{M^2 + \eta^2} \right) \\ \frac{\partial f^*}{\partial q} = P_0' M^2 \left( \frac{2\eta}{M^2 + \eta^2} \right) \\ \frac{\partial f^*}{\partial p'_i} = - \frac{p'}{(P_0')^2} \end{array} \right.$$

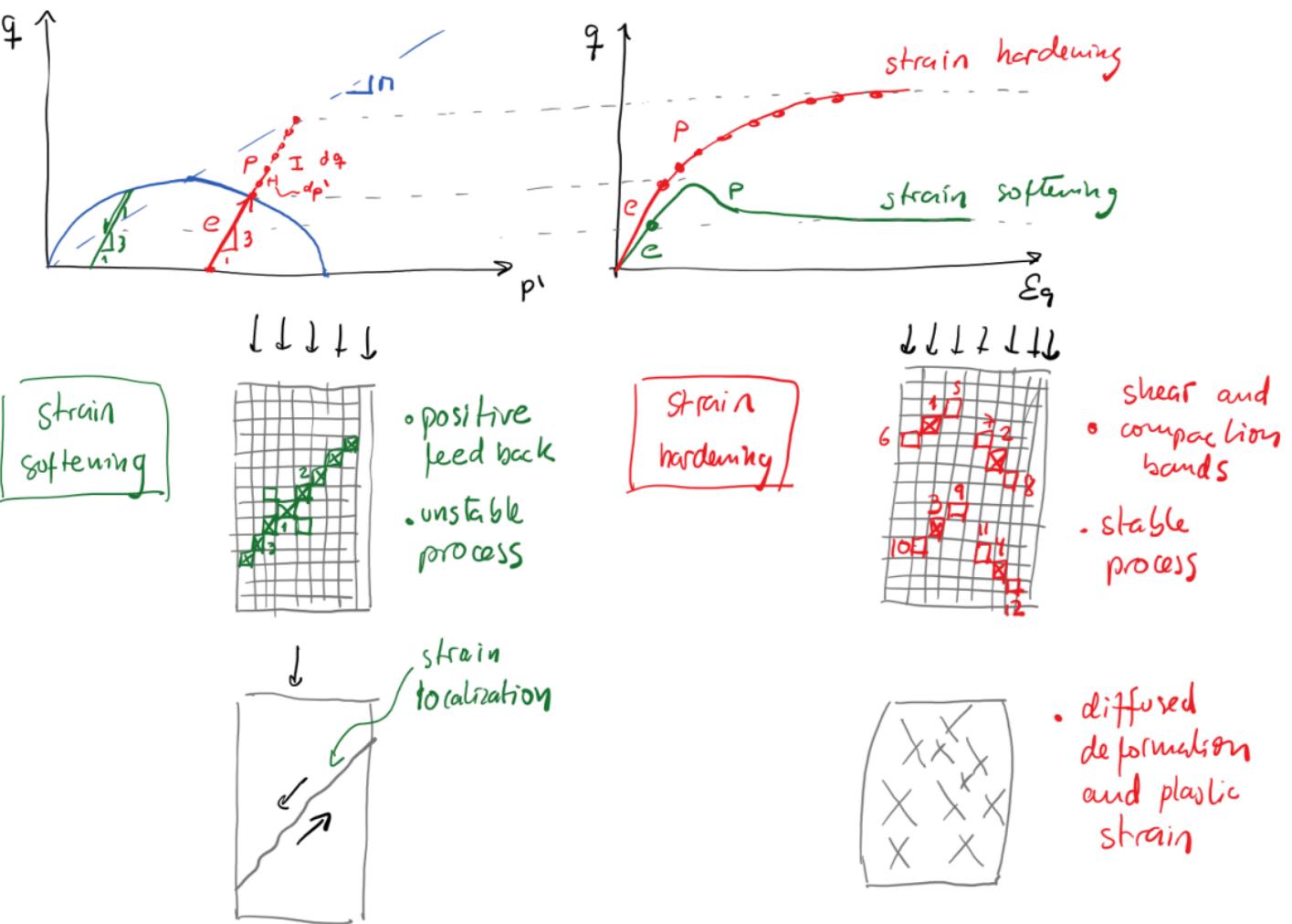
Associated flow rule

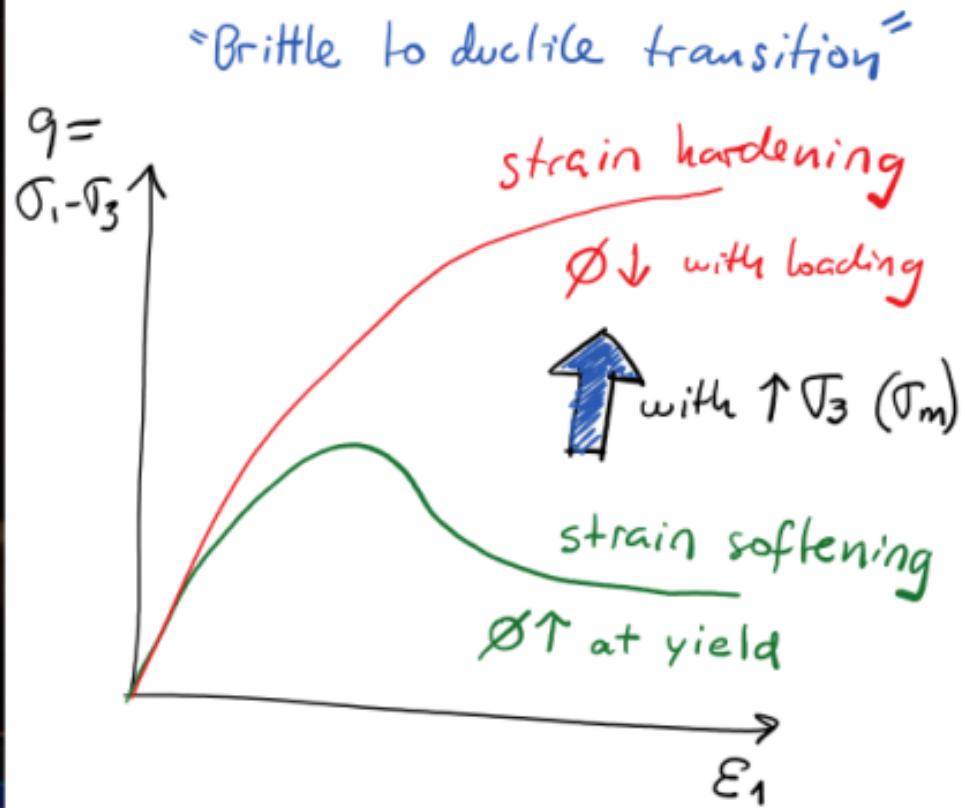
$$\delta \varepsilon_{ij}^p = \delta x \frac{\partial f^*}{\partial \sigma_{ij}} \Rightarrow \left\{ \begin{array}{l} \delta \varepsilon_{p'_i}^p = \delta x \frac{\partial f^*}{\partial p'} \\ \delta \varepsilon_q^p = \delta x \frac{\partial f^*}{\partial q} \end{array} \right. \quad (3)$$

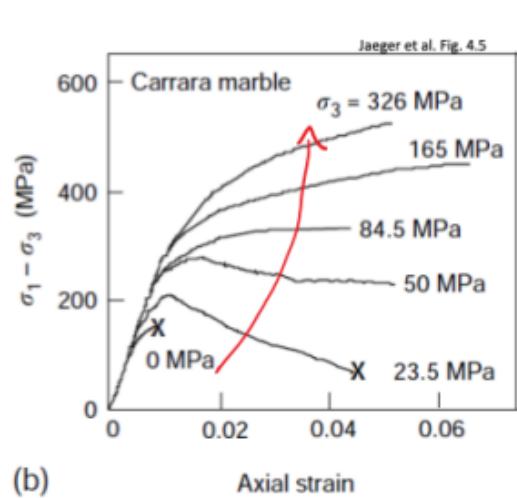
From ICL ④  $\delta \varepsilon_p^p = \frac{\lambda - \kappa}{1+e} \frac{dp'_0}{p'_0}$

③ + ④  $\rightarrow d\chi = \frac{\lambda - \kappa}{(1+e)p'(\eta^2 + n^2)} \leftarrow$  hardening parameter

$$\begin{bmatrix} \delta \varepsilon_{p'}^p \\ \delta \varepsilon_g^p \end{bmatrix} = \frac{\lambda - \kappa}{(1+e)p'(\eta^2 + n^2)} \begin{bmatrix} \eta^2 - n^2 & 2n \\ 2n & \frac{4n^2}{\eta^2 - n^2} \end{bmatrix} \begin{bmatrix} dp' \\ dg \end{bmatrix} \rightarrow \text{Plastic Strains}$$

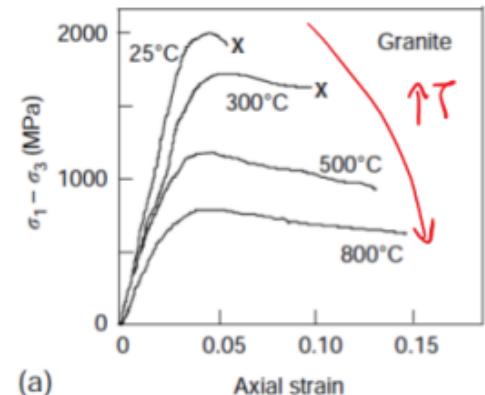
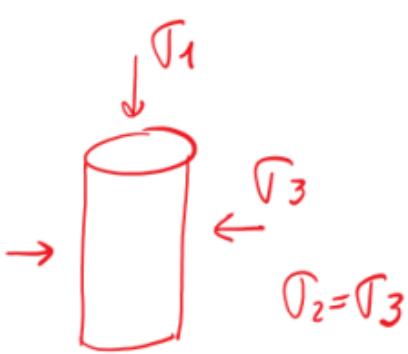






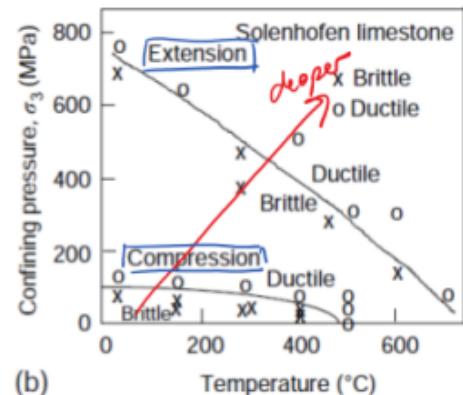
(b)

Axial strain



(a)

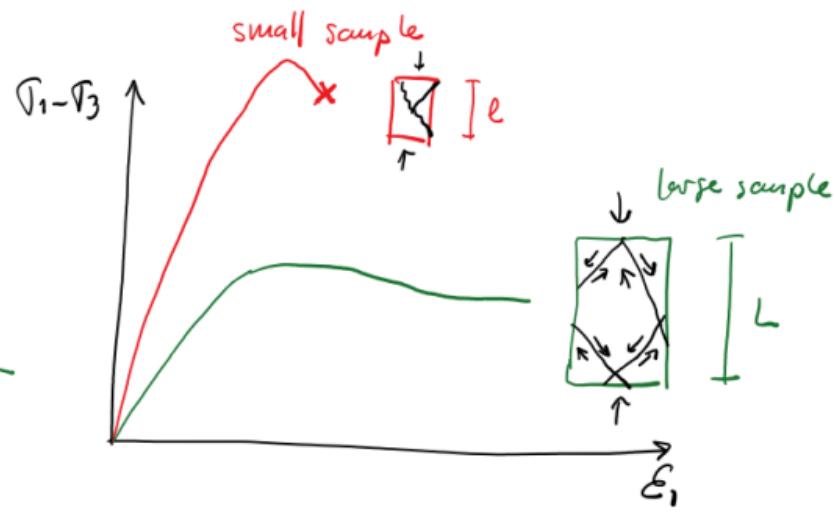
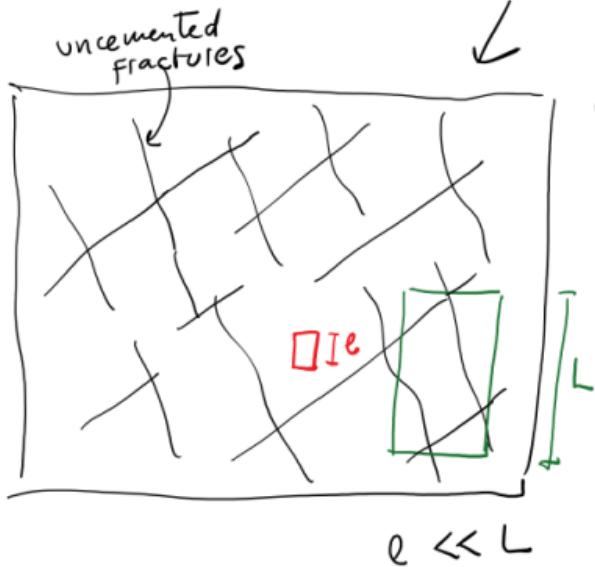
Axial strain



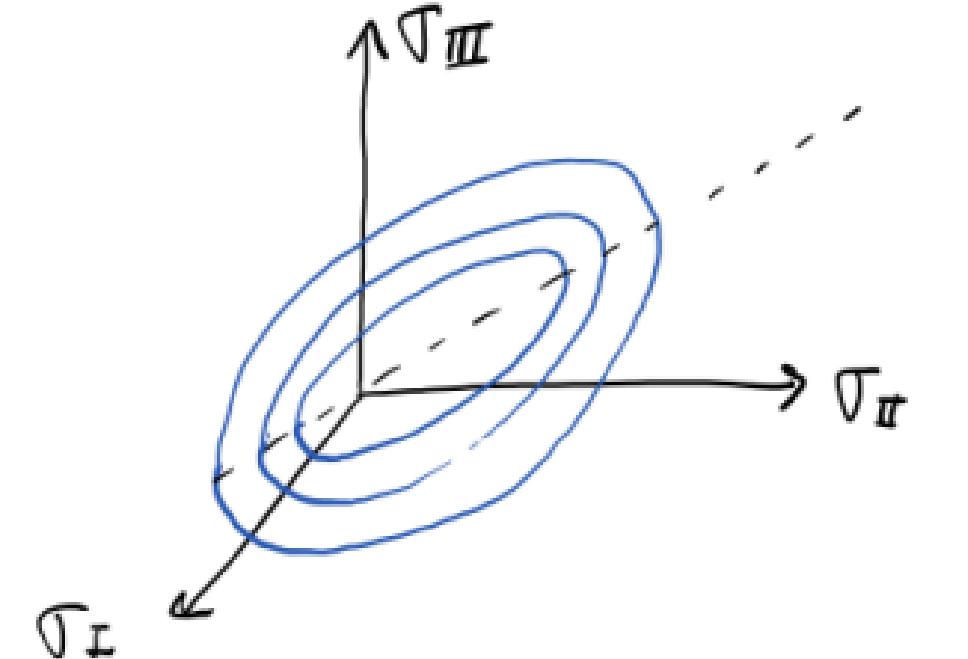
(b)

Temperature ( $^{\circ}$ C)

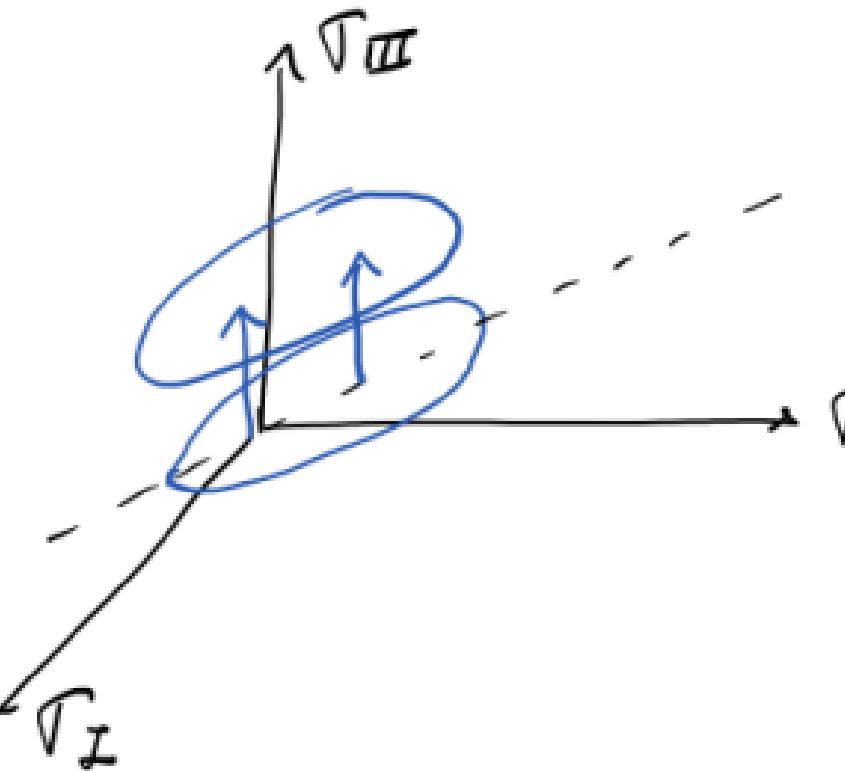
- Brittle to ductile transition
- $\uparrow \bar{\tau}_m (= p')$ ,  $\uparrow \bar{\tau}_3$ , mean stress
  - $\uparrow T$ , temperature
  - $\downarrow$  loading rate
  - mineralogy of rocks, organic shales
  - length scale



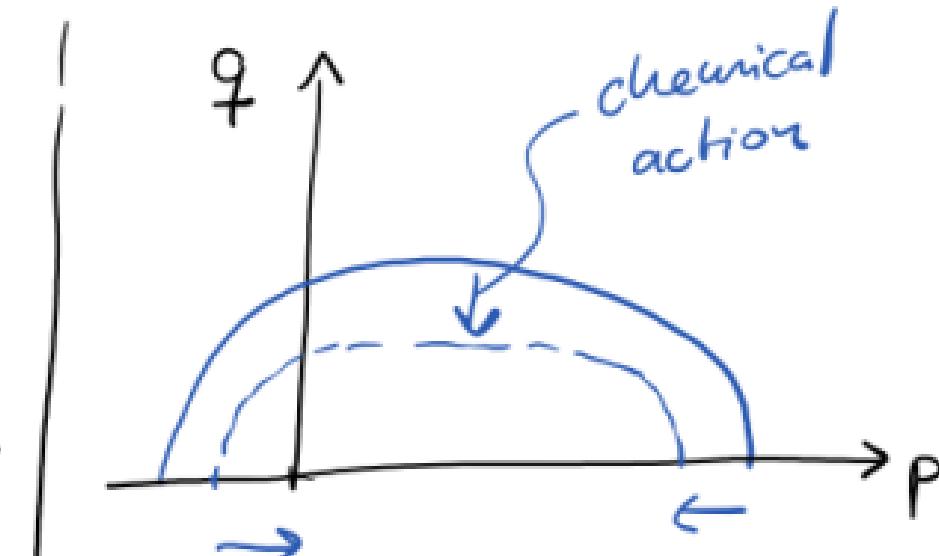
## Isotropic hardening



## Kinematic hardening



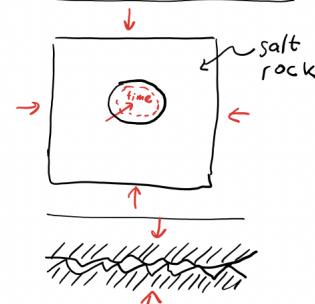
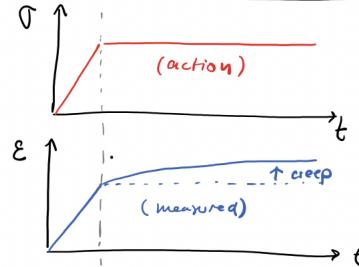
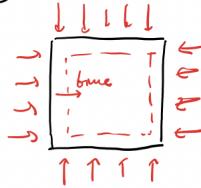
## Chemo-plasticity



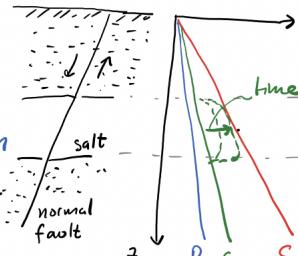
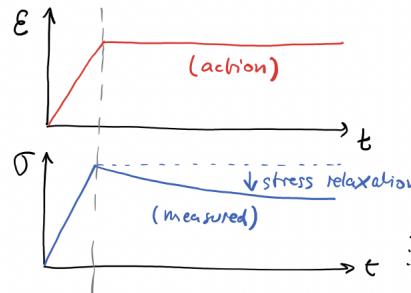
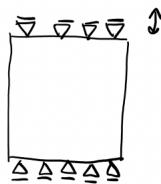
• Manifestations of visco-elastic behavior

Example

① Creep strain



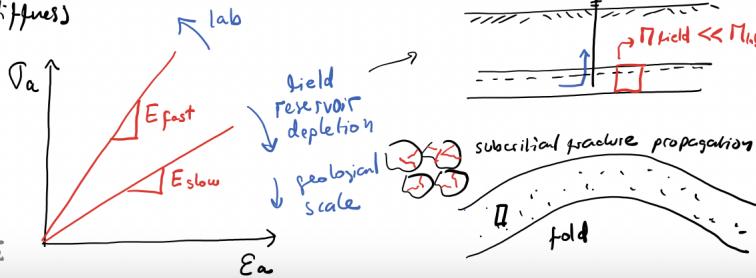
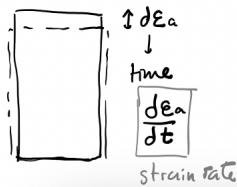
② Stress relaxation

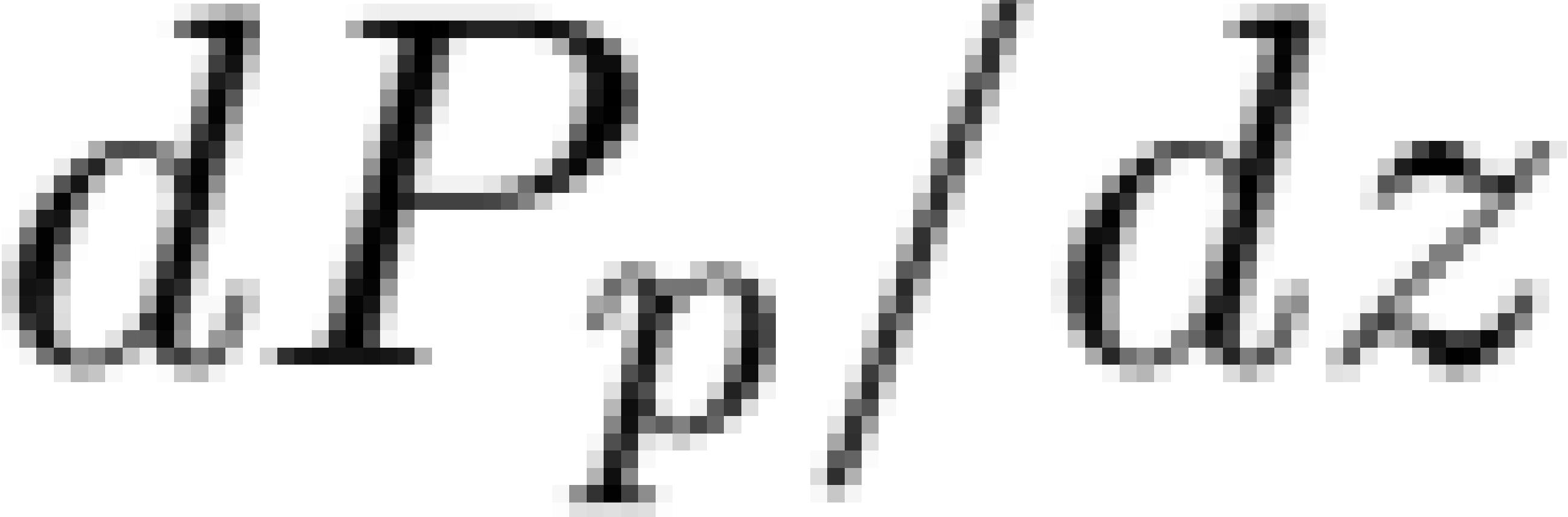


$$q = \frac{1 + \sin \varphi}{1 - \sin \varphi} \approx 3 \quad (\varphi \approx 30^\circ)$$

$$\sqrt{\text{hui}} = \pm \sqrt{q}$$

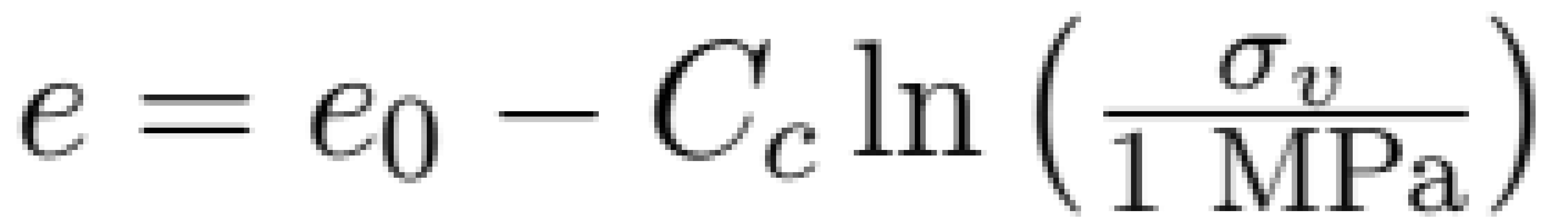
③ strain-rate dependent stiffness





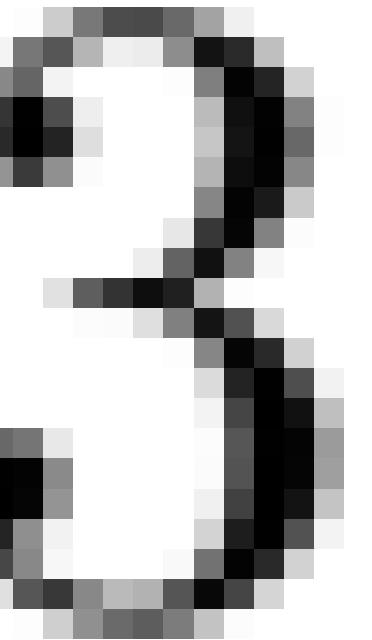


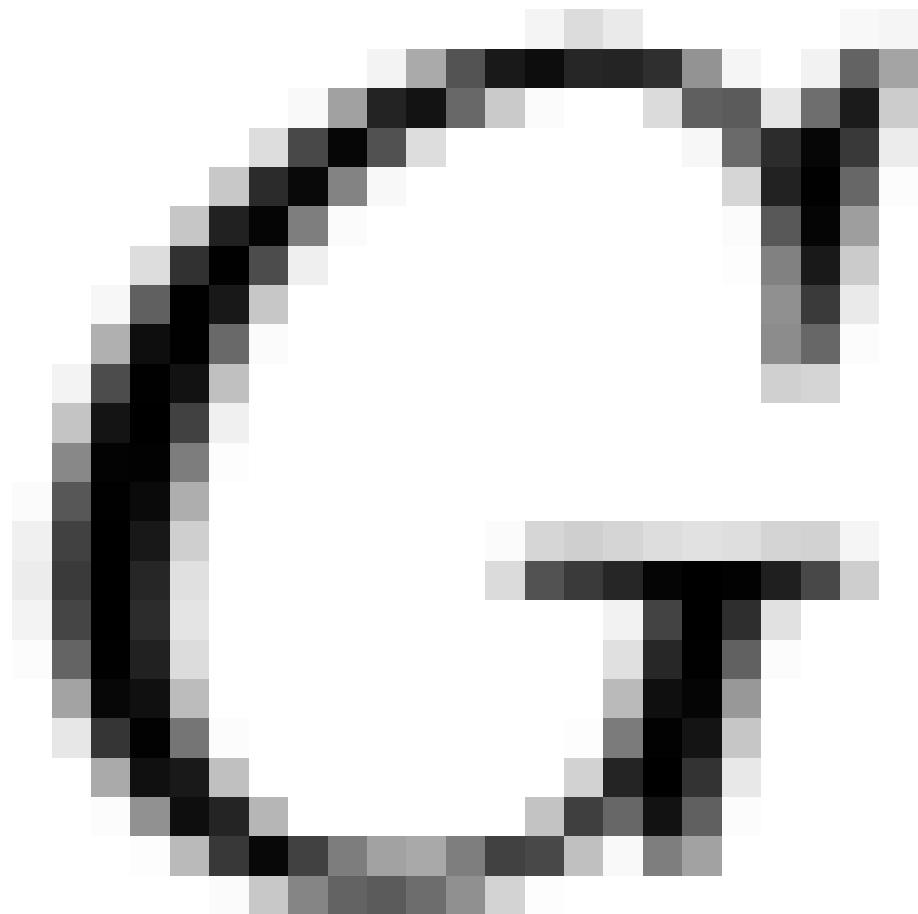




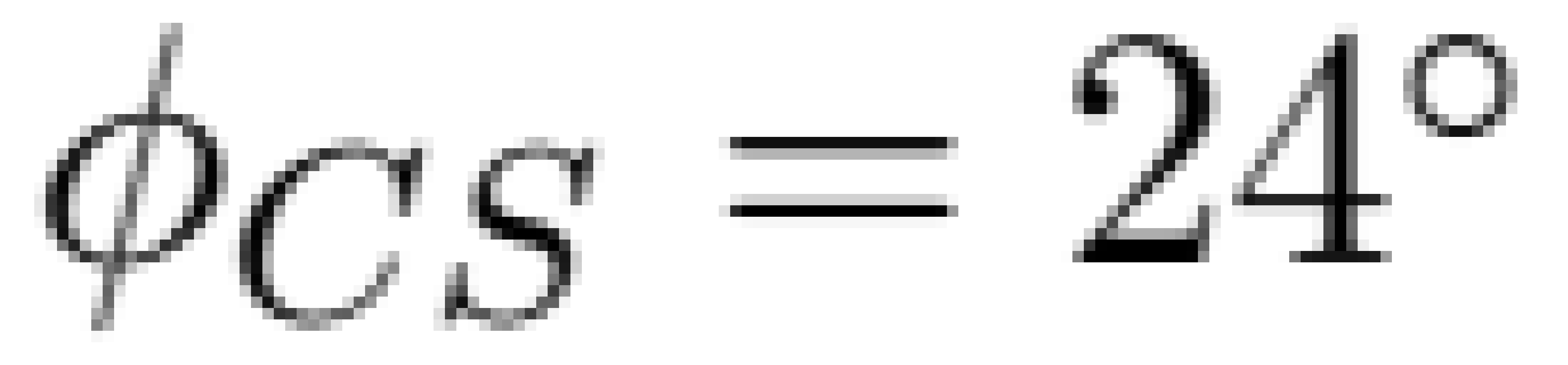


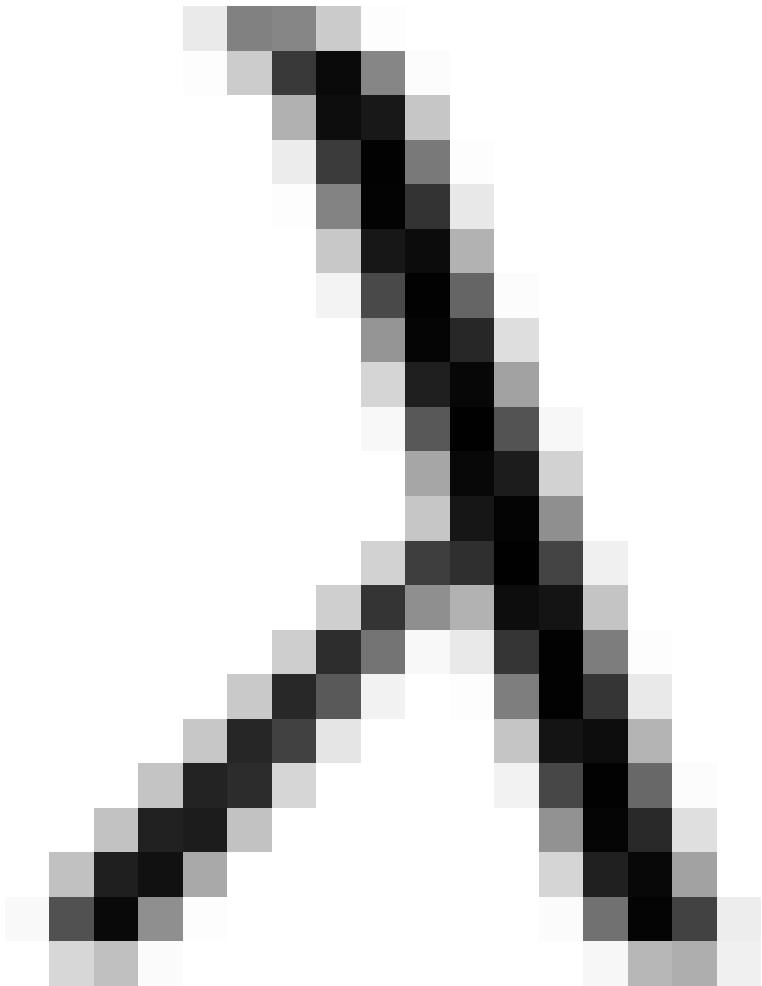


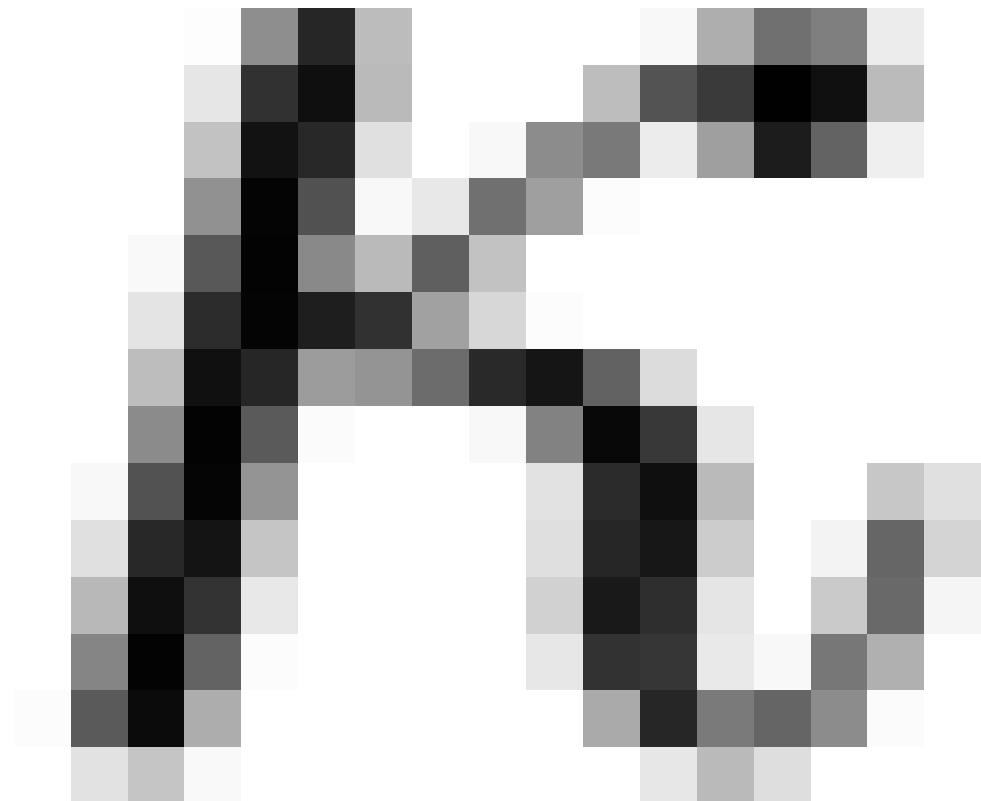


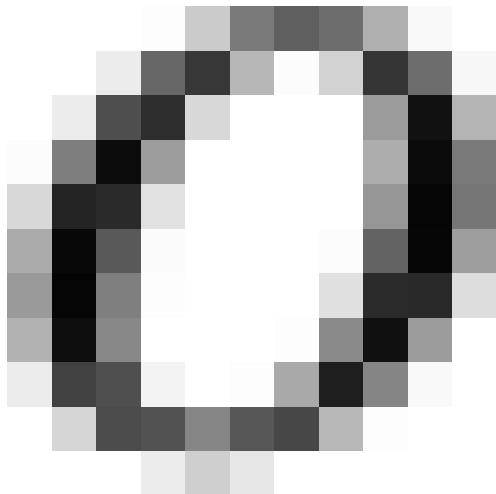


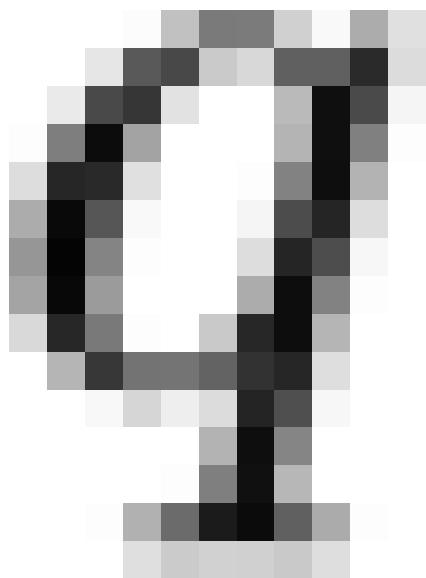
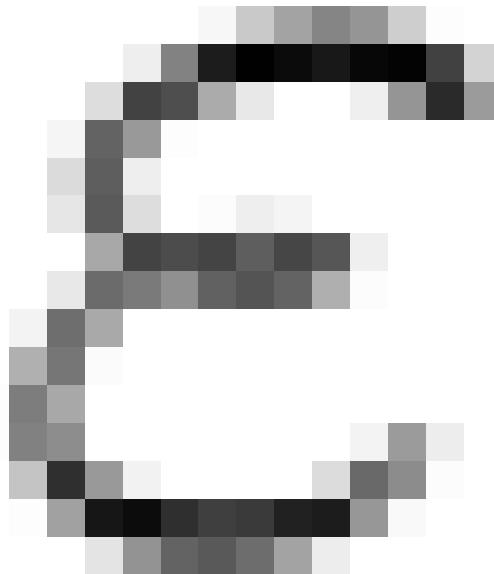


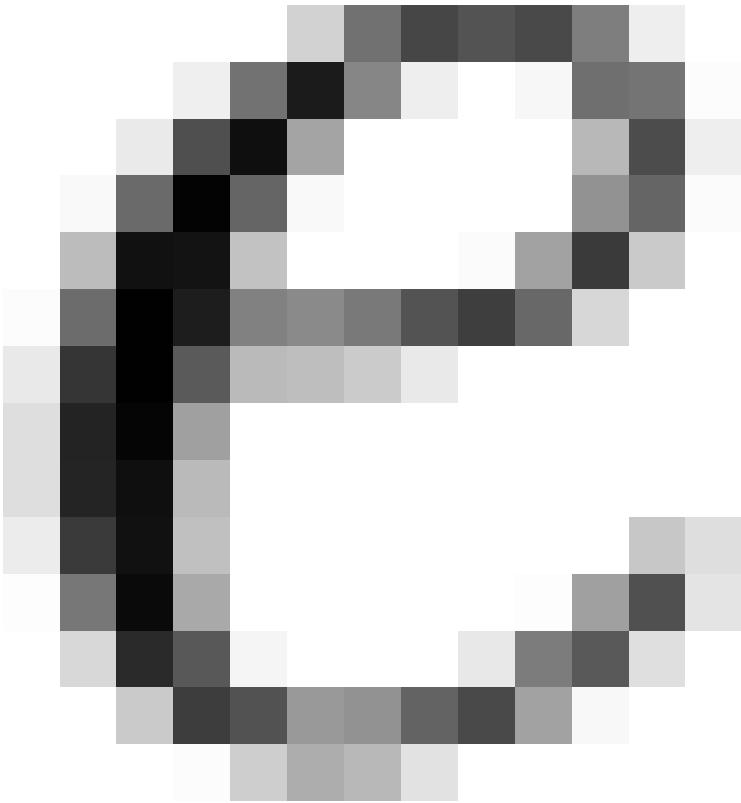


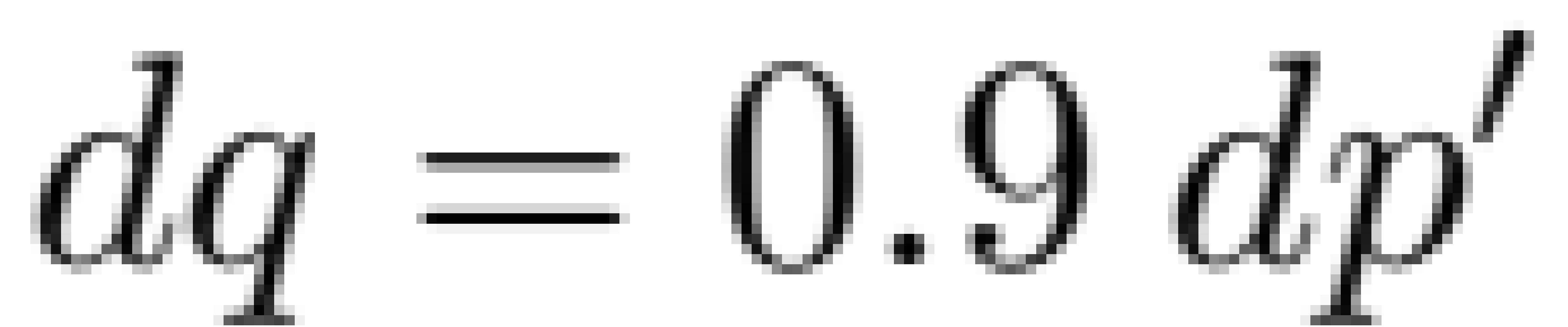


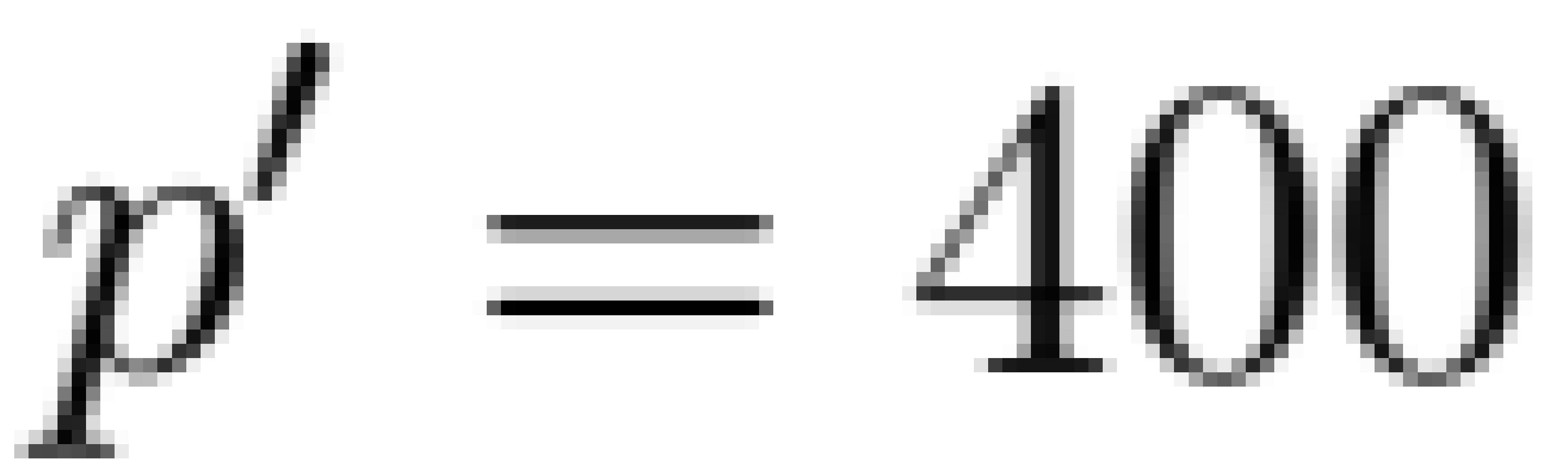


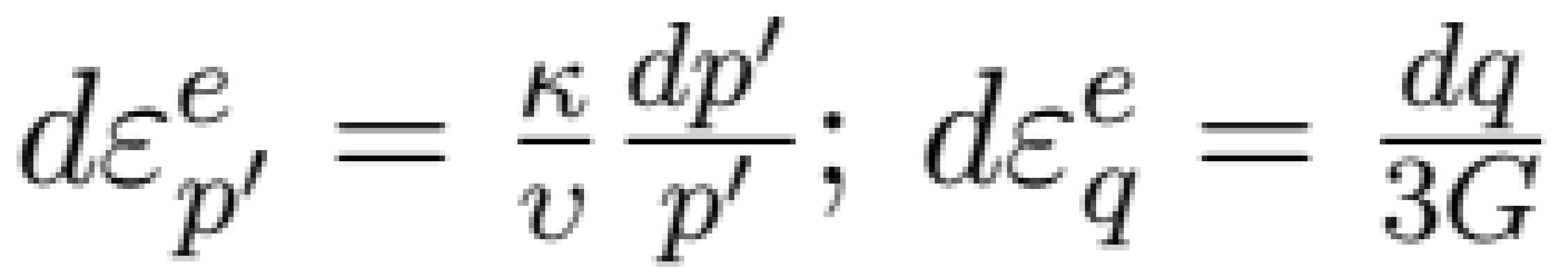






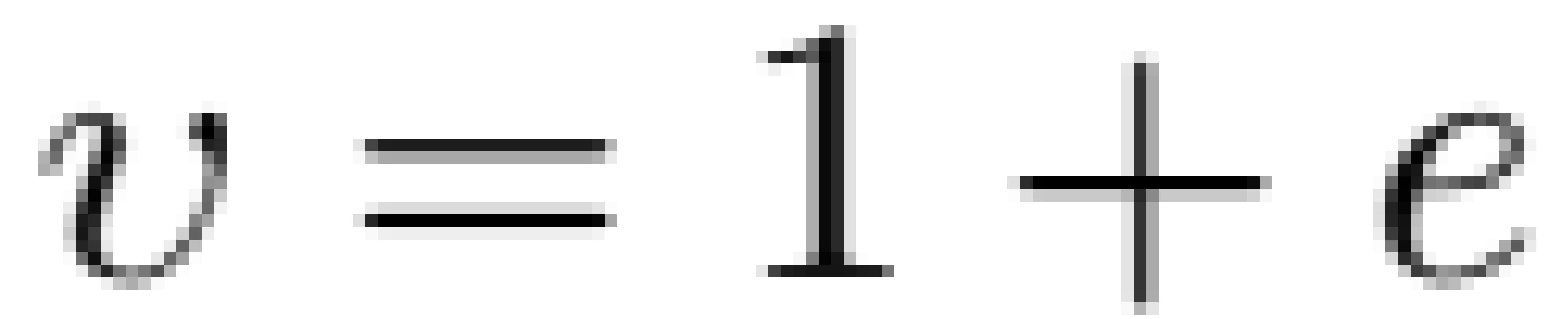


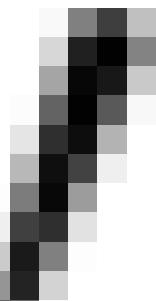
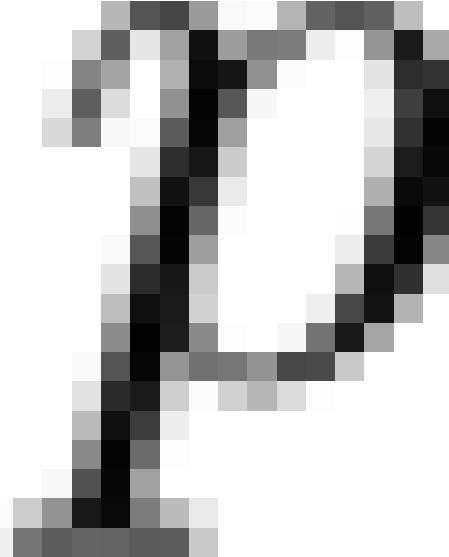
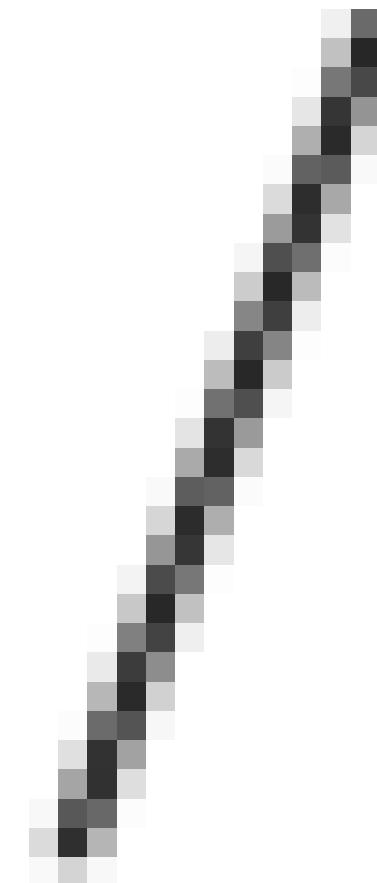
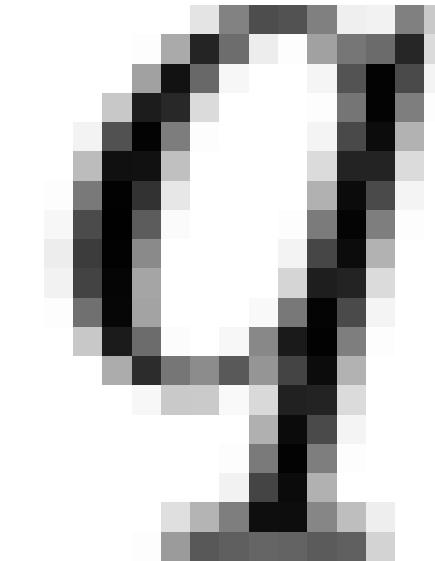
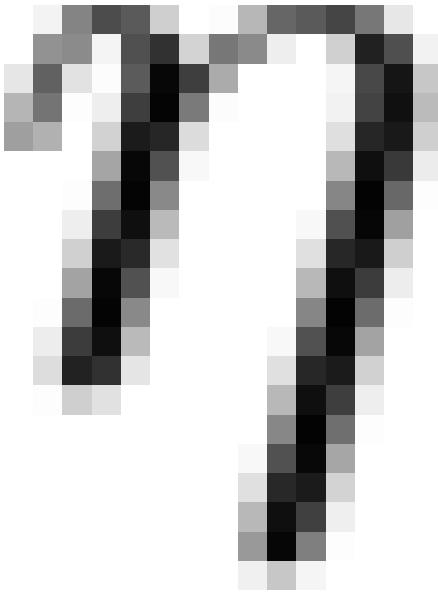


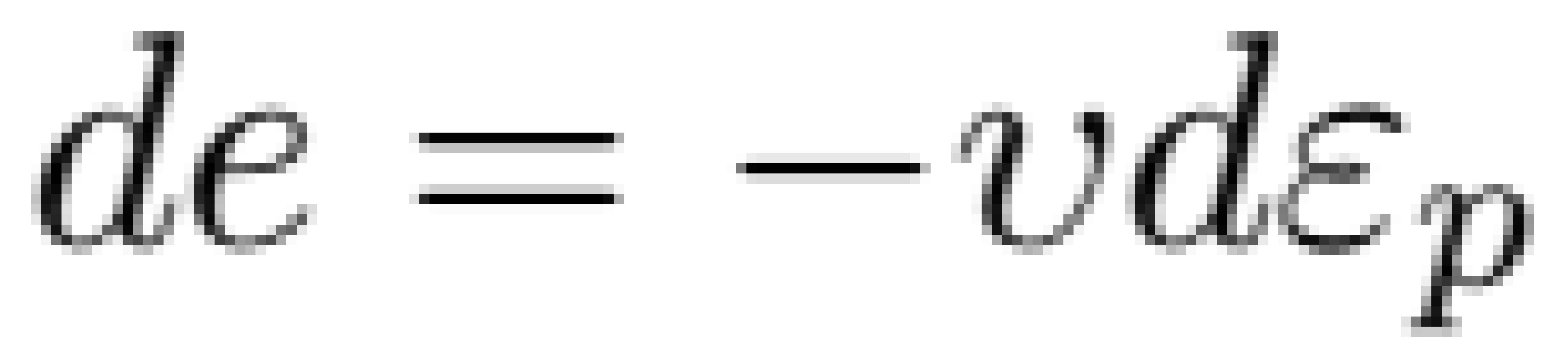


$$\begin{bmatrix} d\epsilon_p^p \\ d\epsilon_q^p \end{bmatrix} = \frac{\lambda - \kappa}{vp(M^2 + \eta^2)} \begin{bmatrix} M^2 - \eta^2 \\ 2\eta \\ 4\eta^2 \\ M^2 - \eta^2 \end{bmatrix}$$

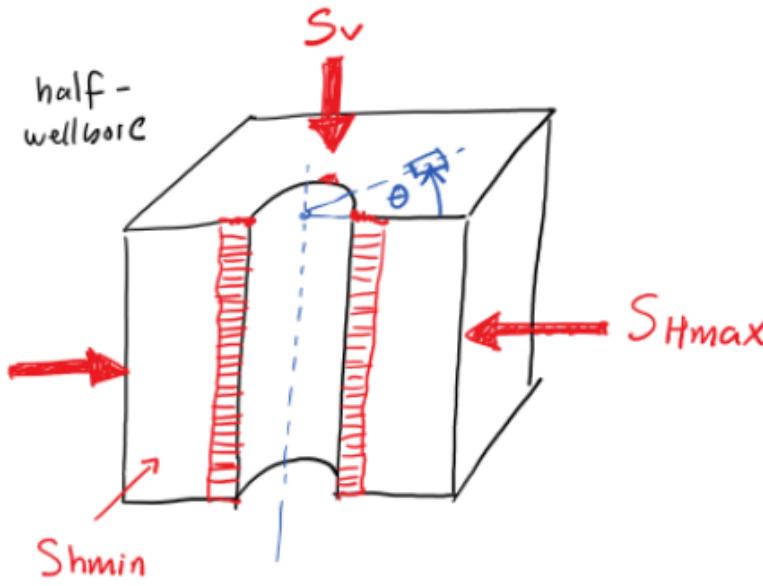
$$\begin{bmatrix} dp \\ dq \end{bmatrix}$$







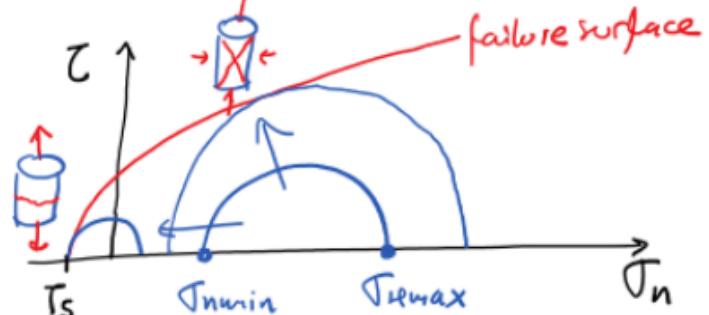




Break down pressure

- Normal faulting ✓  
 $S_v > S_{Hmax} > S_{Hmin}$
- Kirsch solution ✓  
 $\sigma_{\theta\theta}(r=a, \theta) = -(P_w - P_p) + (\sigma_{Hmax} + \sigma_{Hmin}) \dots$   
 $- 2(\sigma_{Hmax} - \sigma_{Hmin}) \cos(2\theta)$

Solve for  $\begin{cases} \theta = 0 \text{ or } \pi \\ \sigma_{\theta\theta} = -T_s \end{cases}$



$$P_w = P_b = P_p - \underbrace{\sigma_{Hmax}}_{\text{Pore pressure}} + \underbrace{3\sigma_{Hmin}}_{\text{Stress Anisotropy}} + \underbrace{T_s}_{\text{Tensile strength}}$$

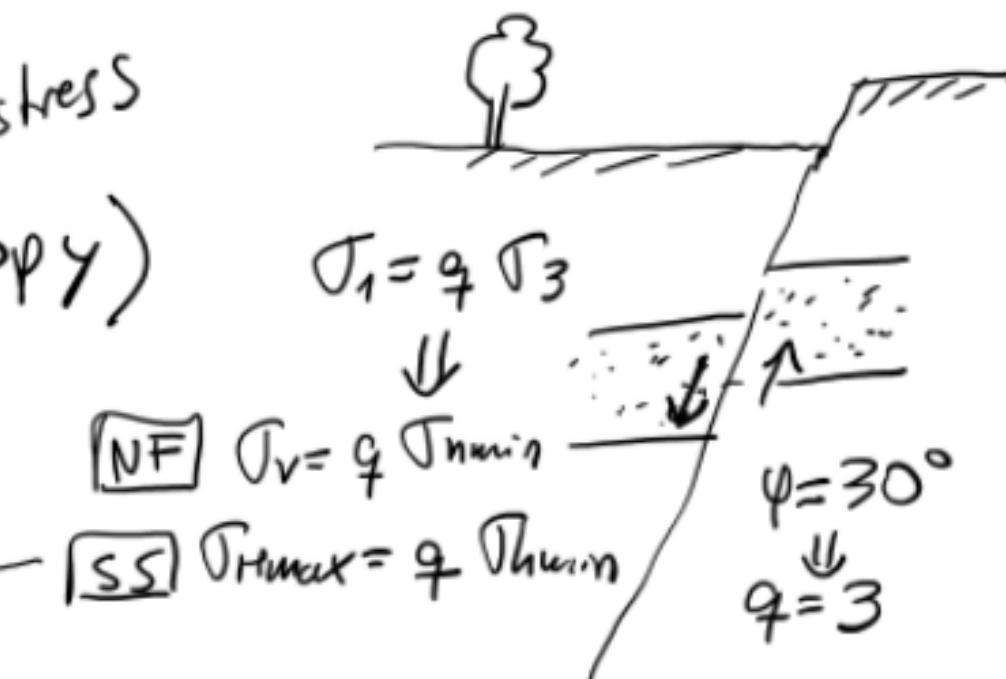
(Case 1)  $\sigma_{H\max} \approx \sigma_{H\min} = \sigma_h$  (isotropic, horizontal stress anisotropy)

$$P_b = P_p + 2\sigma_h + \tau_s$$

(Case 2)  $\sigma_{H\max} \approx 3\sigma_{H\min}$  (maximum anisotropy)

$$P_b = P_p + \tau_s$$

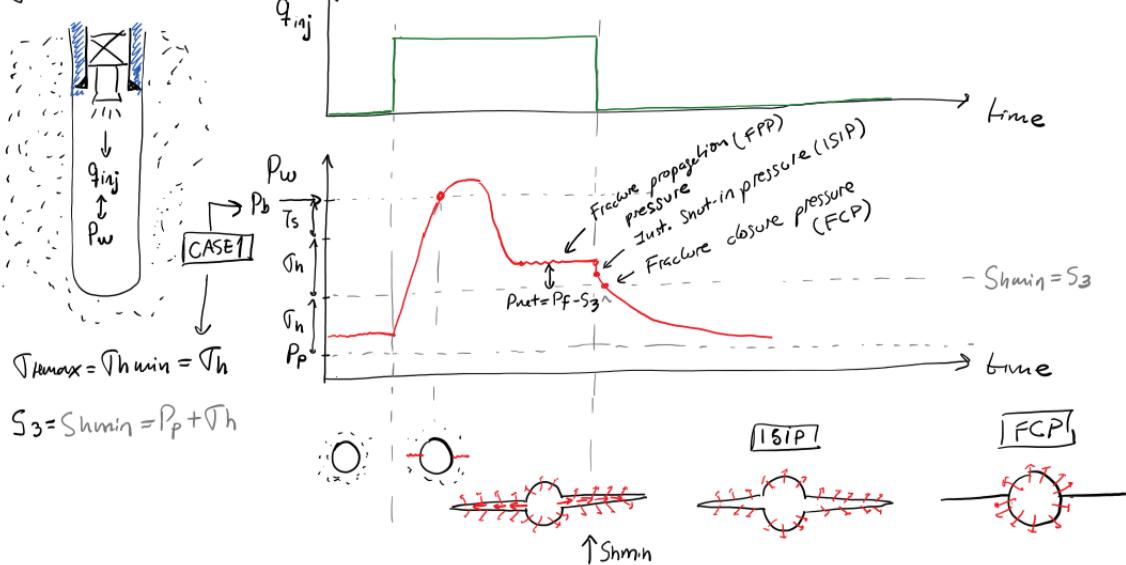
horizontal stress



# Leak-off test $\leftrightarrow$ Diagnostic Fracture Initiation Test (DFIT)

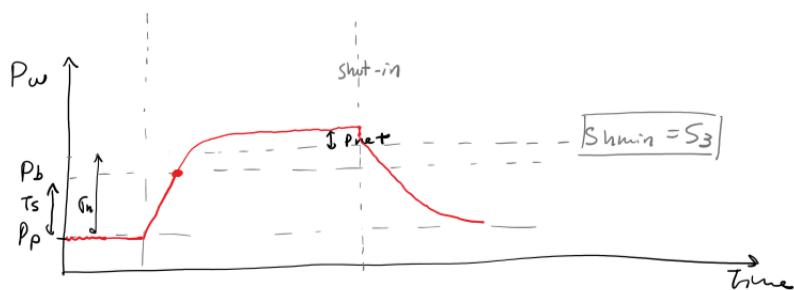
DRILLING

COMPLETIONS AND HF

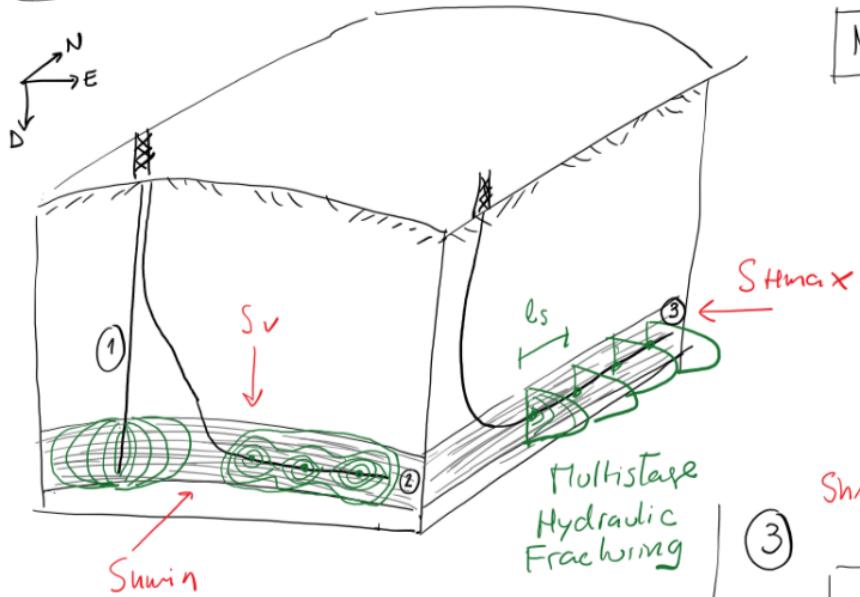


Case 2

$$P_b = P_p + T_s$$

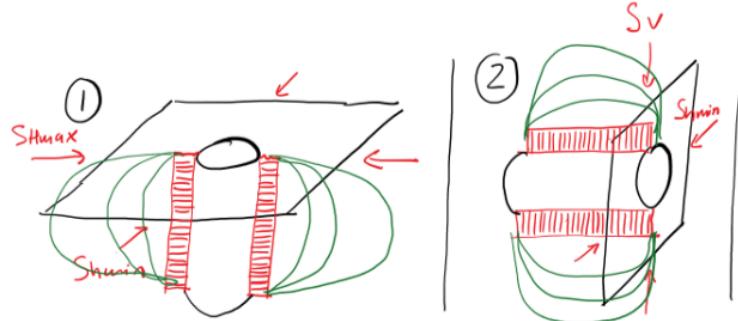
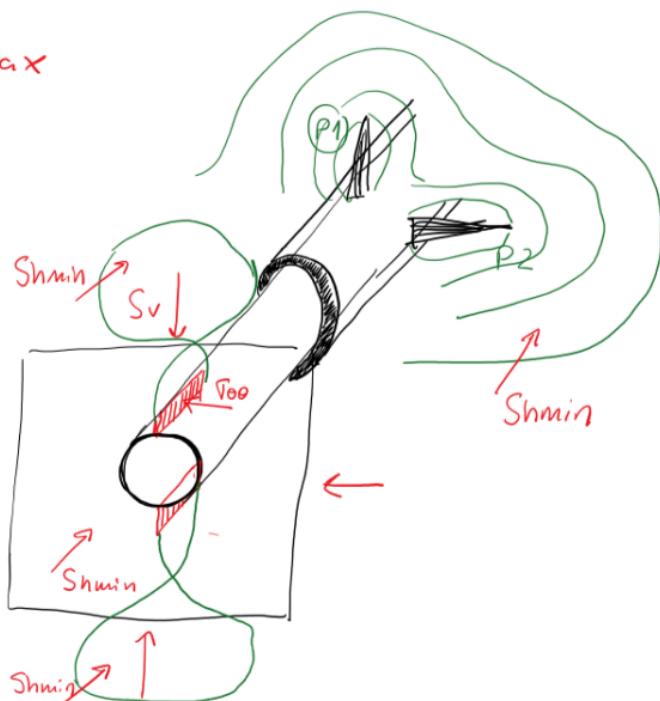


# Ideal orientation of HF growing from wellbores and perforations



$$\boxed{\text{NF}}: S_v > S_{hmax} > \underline{S_{hmin}}$$

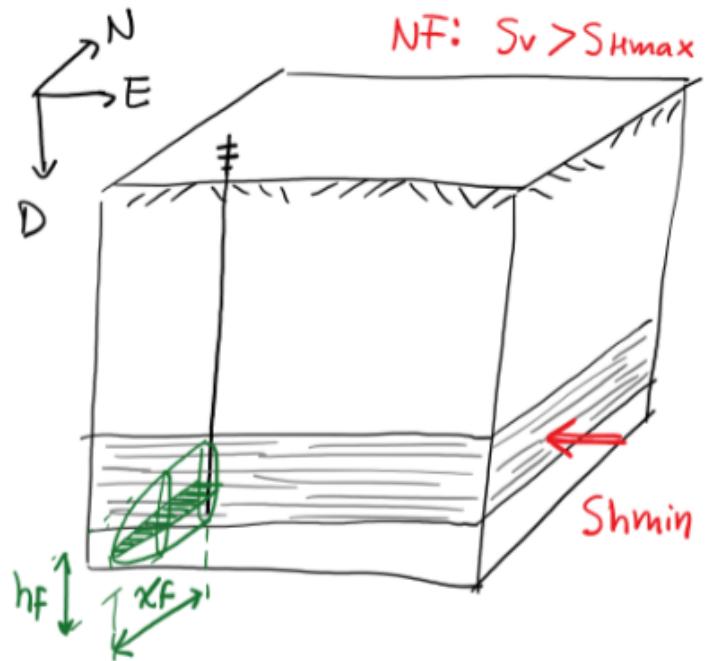
$\underline{S_3}$



5-Coupled fluid-driven fracture problem ← Valko and Economides

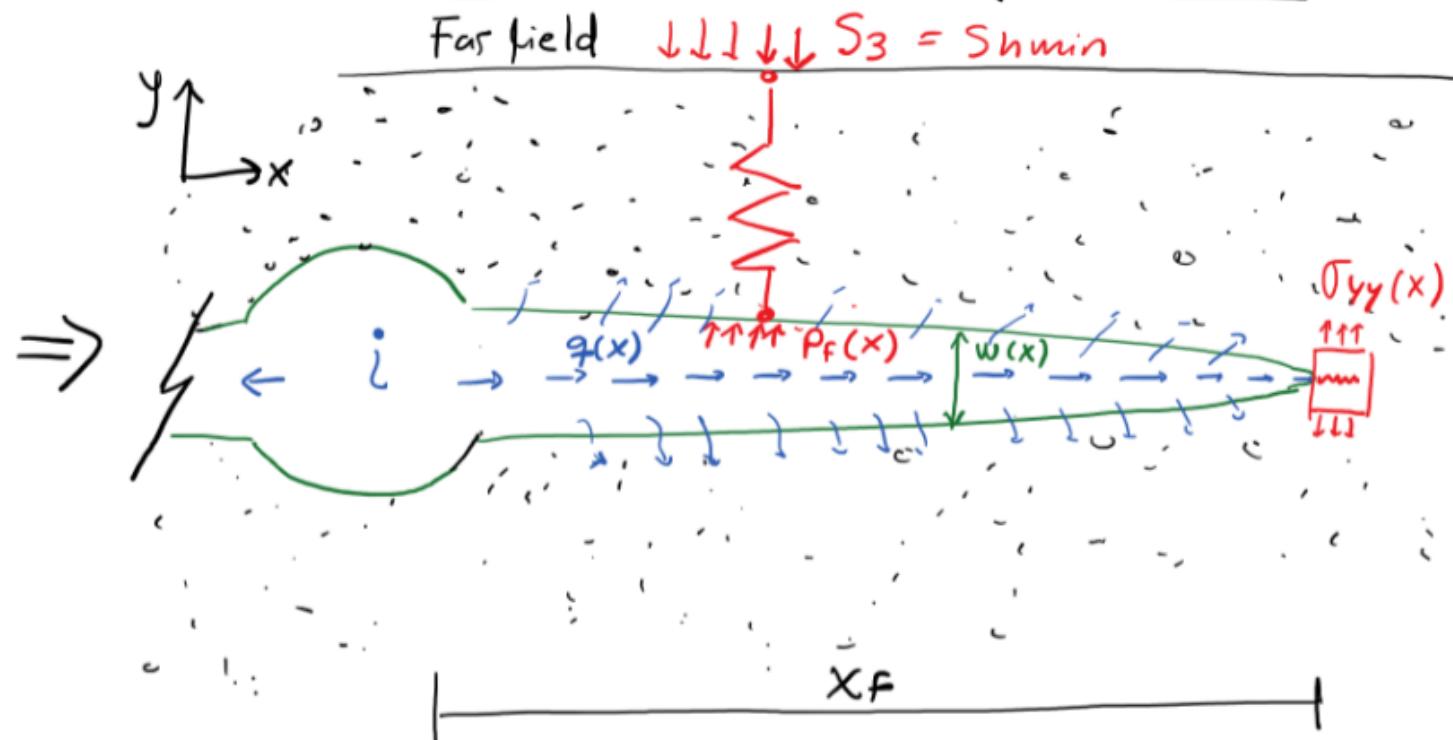
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Hydraulic Fracture Mechanics

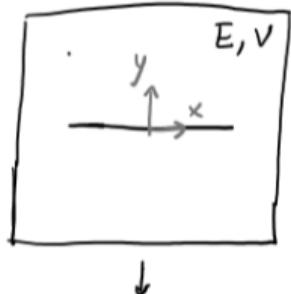


NF:  $S_v > S_{hmax} > S_{hmin}$

Horizontal cross section (plane N-E)



$$\underline{P_{\text{net}}} = P_f - S_3 = P(\text{elastic deformation}) + P(\text{viscous losses}) + P(\text{new rock surface})$$



$$\begin{aligned} \nabla \underline{\underline{\epsilon}} &= 0 \\ \underline{\underline{\epsilon}} &= \underline{\underline{C}} \cdot \underline{\underline{\epsilon}} \quad \rightarrow \\ \underline{\underline{\epsilon}} &= \frac{1}{2} (\nabla \underline{u} + \nabla \underline{u}^T) \end{aligned}$$

Below the plate, a small diagram shows a crack tip with a stress intensity factor  $K_I$  and a stress component  $\sigma_{yy}$ .

$w(x)$

A diagram of a crack tip with a stress intensity factor  $K_I$ . A stress concentration factor  $N$  is shown.

$$q(x) \propto [w(x)]^3 h f \cdot \frac{dP(x)}{dx}$$

$K_I > K_{IC} \Rightarrow$  frac propog  
 $K_I < K_{IC} \Rightarrow$  frac does not propogate

- leak-off

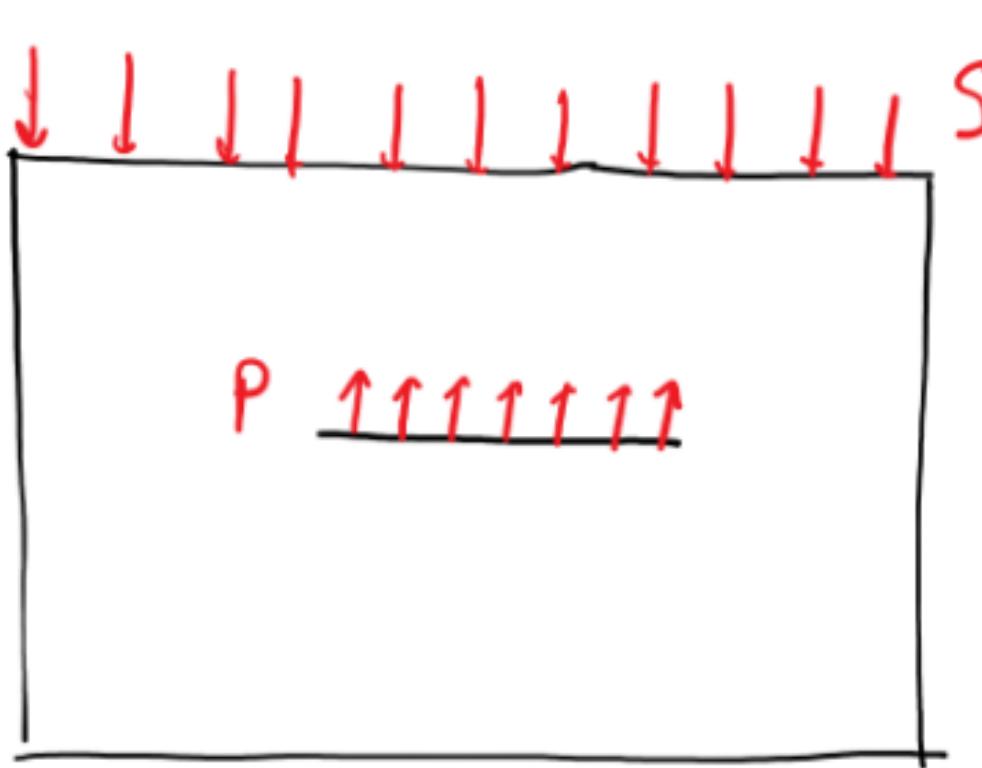
$$q_m = - \frac{K}{N} \nabla P$$

$K_I$ : Fracture Intensity

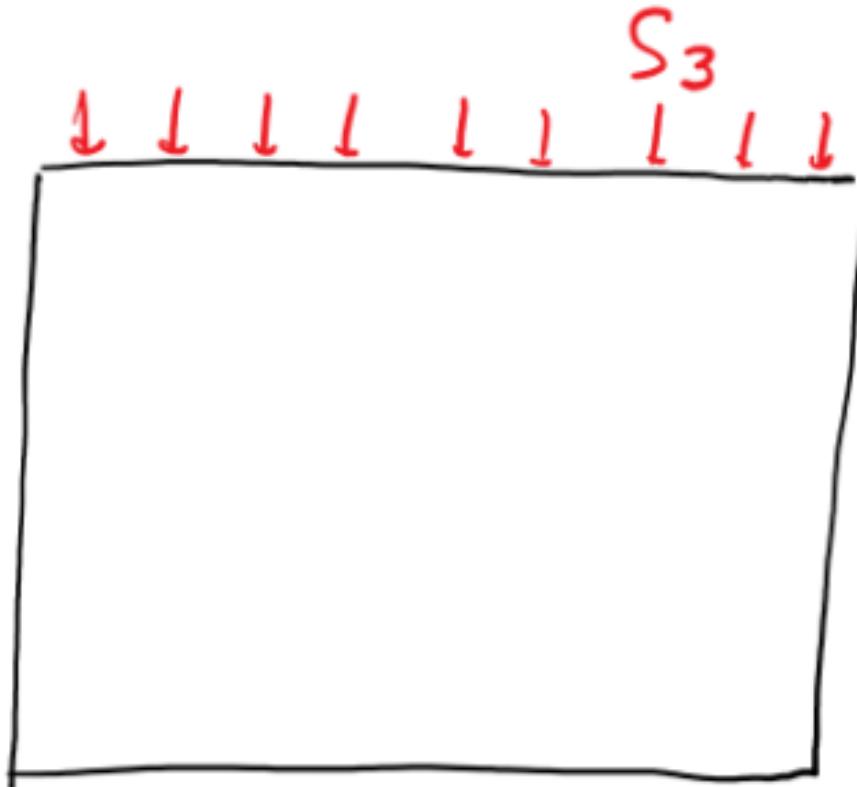
$$K_I = \lim_{r \rightarrow 0^+} \sqrt{2\pi r} \cdot \sigma_{yy}(c+r, 0)$$

A diagram of a crack tip with a stress intensity factor  $K_I$ . A stress concentration factor  $N$  is shown.

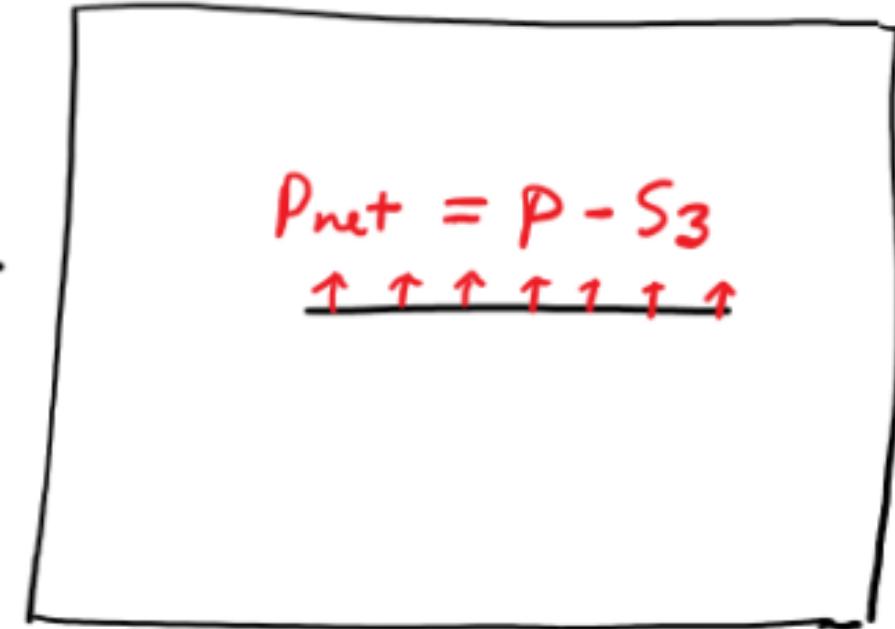
$$K_I = f(P_{\text{net}}, x_f)$$



=



+



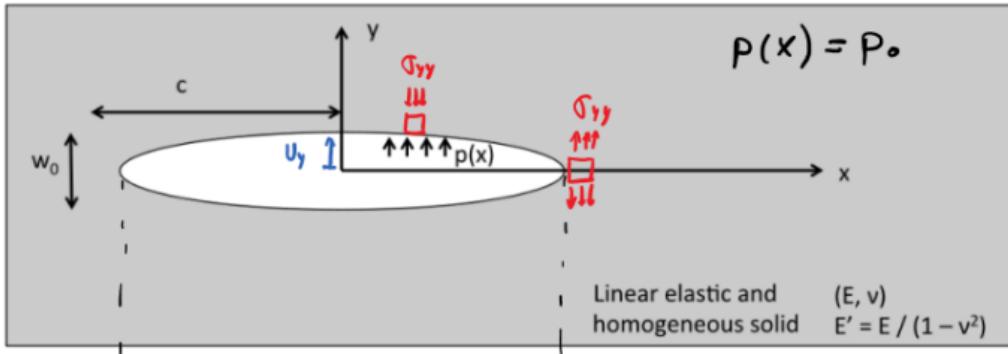
$S_3$ : far-field least principal stress

$P_{\text{net}}$ : net pressure

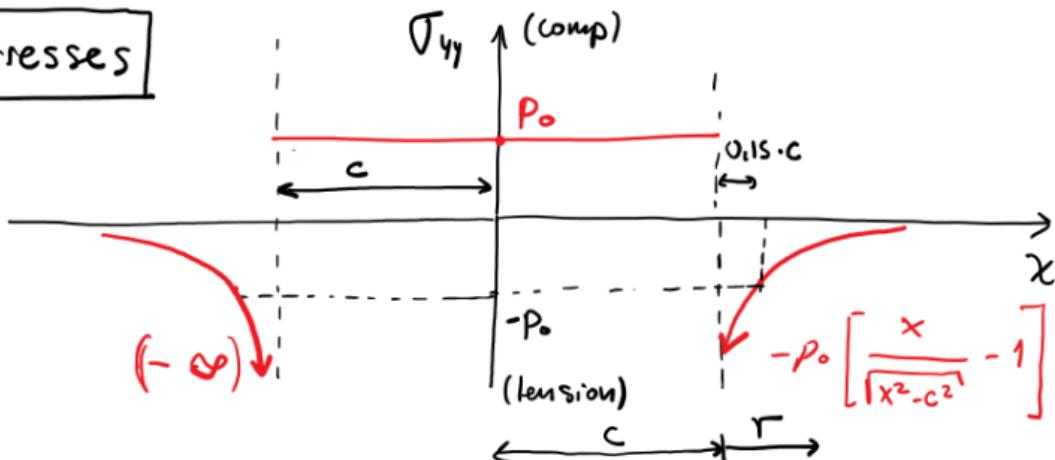
# Displacements and stresses around a linear fracture (plane-strain)

Griffith

Solution for  
Navier's Eq. with  
a Linear Elastic  
Isotropic Solid



Stresses



Displacements

$$u_y(x, 0) = \begin{cases} \frac{2P_0}{E'} \sqrt{c^2 - x^2}; & x < c \\ 0; & x > c \end{cases}$$

$$\downarrow \\ W_0 = 2 u_y(0, 0) = \frac{4P_0}{E'}$$

$$\sigma_{yy} = \begin{cases} P_0; & 0 \leq x < c \\ -P_0 \left[ \frac{x}{\sqrt{x^2 - c^2}} - 1 \right]; & x > c \end{cases}$$

## Stress intensity factor

$$\rightarrow K_I = \lim_{r \rightarrow 0^+} \left[ \sqrt{2\pi r} \cdot (-\sigma_{yy}(c+r, 0)) \right]$$

Fracture Modes

|                          |
|--------------------------|
| I : open-mode            |
| II : shear in-plane      |
| III : shear out-of-plane |



$$= \lim_{r \rightarrow 0^+} \left[ \sqrt{2\pi r} \cdot \left( +P_0 \left[ \frac{c+r}{\sqrt{(c+r)^2 - c^2}} - 1 \right] \right) \right] ; x = c+r$$

$$= P_0 \sqrt{2\pi} \cdot \lim_{r \rightarrow 0^+} \left[ \Gamma^{1/2} \cdot \left( \frac{c+r}{\sqrt{2cr+r^2}} - 1 \right) \right]$$

$$K_I = P_0 \sqrt{2\pi} \left( \frac{c}{\sqrt{rc}} \right) = P_0 \sqrt{\pi c}$$

-  $P_0$ : constant pressure

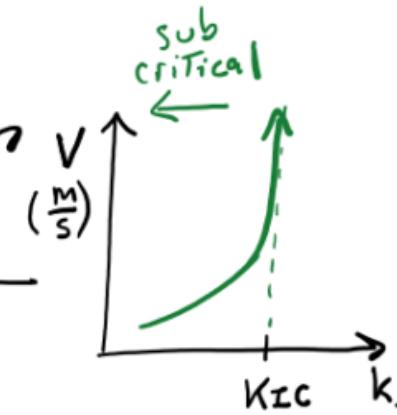
LEFM

criterion for  
fracture propag.

$K_I \geq K_{IC} \Rightarrow$  fracture propagation

$K_I < K_{IC} \Rightarrow$  no fracture propagation

Fracture toughness ( $c$ : critical)  $\leftarrow$



Typical values of

$K_{IC} \sim 0.2 - 2 \text{ MPa} \cdot \text{m}^{1/2}$  (geological materials)

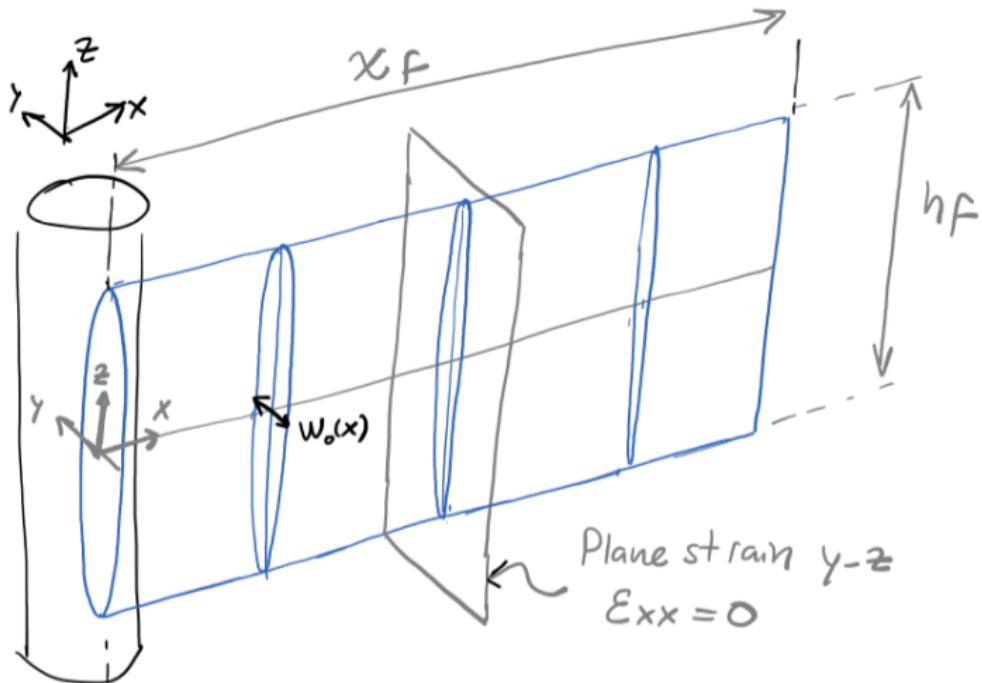
$\tilde{K}_{Mode\ 1}$   $\sim$

measured through "notch experiments"

e.g.: semi circular bending experiment

Objective: understand methodology to solve a fluid-driven fracture problem

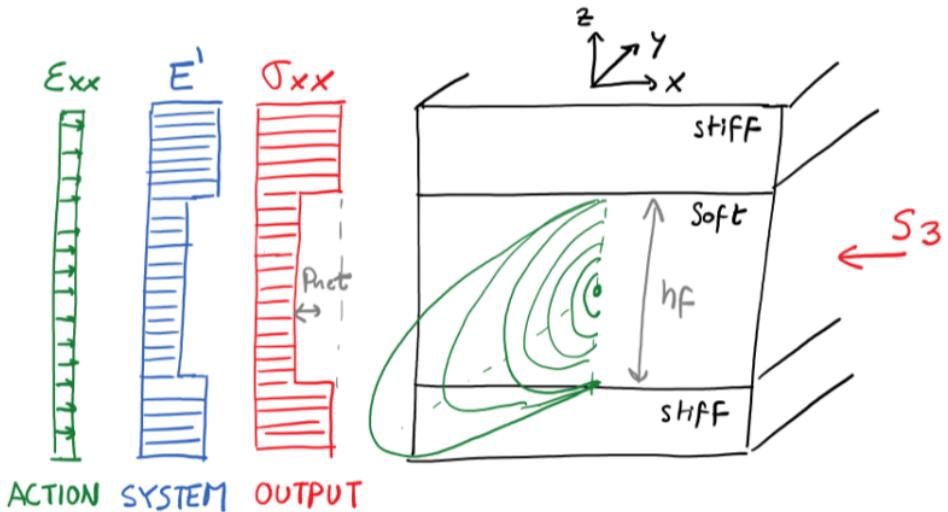
Example: PKN Model (Perkins, Kern, Nordgren)



Main assumptions

- $h_F$  constant ✓
- $x_F > h_F$  ✓
- Plane strain  $y$ - $z$  ✓
- $K_{IC} = 0$  ✓
- Geometry as in figure

① Fracture height  $\leftarrow$  1D stress solution



② Fracture width  $\leftarrow$  Griffith solution

Solution in plane  $y-z$

$$W_o(x) = \frac{4 P_n(x)}{E'} \boxed{C} = \frac{2 P_n(x) hf}{E'} \boxed{\text{Eq. 1}}$$

③ Fluid flow within fracture  $\leftarrow$  laminar flow + Newtonian fluid

Flow along  $x$  through elliptical channel

$$\frac{\partial P}{\partial x} = - \frac{64 N}{\pi} \frac{q(x)}{[w_0(x)]^3 h_f}$$

Eq. 2

④ Mass conservation  $\leftarrow$  fluid in fracture + leak-off

$$V_i = V + \cancel{V_L}^{=0}$$

Eq. 3

$$V_L = 2 A_F (\bar{c}_L) \sqrt{t} \text{ or Darcy } \checkmark$$

$$V_i = i \cdot t$$

Eq. 4

$i$ : injection rate (constant)

$$q(x) = \underbrace{i}_{\text{one-way}}$$

Eq. 5

no leak-off

Eq 1 + 2 + 5

→ Pressure gradient along frac

$$\frac{dP}{dx} = - \frac{64 \cdot N \cdot i}{\pi [2 h_f P_n(x) E^{-1}]^3 \cdot h_f}$$

$$\frac{dP}{dx} = - \frac{8N i E^3}{\pi h_f^4 [P_n(x)]^3}$$

→ Integrate  
 $P_n(x)$

$$\int_{P_n(0)}^{P_n(x_f)} [P_n(x)]^3 dP = - \int_0^{x_f} \frac{8N i E^3}{\pi h_f^4} dx$$

$$\frac{[P_n(x_f)]^4}{4} - \frac{[P_n(0)]^4}{4} = - \frac{8N i E^3}{\pi h_f^4} (x_f - 0)$$

$$\left\{ \begin{array}{l} P_n(x=0) = \left( \frac{32 \pi i E^3 x_F}{\pi h_F^4} \right)^{1/4} \\ W_o(x=0) = \left( \frac{512 \pi i x_F}{\pi E^4} \right)^{1/4} \end{array} \right. \quad \begin{array}{l} \boxed{\text{Eq. 6}} \\ \boxed{\text{Eq. 7}} \end{array} \quad \begin{array}{l} \xrightarrow{\quad} \\ + \end{array} \quad \text{Eq. 1}$$

Introducing time into equations

Geometry :  $V_{\text{frac}} = x_F h_F \bar{w}$

Material balance:  $V_{\text{frac}} = i \cdot t$

$\rightarrow$  where  $\bar{w} = \frac{\pi}{5} W_o(x=0)$

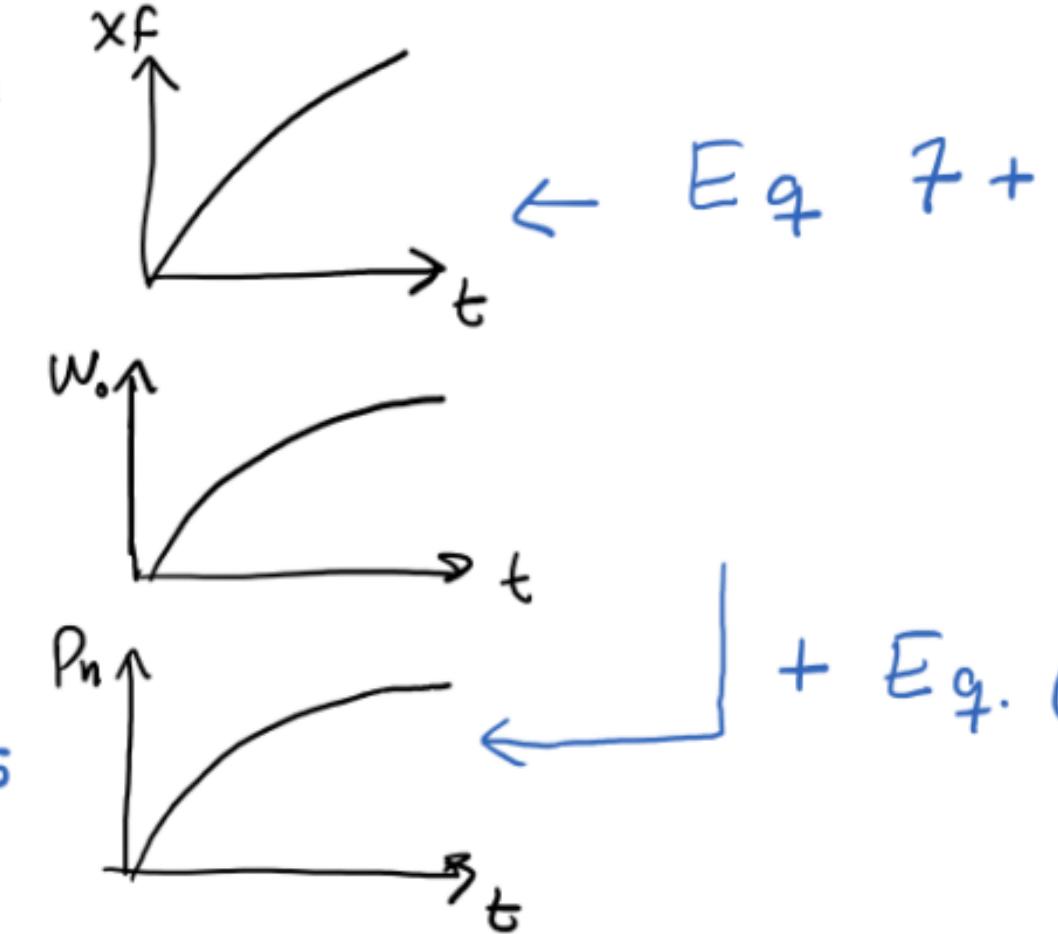
$x_F = \frac{i \cdot t}{h_F \left( \frac{\pi}{5} \cdot W_o(x=0) \right)}$

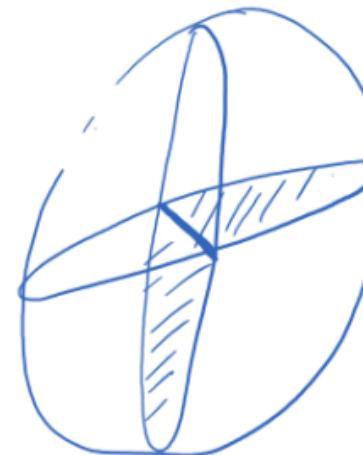
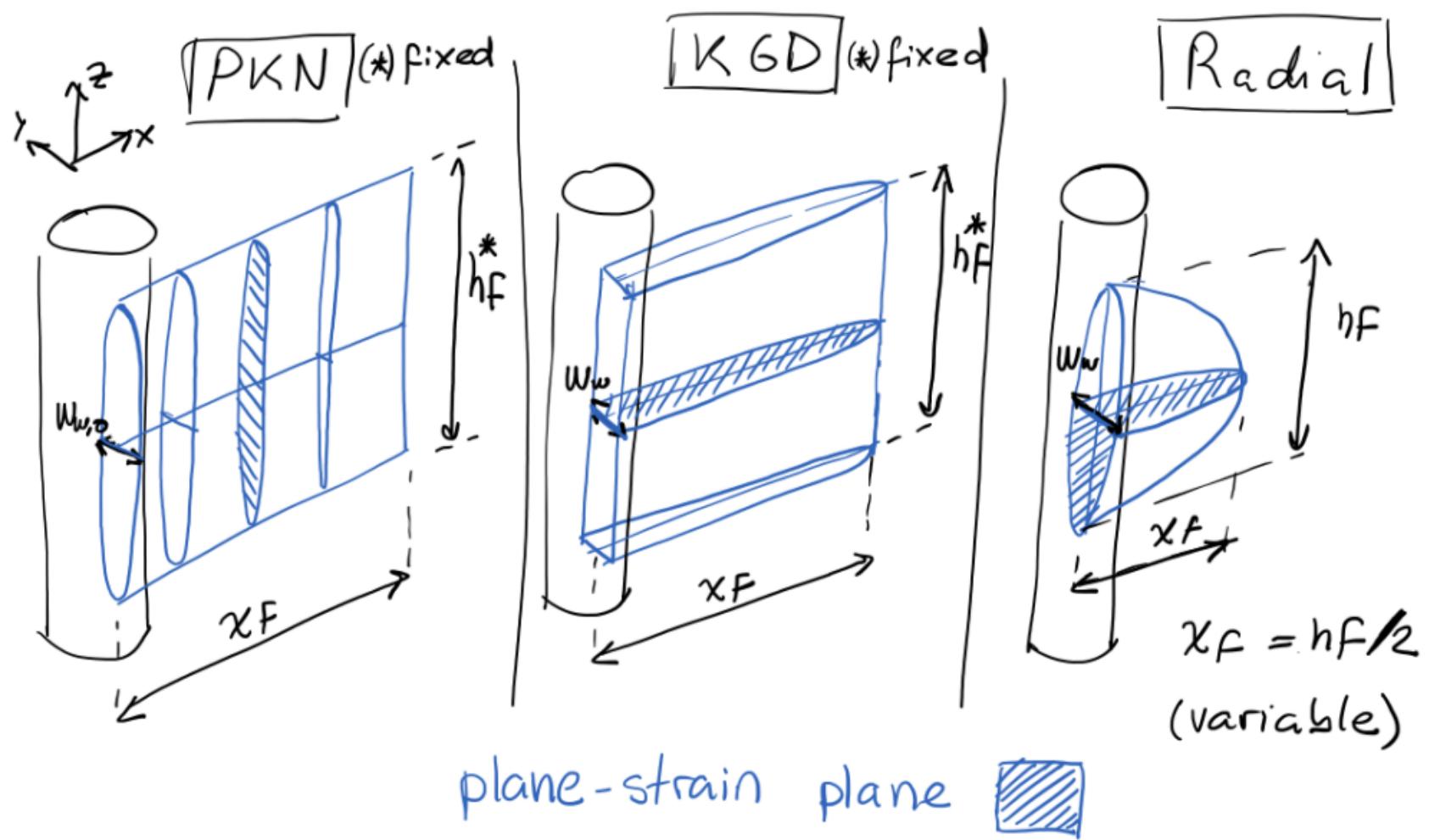
PKN  
Model

$$x_F = \left( \frac{625}{512 \cdot \pi^3} \right)^{1/5} \left( \frac{i^3 \cdot E}{h_f^4 \cdot \mu} \right)^{1/5} \cdot t^{4/5}$$

$$w_0(x=0) = \left( \frac{2560}{\pi^2} \right)^{1/5} \left( \frac{i^2 \cdot \mu}{E \cdot h_f} \right)^{1/5} \cdot t^{1/5}$$

$$P_n(x=0) = \left( \frac{80}{\pi^2} \right)^{1/5} \left( \frac{E^4 \cdot i^2 \cdot \mu}{h_f^6} \right)^{1/5} \cdot t^{1/5}$$



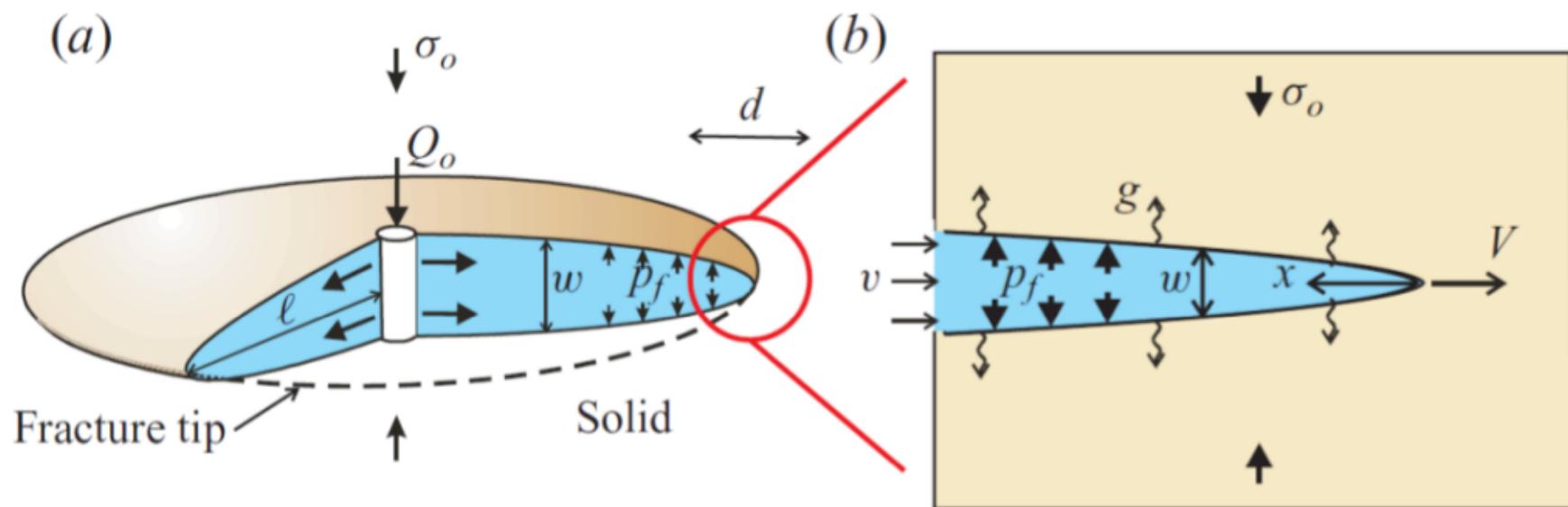


$$x_F = h_F/2 \\ (\text{variable})$$

## 5-Fluid-driven fractures in porous media

Friday, November 20, 2020 4:27 PM

Detournay, Brunger, LeCampion, et al.



Material para

$$E' = \frac{E}{1-\nu^2}; \text{ Plan mo}$$

$$K' = \sqrt{\frac{32}{\pi}} K_{ic}; \text{ Tous}$$

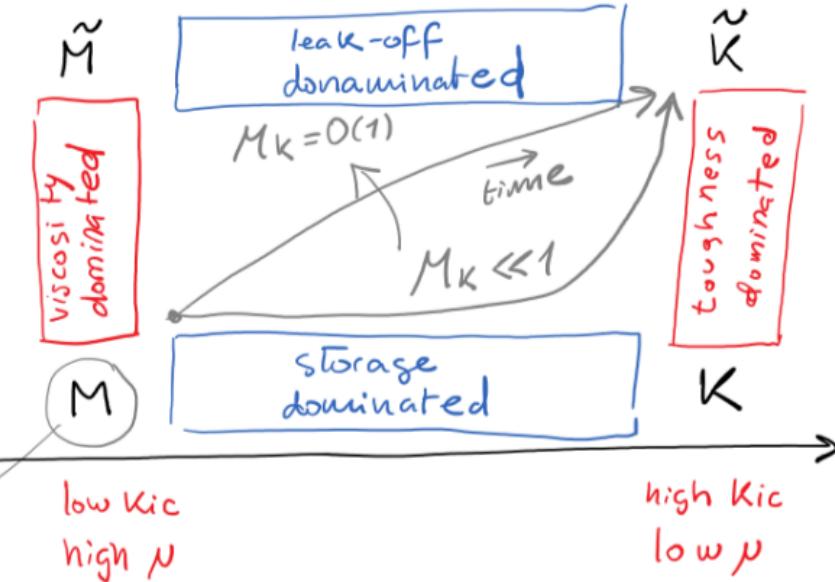
$$\lambda' = 12 N; \text{ Fluid}$$

$$C' = 2 C_L; \text{ Leak-off}$$

Garagash et al. 2011

# Fracture propagation "regimes" (zero-lag)

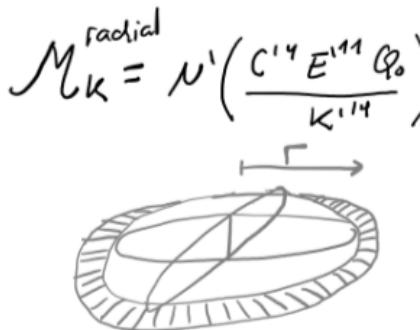
Fluid gone into rock  
Fluid stored in frac.  
 Material balance competition  
 $\uparrow$



$$V_i = V_f + V_{\text{leak-off}}$$

Ex.  $M$  (storage, viscosity) solution for

$$\left. \begin{array}{l} K' = 0 \\ C' = 0 \end{array} \right\} \sim \text{PKN derivation previous lecture}$$



Work creating new rock  
Work circulating fluid  
 Competitive dissipative processes

• Energy release rate  $G = K_I^2$   
 LEFM

Key points: • Subsurface hydraulic fracturing

↑  
viscosity dominated ✓

- scaling must be taken into account when comparing field and lab HF
- asymptotic solutions (extreme cases) are useful to conceptualize complex physical processes ✓
- analytical solutions are useful to define proper meshes for numerical solutions ✓
- consider fluid-lag

