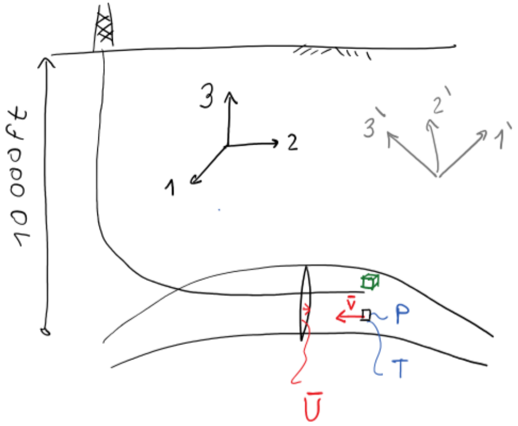
p'

q

K_s

$$P_{mud} = P_W$$

S_{Hmax}



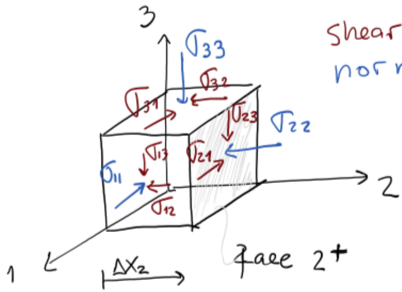
scalar : P , T

vector : $\bar{V} = [0, -0.1, 0] \ \frac{m}{day}$

$\bar{U} = [0, 1, 0]$ cm

(2nd order) tensor :

stress $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$



shear stresses

normal stresses

$\sigma_{ij} = \sigma_{ji}$

$\sigma_{\underbrace{i}_{\text{face}} \underbrace{j}_{\text{direction}}}$

symmetric

$\sigma_{ij} \in \mathbb{R}$

$\rightarrow$ eigenvalues $\in \mathbb{R}$

$\underline{\underline{S}} = \begin{bmatrix} 7000 & 0 & 0 \\ 0 & 6500 & 0 \\ 0 & 0 & 10000 \end{bmatrix}$ psi

in coord system 1,2,3

principal stresses

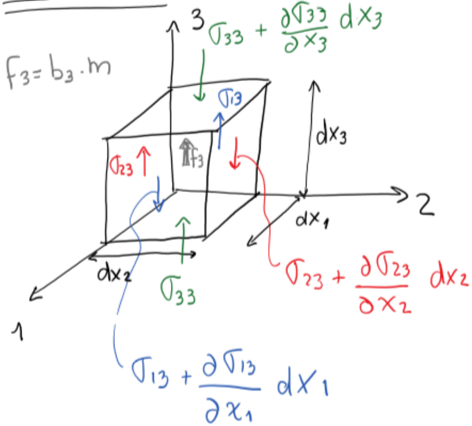
$\updownarrow$

principal directions $\rightarrow$ $\perp$

$\rightarrow \underline{\underline{\sigma}}^P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$\sigma_1 \neq \sigma_2 \neq \sigma_3$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$



Vol = $dx_1\, dx_2\, dx_3$

$$\sum F_3 = m\, \cancel{a_3}^{\,0} = 0$$

$$\cancel{\sigma_{33}}\, dx_1 dx_2 - \left(\cancel{\sigma_{33}} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3\right) dx_1 dx_2 +$$

$$\cancel{\sigma_{23}}\, dx_1\, dx_3 - \left(\cancel{\sigma_{23}} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2\right) dx_1 dx_3 +$$

$$\cancel{\sigma_{13}}\, dx_2\, dx_3 - \left(\cancel{\sigma_{13}} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1\right) dx_2\, dx_3 +$$

$$F_3 = 0$$

$$\frac{\partial \sigma_{33}}{\partial x_3} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{23}}{\partial x_2} dx_1 dx_2 dx_3 +$$

$$\frac{\partial \sigma_{13}}{\partial x_1} dx_1\, dx_2\, dx_3 + b_3\, m = 0$$

$$\underline{\underline{\sum F_3}} \longrightarrow \frac{\partial \sigma_{33}}{\partial x_3} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_1} + b_3 \frac{m}{Vol} = 0$$

Cauchy's equilibrium equations

- $\dfrac{\partial \sigma_{ij}}{\partial x_j} + b_i \rho = 0$

- $\nabla \cdot \underline{\underline{\sigma}} + \underline{b}\, \rho = 0$

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + b_1 \rho = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + b_2 \rho = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + b_3\, \rho = 0 \end{cases}$$

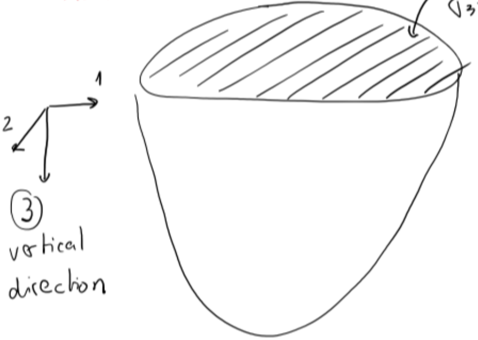
gravity in 3

$b_3 = -g$

$\dfrac{m}{Vol}$

half-space $\rightsquigarrow \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = 0$

free surface $\sigma_{33}(x_3 = 0) = 0$



$$\begin{cases} \partial\sigma_{11} \qquad \partial x_2 + \dfrac{\partial\sigma_{13}}{\partial x_3} + b_1\rho = 0 \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} + \dfrac{\partial\sigma_{23}}{\partial x_3} + b_2\rho = 0 \\ \cancel{\dfrac{\partial\sigma_{31}}{\partial x_1}}^{\,0} + \cancel{\dfrac{\partial\sigma_{32}}{\partial x_2}}^{\,0} + \boxed{\dfrac{\partial\sigma_{33}}{\partial x_3}} + b_3\rho = 0 \end{cases}$$

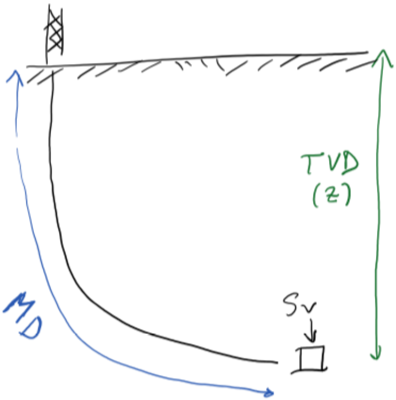
$$\frac{\partial\sigma_{33}}{\partial x_3} = g\cdot\rho$$

$$\int_{\sigma_{33}(x_3=0)}^{\sigma_{33}(x_3)} d\sigma_{33} = \int_{x_3=0}^{x_3} g\cdot\rho\cdot dx_3$$

$$\sigma_{33}(x_3) = \int_0^{x_3} g\cdot\rho(x_3)\,dx_3$$

vertical depth

$$\boxed{S_v(z) = \int_0^{z} g\cdot\rho_{bulk}(z)\,dz}$$

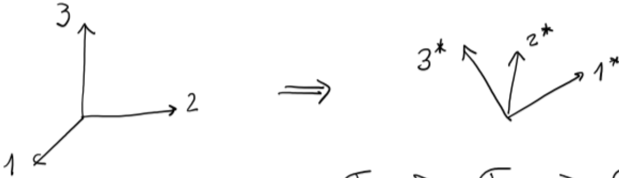


$\rho_{bulk}(z) = \rho_{bulk}$ $\qquad S_v(z) = \rho_{bulk}\cdot g\cdot z$

$$\frac{dS_v}{dz} = \rho_{bulk}\cdot g \begin{cases} 23\ \text{MPa/km} \\ 1\ \text{psi/ft} \end{cases}$$

$2300\ \frac{kg}{m^3}\cdot 9.8\ \frac{m}{s^2}$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

$$\sigma_1 \perp \sigma_2 \perp \sigma_3$$

σ_v is a principal stress:

↳ $\sigma_v \perp \sigma_{Hmax} \perp \sigma_{hmin}$

(*)

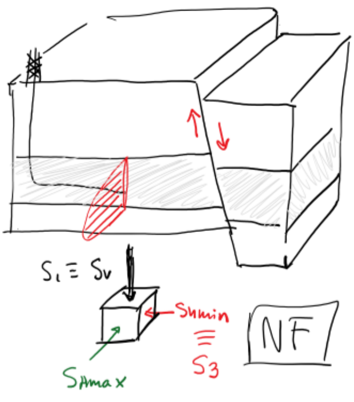
$$\underline{\underline{\sigma}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} - \begin{bmatrix} P_p & 0 & 0 \\ 0 & P_p & 0 \\ 0 & 0 & P_p \end{bmatrix}$$

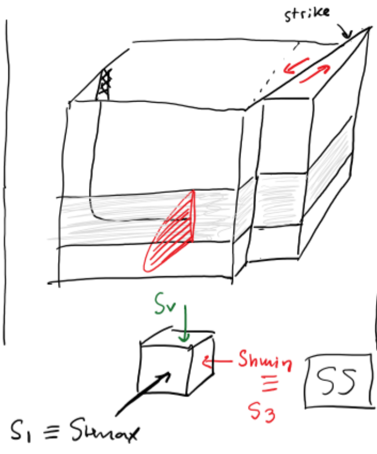
NORMAL FAULTING	STRIKE-SLIP FAULTING	REVERSE F.
$\sigma_v \geq \sigma_{Hmax} \geq \sigma_{hmin}$	$\sigma_{Hmax} \geq \sigma_v \geq \sigma_{hmin}$	$\sigma_{Hmax} \geq \sigma_{hmin} \geq \sigma_v$
$\begin{bmatrix} \sigma_v & 0 & 0 \\ 0 & \sigma_{Hmax} & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$	$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_v & 0 \\ 0 & 0 & \sigma_{hmin} \end{bmatrix}$	$\begin{bmatrix} \sigma_{Hmax} & 0 & 0 \\ 0 & \sigma_{hmin} & 0 \\ 0 & 0 & \sigma_v \end{bmatrix}$

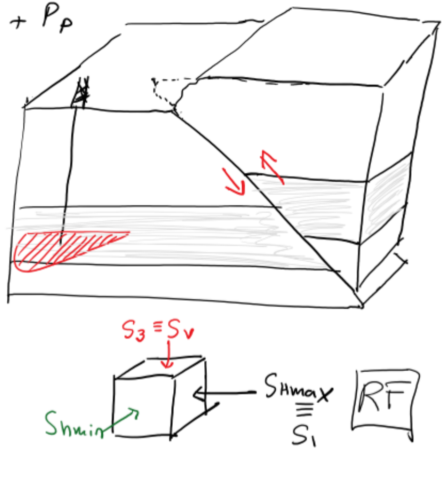
Effective stresses : $\underline{\underline{\sigma}}$; Total stresses $\underline{\underline{S}} = \underline{\underline{\sigma}} + P_p \underline{\underline{I}}$ (*)

$\begin{bmatrix} S_v & 0 & 0 \\ 0 & S_{Hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix}$	$\begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_v & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix}$	$\begin{bmatrix} S_{Hmax} & 0 & 0 \\ 0 & S_{hmin} & 0 \\ 0 & 0 & S_v \end{bmatrix}$

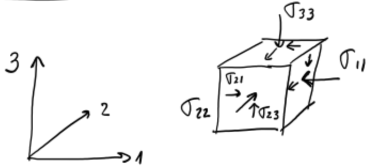
$S_v = \sigma_v + P_p$; $S_{Hmax} = \sigma_{Hmax} + P_p$; $S_{hmin} = \sigma_{hmin} + P_p$



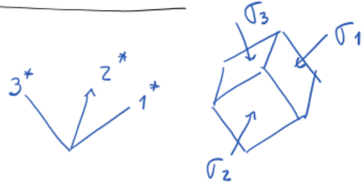




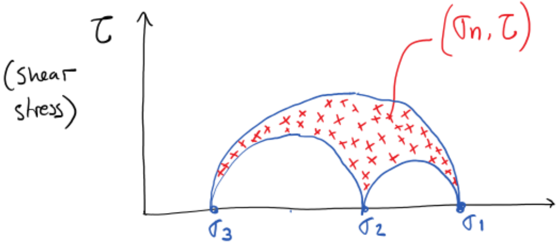
Stress Invariants and graphical representation

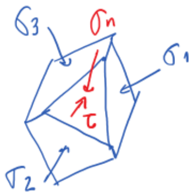


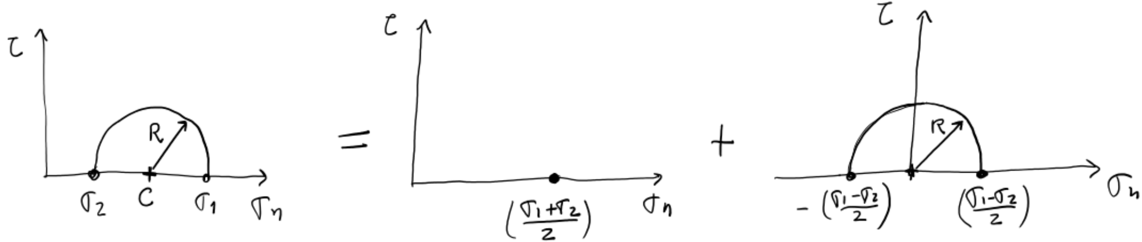
$\Longrightarrow$



$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$







isotropic

deviatoric
(pure shear)

$$\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \underbrace{\begin{bmatrix}\sigma_m & 0 & 0\\ 0 & \sigma_m & 0\\ 0 & 0 & \sigma_m\end{bmatrix}}_{\text{isotropic}} + \underbrace{\begin{bmatrix}\sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m\end{bmatrix}}_{\text{deviatoric}}$$

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$$

$$\boxed{\underline{\underline{\sigma}} = \sigma_m \underline{\underline{I}} + \underline{\underline{S}}_d}$$

Invariants (do not change wrt coord system)

$$\Rightarrow I_1(\underline{\underline{\sigma}}) = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_1 + \sigma_2 + \sigma_3 \quad \Longrightarrow \quad \sigma_m = \frac{I_1(\underline{\underline{\sigma}})}{3}$$

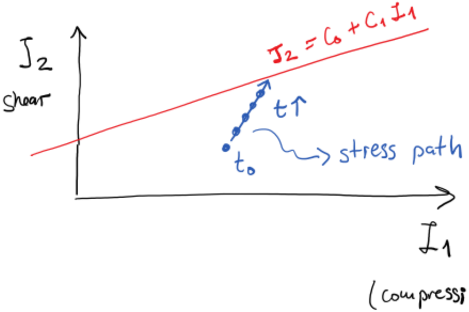
$$I_2(\underline{\underline{\sigma}}) = \sigma_{11}\sigma_{22} + \sigma_{11}\cdot\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$I_3(\underline{\underline{\sigma}}) = \det(\underline{\underline{\sigma}}) = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \leftarrow$$

$$J_1(\underline{\underline{s}}_d) = 0$$

$$\Rightarrow J_2(\underline{\underline{s}}_d) = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]$$

$$J_3(\underline{\underline{s}}_d) = (\sigma_1-\sigma_m)\cdot(\sigma_2-\sigma_m)\cdot(\sigma_3-\sigma_m)$$



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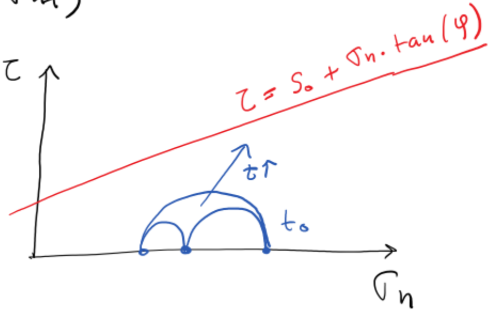
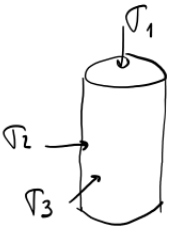


soft sediments (soil mechanics)

$$p' = \underbrace{\sigma_m}_{\text{effective}} = I_1(\underline{\sigma})/3$$

$$q = \sqrt{3 J_2}$$

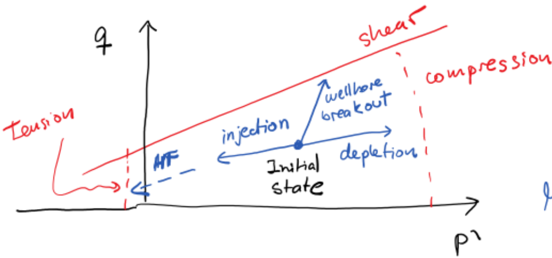


$\Rightarrow$ Axisymmetric stress conditions

$\sigma_2 = \sigma_3$

$$p' = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_1 + 2\sigma_3}{3}$$

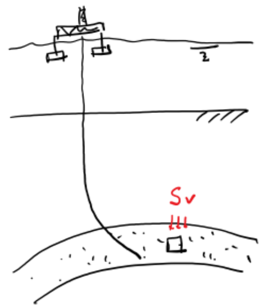
$$q = \sqrt{\cancel{3}\left\{\frac{1}{\cancel{6}_{\cancel{3}}}\left[(\sigma_1-\sigma_3)^2 + \cancel{(\sigma_1-\sigma_3)^2}\right]\right\}} = \sigma_1 - \sigma_3$$

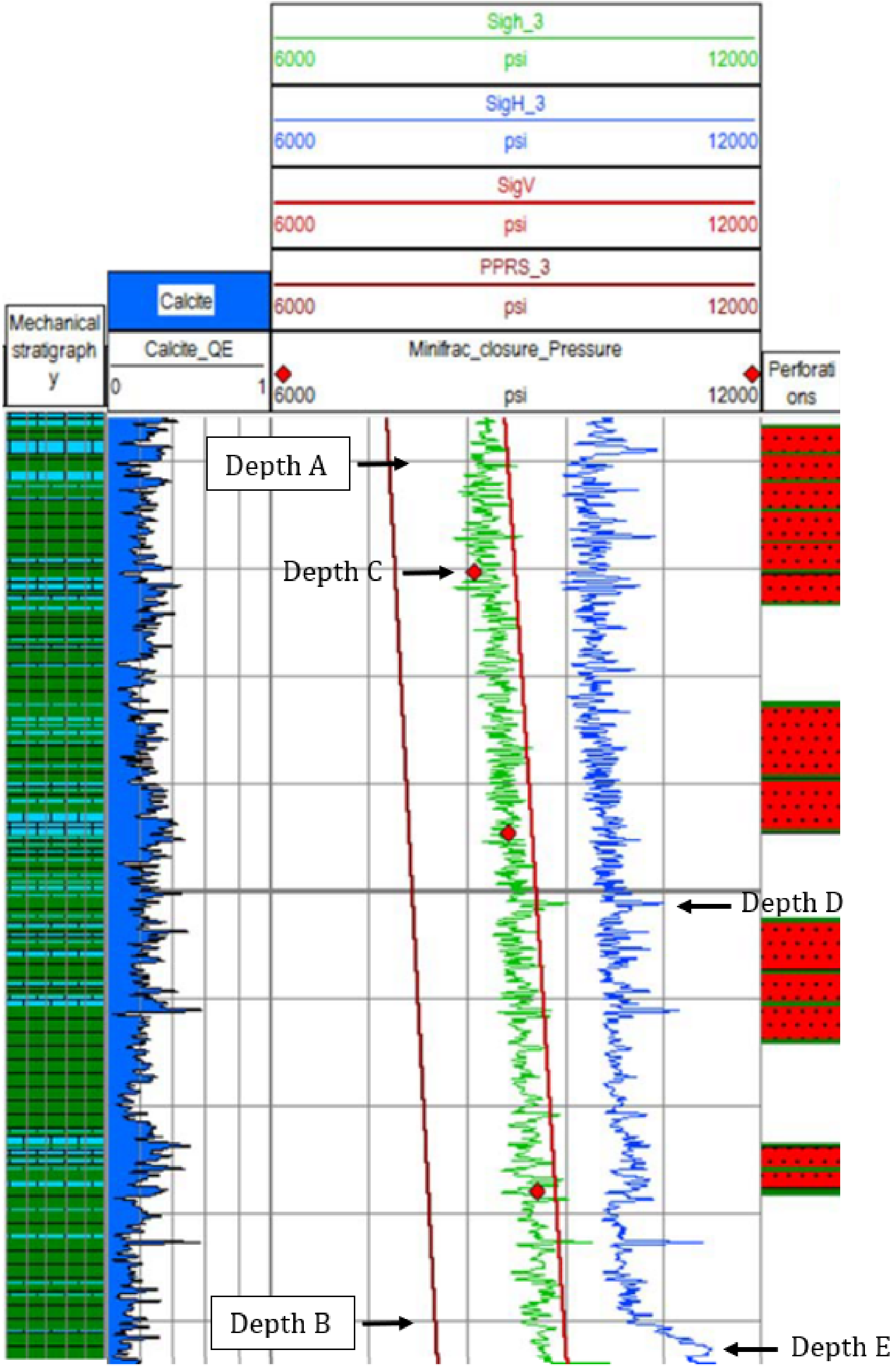


stress path

$\Downarrow$

track state of stress as a function of time





Sigh_3
6000
psi
12000
SigH_3
6000
psi
12000
SigV
6000
psi
12000
PPRS_3
6000
psi
12000
Mechanical stratigraphy
Calcite
Calcite_QE
0
1
Minifrac_closure_Pressure
6000
psi
12000
Perforations
Depth A
Depth C
Depth D
Depth B
Depth E

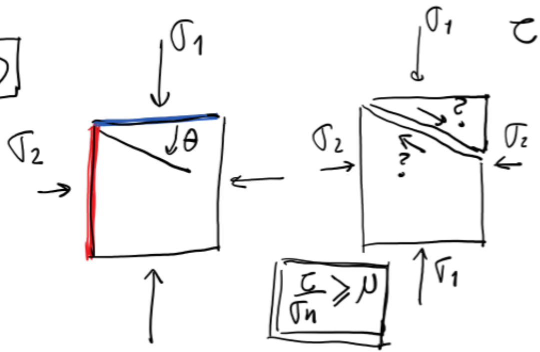
p

I_1

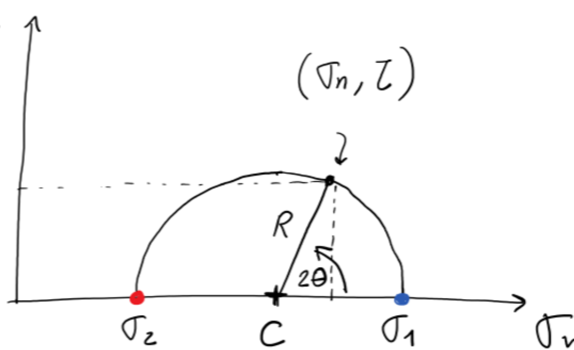
J_2

Stress projection on a plane

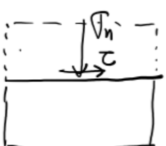
$\boxed{2D}$



$$\boxed{\frac{\tau}{\sigma_n} \geq \mu}$$



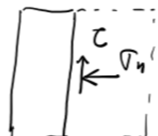
$\theta = 0$



$\underline{\sigma_n = \sigma_1 \; ; \; \tau = 0}$

σ_n, τ

$\theta = 90°$



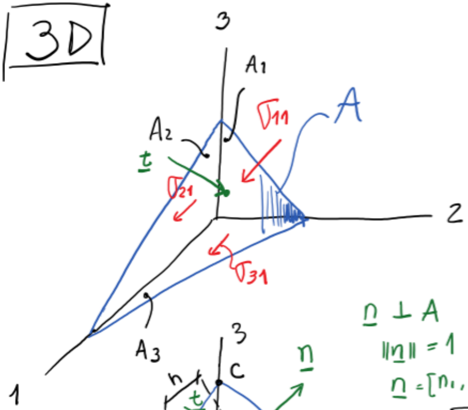
$\sigma_n = \sigma_2 \; ; \; \tau = 0$

$$\begin{cases} \sigma_n = C + R \cdot \cos 2\theta \\ \tau = R \cdot \sin 2\theta \end{cases}$$

$$C = (\sigma_1 + \sigma_2)/2$$

$$R = (\sigma_1 - \sigma_2)/2$$

$\boxed{3D}$



$$\Sigma F_1 = 0$$

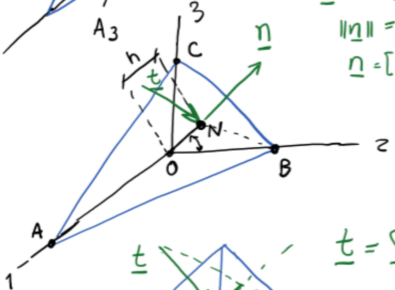
① $\sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3 = t_1 A$

② Vol  $= \frac{1}{3} A \cdot h$

$$\underbrace{= \frac{1}{3} A_1 \cdot \overline{OA}}$$

$$\cancel{\tfrac{1}{3}} A h = \cancel{\tfrac{1}{3}} A_1 \overline{OA}$$

$$A_1 = \frac{h}{\overline{OA}} A$$



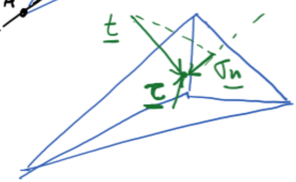
$\underline{n} \perp A$

$\|\underline{n}\| = 1$

$\underline{n} = [n_1, n_2, n_3]$

$$\begin{cases} A_1 = \cos A\hat{O}N \cdot A = n_1 A \\ A_2 = \cos B\hat{O}N \cdot A = n_2 A \\ A_3 = \underbrace{\cos C\hat{O}N}_{\text{cosine directors}} \cdot A = n_3 A \end{cases}$$

$$\boxed{A_i = n_i A}$$



$\underline{t} = \underline{\sigma_n} + \underline{\tau}$

→ ①+② $\sigma_{11} n_1 \cancel{A} + \overset{=\sigma_{12}}{\sigma_{21}} n_2 \cancel{A} + \overset{=\sigma_{13}}{\sigma_{31}} n_3 \cancel{A} = t_1 \cancel{A}$

⇓ $\sigma_{21} = \sigma_{12}$ (angular momentum equil.)

→ $$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

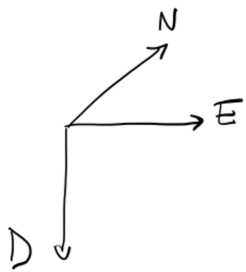
row × column

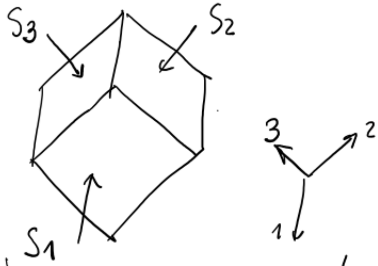
$$\underline{t} = \underline{\sigma} \cdot \underline{n}$$

$$\begin{cases} \sigma_n = \underline{t} \cdot \underline{n} \quad \} \text{ proyection of } \underline{\sigma_n} \text{ on } \underline{n} \\ \tau = \sqrt{\|\underline{t}\|^2 - \|\sigma_n\|^2} \quad \} \Leftarrow \|\underline{t}\|^2 = \|\sigma_n\|^2 + \|\tau\|^2 \end{cases}$$

Geographical coordinate system

N - E - D





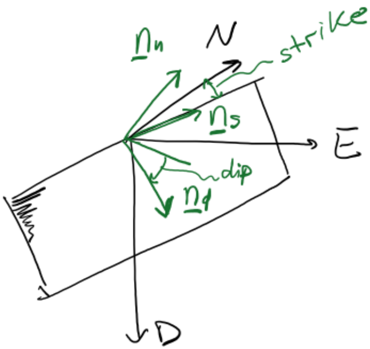
Principal stresses and direction

$$\underline{\underline{S}}_P = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

$$\underline{\underline{S}}_G = \begin{bmatrix} S_{NN} & S_{NE} & S_{ND} \\ S_{EN} & S_{EE} & S_{ED} \\ S_{DN} & S_{DE} & S_{DD} \end{bmatrix}$$

$$\boxed{\underline{\underline{S}}_G = R_{PG}^T \, \underline{\underline{S}}_P \, R_{PG}}$$

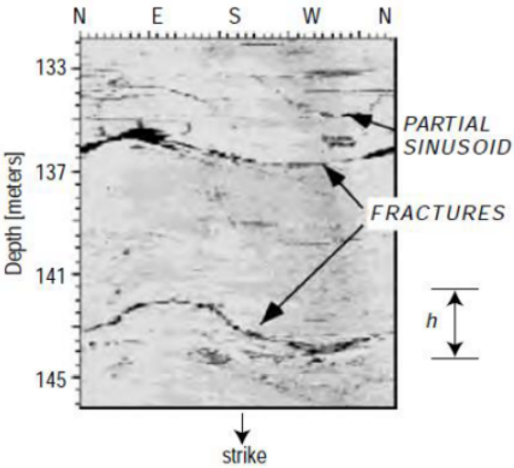
$$R_{PG} = \underbrace{\begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}}_{f(\alpha, \beta, \gamma)}$$



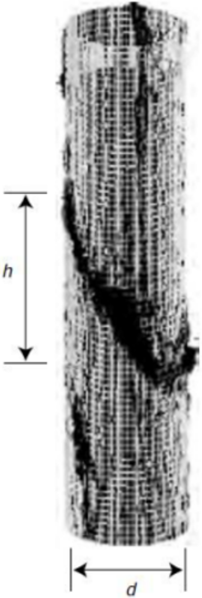
τ

σ_n

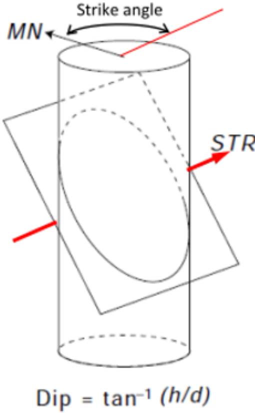
Un-wrapped image
(ultrasonic)



3D-representation



Interpretation



(Zoback 2013, RM, Ch 5.3)

O

(σ_n, τ)

τ / σ_n

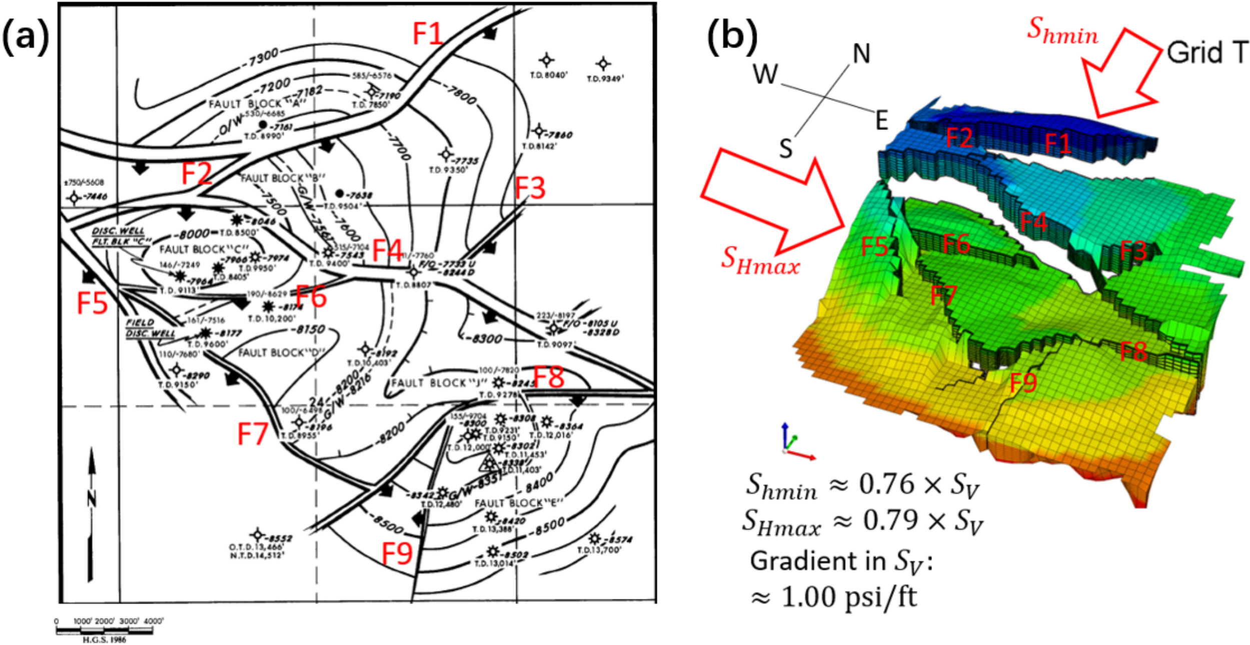
$$\mu = 0.5 \pm 0.1$$

$$60^\circ \pm 10^\circ$$

$$80^\circ \pm 5^\circ$$

$$10^\circ \pm 10^\circ$$

$$60^\circ \pm 5^\circ$$



(a)
F1
F2
F3
F4
F5
F6
F7
F8
F9
FAULT BLOCK "A"
FAULT BLOCK "B"
FAULT BLOCK "C"
FAULT BLOCK "D"
FAULT BLOCK "E"
FAULT BLOCK "J"
DISC. WELL FLT. BLK "C"
FIELD DISC. WELL
-7300
-7200
-7800
-7700
-7500
-7600
-8000
-8150
-8300
-8200
-8400
-8500
N
0 1000' 2000' 3000' 4000'
H.G.S. 1986
(b)
W
N
E
S
S_{hmin}
Grid T
S_{Hmax}
F1
F2
F3
F4
F5
F6
F7
F8
F9
$S_{hmin} \approx 0.76 \times S_V$
$S_{Hmax} \approx 0.79 \times S_V$
Gradient in S_V:
≈ 1.00 psi/ft

S_{hmin}

σ_n/σ_v

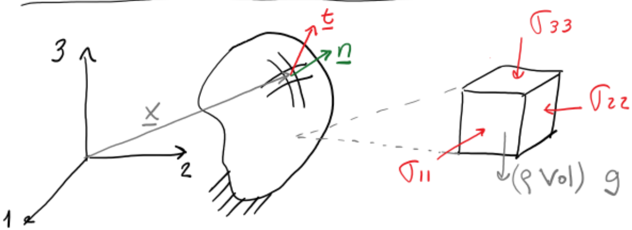
$$\tau / \sigma_v$$

$P_p/$

$$z = 0.44$$

$$\mu = 0.6$$

General solution to a continuum mechanics problem



$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \rho\, \underline{a} & \rightarrow \text{Equilibrium (Cauchy's)} \\ \underline{\underline{\varepsilon}} = F_1(\underline{u}) & \rightarrow \text{Kinematic eq} \\ \underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}}) & \rightarrow \text{Constitutive equations} \end{cases}$$

Kinematic eq:
- small strains ←
- large strains

Constitutive equations:
- linear isotropic elastic solid ←
- TVI (VTI)
- orthorhombic
- visco-elasticity
- plasticity

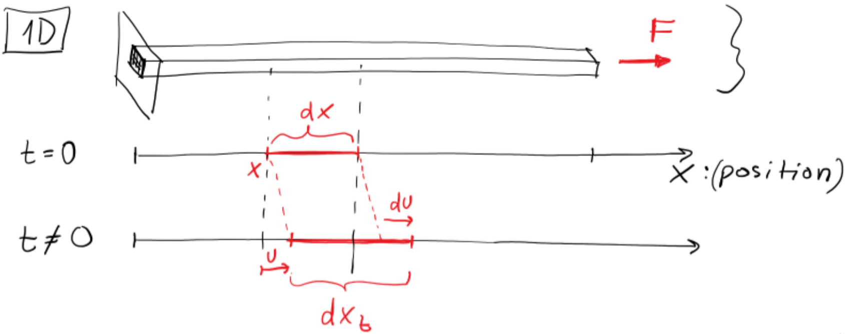
$$\nabla \cdot \left[F_2(\underline{\underline{\varepsilon}}) \right] + \underline{f} = \rho\, \underline{a}$$

$$\boxed{\nabla \cdot \left[F_2 \left[F_1(\underline{u}) \right] \right] + \underline{f} = \rho\, \underline{a}}$$

displacement $\underline{u} \rightarrow \underline{\underline{\varepsilon}} \rightarrow \underline{\underline{\sigma}}$

Kinematic Equations (small strains) $\boxed{\underline{\underline{\varepsilon}} = F_1(\underline{u})}$

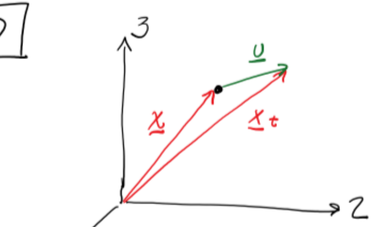
$\boxed{1D}$



$$\varepsilon = \frac{dx_t - dx}{dx} = \frac{[\cancel{x} + \cancel{u} + \cancel{dx} + du - (\cancel{x} + \cancel{u})] - [\cancel{x} + \cancel{dx} - \cancel{x}]}{[\cancel{x} + dx - \cancel{x}]}$$

$$\boxed{\varepsilon = \frac{du}{dx}}$$

$\boxed{3D}$



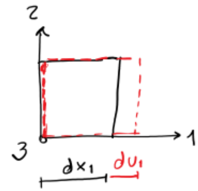
$\underline{x} = [x_1, x_2, x_3]$

$\underline{u} = [u_1, u_2, u_3]$

Jacobian $\updownarrow$ $\underline{\underline{\varepsilon}}$

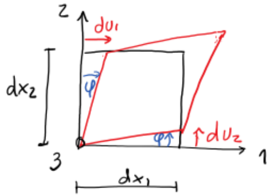
$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial u_2}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial u_3}{\partial x_3}
\end{bmatrix}
$$



$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$
+
\begin{bmatrix}
0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\
\cdots & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\
\cdots & \cdots & 0
\end{bmatrix}
$$

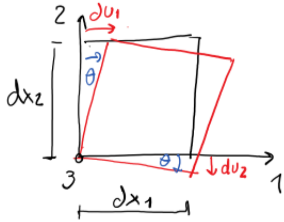


$$\tan\varphi = \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}$$

$$\varepsilon_{12} = \frac{1}{2}(2 \cdot \tan\varphi) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\underline{\underline{\varepsilon}}$$

$$
+
\begin{bmatrix}
0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\
\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\
\cdots & \cdots & 0
\end{bmatrix}
$$



$$\underline{\underline{\omega}}$$

$$\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right) = \omega_{12}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

↳ symmetric ⎫ eigenvalues ⇒ principal strains

↕

↳ real values ⎭ $\underbrace{\text{eigen}}_{\text{own}}$ vectors → principal directions

$$I_1(\underline{\underline{\varepsilon}}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{vol}$$

$$\boxed{\varepsilon_{vol} = \frac{Vol(t) - Vol_0}{Vol_0}}$$

$$\underline{\underline{\sigma}} = F_2(\underline{\underline{\varepsilon}})$$

$\underbrace{\underline{\underline{\sigma}}}_{\text{stress}} = F_2(\underbrace{\underline{\underline{\varepsilon}}}_{\text{strain}})$

Superposition $\left\{ \begin{array}{l} \text{space} \\ \text{time} \end{array} \right\}$ Green's functions

linear relationships $\left\{ \begin{array}{l} F_2(A+B) = F_2(A) + F_2(B) \\ F_2(c \cdot A) = c \cdot F_2(A) \end{array} \right.$

$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}} \longleftrightarrow \underset{\sim}{y} = \underbrace{b}_{\text{constant}} \cdot \underset{\sim}{x}$$

9	81	9
6	36	6

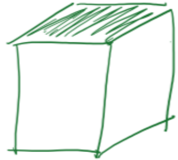
Voigt Notation $\left(\underset{3\times 3}{\underline{\underline{\sigma}}} \rightarrow \underset{6\times 1}{\underline{\sigma}} \right)$

$$\rightarrow \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

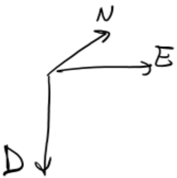
$$\underset{6\times 1}{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}} = \underbrace{\underset{6\times 6}{\left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot \\ \hline \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ C_{61} & C_{62} & \cdot & \cdot & \cdot & C_{66} \end{array}\right]}}_{\text{Stiffness Matrix}} \underset{6\times 1}{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\cdot\varepsilon_{23} \\ 2\cdot\varepsilon_{13} \\ 2\,\varepsilon_{12} \end{bmatrix}}$$

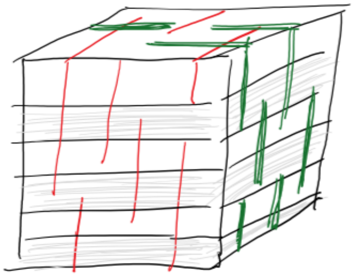
Shear decoupling $\left\langle \begin{array}{l} \text{normal} \leftrightarrow \text{shear} \\ \text{shear} \leftrightarrow \text{shear} \end{array} \right\}$ does not apply for plasticity

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{array}\right]
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\cdot\varepsilon_{23} \\ 2\cdot\varepsilon_{13} \\ 2\,\varepsilon_{12} \end{bmatrix}
$$



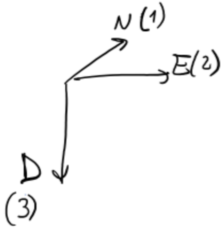
12 coeff → $\boxed{\text{9 coeff}}$ → Orthorhombic

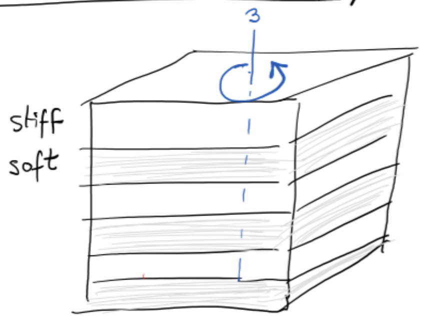




$$E_V \neq E_{H(N\text{-}S)} \neq E_{H(E\text{-}W)}$$

Vertical Transverse Isotropy





$$E_V < E_H$$

$$\rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{array}\right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}}_{\boxed{\text{5 indepent coefficients}}}$$

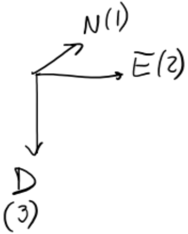
C_{11}, C_{33}

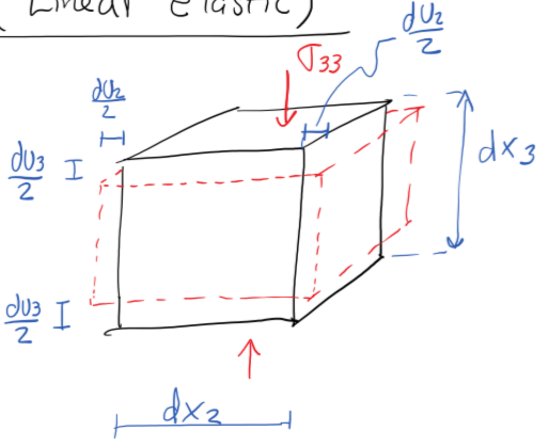
C_{12}, C_{13}

$C_{44}, \cancel{C_{66}}$

$$C_{66} = \frac{C_{11} - C_{12}}{2}$$

Isotropy (Linear elastic)





$$\varepsilon_{33} = \frac{dU_3}{dx_3}$$

$$\varepsilon_{22} = \frac{dU_2}{dx_2}$$

$$\varepsilon_{11} = \frac{dU_1}{dx_2}$$

$$\boxed{E \overset{\text{def}}{=} \frac{\sigma_{33}}{\varepsilon_{33}}}$$

Young's modulus

$$\boxed{\nu \overset{\text{def}}{=} -\frac{\varepsilon_{11}}{\varepsilon_{33}} = -\frac{\varepsilon_{22}}{\varepsilon_{33}}}$$

Poisson's ratio

$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 \\ 0 \\ \sigma_{33} \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\underline{\varepsilon}} = \begin{bmatrix} -(\nu/E)\sigma_{33} \\ -(\nu/E)\sigma_{33} \\ \sigma_{33}/E \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \begin{array}{l} \sigma_{33} \\ \sigma_{22} \\ \sigma_{11} \end{array}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{array}\right] \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \leftarrow G = \frac{E}{2(1+\nu)}$$

Isotropic loading $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$$\rightarrow \varepsilon_{11} = \frac{1-2\nu}{E}\sigma_m$$

$$\varepsilon_{vol} = \frac{3(1-2\nu)}{E}\sigma_m \Rightarrow K = \frac{E}{3(1-2\nu)}$$

$$\underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}}_{\text{compliance matrix}} \cdot \underline{\underline{\sigma}}$$

E, ν (2 indep. coeff)

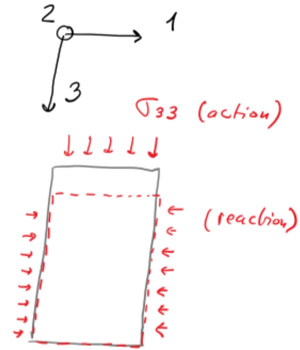
$$\underbrace{\underline{\underline{\underline{\underline{D}}}}^{-1}} \cdot \underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{D}}}}^{-1} \cdot \underline{\underline{\underline{\underline{D}}}}} \cdot \underline{\underline{\sigma}}$$

stiffness matrix

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{2} \end{array}\right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}$$

Uniaxial-strain loading (stress path)



$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \hline \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc|ccc} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ \hline & & & \frac{1-2\nu}{2} & 0 & 0 \\ & 0 & & 0 & \frac{1-2\nu}{2} & 0 \\ & & & 0 & 0 & \frac{1-2\nu}{\nu} \end{array}\right]\begin{bmatrix}\cancel{\varepsilon_{11}}\\ \cancel{\varepsilon_{22}}\\ \varepsilon_{33}\\ \hline \cancel{2\varepsilon_{23}}\\ \cancel{2\varepsilon_{13}}\\ \cancel{2\varepsilon_{12}}\end{bmatrix}$$

$$\begin{cases}\varepsilon_{11} = \varepsilon_{22} = 0\ ;\ \varepsilon_{33} \neq 0\\ \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0\end{cases}$$

$$\boxed{\sigma_{33} = \underbrace{\frac{(1-\nu)\ E}{(1+\nu)(1-2\nu)}} \cdot \varepsilon_{33}}$$

M : constrained modulus
: oedometric modulus
: P-wave modulus

$M \geqslant E$
for $\nu \geqslant 0$

$$\sigma_{11} = \frac{\nu\ E}{(1+\nu)(1-2\nu)} \cdot \varepsilon_{33} = \frac{\nu\ \cancel{E}}{\cancel{(1+\nu)(1-2\nu)}} \cdot \frac{\cancel{(1+\nu)(1-2\nu)}}{(1-\nu)\ \cancel{E}} \cdot \sigma_{33}$$

$$\boxed{\sigma_{11} = \underbrace{\frac{\nu}{1-\nu}}\ \sigma_{33}}$$

lateral effective stress coefficient
K_0

1D Mechanical Earth Model with tectonic strains

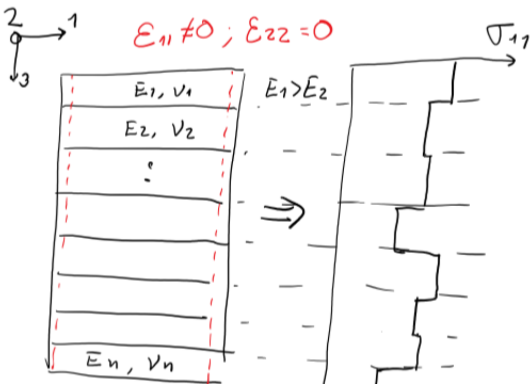
$\varepsilon_{33} \neq 0$; $\varepsilon_{11} \neq 0$; $\varepsilon_{22} \neq 0$; $\varepsilon_{ij} = 0$ for $i \neq j$

$$\left\{ \begin{aligned} \sigma_{11} &= \overbrace{\frac{\nu}{1-\nu}}^{K_0}\sigma_{33} + \frac{E}{1-\nu^2}\varepsilon_{11} + \frac{\nu E}{1-\nu^2}\varepsilon_{22} \\ \sigma_{22} &= \frac{\nu}{1-\nu}\sigma_{33} + \frac{\nu E}{1-\nu^2}\varepsilon_{11} + \frac{E}{1-\nu^2}\varepsilon_{22} \end{aligned} \right.$$

→ tectonic strains ε_{11} , ε_{22}

$$\underbrace{E' = \frac{E}{1-\nu^2}}_{\text{Plane-strain Modulus}}$$

$\sigma_{33} = S_{33} - P_p$



(E, ν) from field and laboratory data

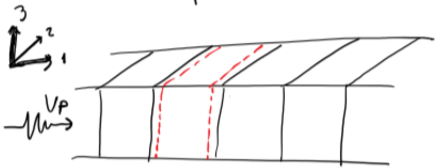
↳ lab: static, dynamic

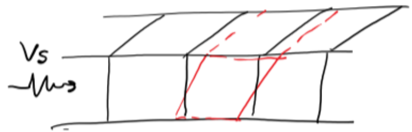
↳ field: dynamic

(E, ν) →

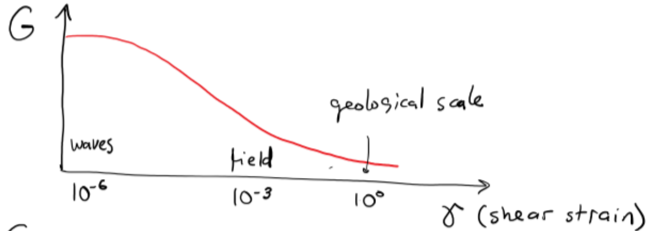
$$V_P = \sqrt{\frac{M}{\rho_{bulk}}} \quad \text{(P-WAVE)}$$

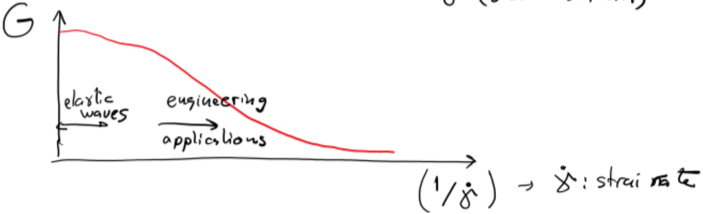
$$V_S = \sqrt{\frac{G}{\rho_{bulk}}} \quad \text{(S-WAVE)}$$





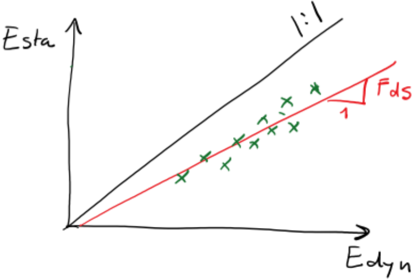
$$\begin{cases} E_{dyn} = \rho_{bulk} V_S^2 \left(\dfrac{3V_P^2 - 4V_S^2}{V_P^2 - V_S^2} \right) \\[2ex] \nu_{dyn} = \dfrac{V_P^2 - 2 \cdot V_S^2}{2(V_P^2 - V_S^2)} \end{cases}$$


G
geological scale
waves
field
10^{-6}
10^{-3}
10^{0}
γ (shear strain)

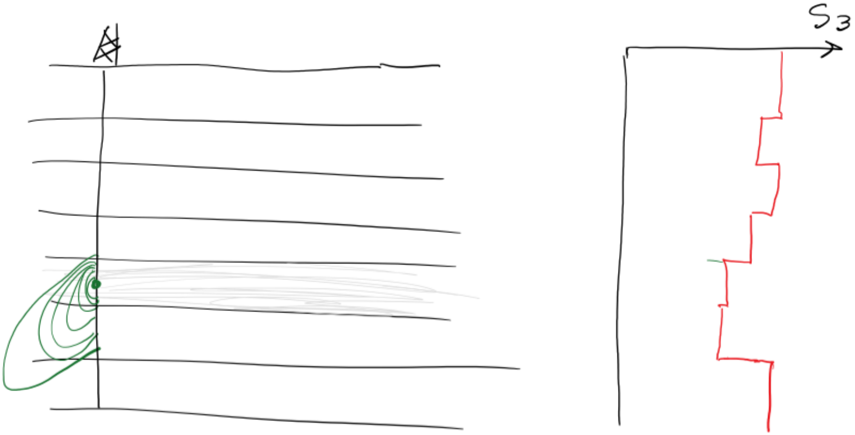

G
elastic waves
engineering applications
$(1/\dot{\gamma}) \rightarrow \dot{\gamma}$: strain rate

$$\boxed{\underbrace{E_{sta}}_{\text{Triaxial Test}} = F_{ds} \cdot \underbrace{E_{dyn}}_{\text{ultrasonic elastic wave velocity}}} \quad ; \quad \boxed{V_{sta} \approx V_{dyn}}$$

LAB	Triaxial Test	ultrasonic elastic wave velocity
FIELD	—	acoustic log



$F_{ds} < 1$


S_3

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}
=
\begin{bmatrix}
+\frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\
-\frac{\nu_h}{E_h} & +\frac{1}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\
-\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & +\frac{1}{E_v} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_v} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_v} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_h}
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}
$$

$$G_h = \frac{E_h}{2(1 + \nu_h)}$$

$$G_v$$

$$E_h = \frac{(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2\,C_{13}^2\right]}{C_{11}C_{33} - C_{13}^2}$$

$$E_v = C_{33} - \frac{2\,C_{13}^2}{C_{11} + C_{12}}$$

$$\nu_h = \frac{C_{12}C_{33} - C_{13}^2}{C_{11}C_{33} - C_{13}^2}$$

$$\nu_v = \frac{C_{13}}{C_{11} + C_{12}}$$

$$G_v = C_{44}$$

$$G_h = C_{66} = \frac{C_{11} - C_{12}}{2}$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}
$$

$$C_{11} = \left[\frac{1}{(1-\nu_h)E_v - 2\nu_v^2 E_h} \right] \left(\frac{E_h E_v - \nu_v^2 E_h^2}{1+\nu_h} \right)$$

$$C_{33} = \left[\frac{1}{(1 - \nu_h) E_v - 2\nu_v^2 E_h} \right] (E_v^2 - \nu_h E_v^2)$$

$$C_{12} = \left[\frac{1}{(1-\nu_h)E_v - 2\nu_v^2 E_h} \right] \left(\frac{\nu_v^2 E_h^2 + \nu_h E_h E_v}{1+\nu_h} \right)$$

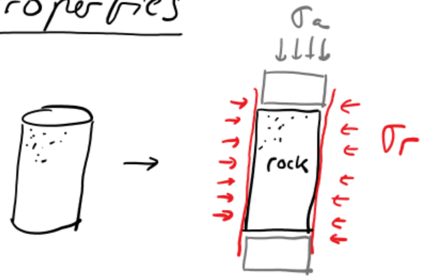
$$C_{13} = \left[\frac{1}{(1-\nu_h)E_v - 2\nu_v^2 E_h} \right] (\nu_v E_h E_v)$$

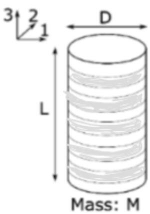
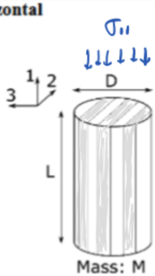
$$C_{66} = \frac{C_{11} - C_{12}}{2} = G_h = \frac{E_h}{2(1 + \nu_h)}$$

C_{44}

VTI Static Elastic Properties

- Conventional Method
  - ↳ Axisymmetric Triaxial Cell
  - ↳ Deviatoric Loading Stress Path
    - ↳ $\sigma_r = cst$
    - ↳ $\Delta(\sigma_a - \sigma_r) = \Delta\sigma_a \quad , \quad \sigma_a \geq \sigma_r$

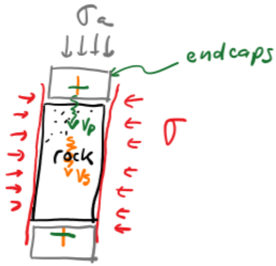


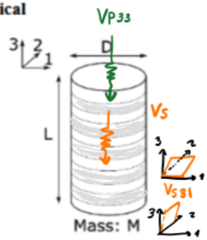
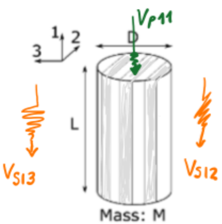
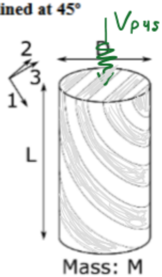
Sample	Quasi-static
Vertical 	Vertical Young modulus $E_v = \left.\dfrac{\Delta\sigma_{33}}{\Delta\varepsilon_{33}}\right\vert_{\sigma_{11},\sigma_{22}}$ Vertical Poisson ratio $\nu_v = -\dfrac{1}{2}\left(\dfrac{\Delta\varepsilon_{11}}{\Delta\varepsilon_{33}} + \dfrac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{33}}\right) = -\overbrace{\dfrac{\Delta\varepsilon_{11}}{\Delta\varepsilon_{33}}}^{\nu_{13}} = -\overbrace{\dfrac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{33}}}^{\nu_{23}}$
Horizontal 	Horizontal Young modulus $E_h = \dfrac{\Delta\sigma_{11}}{\Delta\varepsilon_{11}}$ Vertical Poisson ratio $\nu_v = -\dfrac{\Delta\varepsilon_{33}}{\Delta\varepsilon_{11}} = \nu_{31} \longleftarrow = \nu_{13}$ Horizontal Poisson ratio $\nu_h = -\dfrac{\Delta\varepsilon_{22}}{\Delta\varepsilon_{11}} = \nu_{21} \Rightarrow = \nu_{12}$

4 parameters

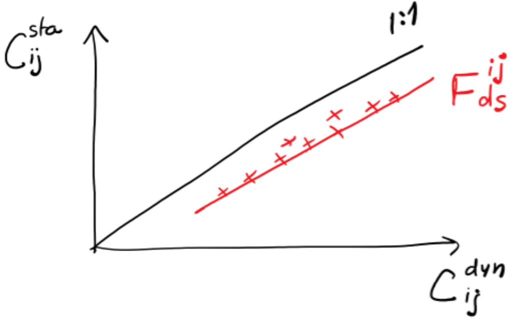
VTI Dynamic Elastic Properties

- Conventional Method



Sample	Dynamic
Vertical 	P-wave stiffness perpendicular to bedding $$C_{33} = \rho(V_{p33})^2$$ S-wave stiffness perpendicular to bedding $$C_{44} = \frac{1}{2}[\rho(V_{s31})^2 + \rho(V_{s32})^2]$$ $$C_{44} = \rho(V_{s31})^2 = \rho(V_{s32})^2$$
Horizontal 	P-wave stiffness parallel to bedding $$C_{11} = \rho(V_{p11})^2$$ S-wave stiffness perpendicular to bedding $$C_{44} = \rho(V_{s13})^2$$ S-wave stiffness in the plane of bedding $$C_{66} = \rho(V_{s12})^2$$
Inclined at 45° 	Off-diagonal stiffness $$C_{13} = -C_{44} + [4\rho^2 V_{p45}^4 - 2\rho V_{p45}^2(C_{11} + C_{33} + 2C_{44}) + (C_{11} + C_{44})(C_{33} + C_{44})]^{1/2}$$

Dynamic to static conversion



Quantification of anisotropy

Static

Young modulus anisotropy

$$\frac{E_h}{E_v}\ ; \quad E_h > E_v$$

Poisson's ratio anisotropy

$$\frac{\nu_v}{1-\nu_h}$$ } effective lateral stress coefficient

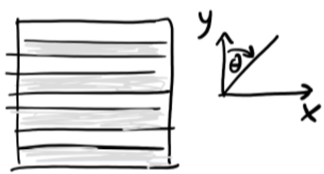
Dynamic

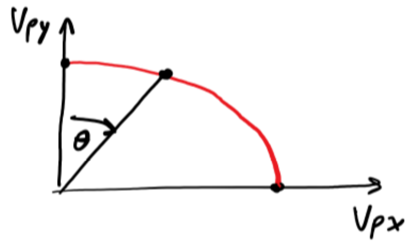
Thomsen Parameters :

$$\varepsilon = \frac{V_{p11}^2 - V_{p33}^2}{2\,V_{p33}^2} = \frac{C_{11} - C_{33}}{2\,C_{33}}$$

$$\gamma = \frac{C_{66} - C_{44}}{2\,C_{44}}$$

$$\delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2\,C_{33}\,(C_{33} - C_{44})}$$

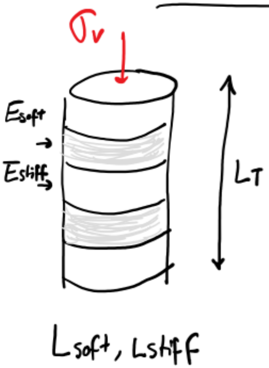




Weak anisotropy

$$V_p(\theta) = V_{p33}\left[1 + \delta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta\right]$$

Iso-stress (Reuss Average)



$$\langle E_v\rangle = \frac{\sigma_v}{\langle\varepsilon_v\rangle}$$

$$\hookrightarrow \langle\varepsilon_v\rangle = \frac{\sigma_v}{\langle E_v\rangle} \quad (1)$$

$$(2)\quad \langle\varepsilon_v\rangle = \left(\frac{\sigma_v}{E_{stiff}}\cdot L_{stiff} + \frac{\sigma_v}{E_{soft}}\cdot L_{soft}\right)\cdot\frac{1}{L_T}$$

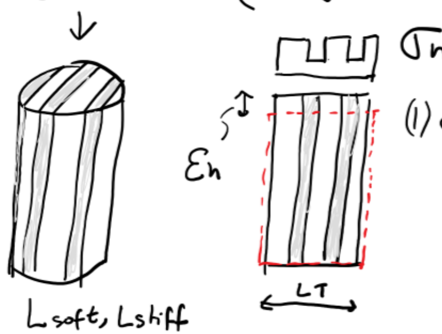
$$= \frac{\sigma_v}{E_{stiff}}\cdot\underbrace{\frac{L_{stiff}}{L_T}}_{f_{stiff}} + \frac{\sigma_v}{E_{soft}}\cdot\underbrace{\frac{L_{soft}}{L_T}}_{f_{soft}}$$

(1,2) →

$$\frac{\cancel{\sigma_v}}{\langle E_v\rangle} = \frac{\cancel{\sigma_v}}{E_{stiff}}\cdot f_{stiff} + \frac{\cancel{\sigma_v}}{E_{soft}}\cdot f_{soft}$$

$$\boxed{\langle E_v\rangle = \left(\frac{f_{stiff}}{E_{stiff}} + \frac{f_{soft}}{E_{soft}}\right)^{-1}}$$

Iso-strain (Voigt Average)



$$(1)\quad \langle E_h\rangle$$

$$\frac{\langle\sigma_n\rangle}{\varepsilon_h}$$

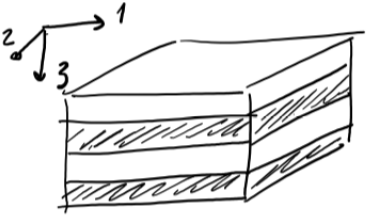
$$(2)\quad \langle\sigma_h\rangle = \sigma_h^{soft}\cdot\underbrace{\frac{L_{soft}}{L_T}} + \sigma_n^{stiff}\cdot\underbrace{\frac{L_{stiff}}{L_T}}$$

(1,2)

$$\cancel{\varepsilon_h}\langle E_h\rangle = \cancel{\varepsilon_h}E_{soft}\cdot f_{soft} + \cancel{\varepsilon_h}\cdot E_{stiff}\cdot f$$

$$\boxed{\langle E_h\rangle = E_{soft}\cdot f_{soft} + E_{stiff}\cdot f_{stiff}}$$

VTI stiffness matrix



$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} \\ -\frac{\nu_h}{E_h} & \frac{1}{E_h} & -\frac{\nu_v}{E_v} \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & \frac{1}{E_v} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix}$$

For isotropic loading $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$

$$\varepsilon_{11} = \varepsilon_{22} = \left(\frac{1-\nu_h}{E_h} - \frac{\nu_v}{E_v} \right) \sigma_m$$

$$\varepsilon_{33} = \left(-\frac{2\nu_v}{E_v} + \frac{1}{E_v} \right) \sigma_m$$

$$\varepsilon_{vol} = \left[2\left(\frac{1-\nu_h}{E_h} - \frac{\nu_v}{E_v} \right) + \left(\frac{1-2\nu_v}{E_v} \right) \right] \sigma_m$$

$$\rightarrow \boxed{K_{VTI} = \left[\frac{2(1-\nu_h)}{E_h} + \frac{1-4\nu_v}{E_v} \right]^{-1}}$$

if $\nu_h = \nu_v$; $E_h = E_v$

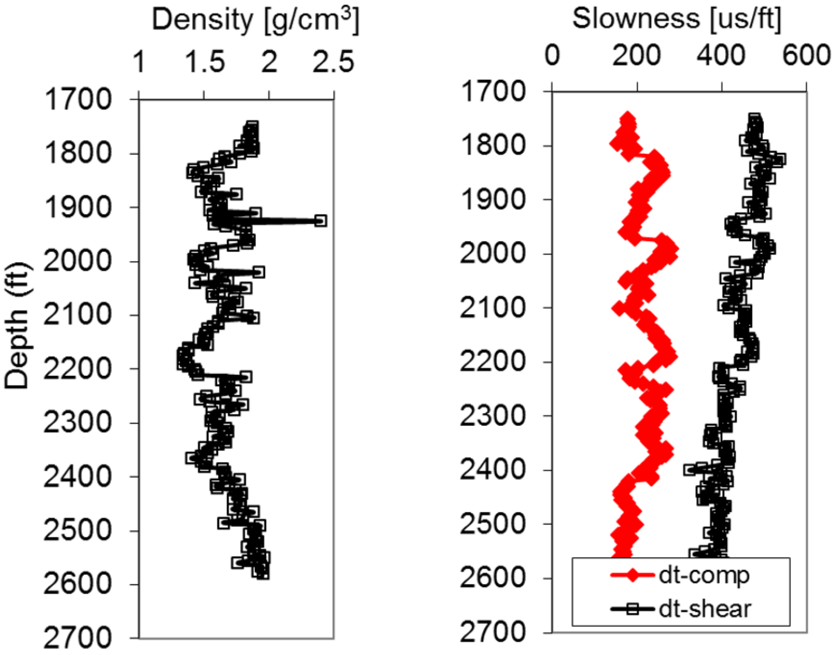
$$K = \left[\frac{2-2\nu+1-4\nu}{E} \right]^{-1} = \frac{E}{3(1-2\nu)} \quad \checkmark$$

$$E_{static} = 0.65 \cdot E_{dynamic}$$

$$E'_{static} = E_{static}/(1 - \nu^2)$$

$$\varepsilon_{Hmax} = 0.0015$$

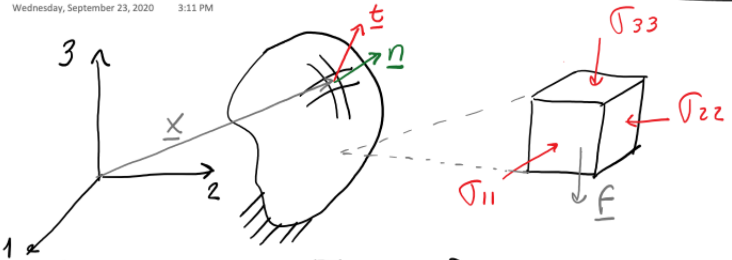
$$\varepsilon_{hmin} = 0$$



Density [g/cm3]
1
1.5
2
2.5
Depth (ft)
1700
1800
1900
2000
2100
2200
2300
2400
2500
2600
2700
Slowness [us/ft]
0
200
400
600
dt-comp
dt-shear

2- General solution for a continuum mechanics problem

Wednesday, September 23, 2020 3:11 PM



① Equilibrium : $\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = 0$

② Kinematic : $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$ → small strains

③ Constitutive : $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$ → linear elasticity

Plan : ① ← ③ ← ②

① Coordinate 1: $\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} \Rightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} ; \quad \lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)} = G$$

$$\frac{\partial}{\partial x_1}\Big(\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + 2\mu\,\varepsilon_{11}\Big) + \frac{\partial}{\partial x_2}\big(2\mu\,\varepsilon_{12}\big) + \frac{\partial}{\partial x_3}\big(2\mu\,\varepsilon_{13}\big) + f_1 = 0$$

$$\frac{\partial}{\partial x_1}\left(\lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\cancel{2}\mu\,\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right) + \frac{\partial}{\partial x_3}\left(\cancel{2}\mu\,\frac{1}{\cancel{2}}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\right) + f_1 = 0$$

$$\lambda\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)\right] + 2\mu\frac{\partial^2 u_1}{\partial x_1^2} + \mu\frac{\partial^2 u_1}{\partial x_2^2} + \mu\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}\right) + \mu\frac{\partial^2 u_1}{\partial x_3^2} + \mu\frac{\partial^2 u_3}{\partial x_3 \partial x_1} + f_1 = 0$$

$\leftarrow \dfrac{\partial^2 f(x,y)}{\partial x\,\partial y} = \dfrac{\partial^2 f(x}{\partial y\,\partial}$

$$\lambda\Bigg[\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}}\Bigg] + \mu\underbrace{\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)}_{\nabla^2 u_1} + \mu\frac{\partial}{\partial x_1}\underbrace{\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)}_{\nabla\cdot\underline{u}} + f_1 = 0$$

$$\lambda\frac{\partial}{\partial x_{\underline{1}}}(\nabla\cdot\underline{u}) + \mu\frac{\partial}{\partial x_{\underline{1}}}(\nabla\cdot\underline{u}) + \mu\nabla^2 u_{\underline{1}} + f_{\underline{1}} = 0 \Rightarrow \text{Coord. } 1$$

$$\boxed{(\lambda+\mu)\,\nabla(\nabla\cdot\underline{u}) + \mu\nabla^2\underline{u} + \underline{f} = \underline{0}} \qquad \underline{u}: \text{unknown}$$

Navier's Equation

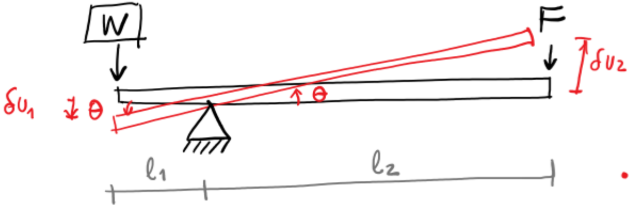
$\boxed{\text{Solution}}$

- Analytical
  ↓
  simple boundary conditions
  - Beltrami-Mitchell (strain compatibility)
    - $\underline{u} \rightarrow \underline{\underline{\varepsilon}}$ ✓
    - $\underline{\underline{\varepsilon}} \rightarrow u$ ✗
  - Airy's functions
    $$\nabla^4 \varphi = 0 \; ; \quad \sigma_{11} = \frac{\partial^2 \varphi}{\partial x_1^2}$$
  - Green functions (convolution) : Boussinesq
  - Examples: Kirsch, Griffith, Sneddon

- Numerical
  - Finite differences ; FLAC-Itasca
  - Finite Element Method (FEM) : Abaqus, FreeFEM++, FEniCS, Comsol

Weak formulation of continuum mechanics equations

Analogy with virtual work



Equil

- Solution #1: Angular Momentum

$$W \cdot l_1 = F \cdot l_2$$

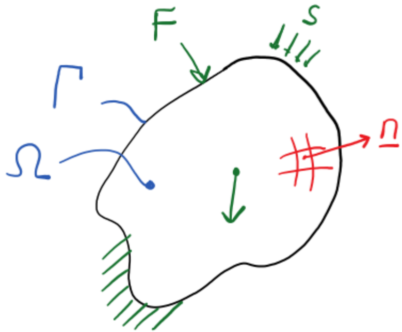
$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$

- Solution #2: Energy Conserv $\Leftrightarrow$ Principle of Virtual Work

$$W \cdot \delta u_1 = F \cdot \delta u_2$$

$$W \cdot (l_1 \cdot \cancel{\tan\theta}) = F \cdot (l_2 \cancel{\tan\theta})$$

$$\boxed{F = \frac{l_1}{l_2} \cdot W}$$



Equil: $\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \underline{0}$

$$-\nabla \cdot \underline{\underline{\sigma}} = \underline{f}$$

virtual displacement

$$\underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \left(-\nabla \cdot \underline{\underline{\sigma}}\right)}_{\text{Green's Theorem}} = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}$$

$$\int_\Omega \nabla \delta \underset{\sim}{u} \cdot \underline{\underline{\sigma}} - \int_\Gamma \delta \underset{\sim}{u} \cdot (\underline{\underline{\sigma}} \cdot \underline{n}) = \int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}$$

- Variational form
- Weak form

$$\boxed{\underbrace{\int_\Omega \varepsilon(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u})}_{\text{strain energy}} = \underbrace{\int_\Gamma \delta \underset{\sim}{u} \cdot \left(\underline{\underline{\sigma}}(\underline{u}) \cdot \underline{n}\right)}_{\text{stress boundary condition}} + \underbrace{\int_\Omega \delta \underset{\sim}{u} \cdot \underline{f}}_{\text{body force}}}$$

unknowns : $\underline{u}$; $\delta \underset{\sim}{u}$

actual displacement ; virtual displacement

$$\underset{\sim}{\varepsilon}(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \underset{\sim}{\varepsilon}_{11} \cdot \sigma_{11} + \underset{\sim}{\varepsilon}_{22} \cdot \sigma_{22} + \underset{\sim}{\varepsilon}_{33} \cdot \sigma_{33} + \underset{\sim}{\varepsilon}_{12} \cdot \sigma_{12} + \ldots$$

$$\rightarrow E = P \cdot V \quad \text{(Energy)}$$

$$E = \sigma \cdot \underbrace{\frac{dV}{V}}_{\varepsilon} \quad \text{(Energy per unit of volume)}$$

$$\int_{\Omega} \underset{\sim}{\varepsilon}(\nabla \delta \underset{\sim}{u}) : \underline{\underline{\sigma}}(\underline{u}) = \int_{\Omega} \underset{\sim}{\varepsilon}(\nabla \delta \underset{\sim}{u}) : \left[\underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}(\underline{u}) \right]$$

constitutive equation

$$-\nabla \cdot \sigma = f$$

Ω

$$\sigma_{ij} = C_{ijkl}\varepsilon_{ij}$$

$$\sigma_{ij} = \lambda \delta_{ij} \nabla u + 2G\varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\int_{\Omega} \sigma(u) : \varepsilon(v) dV + \int_{\partial\Omega_n} \tau \cdot n \, dS = \int_{\Omega} f \cdot v \; dV$$

$$a : b = \sum_{i,j} a_{ij} b_{ij}$$

$\partial\Omega_n$

n

$$d\vec{S}$$

$$\int_{\Omega} \left[2G\varepsilon_{ij}(u)\varepsilon_{ij}(v) + \lambda\varepsilon_{ii}(u)\varepsilon_{jj}(v)\right] dV + \int_{\partial\Omega_n} \tau \cdot n \, dS = \int_{\Omega} f \cdot v \; dV$$

R

$$\begin{cases} -\nabla\sigma = f \\ \sigma_{xx}(x = \pm\infty, y) = S_1 \\ \sigma_{yy}(x, y = \pm\infty) = S_2 \end{cases}$$

$$
\begin{aligned}
\sigma_{rr} =& \frac{1}{2}(S_1 + S_2 - 2P_w)\left[1 - \left(\frac{R}{r}\right)^2\right] + \frac{1}{2}(S_1 - S_2)\left[1 - \left(\frac{2R}{r}\right)^4 + 3\left(\frac{R}{r}\right)^4\right]\cos(2\theta) + P_w\left(\frac{R}{r}\right)^2 \\
\sigma_{\theta\theta} =& \frac{1}{2}(S_1 + S_2 - 2P_w)\left[1 + \left(\frac{R}{r}\right)^2\right] - \frac{1}{2}(S_1 - S_2)\left[1 + 3\left(\frac{R}{r}\right)^4\right]\cos(2\theta) - P_w\left(\frac{R}{r}\right)^2 \\
\sigma_{r\theta} =& \frac{1}{2}(S_1 - S_2)\left[1 + \left(\frac{2R}{r}\right)^4 - 3\left(\frac{R}{r}\right)^4\right]\sin(2\theta)
\end{aligned}
$$

$u1$

$u2$

$v1$

$v2$

$sigmaxx$

$sigmaxy$

$sigmayy$

$$\sigma_{xx}$$

σ_{yy}

E

ν

UCS

T_s

$$\sigma_{rr}$$

$\sigma_{\theta\theta}$

$\sigma_{r\theta}$

x

y

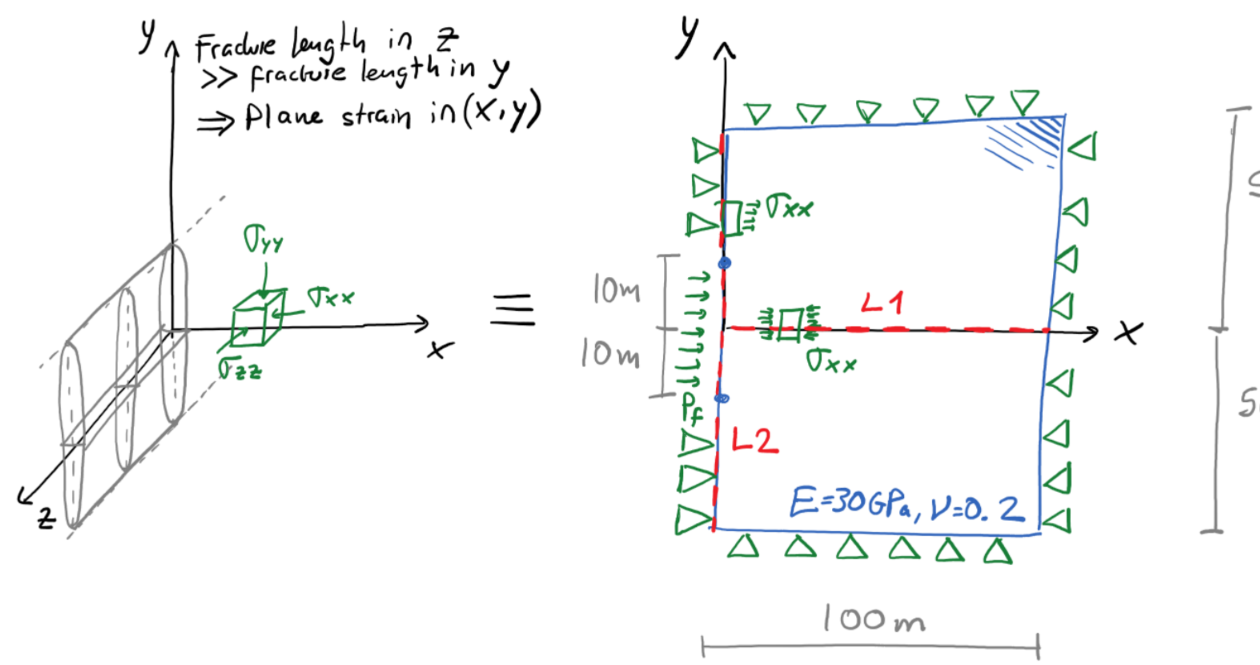
(r, θ)

r

$$\sigma_{xy}$$

c

$(x, y) =$

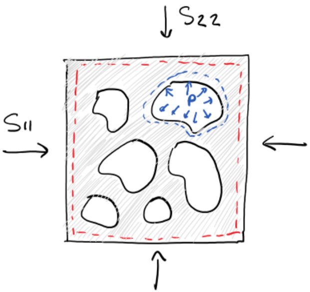


Fracture length in z
>> fracture length in y
⇒ Plane strain in (x,y)
y
x
z
σ_{yy}
σ_{xx}
σ_{zz}
≡
y
x
σ_{xx}
10m
10m
L1
σ_{xx}
p_f
L2
E=30GPa, ν=0.2
100m

$$y =$$

3- Poroelasticity (Coussy, 2010 - Chapters 3 and 4)

Friday, October 2, 2020 5:45 PM



- pores → filled with fluids ↔ pressure
  ↳ no shear stress
- deformable pore walls

Volumetric deformation

$$\varepsilon_{vol} = \epsilon \overset{def}{=} \frac{d\Omega - d\Omega_0}{d\Omega_0}$$

$$\rightarrow \underbrace{d\Omega}_{\text{bulk volume}} = (1+\epsilon)\cdot \underbrace{d\Omega_0}_{\text{initial bulk volume}} \quad (1)$$

Porosity

@ time t $\quad n \overset{def}{=} \dfrac{\boxed{}}{\boxed{} + \boxed{}}$ } Eulerian porosity

$$n \cdot d\Omega \overset{def}{=} \phi \cdot d\Omega_0 \rightarrow \underbrace{\phi}_{\text{Lagrangian Porosity}} = n \frac{d\Omega}{d\Omega_0} \quad (2)$$

Solid strain

$$\epsilon_s \overset{def}{=} \frac{d\Omega^s - d\Omega^s_0}{d\Omega^s_0} \Rightarrow d\Omega^s = (1+\epsilon_s)\, d\Omega^s_0 \quad (3)$$

Combination of ①, ②, ③

$$d\Omega^s = (1+\epsilon_s)\, d\Omega^s_0$$

$$(1-n)\, d\Omega = (1+\epsilon_s)(1-\phi_0)\, d\Omega_0$$

$$\boxed{\underbrace{\epsilon}_{\text{Bulk Vol. Strain}} = (1-\phi_0)\underbrace{\epsilon_s}_{\text{Solid Vol. Strain}} + (\phi_0)\underbrace{\left(\frac{\phi-\phi_0}{\phi_0}\right)}_{\text{Porosity strain}}}$$

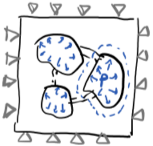
↳ micromechanical Eq.

↳ Volume average

$$\boxed{\epsilon = (1-\phi_0)\epsilon_s + \varphi}$$

$$\varphi \overset{\text{def}}{=} \phi - \phi_0 \; : \; \text{change of porosity}$$

Test 1



$$\boxed{\epsilon = 0}$$

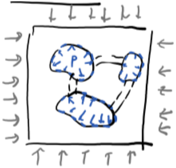
$$0 = (1-\phi_0)\,\epsilon_s + \varphi$$

$$\epsilon_s = -\frac{\varphi}{(1-\phi_0)}$$

if $\varphi > 0 \Rightarrow \underbrace{\epsilon_s < 0}_{\text{contraction}}$

Mechanics convention

Test 2

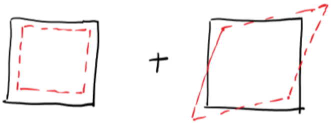


$$\boxed{\varphi = 0}$$

$$\epsilon_s = \frac{\epsilon}{1-\phi_0}$$

Free energy of the porous solid

Non-porous solid



$$\underbrace{dW}_{} = \underbrace{S_m}_{\frac{S_{11}+S_{22}+S_{33}}{3}} \cdot d\epsilon + \underbrace{s_{ij}}_{} \cdot de_{ij}$$

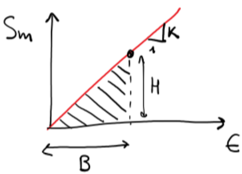
$s_{12} \leftrightarrow e_{12}$

$s_{23} \leftrightarrow e_{23}$

$s_{13} \leftrightarrow e_{13}$

Porous solid

$$\underbrace{dW = S_m \cdot d\epsilon}_{} + s_{ij} \cdot de_{ij} + p\,\underbrace{d\emptyset}_{= d\varphi}$$



$$W = \frac{\epsilon \cdot (\epsilon \cdot K)}{2} \; ; \; K \overset{def}{=} \frac{dS_m}{d\epsilon} \; : \text{Bulk modulus}$$

$$W = \frac{1}{2} K \epsilon^2 \longleftrightarrow \frac{\partial W}{\partial \epsilon} = K \epsilon = S_m$$

$$\eta_s = \underbrace{\frac{1}{2} K \epsilon^2}_{\text{Bulk vol. strain}} + \underbrace{G\, e_{ij} \cdot e_{ij}}_{\text{Shear strains}} \underbrace{- \underbrace{\alpha}_{cst1} \cdot \epsilon \cdot p - \underbrace{\left(\frac{1}{2}\,\frac{1}{N}\right)}_{cst2} p^2}_{\text{pressure}}$$

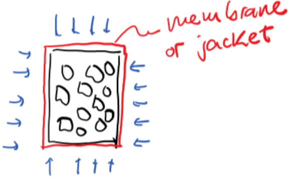
$$
\begin{cases}
\dfrac{\partial \eta_s}{\partial \epsilon} = \overbrace{K\epsilon}^{\text{effective stress}} - \underbrace{\alpha}_{\text{Biot coefficient}} \overbrace{P}^{\text{pressure}} = \overbrace{S_m}^{\text{total stress}} & (1) \\[2ex]
\dfrac{\partial \eta_s}{\partial e_{ij}} = 2 G e_{ij} = s_{ij} & (2) \\[2ex]
-\dfrac{\partial \eta_s}{\partial p} = \alpha \epsilon + \dfrac{P}{\underbrace{N}_{\text{Poroelastic Modulus}}} = \varphi & (3)
\end{cases}
$$

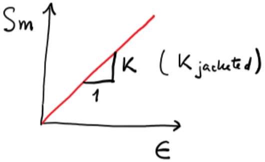
Biot effective stress equation

$\alpha = ?$

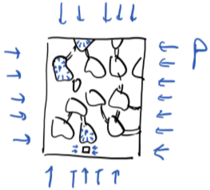
$N = ?$

Jacketed loading





Unjacketed loading



- isotropic elastic solid
- connected porosity

(1) $K\epsilon - \alpha p = S_m = -p$

$$\epsilon = \frac{p(-1+\alpha)}{K}$$

- $\epsilon = -\dfrac{p(1-\alpha)}{K}$
- $\epsilon_s = -\dfrac{p}{K_s}$

$$\epsilon = \epsilon_s$$

bulk modulus porous solid = jacketed =

$$\frac{\cancel{p}(1-\alpha)}{K} = \frac{\cancel{p}}{K_s} \Rightarrow \boxed{\alpha = 1 - \frac{K}{K_s}}$$

bulk modulus solid skeleton

$$\alpha \leq 1 \text{ for } K_s \geq K$$

(3) $\alpha\epsilon + \dfrac{p}{N} = \varphi$

$$\begin{cases} \alpha\left(-\dfrac{p}{K_s}\right) + \dfrac{p}{N} = \varphi \\[2ex] \dfrac{\varphi}{\phi_0} = -\dfrac{p}{K_s} \end{cases}$$

$$\alpha\left(-\frac{\cancel{p}}{K_s}\right) + \frac{\cancel{p}}{N} = -\frac{\cancel{p}}{K_s}\cdot\phi_0$$

$$\boxed{\frac{1}{N} = \frac{\alpha - \phi_0}{K_s}}$$

- second poromechanical modulus
- $N > 0 \Rightarrow \alpha > \phi_0$
- $\phi_0 \leq \alpha \leq 1$

METHOD 1

- Measure K
- Assume or estimate K_s
  - mono-mineral*
  - multimineral
    - ↳ volume average
      - Reuss
      - Voigt
      - Hashin-Shtrikman

$$\boxed{\alpha = 1 - \frac{K}{K_s}}$$

$$\boxed{N = \frac{K_s}{\alpha - \phi_0}}$$

K: jacketed bulk modulus

dry rock, E, ν

↳ $K = \frac{E}{3(1-2\nu)}$

→ $V_p, V_s, \rho \Rightarrow E, \nu, K$

* $\underbrace{K_s = K_{mineral} = K_{SiO_2}}$

Sandstone 100% SiO_2

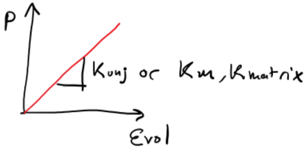
$K_{SiO_2} = 36$ GPa

→ wrong if occluded ϕ

METHOD 2

- Unjacketed test

$$\boxed{\alpha = 1 - \frac{K}{K_{unj}}}$$



- water < clay sensitive / short circuit
- oil – high viscosity
- gas → small vessel
  → not desirable for large vessels
  → non-sorptive

- Increase P_c and P_p simultaneously

constant Terzaghi's effective stress $= P_c - P_p \neq 0$

METHOD 3

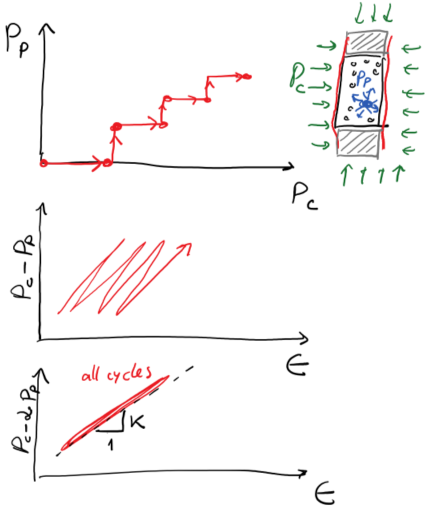
$$S_m = K\epsilon - \alpha P$$

$$K\epsilon = S_m + \alpha P$$

$$\epsilon = \frac{1}{K}\underbrace{\left(S_m + \alpha P\right)}_{\text{Biot effective stress}}$$

$(K, \alpha) \leftarrow$ error minimization

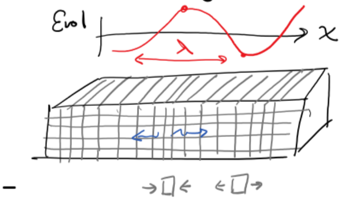
$$\underbrace{K_m}_{\text{matrix}} = \frac{K}{1-\alpha}$$



Drained and Undrained Problems in Poromechanics

Terzaghi's effective stress : $\underline{\underline{\sigma}} = \underline{\underline{S}} - P_p \underline{\underline{I}}$

Poroelasticity



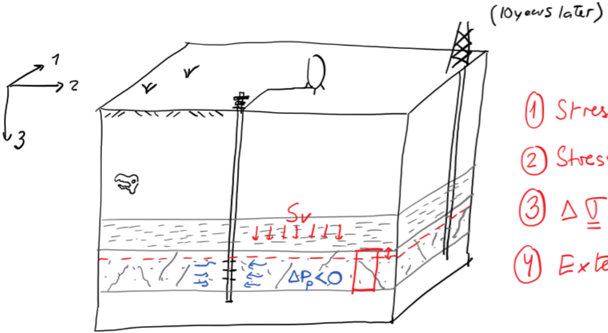
- $\underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$: Biot's effective stress
- φ → changes of porosity
- K_u : undrained bulk modulus
- undrained loading ↔ ΔP_p

  ↕
- "squirt flow" ↔ $\Delta \varepsilon_{vol}$

  ↳ attenuation of elastic waves

  ↳ dispersion of " waves

End-members of actual solution < drained ↕ undrained] → Diffusivity Eq for poroelasticity



1. Stresses in-situ ($\varepsilon_{11} = \varepsilon_{22} = 0$)
2. Stresses in-situ ($\varepsilon_{11} \neq \varepsilon_{22} \neq 0$)
3. $\Delta \underline{\underline{\sigma}}$ with change in P_p?
4. Extend to VTI rock

(1) $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$; $\underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$

$\hookrightarrow \underline{\underline{\varepsilon}} = \underline{\underline{\underline{\underline{D}}}} \cdot \underline{\underline{\sigma}}$

$$\begin{bmatrix} \cancel{\varepsilon_{11}} \\ \cancel{\varepsilon_{22}} \\ \varepsilon_{33} \\ \cancel{2\varepsilon_{23}} \\ \cancel{2\varepsilon_{13}} \\ \cancel{2\varepsilon_{12}} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & 0 & \\ -\nu/E & -\nu/E & 1/E & & & \\ \hline & & & 1/G & 0 & 0 \\ & 0 & & 0 & 1/G & 0 \\ & & & 0 & 0 & 1/G \end{array}\right] \begin{bmatrix} S_{11} - \alpha P_p \\ S_{22} - \alpha P_p \\ S_{33} - \alpha P_p \\ \cancel{S_{23}} \\ \cancel{S_{13}} \\ \cancel{S_{12}} \end{bmatrix}$$

Passive tectonic env

$\varepsilon_{11} = \varepsilon_{22} = \emptyset$

$\varepsilon_{ij} = 0$ for $i \neq j$

$\downarrow$

$S_{11} = S_{22}$

$$\varepsilon_{11} = 0 = \frac{1}{E}(S_{11} - \alpha P_p) - \frac{\nu}{E}(S_{22} - \alpha P_p) - \frac{\nu}{E}(S_{33} - \alpha P_p) \quad ;$$

$$\left(\frac{1-\nu}{\cancel{E}}\right) \cdot (S_{22} - \alpha P_p) = \frac{\nu}{\cancel{E}}(S_{33} - \alpha P_p)$$

$$\underbrace{(S_{22} - \alpha P_p)}_{\sigma_{22}} = \underbrace{\frac{\nu}{1-\nu}} \underbrace{(S_{33} - \alpha P_p)}_{\sigma_{33}}$$

$$S_{hmin} = S_{Hmax} = S_{22} = \alpha P_p + \frac{\nu}{1-\nu}(\underbrace{S_{33}}_{S_v} - \alpha P_p)$$

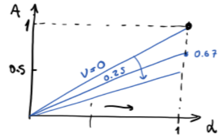
(2)

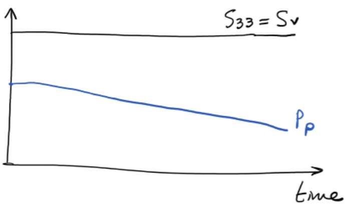
$$\begin{cases} S_{hmin} = \alpha P_p + \dfrac{\nu}{1-\nu}\left(S_v - \alpha P_p\right) + \dfrac{E}{1-\nu}\left(\varepsilon_{hmin} + \nu\,\varepsilon_{Hmax}\right) \\ S_{Hmax} = \alpha P_p + \dfrac{\nu}{1-\nu}\left(S_v - \alpha P_p\right) + \dfrac{E}{1-\nu}\left(\nu\,\varepsilon_{hmin} + \varepsilon_{Hmax}\right) \\ S_v = \displaystyle\int_0^z \rho_{bulk}\, g \, dz \end{cases}$$

③

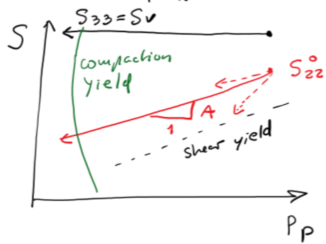
$$\underbrace{S_{22}}_{\text{Total Hz stress}} = \underbrace{\frac{\nu}{1-\nu} S_{33}}_{\text{Contrib Total vert stress}} + \underbrace{\alpha\left(\frac{1-2\nu}{1-\nu}\right)}_{A} \underbrace{P_p}_{\text{Pore pressure}}$$

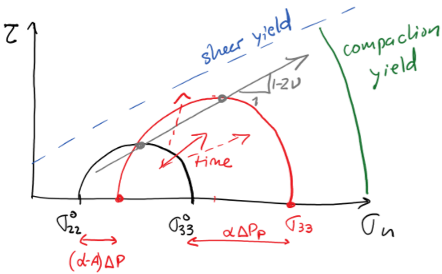
$$\frac{\partial S_{22}}{\partial P_p} = \alpha\left(\frac{1-2\nu}{1-\nu}\right) = A \rightarrow$$





$\Rightarrow$





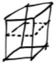
$$\sigma_{33} = S_{33} - \alpha P_p$$

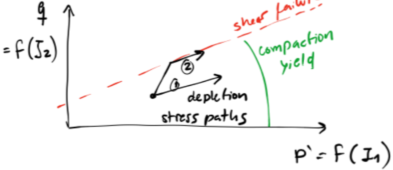
$$\frac{\partial \sigma_{33}}{\partial P_p} = -\alpha$$

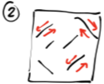
$$\frac{\partial \sigma_{22}}{\partial P_p} = -(\alpha - A) = -\alpha\left(\frac{\nu}{1-\nu}\right)$$

$$\varepsilon_{33} = -\frac{2\nu}{E}\left(S_{22} - \alpha P_p\right) + \frac{1}{E}\left(S_{33} - \alpha P_p\right)$$

$$\varepsilon_{33} = \frac{(1-2\nu)(1+\nu)}{E(1-\nu)}\left(\overbrace{S_{33} - \alpha P_p}^{\sigma_{33}}\right)$$







$$\frac{\partial \varepsilon_{33}}{\partial P_p} = -\frac{\alpha}{M}$$

constrained modulus

$$\Delta \varepsilon_{33} = -\frac{\alpha}{M}\,\Delta P_p$$

$$\boxed{\Delta H = \Delta\varepsilon_{33}\cdot H = \left(-\frac{\alpha}{M}\Delta P_p\right)\cdot H}$$

- subsidence
- casing buckling/shearing

④

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} +\frac{1}{E_h} & -\frac{\nu_h}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_h}{E_h} & +\frac{1}{E_h} & -\frac{\nu_v}{E_v} & 0 & 0 & 0 \\ -\frac{\nu_v}{E_v} & -\frac{\nu_v}{E_v} & +\frac{1}{E_v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_v} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_h} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}$$

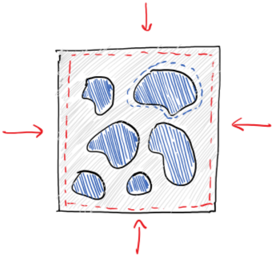
$$\begin{cases} S_{hmin} = \alpha_h P_p + \frac{\nu_v}{1-\nu_h} \frac{E_h}{E_v} (S_v - \alpha_v P_p) + \frac{E_h}{1-\nu_h^2} (\varepsilon_{hmin} + \nu_h \varepsilon_{Hmax}) \\ S_{Hmax} = \alpha_h P_p + \frac{\nu_v}{1-\nu_h} \frac{E_h}{E_v} (S_v - \alpha_v P_p) + \frac{E_h}{1-\nu_h^2} (\nu_h \varepsilon_{hmin} + \varepsilon_{Hmax}) \\ S_v = \int_0^z \rho_{bulk}\, g\, dz \end{cases}$$

$$\begin{cases} \sigma_{hmin} = S_{hmin} - \alpha_h P_p \\ \sigma_{Hmax} = S_{Hmax} - \alpha_h P_p \\ \sigma_v = S_v - \alpha_v P_p \end{cases}$$

$$\begin{cases} \alpha_h = 1 - \frac{C_{11} + C_{12} + C_{13}}{3 K_m} \\ \alpha_v = 1 - \frac{2C_{13} + C_{33}}{3 K_m} \\ \frac{1}{N} = \frac{(2\alpha_h + \alpha_v)/3 - \phi_0}{K_m} \end{cases}$$

3- Pore pressure diffusivity equation coupled with poroelasticity

Wednesday, September 28, 2022 11:28 AM



$$\underbrace{\qquad\qquad\qquad}_{\phi = V_P / V_b^\circ}$$

$$\frac{\text{fluid mass}}{V_b^\circ} = \frac{V_P}{V_b^\circ}\,\rho_F = \phi\,\rho_F$$

$$d(\phi\rho_F) = \rho_F\, d\phi + \phi\, d\rho_F \begin{cases} = 0 \rightarrow \text{undrained loading} \\ \neq 0 \end{cases}$$

$$\frac{d(\phi\rho_F)}{\rho_F} = d\phi + \phi\,\frac{d\rho_F}{\rho_F} \sim c_F = \frac{1}{\rho_F}\frac{d\rho_F}{dP} \; ; \; K_F = c_F^{-1}$$

$$\varphi = d\epsilon + \frac{P}{N}$$

$$\frac{d(\phi\rho_F)}{\rho_F} = \left(\alpha\, d\epsilon + \frac{dp}{N}\right) + \phi\,\frac{dP}{K_F}$$

$d\phi << \phi_0$ (small strains)

$$\frac{d(\phi\rho_F)}{\rho_F} = \alpha\, d\epsilon + \underbrace{\left(\frac{1}{N} + \frac{\phi_0}{K_F}\right)}_{\sim cst} dP$$

Biot Modulus

$$\boxed{\frac{1}{M^*} \overset{def}{=} \left(\frac{\alpha - \phi_0}{K_s} + \frac{\phi_0}{K_F}\right)} \; ; \; \frac{1}{N} = \frac{\alpha - \phi_0}{K_s}$$

Continuity equation

(Mass conservation)

$$\frac{d(\rho_F \phi)}{dt} + \nabla \cdot (\rho_F \underline{q}) = 0$$

$\downarrow$ isotropic Darcy's

$$\frac{1}{dt}\left(\cancel{\rho_F}\left[\alpha\, d\epsilon + \frac{1}{M^*}\, dP\right]\right) + \nabla \cdot \left[\cancel{\rho_F}\left(-\frac{\kappa}{\nu} \nabla P\right)\right] = 0$$

$\downarrow$ homogeneous porous solid

$$\alpha \frac{d\epsilon}{dt} + \frac{1}{M^*}\frac{dP}{dt} + \left(-\frac{\kappa}{\nu} \nabla^2 P\right) = 0$$

$$\boxed{\frac{dP}{dt} = \frac{\kappa M^*}{\nu} \nabla^2 P - \alpha M^* \frac{d\epsilon}{dt}}$$

Diff. Eq.

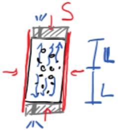
coupled with

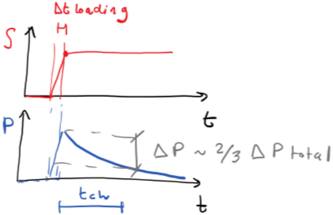
poroelasticity

$$\frac{\partial P}{\partial t} \sim \underset{\sim}{\mathbb{D}} \nabla^2 P$$

$$\hookrightarrow \boxed{t_{ch} = \frac{L^2}{\mathbb{D}}}$$

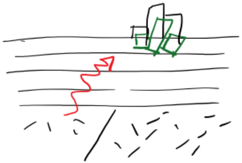
$$\mathbb{D} \sim \frac{K M^*}{N}$$

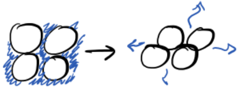




$$\begin{cases} \Delta t_{loading} \ll t_{ch} \Rightarrow \text{undrained loading} \\ \Delta t_{loading} \gg t_{ch} \Rightarrow \text{drained loading} \end{cases}$$

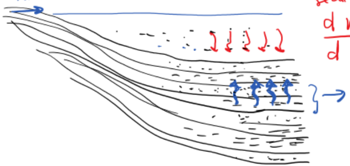
① Liquefaction





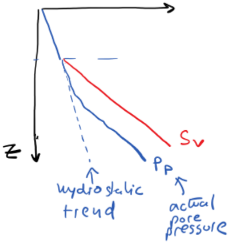
$\uparrow P_p \Rightarrow \downarrow \underline{\sigma} \Rightarrow \downarrow$ shear strength

② Dis-equilibrium compaction

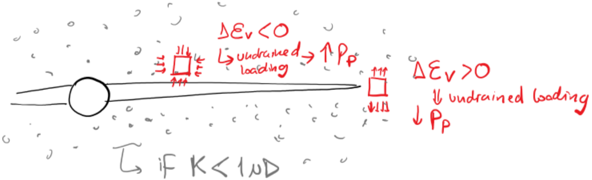


rate of sedimentation

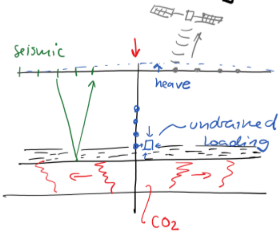
$$\frac{dm}{dt} \gg \frac{dP}{dt}$$



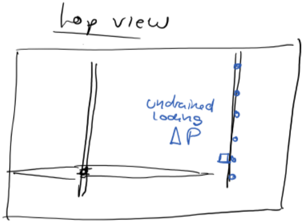
③ Hydraulic fracture



④ Poroelastic monitoring



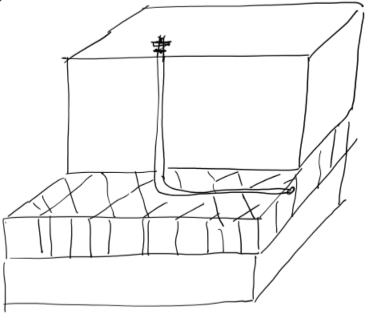
ΔP



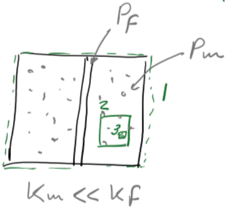
Rousell

Conoco Phillips

⑤



undrained loading of the matrix



During depletion

$$P_F < P_m$$

$$\alpha = 1 - \frac{\langle K \rangle}{\langle K_m \rangle}$$

$\langle \; \rangle$ volume average

① Pore pressure change due to volumetric strain

Diff. Eq. coupled with poroelasticity

$$\frac{dP}{\cancel{dt}} = \cancel{\frac{\kappa M^*}{\mu} \nabla^2 P} - \alpha M^* \frac{d\epsilon}{\cancel{dt}}$$

$$\left.\frac{dP}{d\epsilon}\right|_{d(\rho_F \phi)=0} = -\alpha M^* = -\alpha \left(\frac{\alpha - \phi_0}{K_m} + \frac{\phi_0}{K_F} \right)^{-1}$$

$$\Delta P = (-\alpha M^*) \Delta \epsilon$$

② Undrained Bulk Modulus

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{drained,\,dry} = K \quad \Big\}$$

Bulk modulus
- drained
- dry

$$\left.\frac{\partial S_m}{\partial \epsilon}\right|_{\partial(\rho_F \phi)=0} \overset{def}{=} K_U \quad \Big\}$$

Undrained Bulk modulus

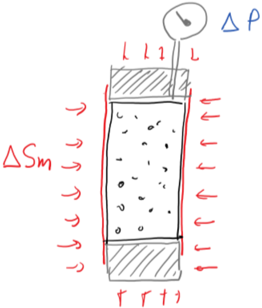
$$S_m = K\epsilon - \alpha P$$

$$\frac{\partial S_m}{\partial \epsilon} = K - \alpha \frac{\partial P}{\partial \epsilon} \Rightarrow \left.\frac{\partial S_m}{\partial \epsilon}\right|_{\partial(\rho_F \phi)=0} = K - \alpha(-\alpha M^*)$$

$$\boxed{K_U = K + \alpha^2 M^*}$$

③ Skempton's parameter

$$B \overset{def}{=} -\left.\frac{\partial P}{\partial S_m}\right|_{d(\rho_f \phi)=0}$$



$$S_m = K\epsilon - \alpha P$$

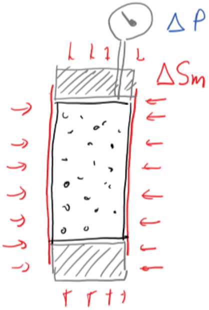
$$\frac{\partial S_m}{\partial P} = K\frac{\partial \epsilon}{\partial P} - \alpha$$

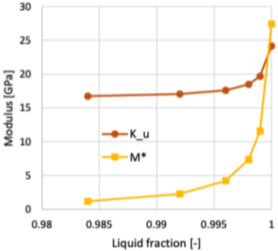
$$\frac{\partial S_m}{\partial P} = K\left(-\frac{1}{\alpha M^*}\right) - \alpha$$

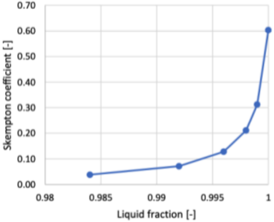
$$\boxed{B^{-1} = \frac{1}{\alpha}\left(\frac{K}{M^*} + \alpha^2\right)}$$

$$\boxed{K_u = K + \alpha^2 M^*}$$

$$\boxed{B^{-1} = \frac{1}{\alpha}\left(\frac{K}{M^*} + \alpha^2\right)}$$







$$V_P = \sqrt{M/\rho}$$

$$\rightarrow K_F^{-1} = \left(\frac{1-S_w}{K_g} + \frac{S_w}{K_w}\right)$$

P_c

P_p

ε_{vol}

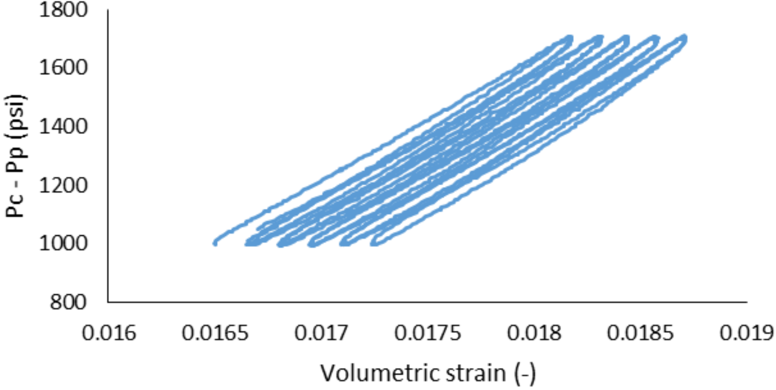
σ_{mean}

α

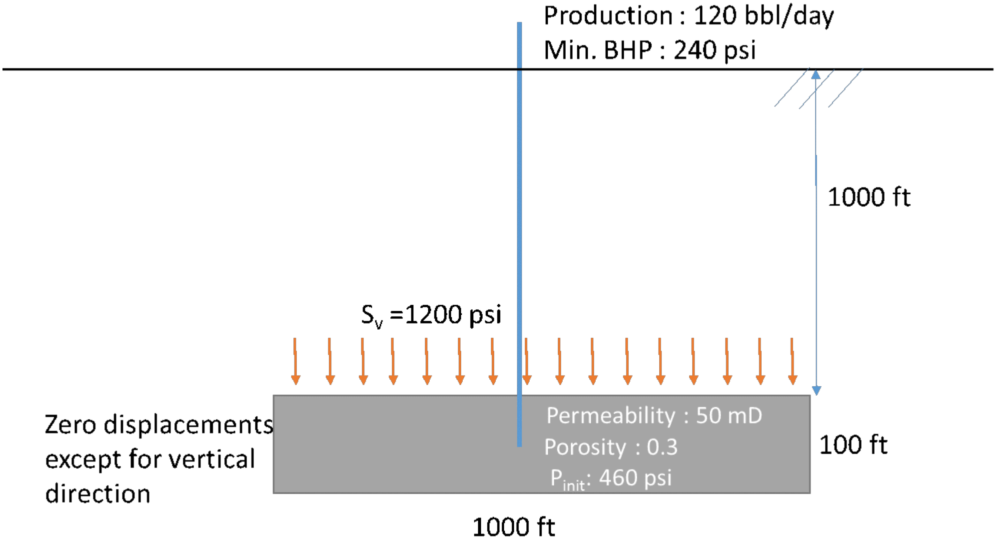
$$T_{ch}$$

α_h

α_v



1800
1600
1400
1200
1000
800
Pc - Pp (psi)
0.016
0.0165
0.017
0.0175
0.018
0.0185
0.019
Volumetric strain (-)

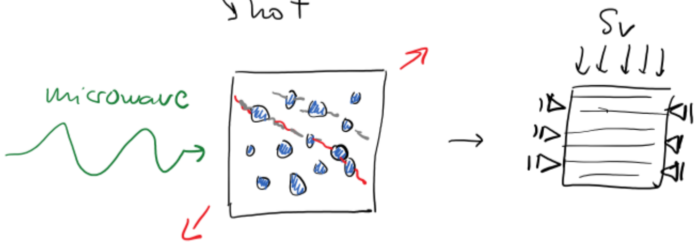

Production : 120 bbl/day
Min. BHP : 240 psi
1000 ft
S_v =1200 psi
Permeability : 50 mD
Porosity : 0.3
P_{init}: 460 psi
Zero displacements except for vertical direction
100 ft
1000 ft

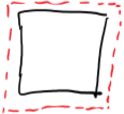
$$\alpha \frac{(1-2\nu)}{(1-\nu)}$$

(p', q)

Applications

- geothermal energy
  - shallow
  - deep { HF, $\tau/\sigma_n \downarrow\uparrow$, $\Delta\sigma$
- drilling — $T_{mud} < T_{form} \Rightarrow \downarrow \sigma_{\theta\theta} \rightarrow$ tensile fractures
- subsurface fluid injection / disposal
  - EOR
    - water
      - liquid
      - steam
    - CO_2
    - chemicals
  - disposal
    - produced water, HF water
    - CO_2
- hydraulic fracturing
  - cold $\rightarrow$ cryogenic fracturing
  - hot



$\Delta T > 0 \Rightarrow$  $\varepsilon_{vol} < 0$ (Dilation)

$\Delta T < 0 \Rightarrow$  $\varepsilon_{vol} > 0$ (Contraction)

$\Delta T \lessgtr 0 \Rightarrow$ no shear strains (isotropic homogeneous) $\neq$  layers $\neq \beta$

Identity matrix

$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} + 3\underbrace{\beta}_{} \underbrace{K}_{} \underbrace{(\Delta T)}_{} \overbrace{\underline{\underline{I}}}^{}}$$

Linear thermal dilation coefficient

Bulk Modulus

Change of temperature

$$\Delta T = T - T_0$$

$$\Delta T = \theta$$

Dilation

$\sim$

- unconstrained
- free dilation

$$\overset{0}{\cancel{\underline{\underline{\sigma}}}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} + 3\beta K (\Delta T)\, \underline{\underline{I}}$$

$$\underline{\underline{\varepsilon}} = - \underline{\underline{\underline{\underline{D}}}} \left(3\beta K \theta\, \underline{\underline{I}} \right)$$

$$\begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{vmatrix} = \begin{vmatrix} 1/E & -\nu/E & -\nu/E \\ -\nu/E & 1/E & -\nu/E \\ -\nu/E & -\nu/E & 1/E \end{vmatrix} \begin{vmatrix} -3\beta K\theta \\ -3\beta K\theta \\ -3\beta K\theta \end{vmatrix}$$

$$\varepsilon_{11} = -\boxed{\frac{(1-2\nu)}{\cancel{E}}\left[+3\right.}\left.\beta \cancel{K} \theta\right]$$

$1/K$

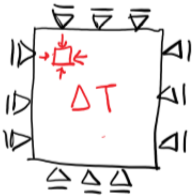
$$\boxed{-\frac{\partial \varepsilon_{11}}{\partial \theta} \overset{def}{=} \beta} \longleftrightarrow \boxed{\varepsilon_{11} = -\beta\theta}$$

$\theta = \Delta T > 0 \Rightarrow \Delta\varepsilon < 0$ (Dilati

$\theta = \Delta T < 0 \Rightarrow \Delta\varepsilon > 0$ (Contra

$$\left.\begin{array}{l} \beta = 10^{-6} \text{ to } 10^{-5} \dfrac{1}{°K} \\ \Delta T \sim 100\,°C \end{array}\right\} \varepsilon \sim 10^{-4} \text{ to } 10^{-3}$$

Constrained dilation



$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\ \cancel{\underline{\underline{\varepsilon}}}^{\,0} + 3\beta K \Delta T\, \underline{\underline{I}}$$

$\theta = \Delta T > 0$

$\Downarrow$

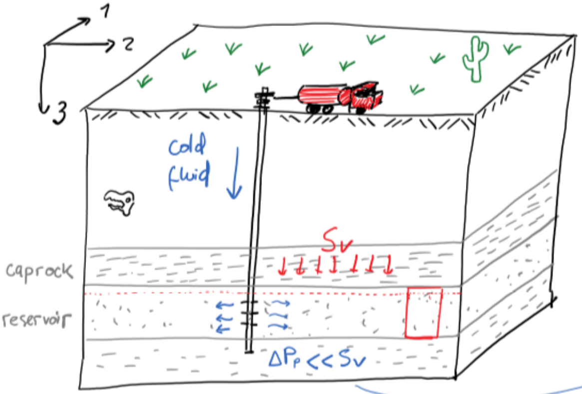
$\Delta\sigma > 0$

$$\begin{vmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{vmatrix} = \begin{vmatrix} +3\beta K\theta \\ +3\beta K\theta \\ +3\beta K\theta \end{vmatrix}$$

$$\sigma_{11} = +3\beta K\theta$$

$$\left.\begin{aligned} \beta &= 10^{-5}\ 1/{}^\circ K \\ \theta &= 100\,{}^\circ K \\ K &= 10\ GPa \end{aligned}\right\} \quad \sigma_{11} = 3 \cdot 10^{-5} \frac{1}{\cancel{{}^\circ K}} \cdot 10^{10}\,Pa \cdot 10^{2}\,\cancel{{}^\circ K}$$

$$= 3 \cdot 10^{7}\,Pa = \underbrace{30\,MPa \cong 4400\,psi}_{\text{compression}}$$



$$\boxed{\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{\varepsilon}} + 3\beta K \underbrace{\theta}_{} \underline{\underline{I}}}$$

$$= \Delta T = T - T_0$$

$$\Downarrow$$

Uniform cooling in reservoir

No horizontal strains

$$\varepsilon_{11} = \varepsilon_{22} = 0$$

$\sigma_{33} \sim$ constant

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{bmatrix} \begin{bmatrix}\cancel{\varepsilon_{11}}\\ \cancel{\varepsilon_{22}}\\ \varepsilon_{33}\end{bmatrix} + \begin{bmatrix}3\beta K\theta\\ 3\beta K\theta\\ 3\beta K\theta\end{bmatrix}$$

$$\underline{\underline{\sigma}} = \underline{\underline{C}} \cdot \underline{\underline{\varepsilon}} + 3\alpha_L K \theta \underline{\underline{I}}; \quad \theta = T - T_0$$

$$\varepsilon_{11} = \varepsilon_{22} = 0$$

$\sigma_{33} =$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{33} \end{bmatrix} + \begin{bmatrix} 3\alpha_L K\theta \\ 3\alpha_L K\theta \\ 3\alpha_L K\theta \end{bmatrix}$$

σ_{11}

σ_{33}

$$\begin{cases} \sigma_{11} = \dfrac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{33} + 3\alpha_L K\theta \\ \sigma_{33} = \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{33} + 3\alpha_L K\theta \end{cases}$$

ε_{33}

$$\sigma_{11} = \left(\frac{\nu}{1-\nu}\right)\sigma_{33} + \left(\frac{1-2\nu}{1-\nu}\right)3\alpha_L K\theta$$

$$K = E/[3(1 - 2\nu)]$$

$$\sigma_{11} = \left(\frac{\nu}{1-\nu}\right)\sigma_{33} + \frac{\alpha_L E}{1-\nu}\theta$$

$$\frac{\partial \sigma_{11}}{\partial \theta} = +\frac{\alpha_L E}{1-\nu}$$

$$\frac{\partial \sigma_{11}}{\partial \theta} = 0.1 \frac{\mathrm{MPa}}{^\circ\mathrm{C}}$$

$$\alpha_L = 10^{-5}$$

$1/^{\circ}$

$$E = 10$$

$$\nu = 0.2$$

$$\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{33} + 3\beta K \theta$$

$$\uparrow$$

$$\sigma_{33} = \frac{(1-\nu) E}{(1+\nu)(1-2\nu)} \varepsilon_{33} + 3\beta K \theta$$

$$\sigma_{11} = \frac{\nu \cancel{E}}{\cancel{(1+\nu)(1-2\nu)}} \left[\frac{\cancel{(1+\nu)(1-2\nu)}}{(1-\nu) \cancel{E}} \left(\sigma_{33} - 3\beta K \theta \right) \right] + 3\beta K \theta$$

$$\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33} + \left(1 - \frac{\nu}{1-\nu}\right) 3\beta K \theta$$

$$\boxed{\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33} + \left(\frac{1-2\nu}{1-\nu}\right) 3\beta K \theta}$$

$$\left.\frac{\partial \sigma_{11}}{\partial \theta}\right|_{\sigma_{33}} = \left(\frac{1-2\nu}{1-\nu}\right) 3\beta K = \frac{\cancel{1-2\nu}}{1-\nu} \cdot \frac{E}{\cancel{3}\cancel{(1-2\nu)}} \cdot \cancel{3}\beta$$

$$\boxed{\left.\frac{\partial \sigma_{11}}{\partial \theta}\right|_{\sigma_{33}} = + \frac{\beta E}{1-\nu}}$$

$$\begin{cases} \downarrow \theta \Rightarrow \downarrow \sigma_{11} \\ \uparrow \theta \Rightarrow \uparrow \sigma_{11} \end{cases}$$

just mechanics

$$\sigma_{11} = \frac{\nu}{1-\nu} \sigma_{33}$$

ΔP_p : pore pressure

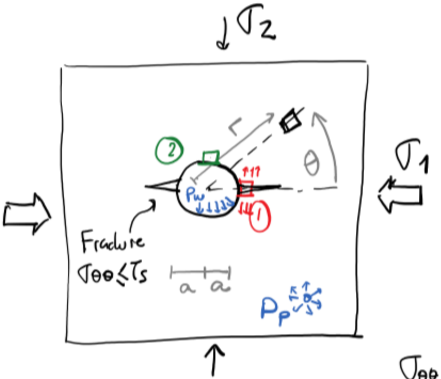
$$\frac{\partial \sigma_{11}}{\partial P_p} = \alpha \left(\frac{1-2\nu}{1-\nu}\right)$$

Δ Temp

$$\frac{\partial \sigma_{11}}{\partial \theta} = \frac{\beta E}{1-\nu}$$

$$\begin{cases} \beta = 10^{-5} \ 1/{}^\circ K \\ E = 10 \text{ GPa} \\ \nu = 0.2 \end{cases} \Rightarrow \frac{\partial \sigma_{11}}{\partial \theta} = 0.1 \frac{\text{MPa}}{{}^\circ K}$$

Changes of stresses due to temperature around a wellbore

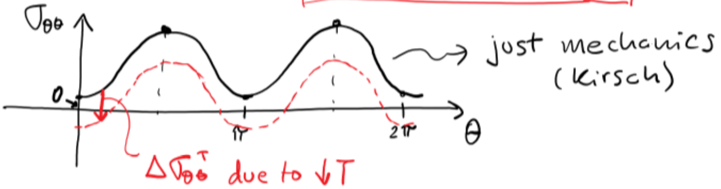


$$① \ \sigma_{\theta\theta}(r=a,\theta=0°) = \underbrace{-(P_w-P_p)-\sigma_1+3\sigma_2}_{\text{Kirsch}} + \underbrace{\Delta\sigma_{\theta\theta}^T}_{\text{Thermal stress}}$$

$$② \ \sigma_{\theta\theta}(r=a,\theta=90°) = -(P_w-P_p)+3\sigma_1-\sigma_2+\Delta\sigma_{\theta\theta}^T$$

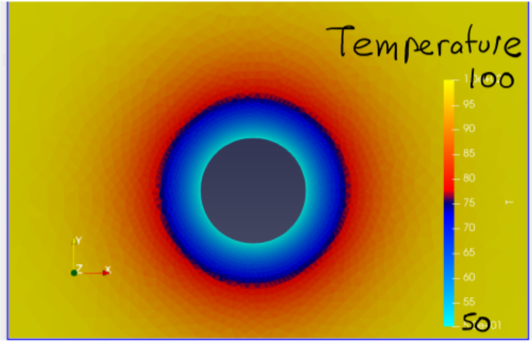
steady state solution

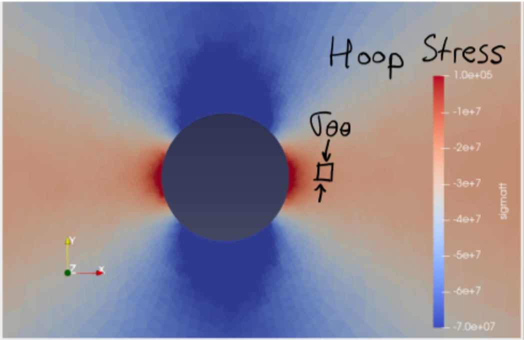
$$\boxed{\Delta\sigma_{\theta\theta}^T = \frac{\beta E}{1-\nu}\cdot\Delta T}$$



$$
\begin{cases}
\nabla \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{Equilibrium}\\[2ex]
\underline{\underline{\varepsilon}} = \dfrac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) & \text{Kinematic}\\[2ex]
\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} + 3\alpha_L K \theta \underline{\underline{I}}; \;\; \theta = T - T_0 & \text{Constitutive}\\[2ex]
\dfrac{\partial \theta}{\partial t} = \dfrac{k_T}{\rho c_v} \nabla^2 \theta + \dfrac{3\beta K T_0}{\rho c_v} \dfrac{\partial \varepsilon_{vol}}{\partial t} & \text{Diffusivity}
\end{cases}
$$

$$
\begin{cases}
\nabla \underline{\underline{\sigma}} + \underline{F} = \underline{0} & \rightarrow \text{Equilibrium} \\
\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) & \rightarrow \text{Kinematic} \\
\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} + 3\beta K \theta \underline{\underline{I}} & \rightarrow \text{Constitutive} \quad (\theta = \Delta T = T - T_0) \\
\frac{\partial \theta}{\partial t} = \frac{k_T}{\rho C_v} \nabla^2 \theta + \frac{3\beta K T_0}{\rho C_v} \cdot \frac{\partial \varepsilon_v}{\partial t} & \rightarrow \text{Diffusivity}
\end{cases}
$$





$$
\begin{cases}
\dfrac{\partial p}{\partial t} - \dfrac{k}{\mu} M \nabla^2 p = -\alpha M \dfrac{\partial \varepsilon_{vol}}{\partial t} + \beta_e M \dfrac{\partial T}{\partial t} & \text{Pore pressure diffusivity} \\
\dfrac{\partial T}{\partial t} - \kappa_T \nabla^2 T = -\dfrac{\alpha_d}{m_d} \dfrac{\partial \varepsilon_{vol}}{\partial t} + \dfrac{\beta_e}{m_d} \dfrac{\partial p}{\partial t} & \text{Temperature diffusivity}
\end{cases}
$$

$$\beta_e = \alpha\beta_d + \phi(\beta_f - \beta_s)$$

$$\beta_f > \beta_s$$

$$\beta_d = \beta_s;$$

K

$$\alpha_d = K\ \beta_d$$

$$m_d = \frac{c_d}{T_0}$$

Chemo-mechanical coupled processes

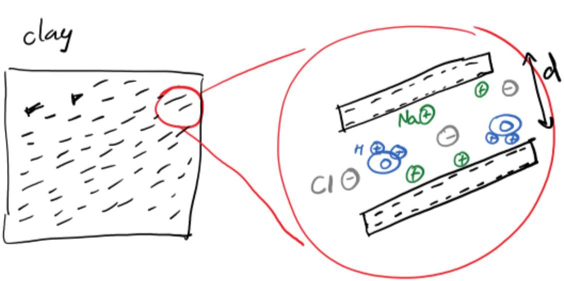
- chemo-plasticity : inelastic deform assisted by chem processes
- chemo-elasticity

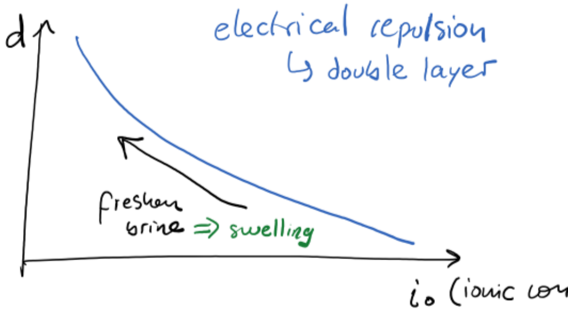
$$\underline{\underline{S}} = \underline{\underline{\underline{\underline{C}}}}\;\underline{\underline{\varepsilon}} + \alpha P \underline{\underline{I}} + 3\beta K \theta \underline{\underline{I}} + \underbrace{\gamma}_{\text{coeff}} \overbrace{\mu}^{\text{chem potential}} K \underline{\underline{I}}$$

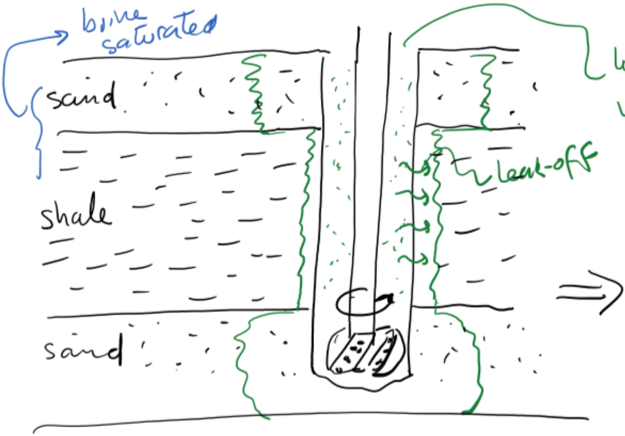
$$\underbrace{\text{Mech} \qquad \begin{matrix}\text{poro}\\ \text{hydro}\end{matrix} \qquad \text{thermal} \qquad \text{chemical}}$$

$\underbrace{\text{THCM}}$ coupled processes $\Longrightarrow$ emergent phenomena

1) Chemical sensitivity of shales



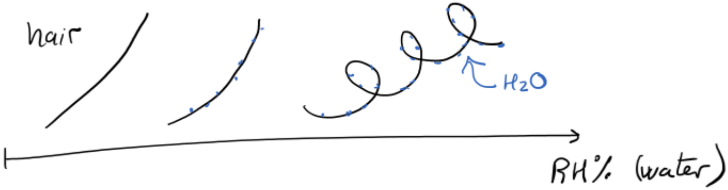


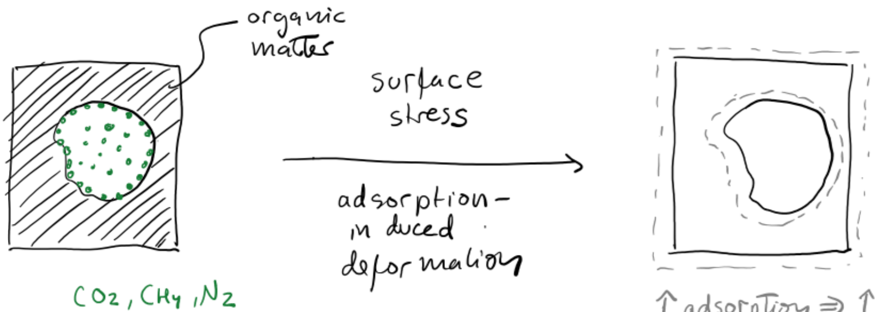


cross section

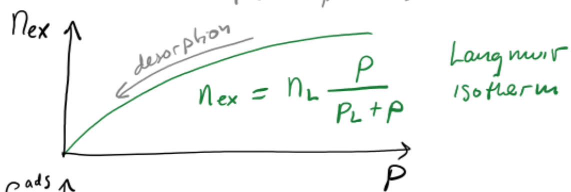
- drilling mud
  - water-based
    ↳ ↑ salinity
  - oil-based m
    ↳ non-polar fl
  - under balance drilling
    $P_{mud} < P_p$

2) Adsorption-induced deformation

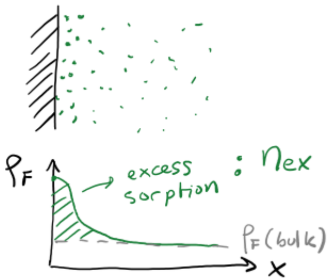




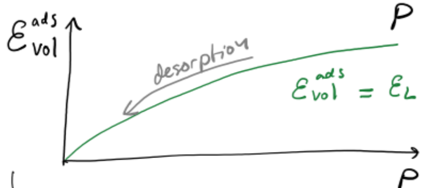
↑ adsorption ⇒ ↑ volume



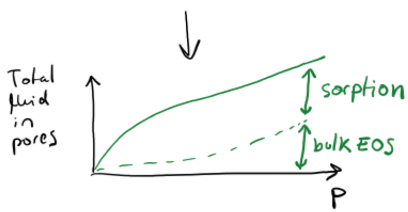
$$n_{ex} = n_L \frac{p}{p_L + p}$$

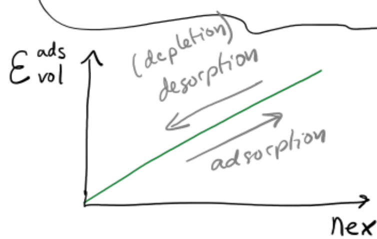


: n_{ex}



$$\varepsilon_{vol}^{ads} = \varepsilon_L \frac{p}{p_L + p}$$

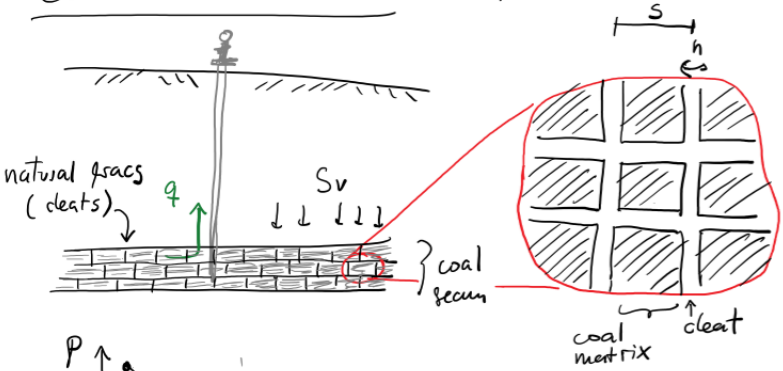




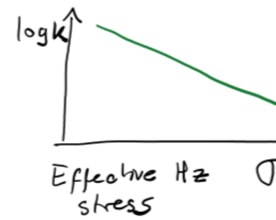
$$\underline{\underline{S}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} + \alpha p \underline{\underline{I}} + (1-\alpha)\, S_a(p)\, \underline{\underline{I}}$$

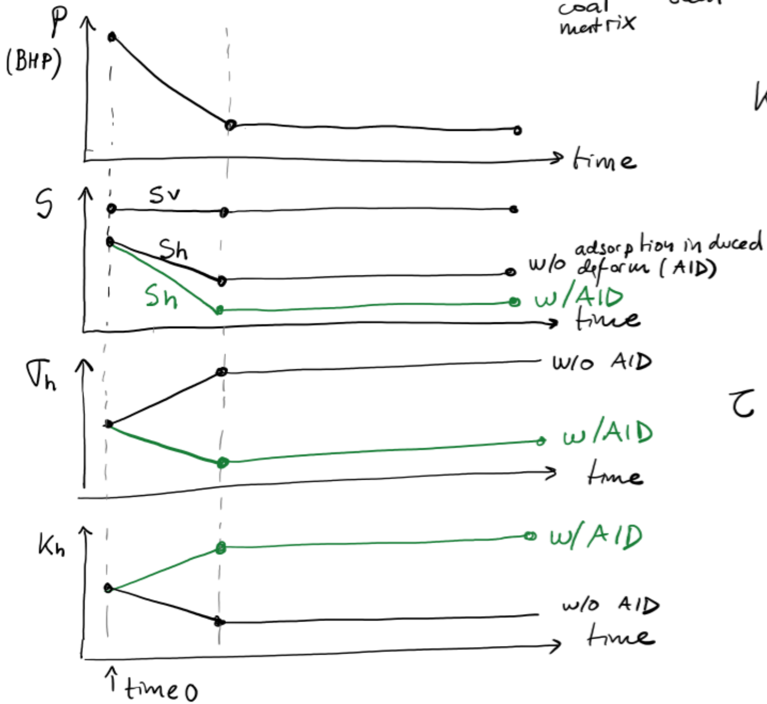
→ Adsorption stress: $S_a(p)$

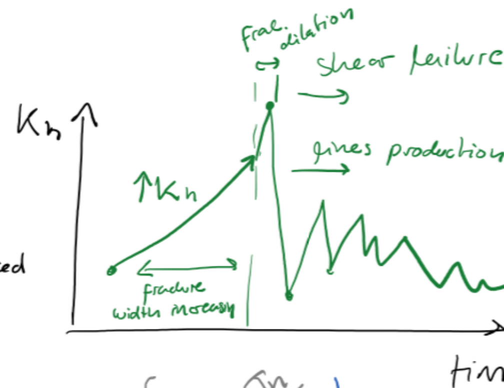
Coal bed methane (desorption-induced shrinkage and stress relaxation)

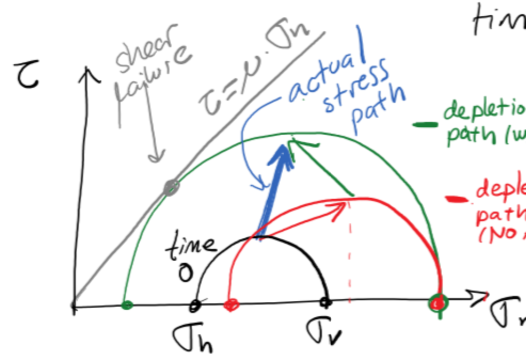


$$K = \frac{h^3}{12 \cdot s}$$



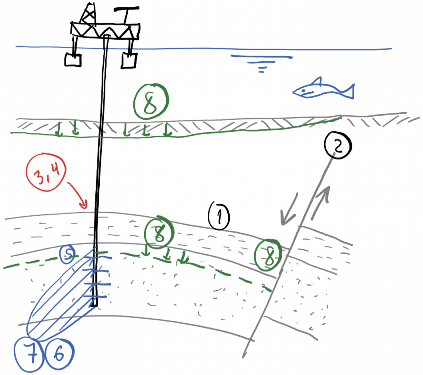






4-Inelasticity in subsurface engineering applications

Monday, October 26, 2020 5:17 PM



Inelasticity
- limits max stresses
- strength passed yield stress
- permanent strain

• Exploration
- • folding (1)
- • faulting (2)

$$\sigma_1 = \frac{1+\sin\varphi}{1-\sin\varphi}\,\sigma_3$$

• Drilling
- • well bore cuttings
- • breakouts (3) and tensile fractures (4)

• Production
- • reservoir compaction (8)
  - → fault reactivation
- • injection
- • sand production

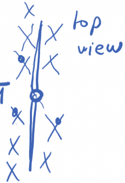
• Completion
- • perforation (5)
- • hydraulic fracture (6)
- • fracture react. → microseism. (7)

(7)

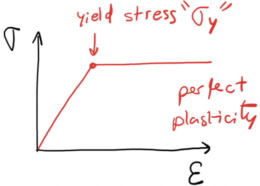
• micro-seismicity

↕

shear failure



Yield criteria insensitive to mean (effective) stress



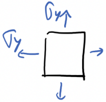
$$f(\underline{\underline{\sigma}}) = K \rightarrow \text{yield criterion, failure criterion}$$

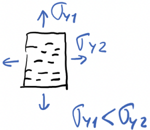
$$f(\sigma_1, \sigma_2, \sigma_3, \text{direction } \sigma_i) = K$$

$\Downarrow$ isotropy

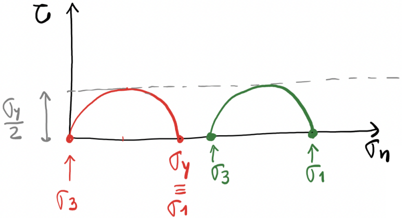
$$f(\sigma_1, \sigma_2, \sigma_3) = K$$

$$f(I_1, I_2, I_3) = K^*$$

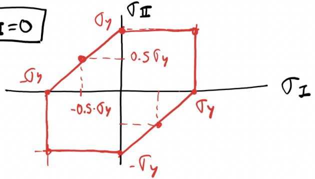




Tresca

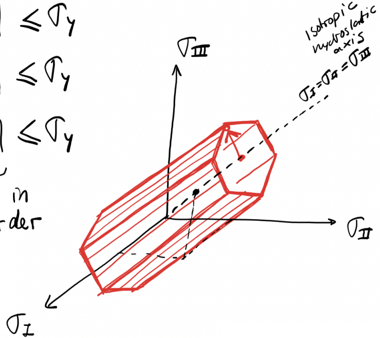


$\boxed{\sigma_{III}=0}$

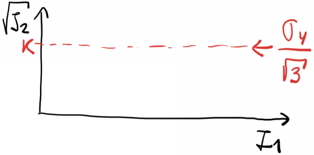


$$\boxed{\sigma_1 - \sigma_3 \leq \sigma_y}$$

$$\underbrace{\begin{array}{l} |\sigma_I - \sigma_{II}| \leq \sigma_y \\ |\sigma_I - \sigma_{III}| \leq \sigma_y \\ |\sigma_{II} - \sigma_{III}| \leq \sigma_y \end{array}}_{\text{not ordered in particular order}}$$



Von Mises

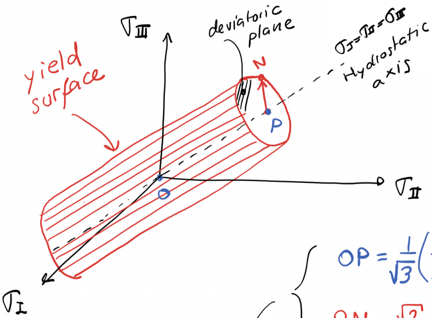


$$J_2 = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]$$

$$\sigma_1 = \sigma_y \; ; \quad \sigma_2 = \sigma_3 = 0$$

$$J_2 = \frac{1}{6}\left[2\sigma_y^2\right]$$

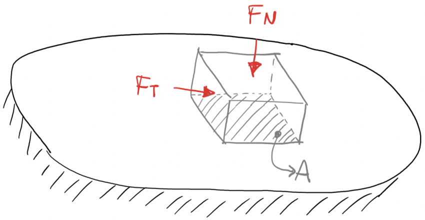
$$\boxed{\sqrt{J_2} = \frac{\sigma_y}{\sqrt{3}}} \longrightarrow PN = \underbrace{\sqrt{\frac{2}{3}}\cdot\sigma_y}_{\text{Von Mises stress}}$$



$$\left\{\begin{array}{l} OP = \frac{1}{\sqrt{3}}\left(\frac{I_1}{3}\right) \\ PN = \sqrt{2}\,\sqrt{J_2} \end{array}\right.$$

Stress sensitive yield criteria

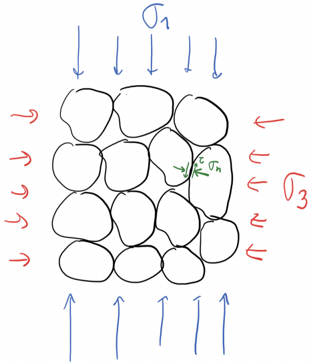
↳ Frictional strength

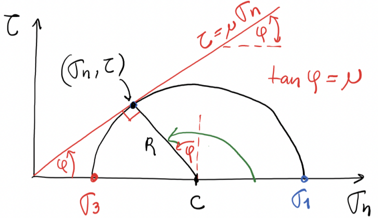


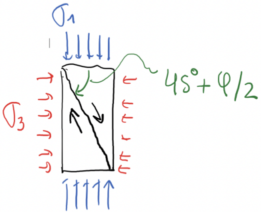
Friction coefficient (~0.3 to 1.0)

$$\tau = \mu \cdot \sigma_n$$

$$\frac{F_T}{A} = \mu \frac{F_N}{A}$$



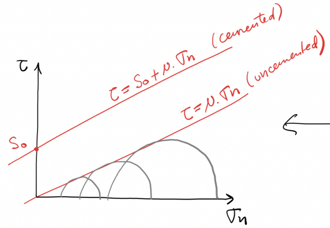


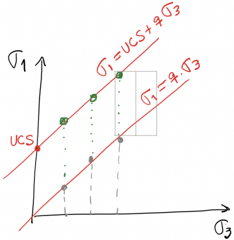


$$\frac{\sigma_1}{\sigma_3} = \frac{C + R}{C - R}$$

$$\frac{\sigma_1}{\sigma_3} = \frac{\cancel{C} + \sin\varphi \cdot \cancel{C}}{\cancel{C} - \sin\varphi \cdot \cancel{C}} \quad \longleftarrow \quad \sin\varphi = \frac{R}{C}$$

$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin\varphi}{1 - \sin\varphi} = q; \qquad \varphi \sim 30^\circ \Rightarrow q = 3$$

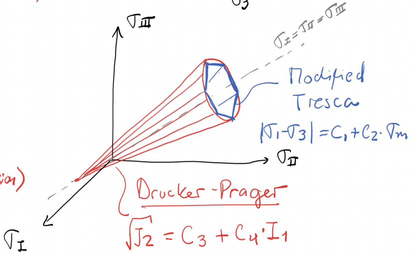




$$\mu = \frac{q-1}{2\cdot\sqrt{q}}$$

$$UCS = 2\,S_0\sqrt{q}$$

Extension to 3D

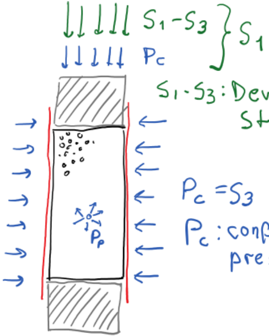


Drucker-Prager

$$\sqrt{J_2} = C_3 + C_4 \cdot I_1$$

Mohr Coulomb $\Rightarrow$
$$\begin{cases} C_3 = \dfrac{6 \cdot S_0 \cdot \cos\varphi}{\sqrt{3}\,(3-\sin\varphi)} \\[2ex] C_4 = \dfrac{2}{\sqrt{3}}\,\dfrac{\sin\varphi}{(3-\sin\varphi)} \end{cases}$$

Triaxial tests (axisymmetric)



$S_1 - S_3$: Deviatoric Stress

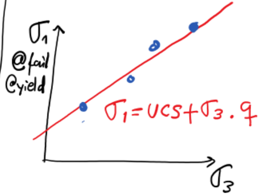
$P_c = S_3$

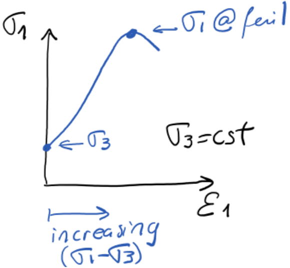
P_c: confining pressure

$$S_1 = (S_1 - S_3) + P_c$$

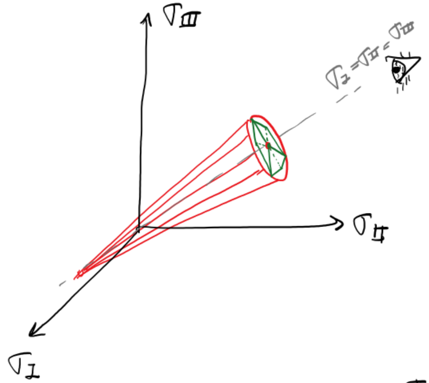
$$\sigma_1 = S_1 - P_p$$

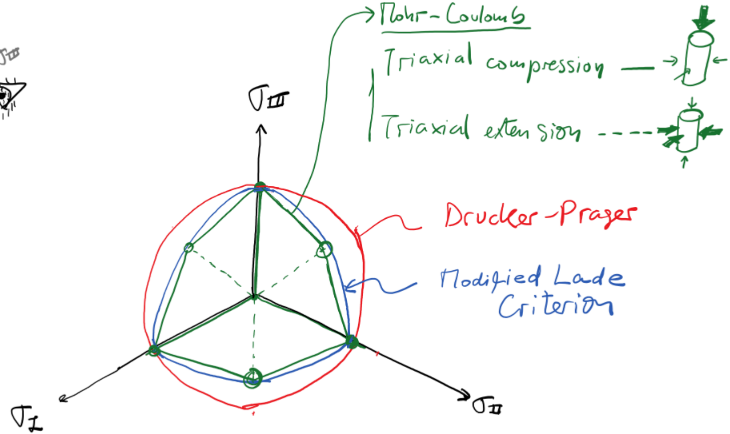
$$\sigma_3 = S_3 - P_p$$





Modified Lade criterion





$$f(I_1, I_3) = K$$

$$\boxed{\frac{(I_1^*)^3}{I_3^*} = 27 + \eta}$$

$$I_1^* = \sigma_1^* + \sigma_2^* + \sigma_3^*$$

$$I_3 = \sigma_1^* \cdot \sigma_2^* \cdot \sigma_3^*$$

$$\sigma_i^* = \sigma_i + S$$

$$S = S_0 / \tan\varphi$$

$$\eta = \frac{4(\tan\varphi)^2(9 - 7\sin\varphi)}{1 - \sin\varphi}$$

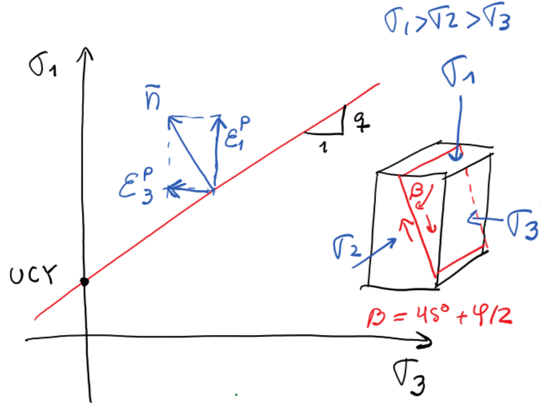
Beyond the yield point: determination of plastic strains

- determination of strains
  - elastic
    $$\underline{\underline{\varepsilon}} = \underline{\underline{\underline{\underline{D}}}} \cdot \underline{\underline{\sigma}}$$
    $$\varepsilon_{ij} = D_{ijmn}\,\sigma_{mn}$$
    $$d\varepsilon_{ij} = D_{ijmn}\,d\sigma_{mn}$$
  - plastic – $d\varepsilon_{ij}^{p} \leftrightarrow d\sigma_{ij}$

- · small strain
  · continuous strain field
  · strain-rate independent

1. Yield criterion
   $$f(\sigma_{ij}) = Y$$
2. Strain-hardening rule
   $$Y = F^{*}(\delta\varepsilon^{p})$$
3. Strain decomposition
   $$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}$$
4. Plastic flow rule
   $$d\varepsilon_{ij}^{p} \leftrightarrow d\sigma_{ij}$$
5. Elastic unloading criterion

Example: Mohr - Coulomb



$$\tau = S_0 + \mu \cdot \sigma_n \quad ; \quad \mu = \tan\varphi$$

$$\updownarrow$$

$$\boxed{\sigma_1 = UCY + q\,\sigma_3} \quad ; \quad q = \frac{1+\sin\varphi}{1-\sin\varphi}$$

$$\varphi = 30° \Rightarrow q =$$

$$\downarrow$$

$$f = \sigma_1 - UCY - q\,\sigma_3$$

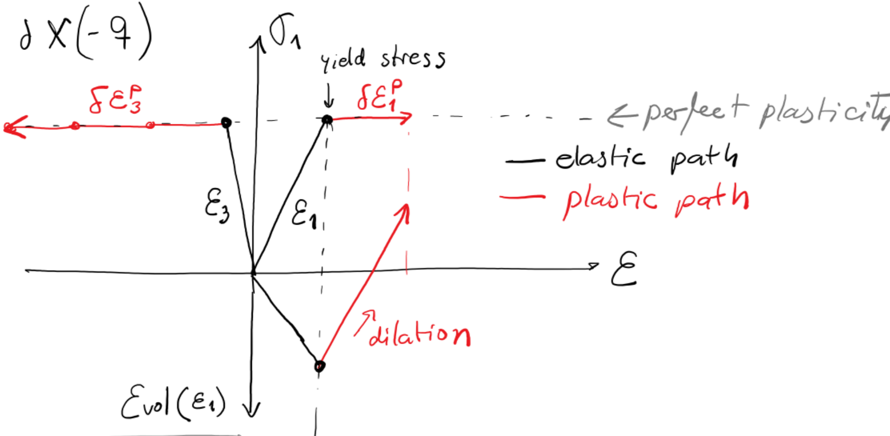
$$\downarrow$$

$$\bar{n} = \left(\frac{\partial F}{\partial \sigma_1}, \frac{\partial F}{\partial \sigma_2}, \frac{\partial F}{\partial \sigma_3} \right)$$

$$\bar{n} = (1, 0, -q)$$

Flow rule: $$d\varepsilon_{ij}^{p} = \underbrace{d\lambda}_{cst} \cdot \frac{\partial F}{\partial \sigma_{ij}}$$

$$\begin{cases} d\varepsilon_1^p = d\chi \cdot 1 \\ d\varepsilon_2^p = d\chi \cdot 0 \\ d\varepsilon_3^p = d\chi(-q) \end{cases} \Longrightarrow \delta\varepsilon_{vol}^p = d\chi(1-q)$$



$$\boxed{d\varepsilon_{ij}^p = d\chi \frac{\partial F}{\partial \sigma_{ij}}}$$

Yield surface

⇒ Associated flow rule

↳ yield surface

$$\boxed{d\varepsilon_{ij}^p = d\chi \frac{\partial g}{\partial \sigma_{ij}}}$$

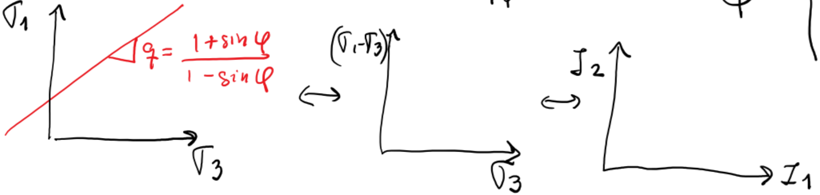
Plastic potential function, $g \neq F$

⇒ Non-associated flow rule

ψ ← dilation angle

$$g = \sigma_1 - UCY - \underbrace{\frac{1+\sin\psi}{1-\sin\psi}}_{q_\psi}\sigma_3 \qquad \psi < \varphi$$

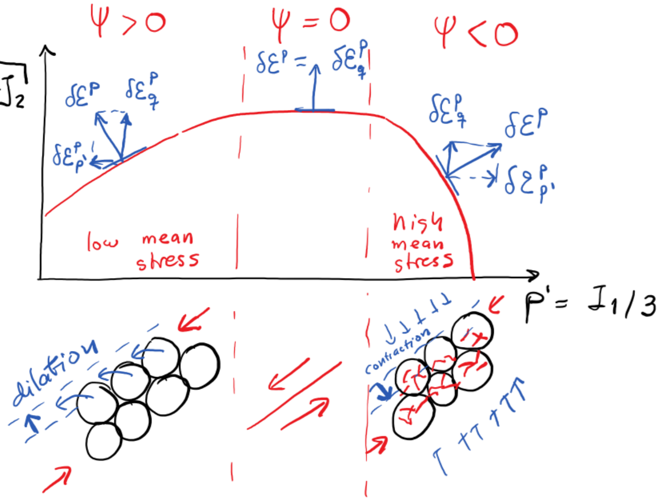
$$\psi = \begin{cases} \psi > 0 \Rightarrow \text{dilation} \\ \psi = 0 \Rightarrow \text{iso-choric} \\ \psi < 0 \Rightarrow \text{contraction} \end{cases}$$



$\boxed{p'-q \text{ space}}$

$$q = \sqrt{3 \cdot J_2}$$

$$\boxed{\delta\varepsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}}}$$



S_1

S_3

$$P_{pore\ fluid}$$

$$\sigma_1 = UCS + q\sigma_3$$

S_0

φ

$$(J_2)^{1/2}$$

$$P_W = P_p$$

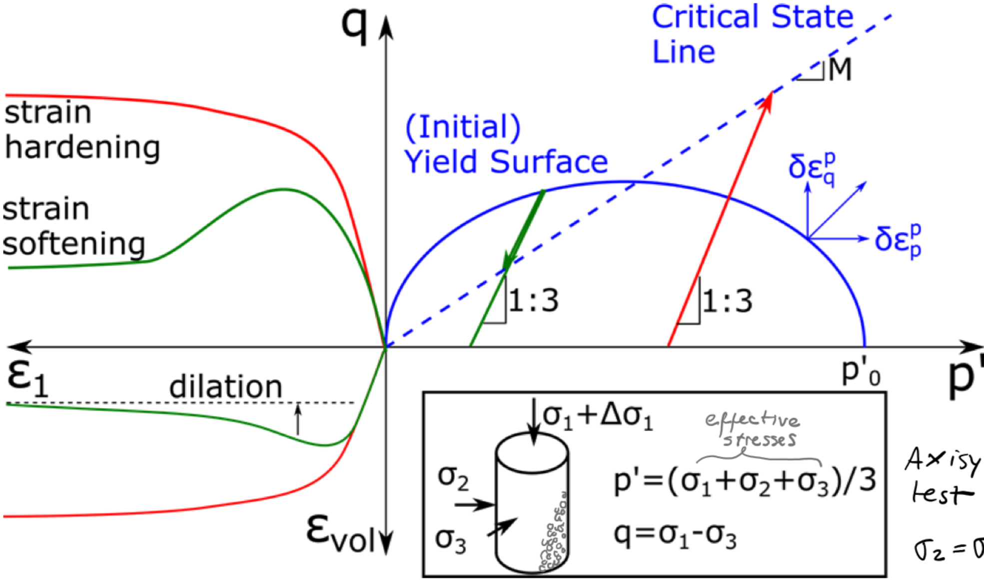
w_{BO}

$$0^\circ < w_{BO} < 60^\circ$$

$$w_{BO} > 60^\circ$$

$$P_W$$

→ Cam-clay Model (uncemented sediments)



Axisymmetric triaxial test (drained)

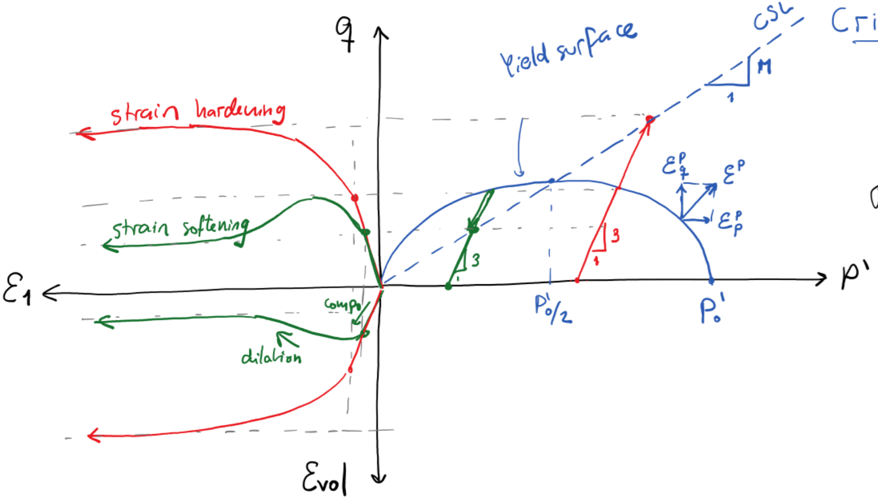
$\sigma_2 = \sigma_3 = cst$ for a given test

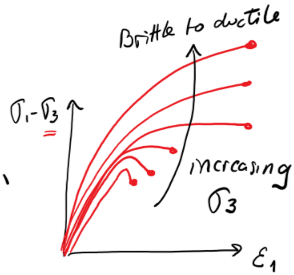
$\Delta\sigma_1 \rightarrow$ until yield

Deviatoric loading

$$\left\{ \begin{aligned} \Delta p' &= \frac{\Delta\sigma_1}{3} \\ \Delta q &= \Delta\sigma_1 \end{aligned} \right\} \quad \frac{\Delta q}{\Delta p'} = 3$$

Critical state line





$q \stackrel{def}{=} M \cdot p'$ (@ CSL)

$$M = \frac{q}{p'} = \frac{\sigma_1 - \sigma_3}{\frac{\sigma_1 + 2\sigma_3}{3}} \Longrightarrow M(\varphi_{cs} = 30^\circ) = \frac{2\cancel{\sigma_3}}{\frac{5\cancel{\sigma_3}}{3}} = \frac{6}{5} = 1.2$$

$$\Downarrow$$

$$\sigma_1 = 3\sigma_3$$

$$\boxed{M = \frac{3\left(\frac{1+\sin\varphi_{cs}}{1-\sin\varphi_{cs}} - 1\right)}{2 + \left(\frac{1+\sin\varphi_{cs}}{1-\sin\varphi_{cs}}\right)}}$$

Yield surface

$$f(q, p', \underset{\sim}{p_0'}) = q^2 - M^2 p'(p_0' - p') = 0$$

$q = 0 \Rightarrow \begin{cases} p' = 0 \\ p' = p_0' \end{cases}$

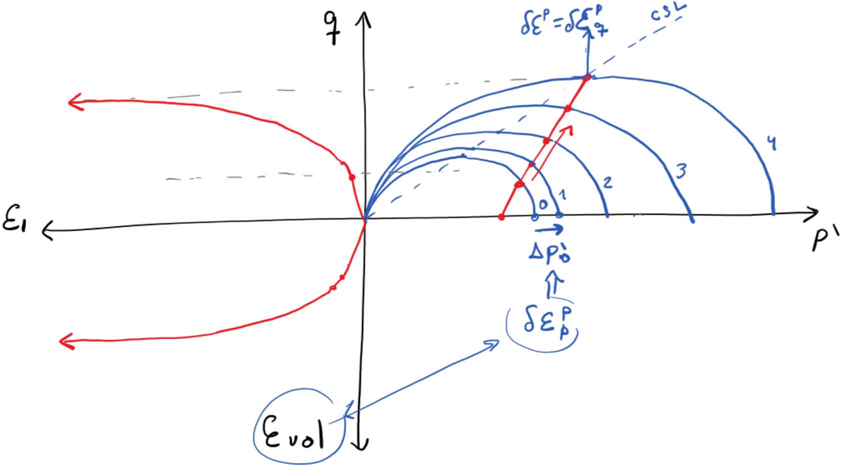
$q = Mp' \Rightarrow p' = \dfrac{p_0'}{2}$

$\rightarrow$ pre-consolidation pressure (stress)

↳ variable

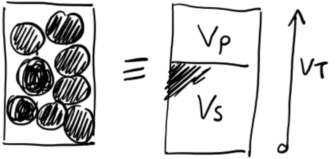
↳ size of the yield surface

↳ hardening parameter $\leftrightarrow f(\underline{\underline{\delta \varepsilon_p^p}}) \sim f(\emptyset)$

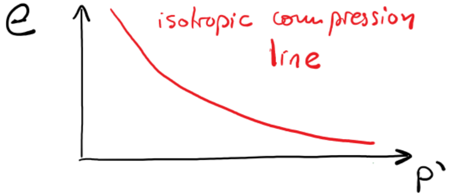


q
CSL
$\delta\varepsilon^p = \delta\varepsilon^p_q$
4
3
2
1
0
ε_1
p'
$\Delta p'_0$
$\Uparrow$
$\delta\varepsilon^p_p$
ε_{vol}

$$\underline{p_0'} = f(\delta \varepsilon_p^p) = f^*\Big(\overset{\text{void ratio}}{e} = \frac{\phi}{1-\phi}\Big)$$



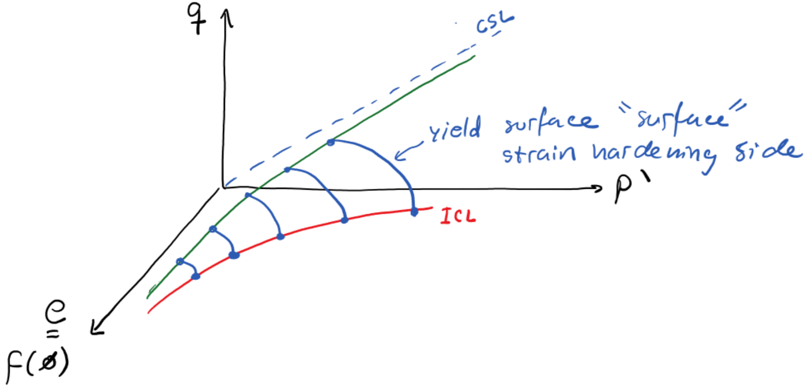
$$\left.\begin{array}{l} \phi \overset{def}{=} \dfrac{V_p}{V_T} \\[2ex] e \overset{def}{=} \dfrac{V_p}{\underline{\quad}} \end{array}\right| \Rightarrow \quad V_T = 1 \qquad e = \frac{V_p}{1-V_p} = \frac{\phi}{1-\phi}$$



$\sigma_1 = \sigma_2 = \sigma_3$

Isotropic Compression Test

$$\Rightarrow \downarrow \phi \approx \downarrow e \Rightarrow \uparrow p_0'$$



q
CSL
yield surface "surface"
strain hardening side
p'
ICL
e = f(ø)

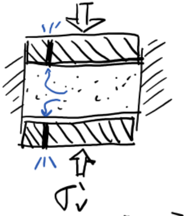
Calculation of elastic and plastic strains

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^e + \underline{\underline{\varepsilon}}^p \leftarrow \text{strain decomposition}$$

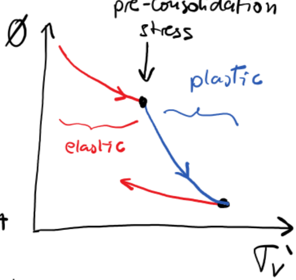
$$d\underline{\underline{\varepsilon}} = d\underline{\underline{\varepsilon}}^e + d\underline{\underline{\varepsilon}}^p \quad \text{(incremental version)}$$

$$d\underline{\underline{\varepsilon}} = \underbrace{\underline{\underline{\underline{\underline{C}}}}^e \, d\underline{\underline{\sigma}}}_{\text{elastic}} + \underbrace{\underline{\underline{\underline{\underline{C}}}}^p \, d\underline{\underline{\sigma}}}_{\text{plastic}}$$

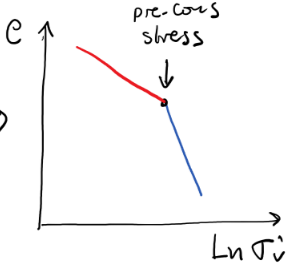
① Elastic strains

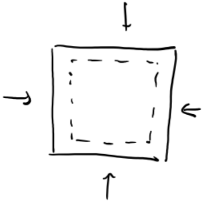


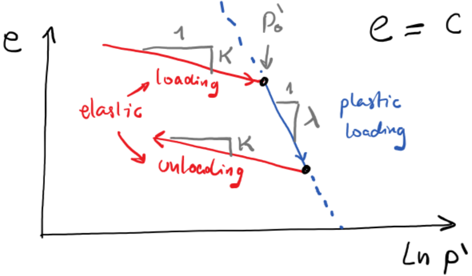
Consolidation Test
(uniaxial strain)
Oedometric condition



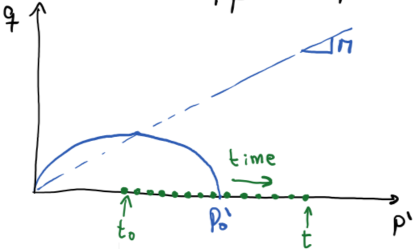
⇒







$$e = c - \kappa \cdot \ln p' \rightarrow \frac{de}{dp'} = -\frac{\kappa}{p'} \qquad (1)$$



$$e \overset{\text{def}}{=} \frac{V_p}{V_b - V_p} \longrightarrow de = \ldots dV_p + \ldots dV_b$$

assumption $dV_b = dV_p \quad (\alpha \approx 1)$

$$\delta \varepsilon_{p'} \equiv \frac{\partial V_b}{V_b} = \frac{\partial V_p}{V_b}$$

$$de = -(1+e)\,\delta \varepsilon_{p'} \qquad (2)$$

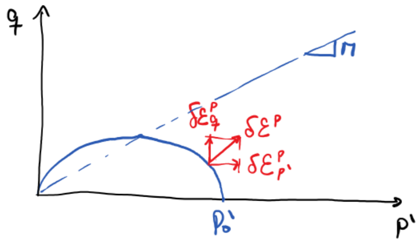
① + ②

$$\begin{cases} \delta \varepsilon_{p'}^{e} = \dfrac{K}{1+e} \cdot \dfrac{dp'}{p'} \\ \delta \varepsilon_{q}^{e} = \dfrac{1}{3G} \cdot dq \end{cases} \quad ; \quad \varepsilon_q \overset{def}{=} \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$$

$$\begin{bmatrix} \delta \varepsilon_{p'}^{e} \\ \delta \varepsilon_{q}^{e} \end{bmatrix} = \begin{bmatrix} \frac{K}{1+e} \cdot \frac{1}{p'} & 0 \\ 0 & \frac{1}{3G} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

Elastic strains (before yield)

(2) Plastic strains



$$f = q^2 - M^2 p'(p'_0 - p')$$

$$f^* = \frac{p'}{p'_0} - \frac{M^2}{\eta^2 + M^2} \quad ; \quad \eta = \frac{q}{p'}$$

derivatives

$$\begin{cases} \dfrac{\partial f^*}{\partial p'} = p'_0 M^2 \left(\dfrac{M^2 - \eta^2}{M^2 + \eta^2} \right) \\ \dfrac{\partial f^*}{\partial q} = p'_0 M^2 \left(\dfrac{2\eta}{M^2 + \eta^2} \right) \\ \dfrac{\partial f^*}{\partial p'_0} = -\dfrac{p'}{(p'_0)^2} \end{cases}$$

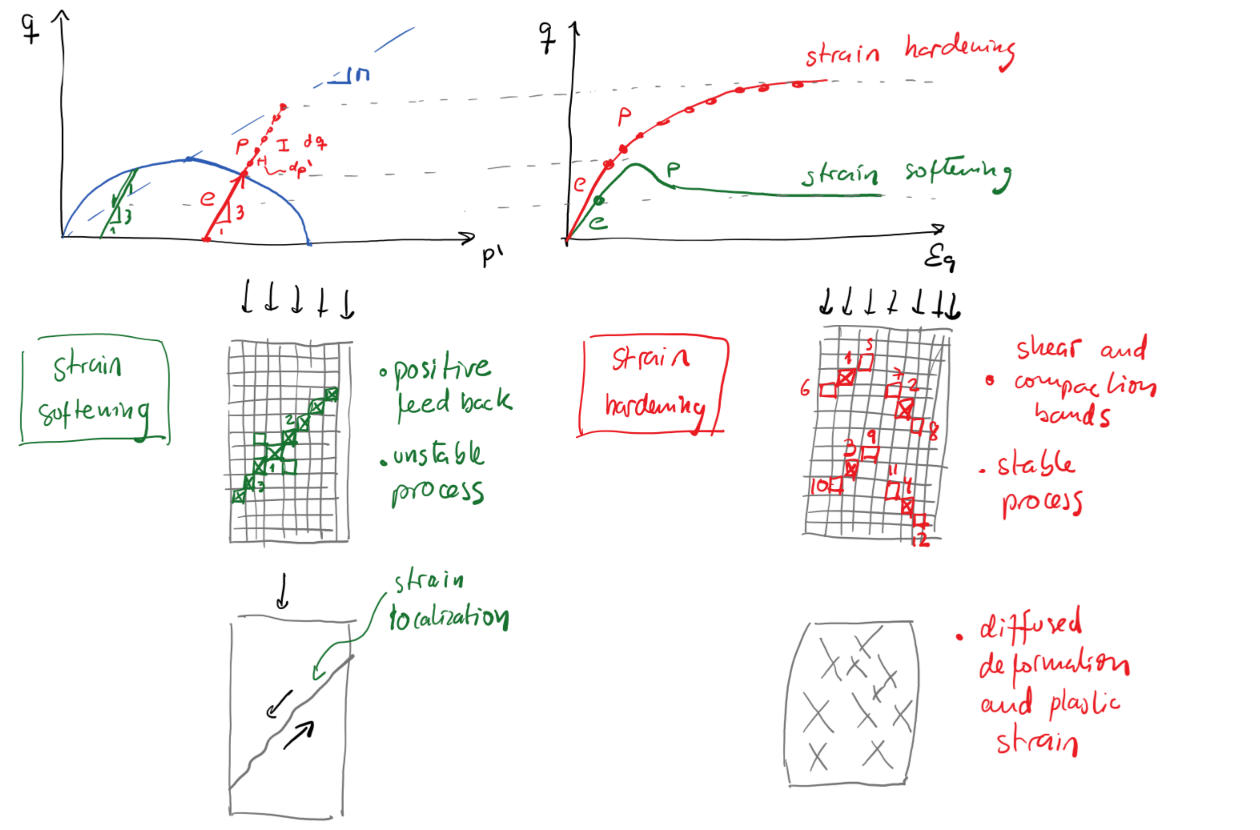
Associated flow rule

$$\delta\varepsilon^p_{ij} = d\chi \frac{\partial f^*}{\partial \sigma_{ij}} \Rightarrow \begin{cases} \delta\varepsilon^p_{p'} = d\chi \dfrac{\partial f^*}{\partial p'} \\ \delta\varepsilon^p_q = d\chi \dfrac{\partial f^*}{\partial q} \end{cases} \quad (3)$$

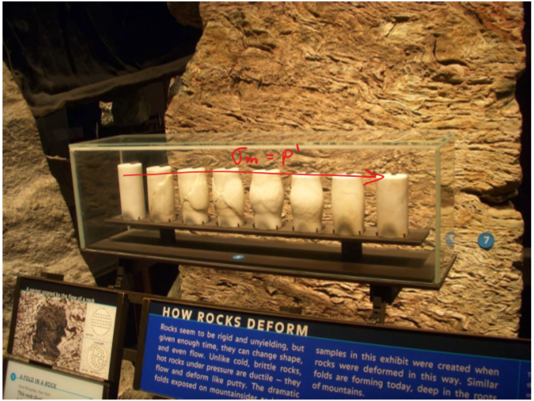
From ICL (4) $\delta\varepsilon^p_{p'} = \frac{\lambda-\kappa}{1+e}\frac{dp_0'}{p_0'}$

(3)+(4) $\rightarrow$ $d\chi = \frac{\lambda-\kappa}{(1+e)p'(M^2+\eta^2)}$ $\leftarrow$ hardening parameter

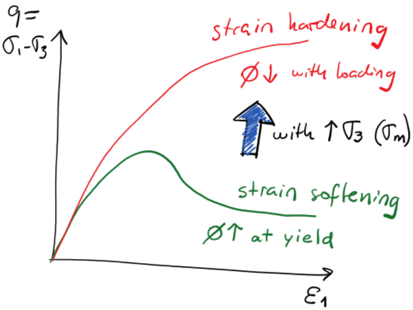
$$\begin{bmatrix}\delta\varepsilon^p_{p'}\\ \delta\varepsilon^p_q\end{bmatrix} = \frac{\lambda-\kappa}{(1+e)p'(M^2+\eta^2)}\begin{bmatrix}M^2-\eta^2 & 2\eta\\ 2\eta & \frac{4\eta^2}{M^2-\eta^2}\end{bmatrix}\begin{bmatrix}dp'\\ dq\end{bmatrix}$$ $\rightarrow$ Plastic Strains



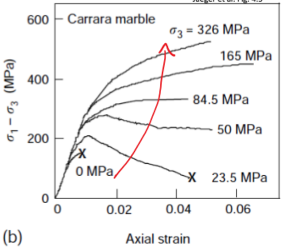
q
p'
e
p
dq
dp'
q
strain hardening
strain softening
ε_q
strain softening
• positive feed back
• unstable process
strain localization
strain hardening
• shear and compaction bands
• stable process
• diffused deformation and plastic strain

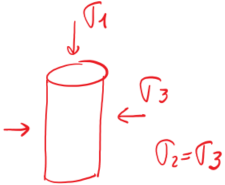


"Brittle to ductile transition"

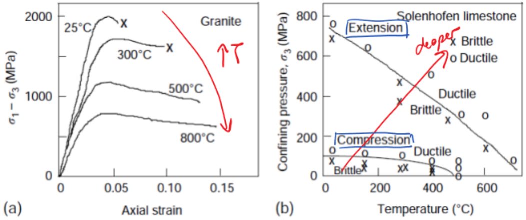


Jaeger et al. Fig. 4.5



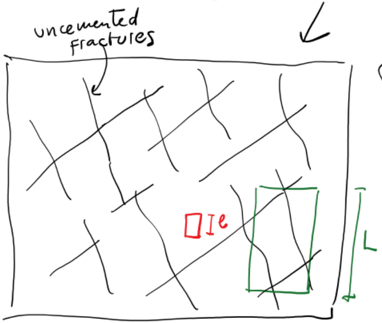


Jaeger et al., Fig. 4.6

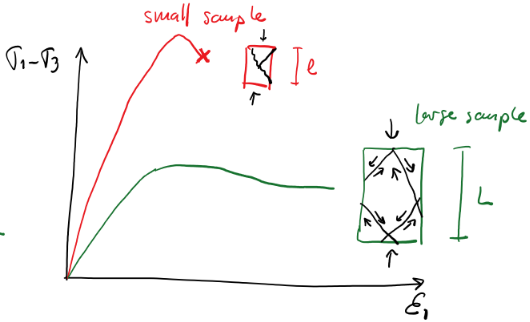


Britte to ductile transition

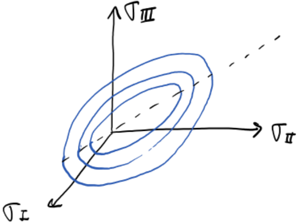
- $\uparrow \sigma_m \ (= p')$, $\uparrow \sigma_3$, mean stress
- $\uparrow T$, temperature
- $\downarrow$ loading rate
- mineralogy of rocks, organic shales $\left\{ \begin{array}{l} \uparrow TOC \\ \uparrow clay \end{array} \right.$
- length scale



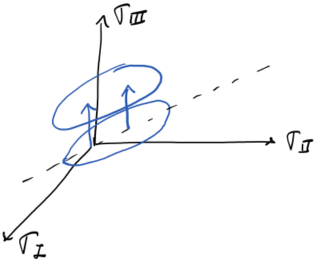
$l \ll L$



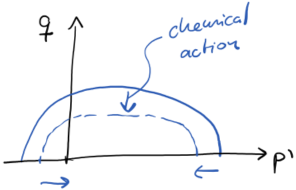
Isotropic hardening



Kinematic hardening



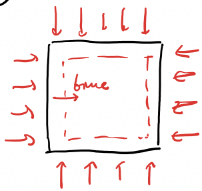
Chemo-plasticity

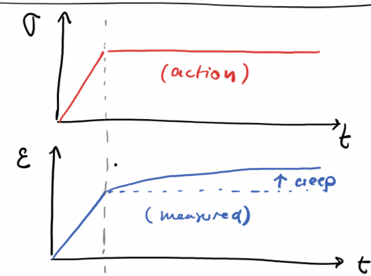


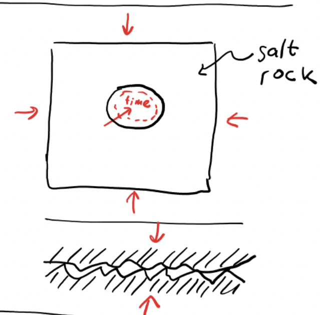
- Manifestations of visco-elastic behavior

Example

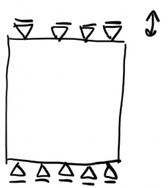
(1) Creep strain

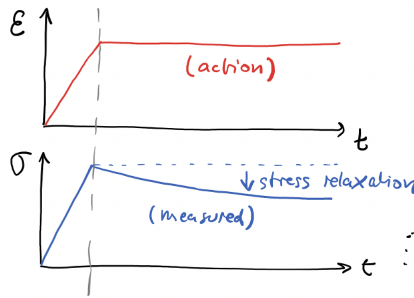


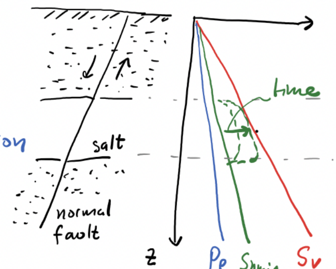




(2) Stress relaxation



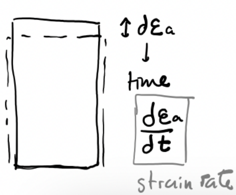


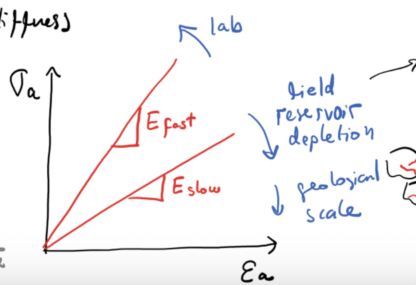


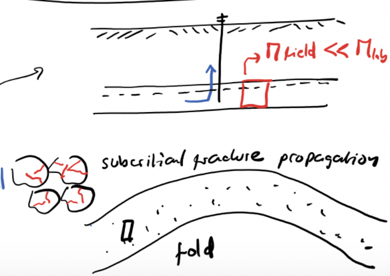
$$q = \frac{1+\sin\varphi}{1-\sin\varphi} \sim 3 \quad (\varphi = 30^\circ)$$

$$\boxed{\sigma_{hmin} = \frac{1}{q}\sigma_v}$$

(3) strain-rate dependent stiffness







$$dP_p/dz$$

$$dS_v/dz$$

$$\phi = \phi_0 \exp\left(-\beta \sigma_v\right)$$

$$e = e_0 - C_c \ln\left(\frac{\sigma_v}{1\ \mathrm{MPa}}\right)$$

λ_p

S_v

$$dq = 3dp'$$

G

$$p'_o$$

$$\phi_{CS} = 24^\circ$$

λ

κ

e_o

ε_q

e

$$dq = 0.9\, dp'$$

$$p' = 400$$

$$d\varepsilon_{p'}^{e} = \frac{\kappa}{\upsilon}\frac{dp'}{p'};\; d\varepsilon_{q}^{e} = \frac{dq}{3G}$$

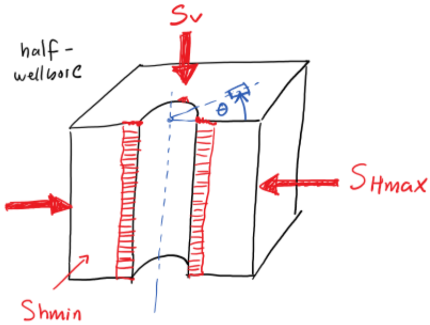
$$\begin{bmatrix} d\varepsilon^p_{p'} \\ d\varepsilon^p_q \end{bmatrix} = \frac{\lambda-\kappa}{\upsilon p'(M^2+\eta^2)} \begin{bmatrix} M^2-\eta^2 & 2\eta \\ 2\eta & \frac{4\eta^2}{M^2-\eta^2} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$v = 1 + e$$

$$\eta = q/p'$$

$$de = -\upsilon d\varepsilon_p$$

$$dp'_o = d\varepsilon^p_{p'} \frac{\upsilon}{\lambda - \kappa} p'_o$$

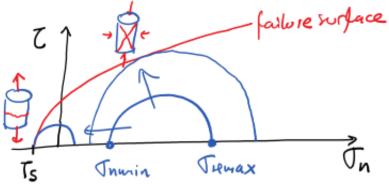


- Normal faulting ✓

$$S_v > S_{Hmax} > S_{hmin}$$

- Kirsch solution ✓

$$\sigma_{\theta\theta}(r=a,\theta) = -(P_w - P_p) + (\sigma_{Hmax} + \sigma_{hmin}) \dots$$
$$- 2(\sigma_{Hmax} - \sigma_{hmin}) \cos(2\theta)$$



Solve for $\begin{cases} \theta = 0 \text{ or } \pi \\ \sigma_{\theta\theta} = -T_s \end{cases}$

$$\Downarrow$$

Break down pressure

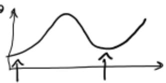
$$\boxed{P_w = P_b = \underbrace{P_p}_{\text{Pore pressure}} - \underbrace{\sigma_{Hmax} + 3\sigma_{hmin}}_{\text{Stress Anisotropy}} + \underbrace{T_s}_{\text{Tensile strength}}}$$

(Case 1) $\sigma_{Hmax} \approx \sigma_{hmin} = \sigma_h$ (isotropic, horizontal stress anisotropy)

$$\boxed{P_b = P_p + 2\sigma_h + T_s}$$

(Case 2) $\sigma_{Hmax} \approx 3\sigma_{hmin}$ (maximum horizontal stress anisotropy)

$$\boxed{P_b = P_p + T_s}$$

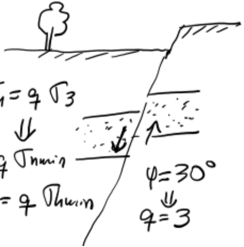


$\sigma_1 = q\,\sigma_3$

$\Downarrow$

NF $\sigma_v = q\,\sigma_{hmin}$

SS $\sigma_{Hmax} = q\,\sigma_{hmin}$



$\varphi = 30°$

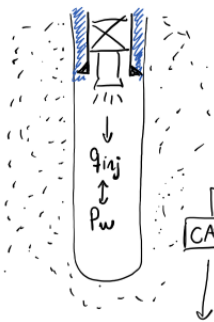
$\Downarrow$

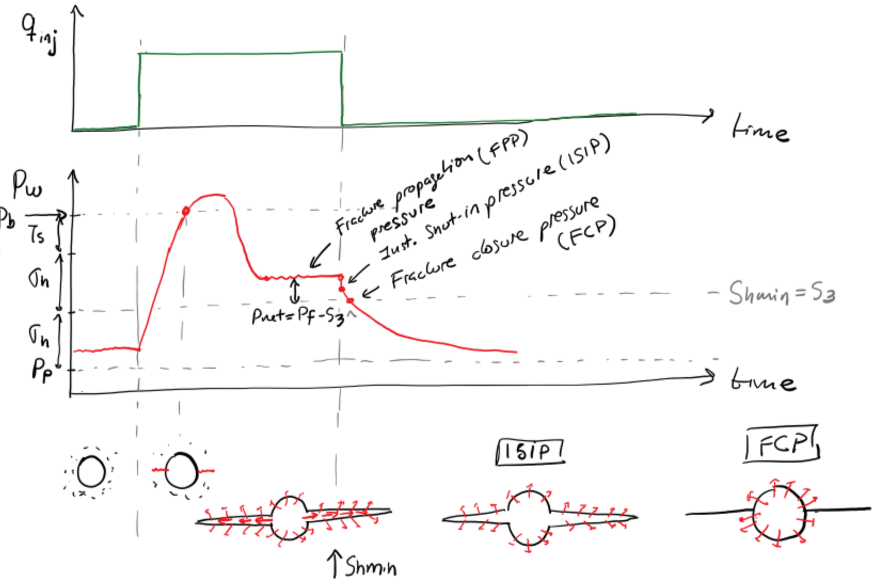
$q = 3$

Leak-off test ⟷ Diagnostic Fracture Initiation Test (DFIT)

DRILLING

COMPLETIONS AND HF

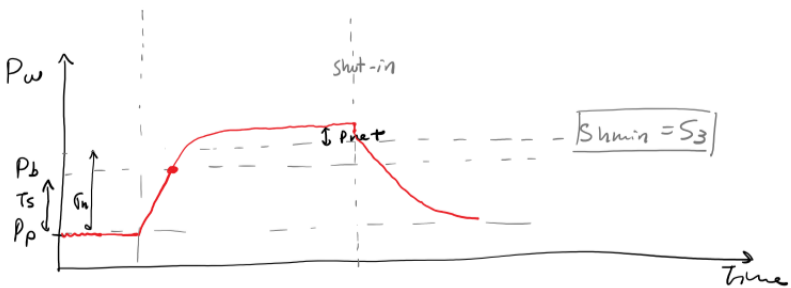




$\sigma_{Hmax} = \sigma_{hmin} = \sigma_h$

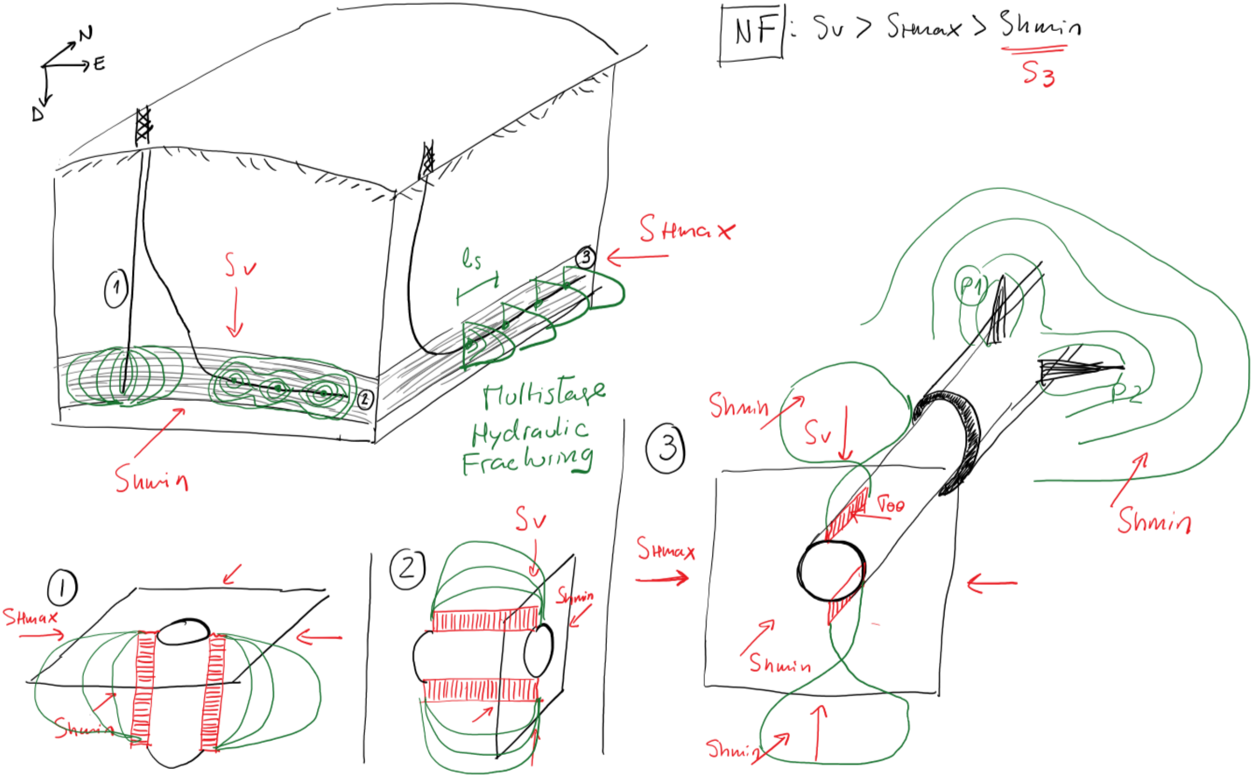
$S_3 = S_{hmin} = P_p + \sigma_h$

Case 2



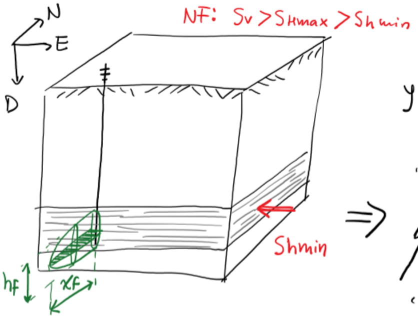
$P_b = P_p + T_s$

Ideal orientation of HF growing from wellbores and perforations



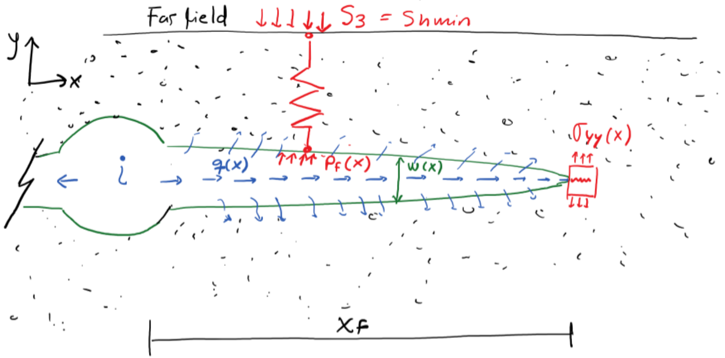
5- Coupled fluid-driven fracture problem ← Valko and Economides Hydraulic Fracture Mechanics

Thursday, November 12, 2020 2:30 PM

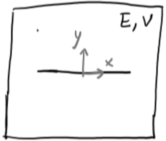


⇒

Horizontal cross section (plane N-E)



$$\underline{P_{net}} = P_f - S_3 = P\left(\begin{array}{c}\text{elastic}\\ \text{deformation}\end{array}\right) + P\left(\begin{array}{c}\text{viscous}\\ \text{losses}\end{array}\right) + P\left(\begin{array}{c}\text{new rock}\\ \text{surface}\end{array}\right)$$



$$\nabla \underline{\underline{\sigma}} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}} \rightarrow$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$w(x) \propto \frac{P_{net} \cdot x_f}{E'}$$

- $q(x) \propto \dfrac{[w(x)]^3 \, h_f}{\mu} \cdot \dfrac{dP(x)}{dx}$

- leak-off

$$q_m = -\frac{k}{\mu} \nabla P$$

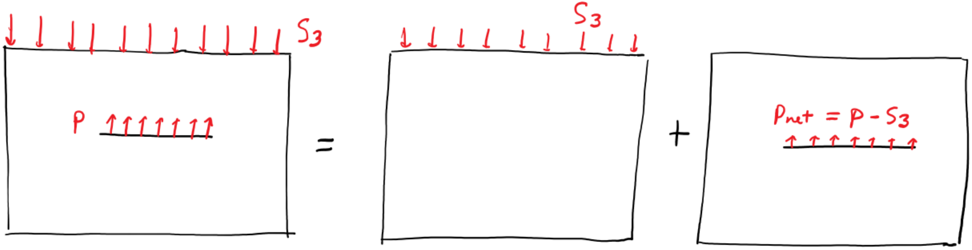
K_{IC} : Fracture toughness

- $K_I > K_{IC} \Rightarrow$ Frac propag
- $K_I < K_{IC} \Rightarrow$ Frac does not propagate

K_I : Fracture Intensity

$$K_I = \lim_{r \to 0^+} \left[\sqrt{2\pi r} \cdot \sigma_{yy}(c+r, 0)\right]$$

$$K_I = f(P_{net}, x_f)$$

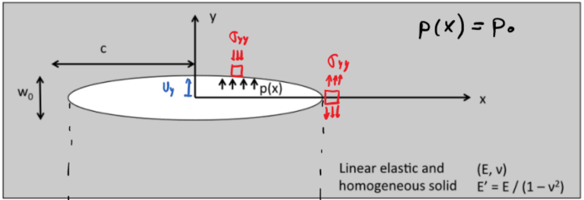


S_3: far-field least principal stress

P_{net}: net pressure

Displacements and stresses around a linear fracture (plane-strain)

Griffith solution for Navier's Eq. with a Linear Elastic Isotropic Solid



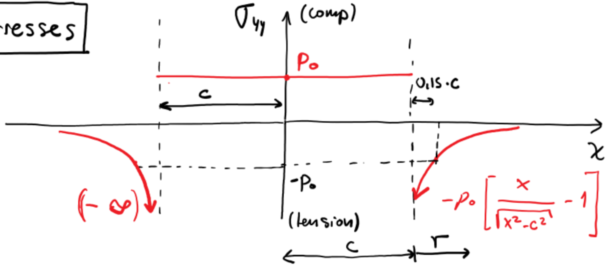
Displacements

$$U_y(x,0) = \begin{cases} \dfrac{2P_0}{E'}\sqrt{c^2 - x^2} & ;\ 0 \leq x \leq c \\ 0 & ;\ x > c \end{cases}$$

$$\Downarrow$$

$$W_0 = 2\,U_y(0,0) = \frac{4P_0 c}{E'}$$

Stresses



$$\sigma_{yy} = \begin{cases} P_0 & ;\ 0 \leq x < c \\ -P_0\left[\dfrac{x}{\sqrt{x^2 - c^2}} - 1\right] & ;\ x \geq c \end{cases}$$

Stress intensity factor

$$\rightarrow K_I = \lim_{r \to 0^+} \left[\sqrt{2\pi r} \cdot \left(-\sigma_{yy}(c+r, 0) \right) \right] \leftarrow$$

Fracture Modes
- I : open-mode
- II : shear in-plane
- III : shear out-of-plane

$$= \lim_{r \to 0^+} \left[\sqrt{2\pi r} \cdot \left(+p_0 \left[\frac{c+r}{\sqrt{(c+r)^2 - c^2}} - 1 \right] \right) \right] \quad ; x = c+r$$

$$= p_0 \sqrt{2\pi} \cdot \lim_{r \to 0^+} \left[r^{1/2} \cdot \left(\frac{c+r}{\sqrt{2cr + r^2}} - 1 \right) \right]$$

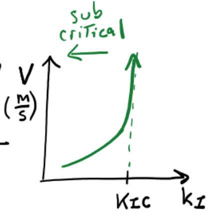
$$\boxed{K_I = p_0 \cancel{\sqrt{2}} \sqrt{\pi} \left(\frac{c}{\cancel{\sqrt{2}} \sqrt{c}} \right) = p_0 \sqrt{\pi c}}$$

p_0 : constant pressure

LEFM criterion for fracture propag.

$$\begin{cases} K_I \geq K_{IC} \Rightarrow \text{fracture propagation} \\ K_I < \underbrace{K_{IC}} \Rightarrow \text{no fracture propagation} \end{cases}$$

Fracture toughness (c: critical)



Typical values of

$K_{IC} \sim 0.2 - 2 \ \underbrace{MPa \cdot m^{1/2}}$ (geological materials)

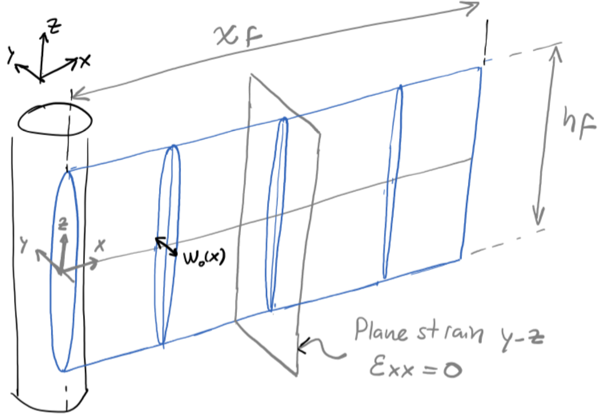
Mode 1

measured through "notch experiments"

e.g.: semicircular bending experiment

Objective: understand methodology to solve a fluid-driven fracture problem

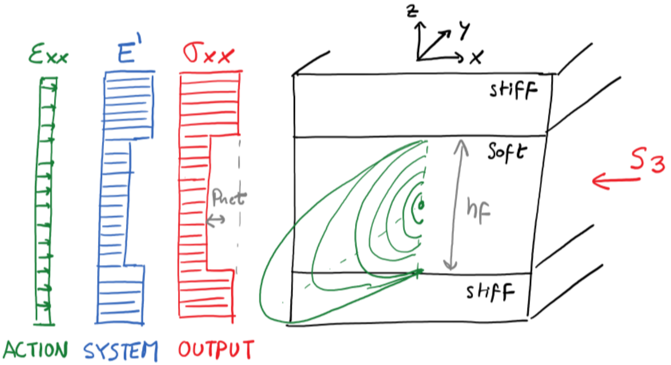
Example: PKN Model (Perkins, Kern, Nordgren)



Main assumptions

- h_f constant ✓
- $x_f > h_f$ ✓
- Plane strain y-z ✓
- $K_{Ic} = 0$ ✓
- Geometry as in figure ✓

① Fracture height ← 1D stress solution



② Fracture width ← Griffith solution

Solution in plane y-z

$$W_o(x) = \frac{4\,P_n(x)\,C}{E'} = \frac{2\,P_n(x)\,h_f}{E'} \quad \boxed{\text{Eq. 1}}$$

hf/2

(3) Fluid flow within fracture ← laminar flow + Newtonian fluid

Flow along x through elliptical channel

$$\frac{dP}{dx} = - \frac{64\,\mu\, q(x)}{\pi\, [W_0(x)]^3\, h_f} \qquad \boxed{Eq.\ 2}$$

(4) Mass conservation ← fluid in fracture + leak-off

$$V_i = V + \cancel{V_L}^{\,=0} \qquad \boxed{Eq.\ 3}$$

$V_L = 2 A_f\, C_L \sqrt{t}$ or Darcy ✓

$$V_i = i \cdot t \qquad \boxed{Eq.\ 4}$$

i: injection rate (constant) ✓

$$q(x) = \underbrace{i}_{\text{one-wing}} \qquad \boxed{Eq.\ 5}$$

↲ no leak-off

$\boxed{\text{Eq } 1+2+5}$

→ Pressure gradient along frac

$$\frac{dP}{dx} = -\frac{64 \cdot \nu \cdot i}{\pi \left[2 h_f \, P_n(x) \, E'^{-1}\right]^3 \cdot h_f}$$

$$\frac{dP}{dx} = -\frac{8 \nu \, i \, E'^3}{\pi \, h_f^4 \left[P_n(x)\right]^3}$$

→ Integrate $P_n(x)$

$$\int_{P_n(0)}^{P_n(x_f)} \left[P_n(x)\right]^3 dP = -\int_0^{x_f} \frac{8 \nu \, i \, E'^3}{\pi \, h_f^4} \, dx$$

$$\frac{\cancel{\left[P_n(x_f)\right]^4}}{4}^{=0} - \frac{\left[P_n(0)\right]^4}{4} = -\frac{8 \nu \, i \, E'^3}{\pi \, h_f^4} \left(x_f - \cancel{0}\right)$$

$$\begin{cases} P_n(x=0) = \left(\dfrac{32\,\nu\, i\, E'^3\, x_f}{\pi\, h_f^4}\right)^{1/4} & \boxed{Eq.6} \\[2ex] W_o(x=0) = \left(\dfrac{512\,\nu\, i\, x_f}{\pi\, E'}\right)^{1/4} & \boxed{Eq.7} \end{cases} \quad + Eq.1$$

Introducing time into equations

Geometry : $V_{frac} = x_f\, h_f\, \bar{w}$

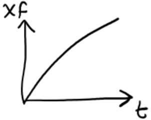
→ where $\bar{w} = \dfrac{\pi}{5} W_o(x=0)$

Material balance: $V_{frac} = i \cdot t$

$$\boxed{Eq.8} \quad x_f = \frac{i \cdot t}{h_f\left(\frac{\pi}{5} \cdot W_o(x=0)\right)}$$

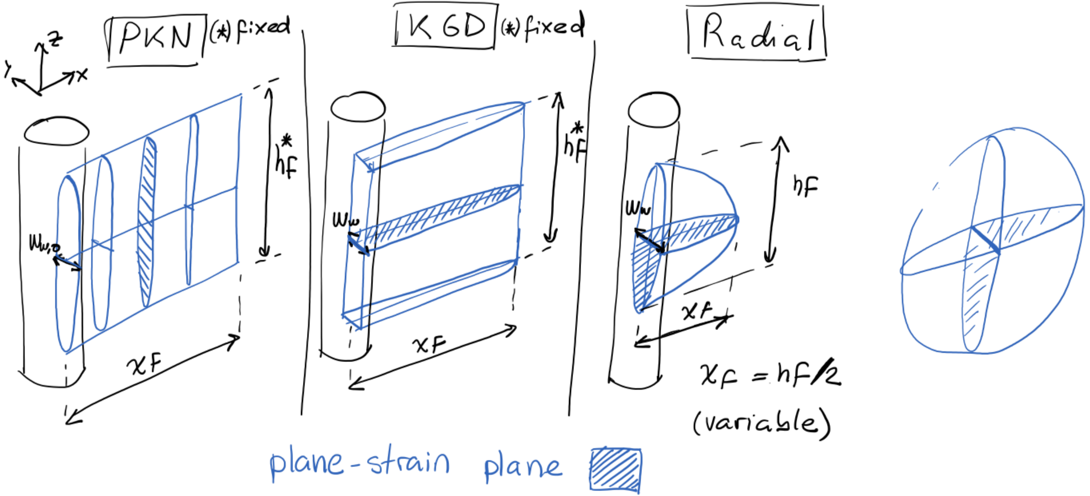
PKN Model

$$
\begin{cases}
x_f = \left(\dfrac{625}{512 \cdot \pi^3}\right)^{1/5} \cdot \left(\dfrac{i^3 \cdot E'}{h_f^4 \cdot \mu}\right)^{1/5} \cdot t^{4/5} \\[2ex]
W_0(x=0) = \left(\dfrac{2560}{\pi^2}\right)^{1/5} \left(\dfrac{i^2 \mu}{E' h_f}\right)^{1/5} \cdot t^{1/5} \\[2ex]
P_n(x=0) = \left(\dfrac{80}{\pi^2}\right)^{1/5} \left(\dfrac{E'^4 i^2 \mu}{h_f^6}\right)^{1/5} \cdot t^{1/5}
\end{cases}
$$



← Eq 7 + 8

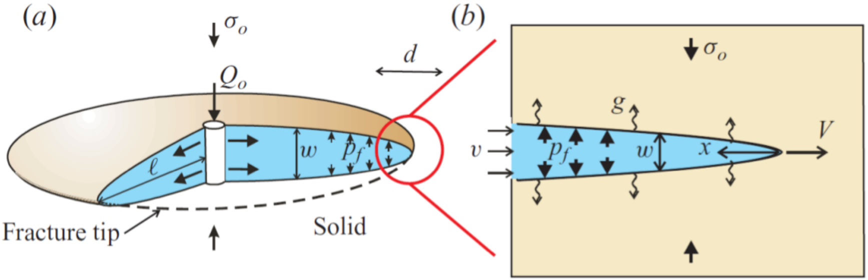
+ Eq. 6


PKN
(*) fixed
KGD
(*) fixed
Radial
z
x
y
h_F^*
$w_{w,0}$
x_F
h_F^*
w_w
x_F
h_F
w_w
x_F
$x_F = h_F/2$
(variable)
plane-strain plane

5- Fluid-driven fractures in porous media

Friday, November 20, 2020 4:27 PM

Detournay, Bunger, Lecampion, et al.



$\boxed{\text{Garagash et al. 2011}}$

Material parameters

$E' = \dfrac{E}{1-\nu^2}$; Plane-strain modulus

$K' = \sqrt{\dfrac{32}{\pi}}\, K_{Ic}$; Toughness

$\mu' = 12\,\mu$; Fluid viscosity

$C' = 2\,C_L$; Leak-off coeff.

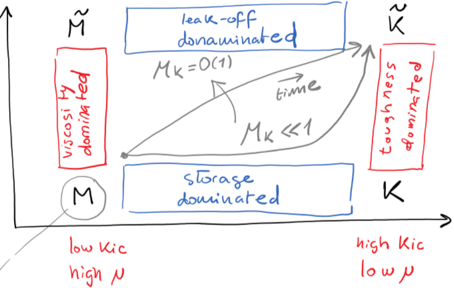
Fracture propagation "regimes" (zero-lag)

$$\underbrace{\frac{\text{Fluid gone into rock}}{\text{Fluid stored in frac.}}}$$

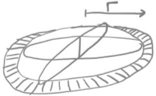
Material balance competition

$\updownarrow$

$V_i = V_f + V_{leak\text{-}off}$



$$\mathcal{M}_K^{radial} = \mu' \left(\frac{C'^4 E'^{11} Q_0}{K'^{14}} \right)^{1/3}$$



$$\underbrace{\frac{\text{Work creating new rock surface}}{\text{Work circulating fluid}}}$$

Competitive dissipative processes

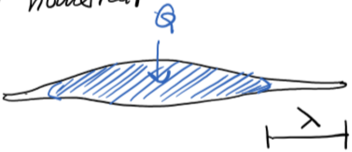
Ex. M (storage, viscosity) solution for

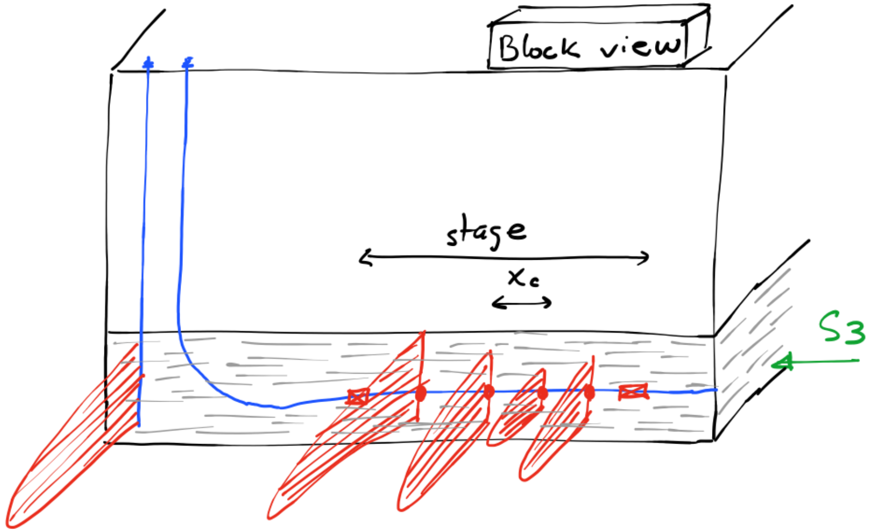
$K' = 0$;
$C' = 0$; } ~ PKN derivation previous lecture

• Energy release rate $G = K_I^2 / E'$
LEFM

Key points:

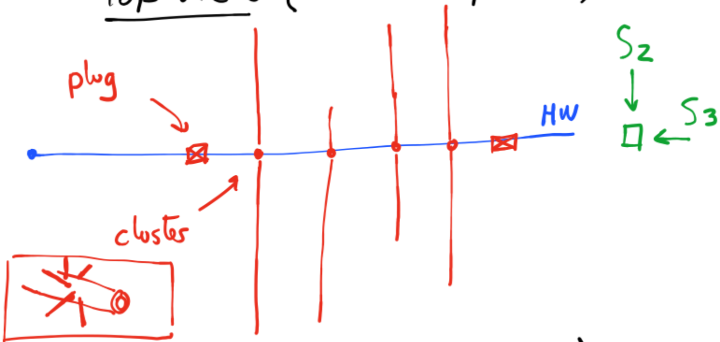
- subsurface hydraulic fracturing $\updownarrow$ viscosity dominated ✓
- scaling must be taken into account when comparing field and lab HF
- asymptotic solutions (extreme cases) are useful to conceptualize complex physical processes ✓
- analytical solutions are useful to define properly meshes for numerical solutions ✓
- consider fluid-lag



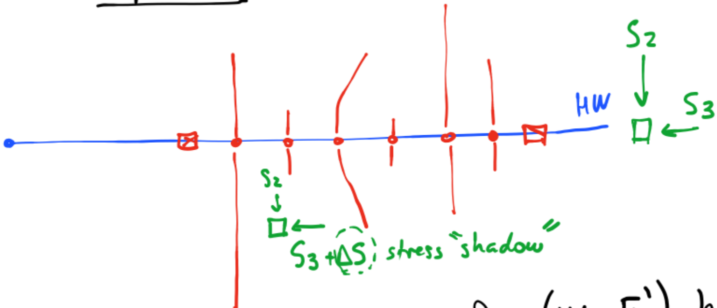


Block view
stage
x_c
S_3

Top view (minor interference)



Top view (large interference)



$$\text{Stress shadow (Interference)} \propto \frac{P_{net}(w_0, E') \cdot h_f}{S_2 - S_3}$$

PERMIAN BASIN

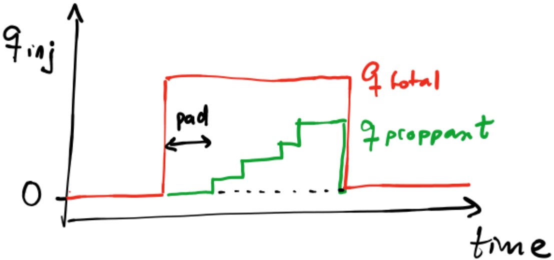
- lateral length $\gtrsim$ 10,000 ft
- ~40 stages → stage length ~250 ft
- 4 to 15 clusters per stage
- 6 to 20 perforations
- 2000 $\dfrac{\text{lb (proppant)}}{\text{LF}}$
- 2500 $\dfrac{\text{gallons (frac fluid)}}{\text{LF}}$

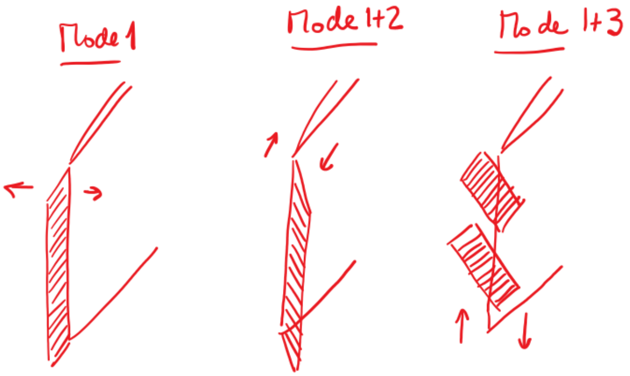
$0.8 \ \dfrac{\text{lb}}{\text{gall}} \ \dfrac{\text{(proppant)}}{\text{(frac fluid)}}$

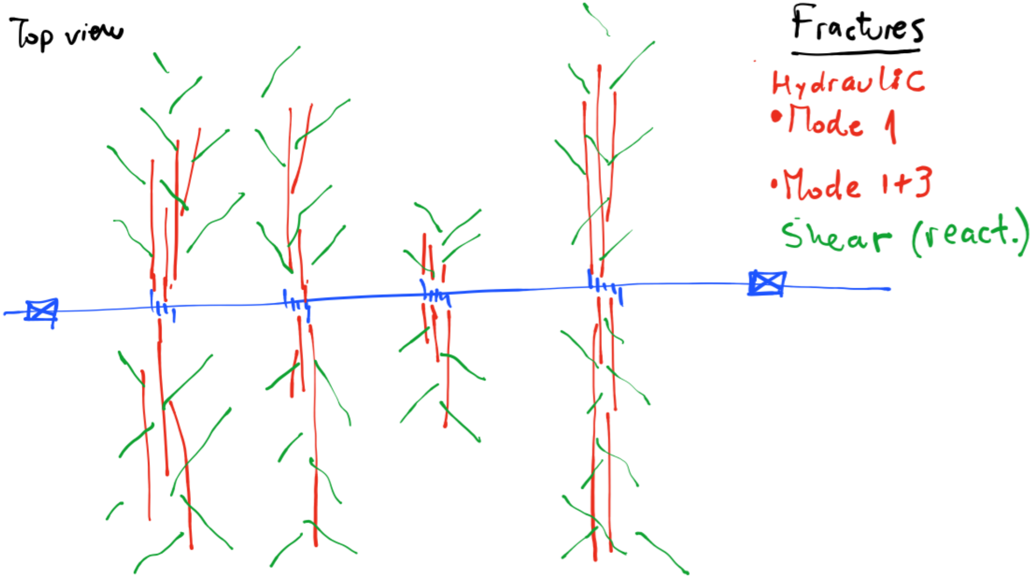
$\sim 7\% \ \dfrac{\text{vol (proppant)}}{\text{vol (frac fluid)}}$

LF: linear foot

Injection Schedule




Mode 1
Mode 1+2
Mode 1+3



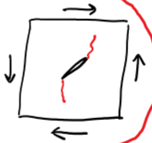
Top view
Fractures
Hydraulic
• Mode 1
• Mode 1+3
Shear (react.)

5- Subcritical fracture propagation

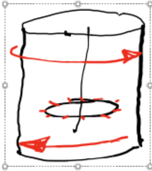
Wednesday, December 2, 2020 11:46 AM

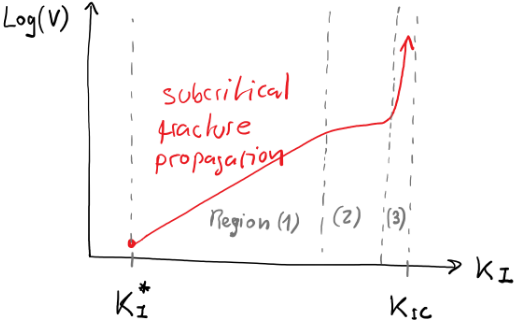
Mode I

Mode II



Mode III





$$\log V \propto K_I$$

$$\boxed{V = A\left(\frac{K_I}{K_{IC}}\right)^n}$$

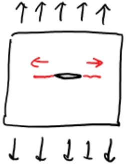
Mode I

$$\log V = \log A + n \cdot \log \frac{K_I}{K_{IC}}$$

↳ main parameters

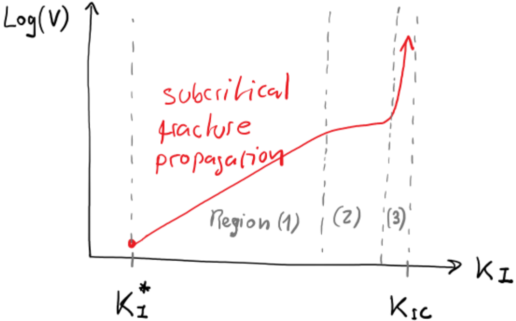
- n : subcritical index (SCI
- K_{IC} : (critical) fracture toughnes

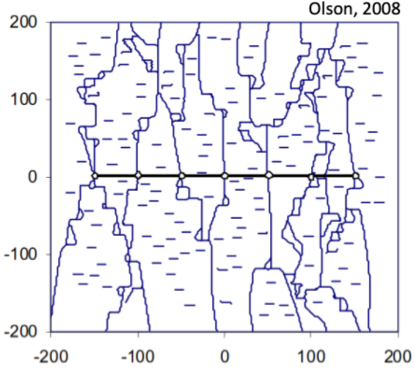
Mode I



$$\boxed{V = A\left(\frac{K_I}{K_{Ic}}\right)^n}$$

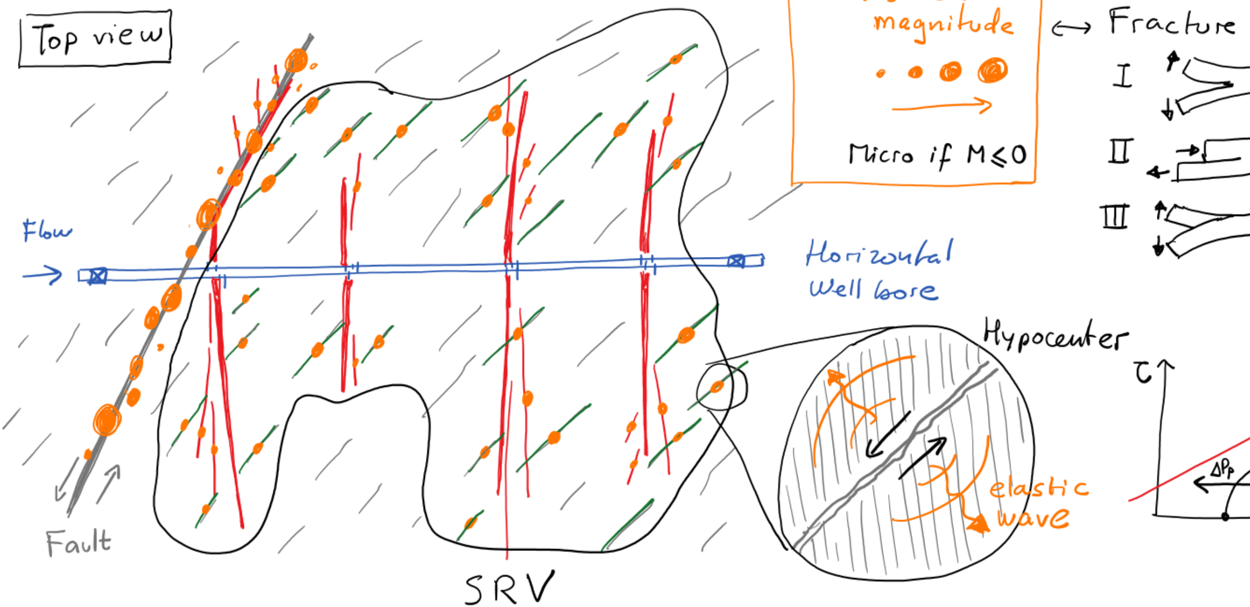
Mode I

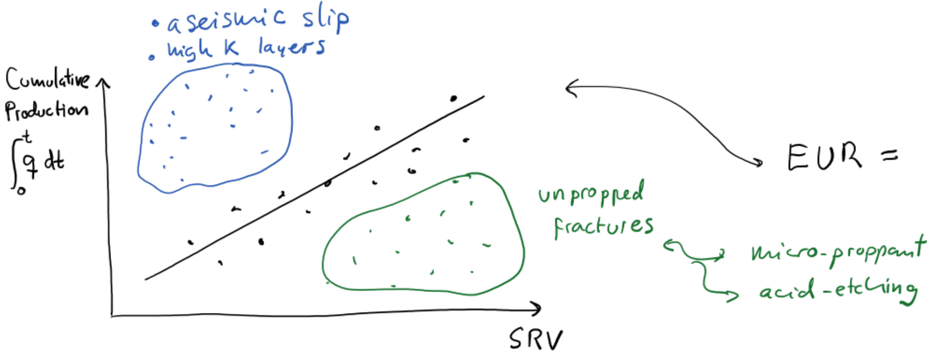




5- Micro-seismicity

Monday, November 30, 2020 5:18 PM

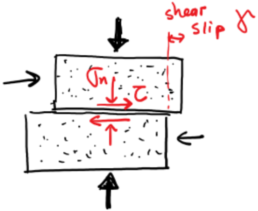




$$EUR = \frac{SRV \cdot \phi \cdot (1-S_w)}{B_o} \cdot RF$$

Seismic and aseismic slip

Direct shear experiment

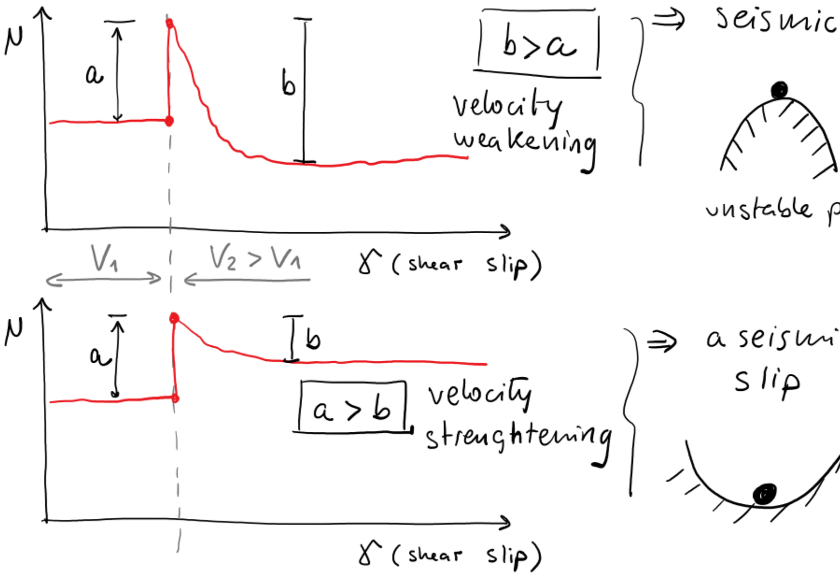


$$\tau = \mu \cdot \sigma_n$$

Apparent Friction Coefficient

$$\mu = f(\text{rock, slip velocity}, P, T)$$

$$V = \frac{d\gamma}{dt}$$



$$\Delta S_{yy} = \nu(\Delta S_{xx} + \Delta S_{zz})$$

h_f

$$E' = 8.9 \times 10^6$$

x_f

$$w_{w,0}$$

p_n

p_n

-6

$$p' - q$$

10^{-6}

$$E_{sta}/E_{dyn}$$