S_1

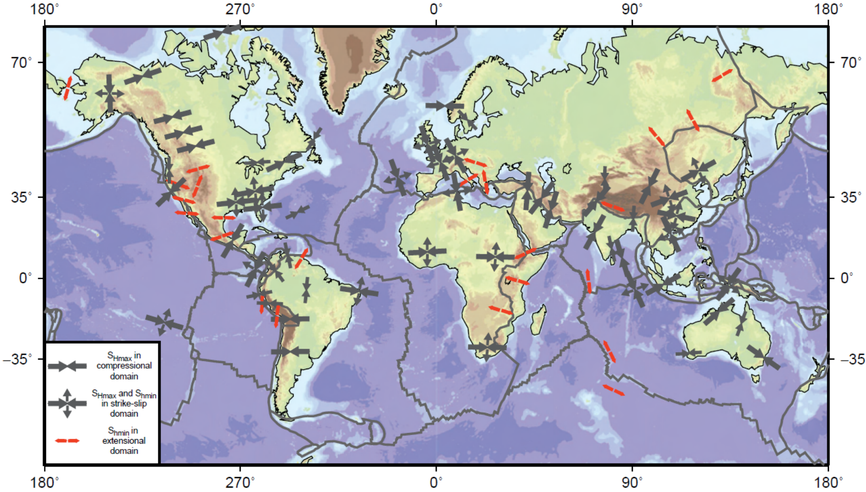
S_3

$$S_{ij}$$

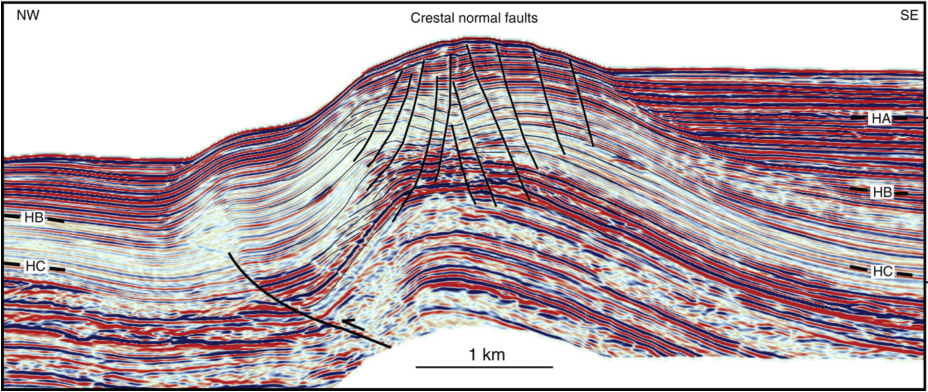
$\underline{e}_i$

$\underline{e}_j$

< 10


180°
270°
0°
90°
180°
70°
35°
0°
−35°
S_{Hmax} in compressional domain
S_{Hmax} and S_{hmin} in strike-slip domain
S_{hmin} in extensional domain

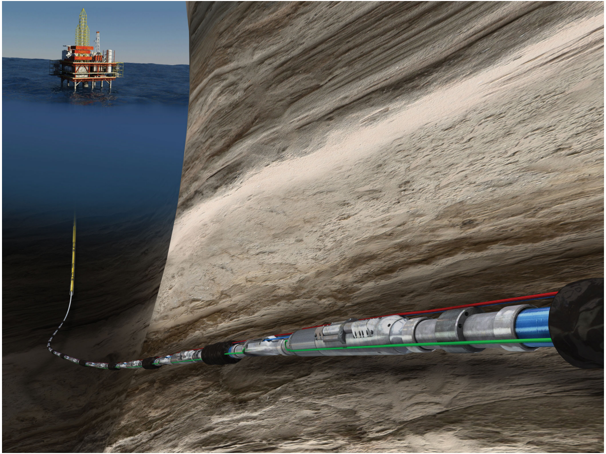
[Zoback 2010]



http://geosphere.gsapubs.org/content/5/2/59/F6.large.jpg

~1850

~2010



http://www.drillingcontractor.org

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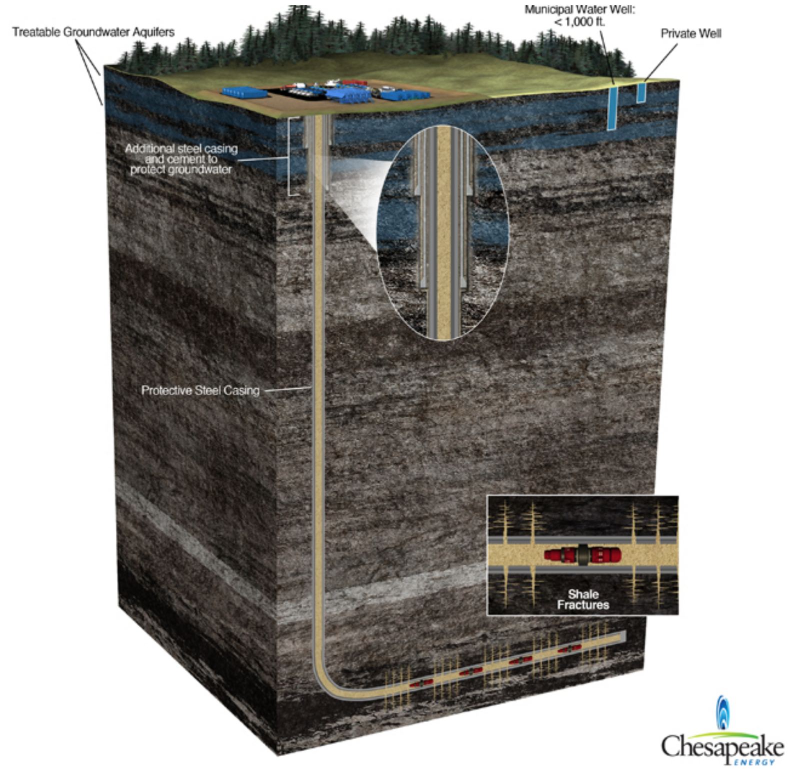
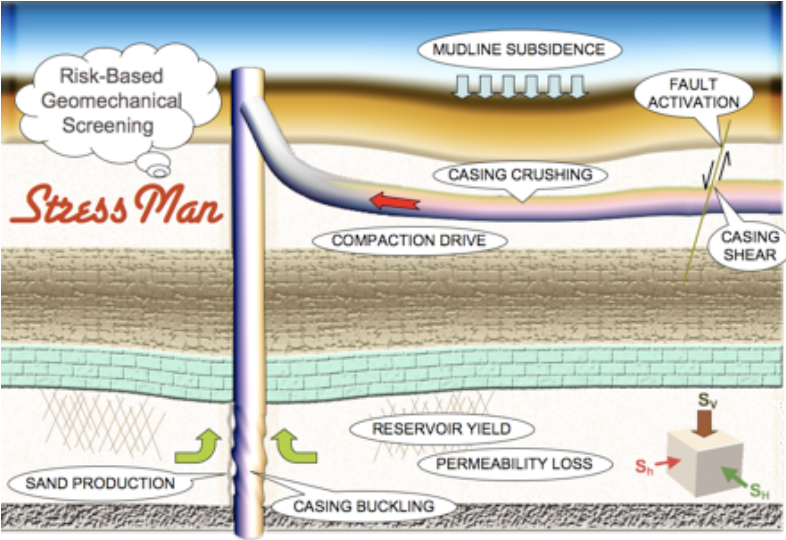


Treatable Groundwater Aquifers
Municipal Water Well:
< 1,000 ft.
Private Well
Additional steel casing
and cement to
protect groundwater
Protective Steel Casing
Shale
Fractures
Chesapeake
ENERGY



http://www.higgs-palmer.com

S_v

$$\underline{F}_1$$

$$\underline{F}_2$$

$\mathbb{R}^3$

$$\underline{F}_1 = \underline{F}_2$$

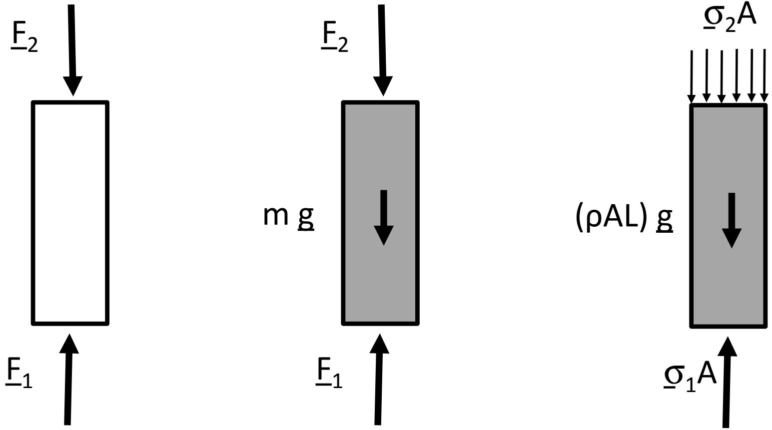
$$\underline{F}_1 = \underline{F}_2 + m\underline{g}$$

$m\underline{g}$

$\underline{F}$

A

$$\underline{\sigma} = \frac{F}{A}$$



$\Sigma F_z = + F_1 - F_2 = 0 \qquad \Sigma F_z = + F_1 - m\ g - F_2 \qquad \Sigma F_z = +\sigma_1 A - (\rho AL)g - \sigma_2 A$

$$\Delta x \Delta y \Delta x$$

$$\sum \underline{F} = 0$$

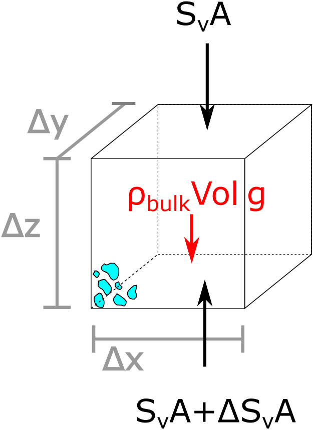
$\underline{g}$

$$\sum F_z = 0$$

$$S_v A + \rho_{bulk} Volg = S_v A + \Delta S_v A$$

$$\rho_{bulk}\, \Delta x \Delta y \Delta z\, g = \Delta S_v \Delta x \Delta y$$

$$\frac{\Delta S_v}{\Delta z} = \rho_{bulk}\, g$$



S_vA
Δy
Δz
$\rho_{bulk} Vol\, g$
Δx
$S_vA+\Delta S_vA$

$$\int_0^{S_v(z)} dS_v = \int_0^z \rho_{bulk}(z) g \, dz$$

$\rho_{bulk}(z)g$

$$S_v(z = 0) = 0$$

$$\rho_{bulk} = cst$$

z

$$S_v(z) = \rho_{bulk} g\, z.$$

$$\rho_{SiO_2}$$

ρ_w

ρ_{bulk}

ϕ

$$\rho_{bulk} = (1 - \phi)\rho_{SiO_2} + \phi\rho_w$$

$$\phi = 0.2$$

$$\rho_{SiO_2} = 2650$$

3

$$\rho_w = 1000$$

$$\rho_{bulk} = (1 - 0.2)2650$$

$^{3} + (0.2)1000$

$^{3} = 2320$

3
.

$$\rho_{bulk} g = 2320$$

^{3} 9.8

$$^2 = 22.7 \times 10^3$$

$$= 22.7$$

• ■

~ 1

~ 23

$$\phi = 0.20$$

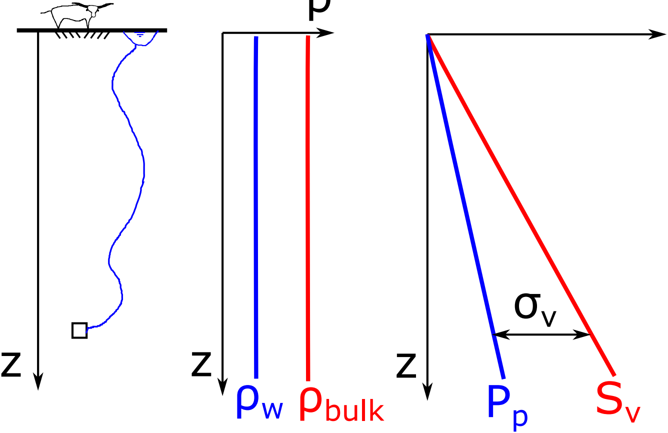
~ 0.44

~ 10

S_w, S_o, S_g

$$P_p = \rho_w g\, z$$

P_p

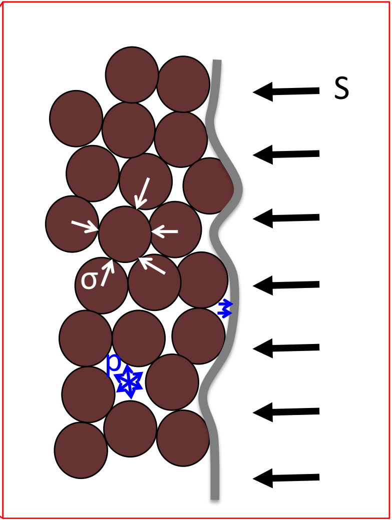


z
ρ
z
ρ_w
ρ_{bulk}
σ_v
z
P_p
S_v

$$\sigma_v = S_v - P_p$$

Effective stress =
Total stress – Pore pressure

$$\sigma = S - p$$



P_w

$$\frac{dP_p}{dz} = \rho_w g = 1040$$

$^{2} = 10.19$

$$\frac{dS_v}{dz} = \rho_{bulk} g = 2350$$

$^{2} = 23.03$

$$P_p = \frac{dP_p}{dz} z = 10.19$$

1.219

$$= 12.43$$

$= 1801$

$$\frac{dS_v}{dz}z = 23.03$$

$$= 28.07$$

$$= 4070$$

z_w

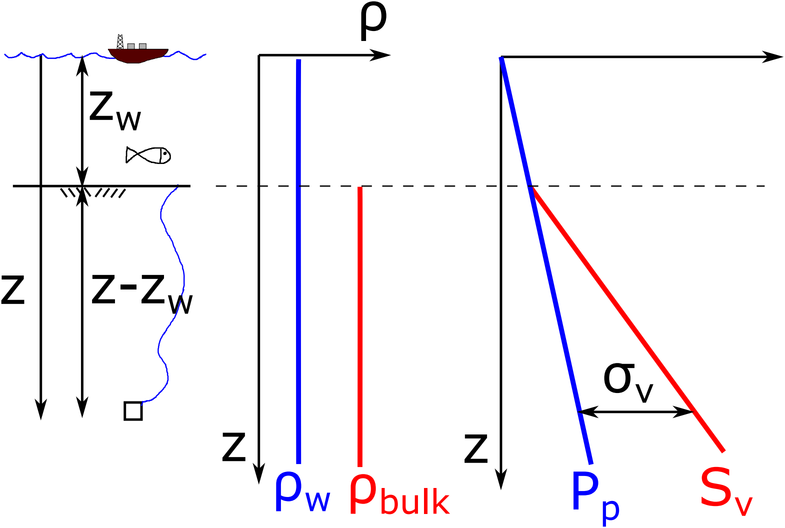
$$z > z_w$$

$$S_v = \rho_w g\, z_w + \rho_{bulk} g (z - z_w)$$

$$z = z_w$$

$$\sigma_v = S_v - P_p = \rho_{bulk} g(z - z_w)$$

$$z \geq z_w$$



ρ
z_w
z
z-z_w
z
ρ_w
ρ_bulk
z
σ_v
P_p
S_v

$$P_p = \rho_w g z_w + \frac{dP_p}{dz}(z - z_w) = 0.44$$

2000

$+0.44$

$$(9000 - 2000)$$

$$= 3960$$

$$S_v = \rho_w g z_w + \frac{dS_v}{dz}(z - z_w) = 0.44$$

$+1.00$

$$= 7880$$

,

$$\sigma_v = S_v - P_p = 7880$$

-3960

$$= 3920$$

$$P_p = 27.31$$

$$S_v = 54.34$$

$$\sigma_v = 27.03$$

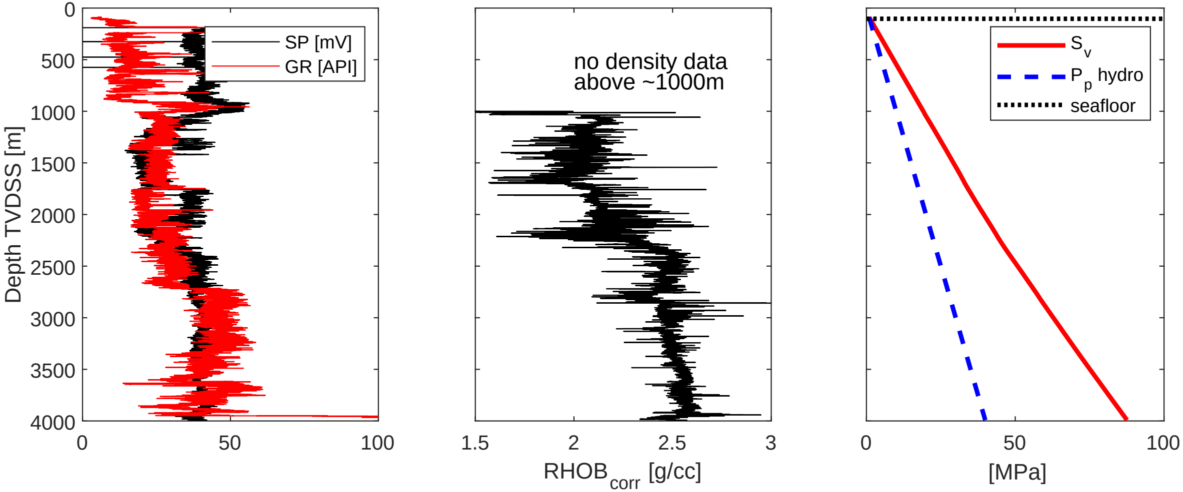
$$S_v(z) = \int_0^z \rho_{bulk}(z) g \, dz$$

$$\rho_{bulk}(z)$$

$$S_v(z_i) = \sum_{1}^{i} \rho_{bulk}(z_i) g \Delta z_i$$

i

z_i

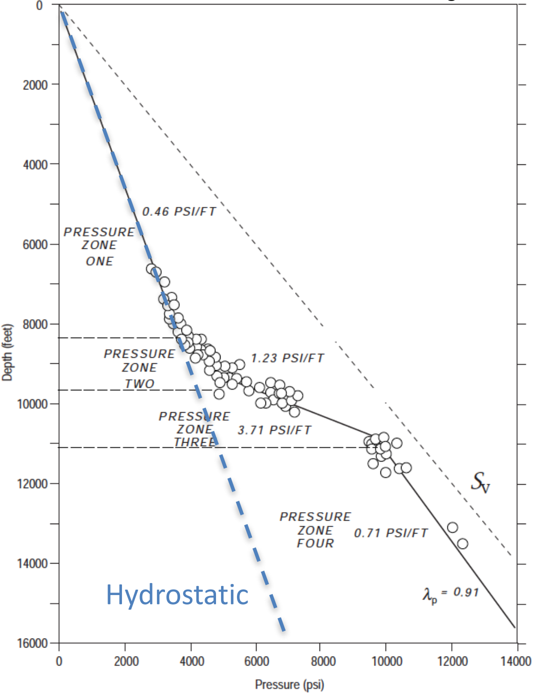


0
500
1000
1500
2000
2500
3000
3500
4000
Depth TVDSS [m]
SP [mV]
GR [API]
0
50
100
no density data
above ~1000m
1.5
2
2.5
3
RHOB corr [g/cc]
S v
P p hydro
seafloor
0
50
100
[MPa]

$$z \sim 8,500$$

$$z \sim 11,000$$

Monte Cristo field Onshore - Zoback 2013 – Figure 2.2



λ_p

$$\lambda_p(z) = \frac{P_p(z)}{S_V(z)}$$

S_V

$\lambda_p \leq 1$

$$\lambda_p \sim 0.44$$

$$\lambda_p \neq 0.44$$

$$\lambda_p \to 1$$

$$P_p > P_p^{\text{hydrostatic}}$$

$$\Delta P_p$$

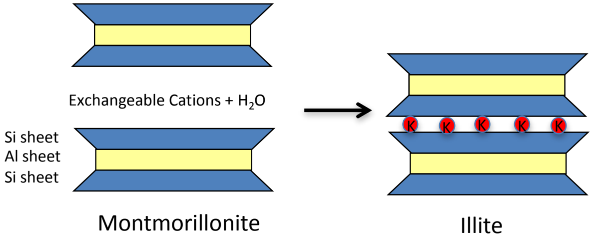
h

$$\rho_{brine} - \rho_{hydrocarbon}$$

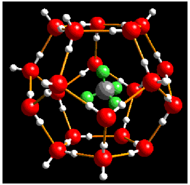
$$\Delta P = (\rho_{brine} - \rho_{hydrocarbon})gh$$

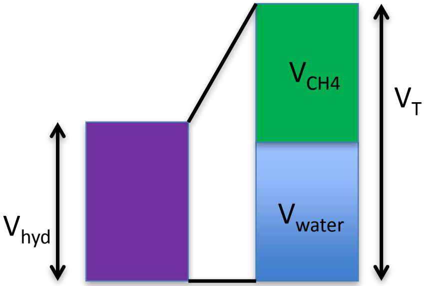
- **Aquathermal pressurization**
  - $\Delta T \rightarrow \Delta P$

- **Dehydration reactions**
  - $\Delta V \rightarrow \Delta P$
  - Montmorillonite to Illite (frees water)



- **Hydrocarbon generation**
  - $\Delta V \rightarrow \Delta P$





V_T/V_{hyd} ~ 1.3 to 6 at cst. P

W

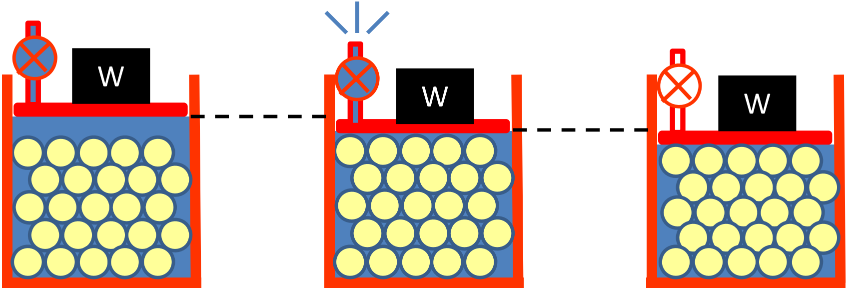
$$\Delta P = W/A$$

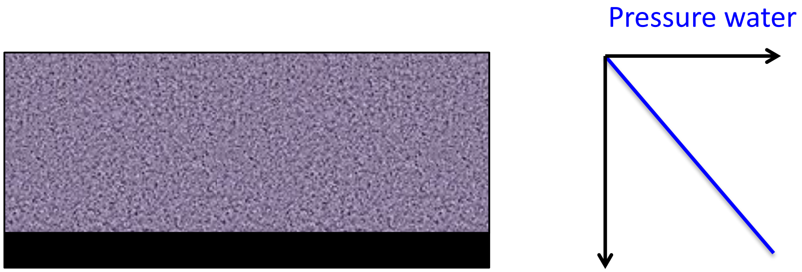
$$\sigma_v = W/A$$

$$\Delta P = 0$$

- **Disequilibrium compaction (Underconsolidation)**
  - $\Delta S \rightarrow \Delta P$ (Vertical)

- **Tectonic compression**
  - $\Delta S \rightarrow \Delta P$ (Horizontal)




Pressure water

Δz

$$D_h = \frac{Mk}{\mu}$$

M

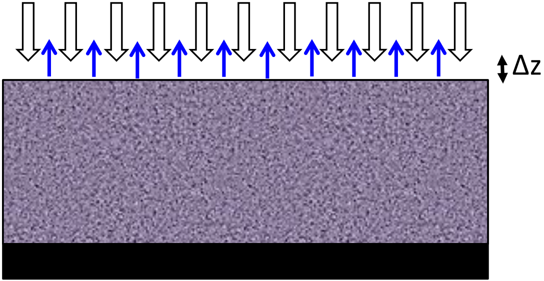
$$C_{bp}$$

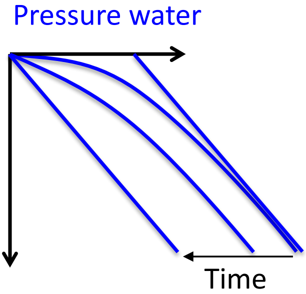
k

$\boldsymbol{\mu}$

$$\frac{\partial P_p}{\partial t} = D_h \frac{d^2 P_p}{dz^2}$$

Rate of sedimentation (loading) and rate of fluid “escape”





T_{ch}

$\sim 2/3$

$$T_{ch} = \frac{L^2}{D_h}$$

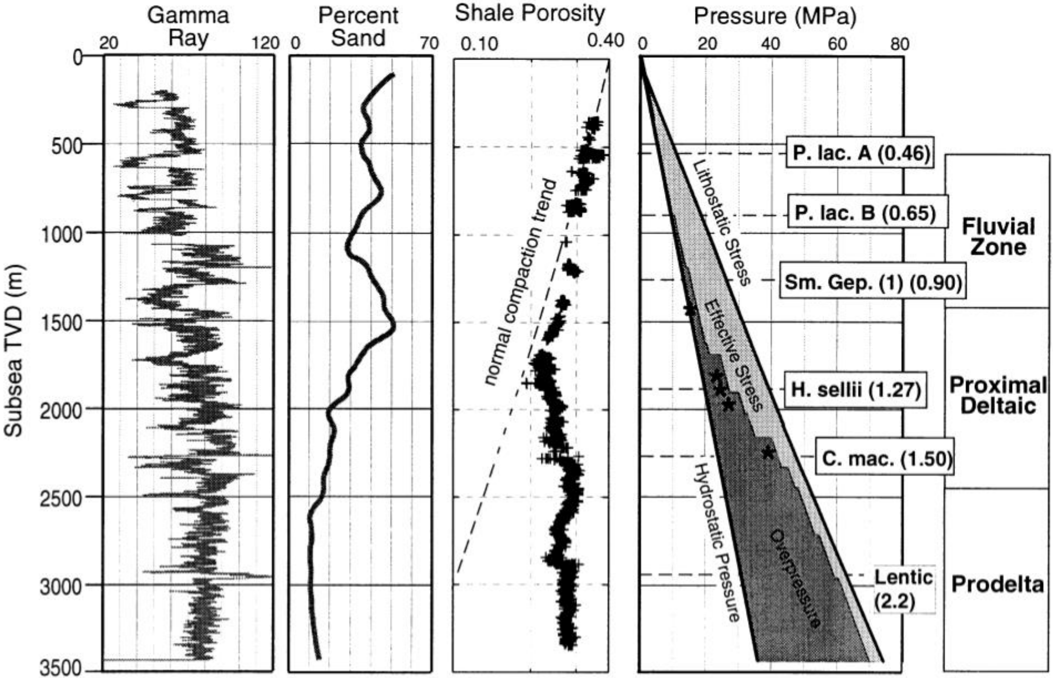
L

$k =$

$M =$

$T_{ch} \sim$

$$\phi = \phi_o \exp(-\beta\sigma_v)$$



Off-shore Louisiana – Gordon and Flemings (1998) Basin Research

$$\phi = 0.298$$

$$\beta = 3.2 \times 10^{-2}$$

-1

$$\phi_0 = 0.38$$

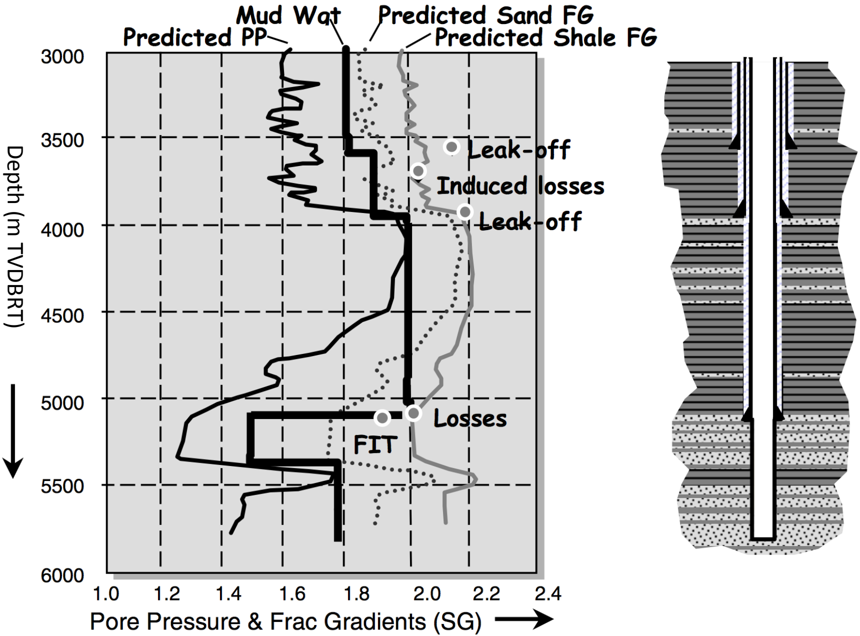
$$S_v =$$

$$P_p = S_v + \frac{\ln\left(\frac{\phi}{\phi_o}\right)}{\beta} = 38$$

-7.6

$$= 30.40$$

$$\lambda_p = \frac{P_p}{S_v} = \frac{30.40 \text{ MPa}}{38 \text{ MPa}} = 0.8. \blacksquare$$

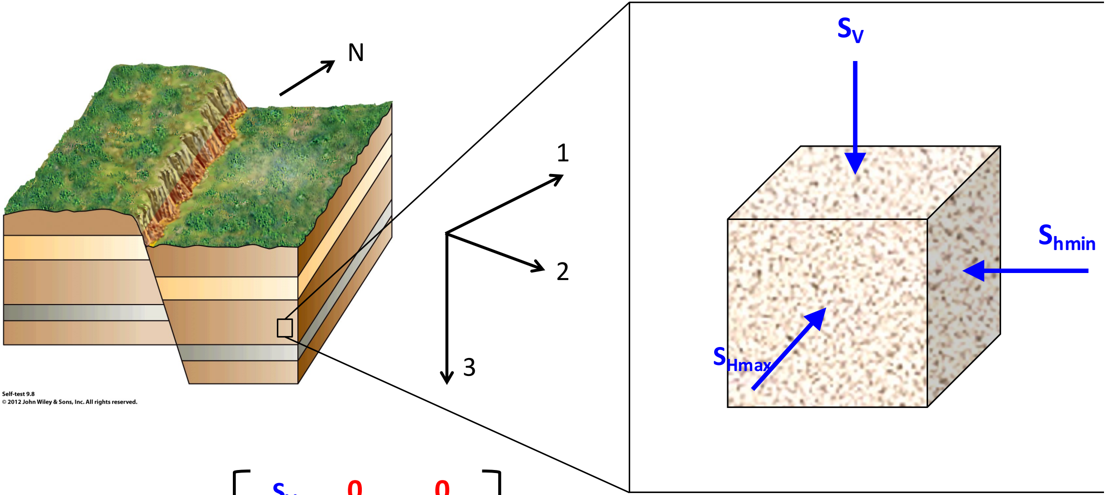


[Caspian Sea, Alberty and McLean – SPE67740]

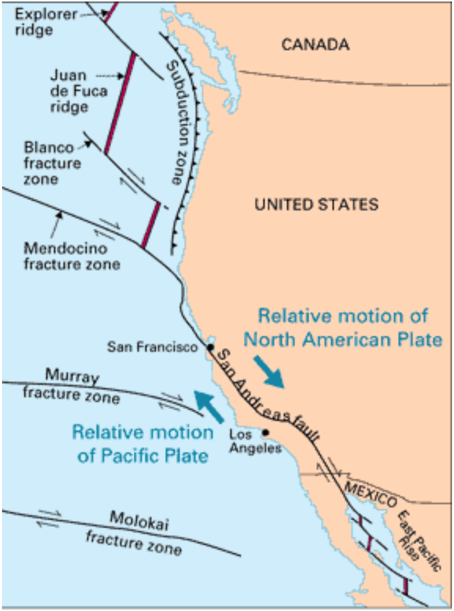
S_{Hmax}

S_{hmin}

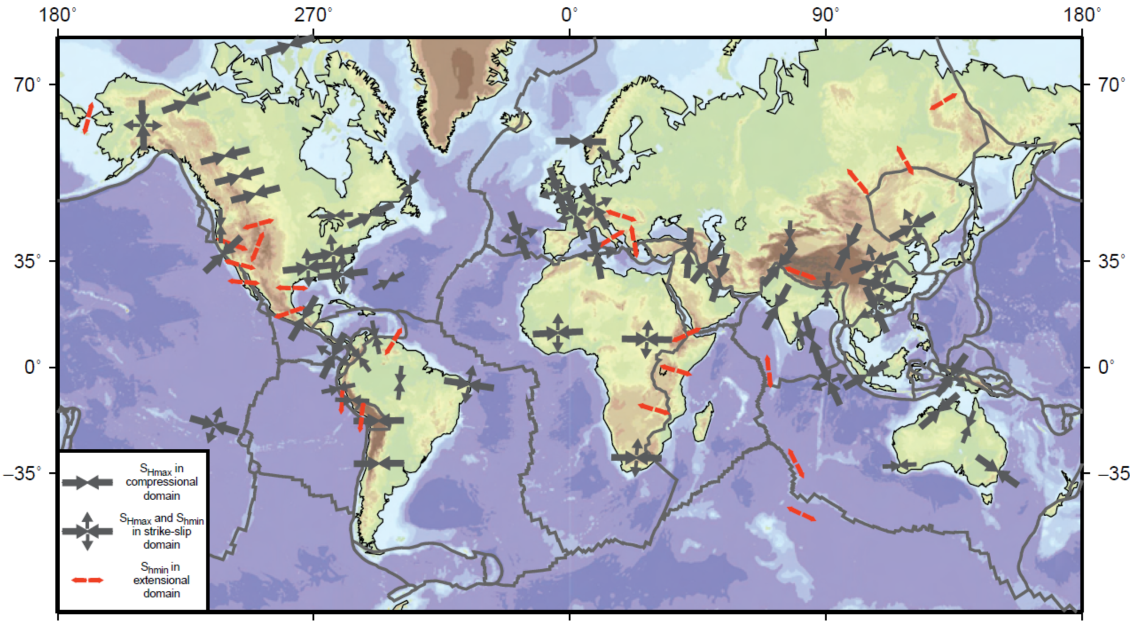
$$S_{Hmax} \geq S_{hmin}$$



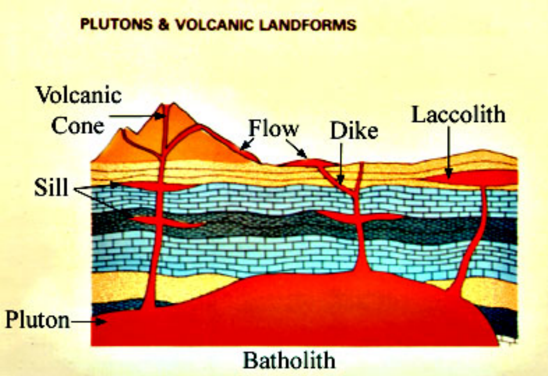
$$\underline{\underline{S}} = \begin{bmatrix} S_V & 0 & 0 \\ 0 & S_{Hmax} & 0 \\ 0 & 0 & S_{hmin} \end{bmatrix}$$

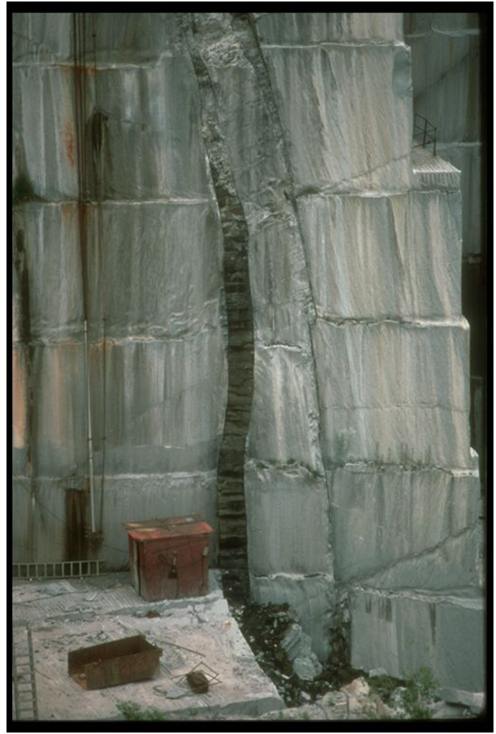


http://pubs.usgs.gov/gip/dynamic/understanding.html#anchor5798673
http://en.wikipedia.org/wiki/File:Aerial-SanAndreas-CarrizoPlain.jpg


180°
270°
0°
90°
180°
70°
35°
0°
–35°
S_{Hmax} in compressional domain
S_{Hmax} and S_{hmin} in strike-slip domain
S_{hmin} in extensional domain

[Zoback 2010]





http://www.indiana.edu/~geol105/1425chap5.htm
http://geophysics.ou.edu/geol1114

$$S_V > S_{hmin} = S_3$$

S_2

$$S_V > S_{Hmax} > S_{hmin}$$

$$S_1 = S_V$$

$$S_2 = S_{Hmax}$$

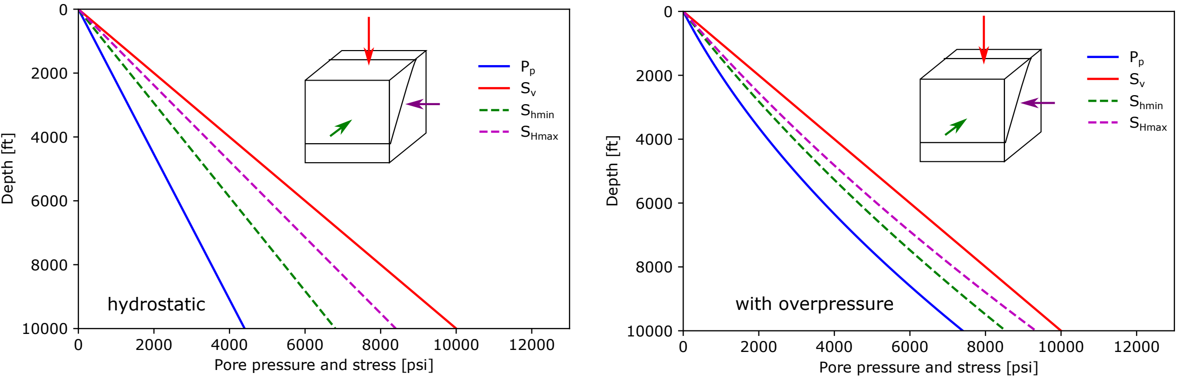
$$S_3 = S_{hmin}$$

~ 60

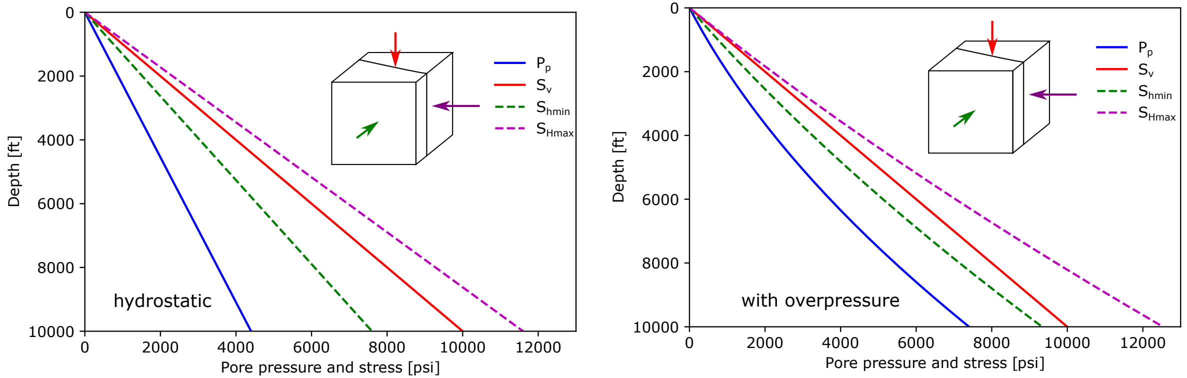
$$\sigma_1 = S_1 - P_p$$

$$\sigma_3 = S_3 - P_p$$

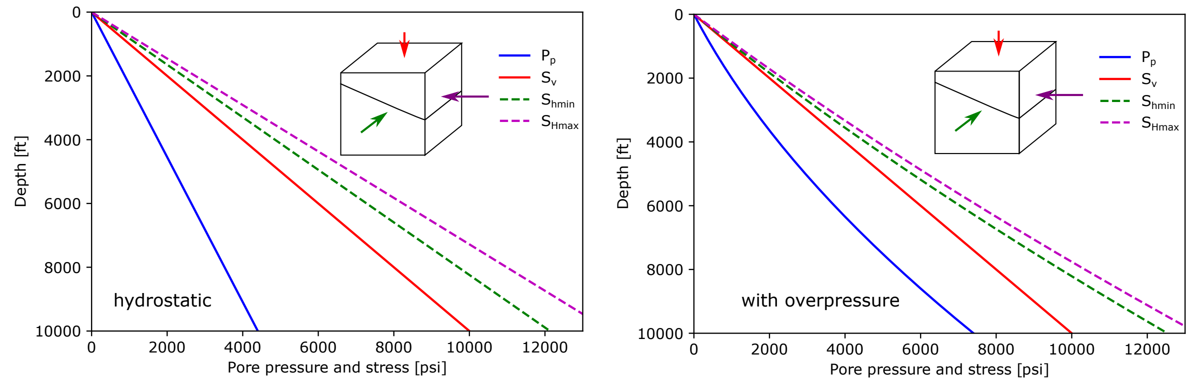
Normal faulting: $S_v > S_{Hmax} > S_{hmin}$



Strike slip faulting: $S_{Hmax} > S_v > S_{hmin}$



Reverse faulting: $S_{Hmax} > S_{hmin} > S_v$



$$S_{Hmax} > S_V > S_{hmin}$$

$$S_1 = S_{Hmax}$$

$$S_2 = S_V$$

$$S_{Hmax} > S_{hmin} > S_V$$

$$S_2 = S_{hmin}$$

$$S_3 = S_V$$

$$\beta = 3 \times 10^{-2}$$

σ_v

$\underline{e}_1$

$\underline{e}_2$

$\underline{e}_3$

T

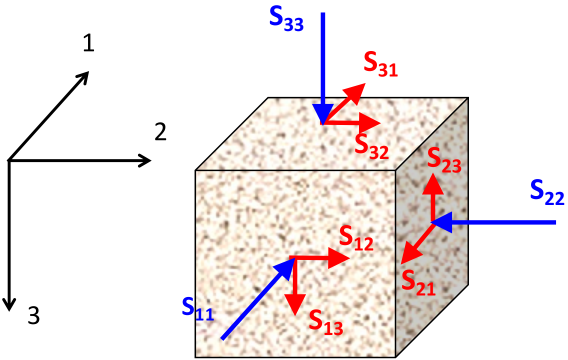
$$(x_1, x_2, x_3)$$

$\underline{v}$

$$dx_i$$

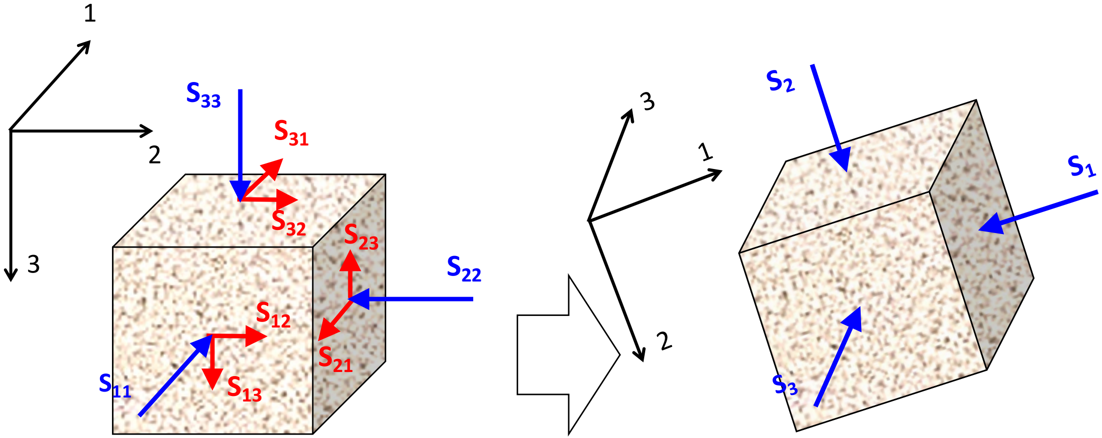
$$S_{ij} = S_{ji}$$

$$i \neq j$$



$$\underline{\underline{S}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$S_1 \geq S_2 \geq S_3$$



$$\underline{\underline{S}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \qquad \underline{\underline{S}} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix}$$

$$m\underline{a} = 0$$

$\rho V b_1$

ρ

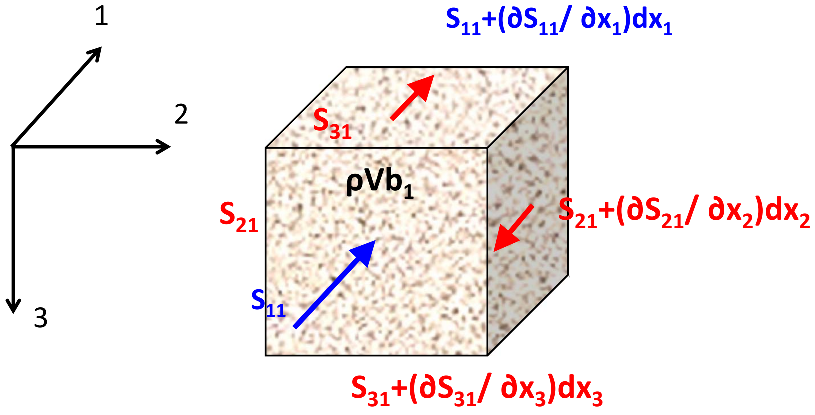
V

b_1

$$
\begin{aligned}
\sum F_1 &= 0 \\
\sum F_1 &= +S_{11}dx_2dx_3 - \left[S_{11} + (\tfrac{\partial S_{11}}{\partial x_1})dx_1\right]dx_2dx_3 \\
&\quad +S_{21}dx_1dx_3 - \left[S_{21} + (\tfrac{\partial S_{21}}{\partial x_2})dx_2\right]dx_1dx_3 \\
&\quad +S_{31}dx_1dx_2 - \left[S_{31} + (\tfrac{\partial S_{31}}{\partial x_3})dx_3\right]dx_1dx_2 \\
&\quad -\rho(dx_1dx_2dx_3)b_1 = 0
\end{aligned}
$$

$$(dx_1 dx_2 dx_3)$$

$$\frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{21}}{\partial x_2} + \frac{\partial S_{31}}{\partial x_3} - \rho b_1 = 0$$


1
2
3
$S_{11}+(\partial S_{11}/ \partial x_1)dx_1$
S_{31}
$\rho V b_1$
S_{21}
$S_{21}+(\partial S_{21}/ \partial x_2)dx_2$
S_{11}
$S_{31}+(\partial S_{31}/ \partial x_3)dx_3$

$$\left\{\begin{array}{rcl} \dfrac{\partial S_{11}}{\partial x_1} + \dfrac{\partial S_{21}}{\partial x_2} + \dfrac{\partial S_{31}}{\partial x_3} - \rho b_1 & = & 0 \\ \dfrac{\partial S_{12}}{\partial x_1} + \dfrac{\partial S_{22}}{\partial x_2} + \dfrac{\partial S_{33}}{\partial x_3} - \rho b_2 & = & 0 \\ \dfrac{\partial S_{13}}{\partial x_1} + \dfrac{\partial S_{23}}{\partial x_2} + \dfrac{\partial S_{33}}{\partial x_3} - \rho b_3 & = & 0 \end{array}\right.$$

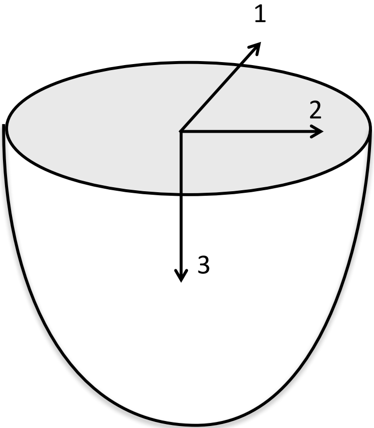
g

$$b_3 = g$$

$$\partial()/\partial x_1 = \partial()/\partial x_2 = 0$$

S_{33}

$$S_{33}(x_3) = \int_0^{x_3} \rho(x_3) g \, dx_3$$



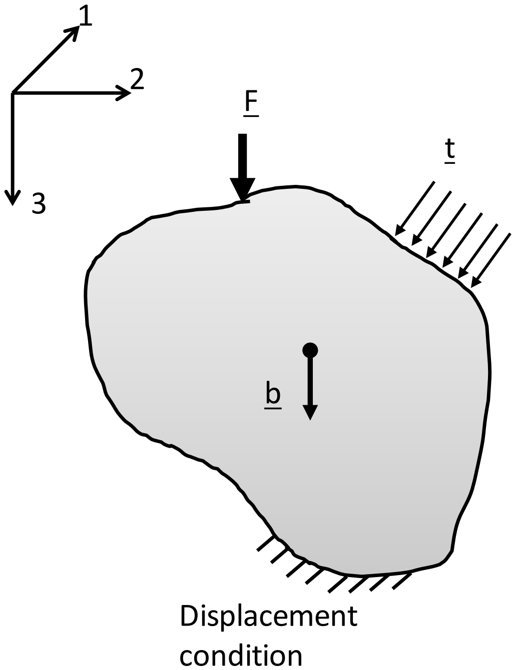
$$\begin{cases} \cancel{\dfrac{\partial S_{11}}{\partial x_1}} + \cancel{\dfrac{\partial S_{21}}{\partial x_2}} + \dfrac{\partial S_{31}}{\partial x_3} + \cancel{\rho b_1} = \cancel{\dfrac{\partial^2 (\rho u_1)}{\partial t^2}} \\ \cancel{\dfrac{\partial S_{12}}{\partial x_1}} + \cancel{\dfrac{\partial S_{22}}{\partial x_2}} + \dfrac{\partial S_{32}}{\partial x_3} + \cancel{\rho b_2} = \cancel{\dfrac{\partial^2 (\rho u_2)}{\partial t^2}} \\ \cancel{\dfrac{\partial S_{13}}{\partial x_1}} + \cancel{\dfrac{\partial S_{23}}{\partial x_2}} + \dfrac{\partial S_{33}}{\partial x_3} + \rho b_3 = \cancel{\dfrac{\partial^2 (\rho u_3)}{\partial t^2}} \end{cases}$$

$$\frac{\partial S_{33}}{\partial x_3} - \rho(x_3) g = 0$$

$$S_{33} = \int_0^{x_3} \rho(x_3) g \; dx_3$$

$$S_{11}$$

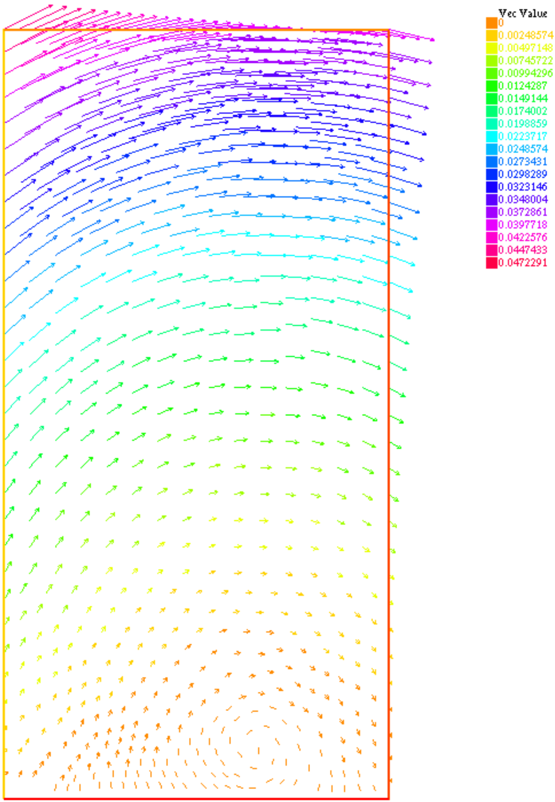
S_{22}



$$\begin{cases} \dfrac{\partial S_{11}}{\partial x_1} + \dfrac{\partial S_{21}}{\partial x_2} + \dfrac{\partial S_{31}}{\partial x_3} + \rho b_1 = \dfrac{\partial^2 (\rho u_1)}{\partial t^2} \\ \dfrac{\partial S_{12}}{\partial x_1} + \dfrac{\partial S_{22}}{\partial x_2} + \dfrac{\partial S_{32}}{\partial x_3} + \rho b_2 = \dfrac{\partial^2 (\rho u_2)}{\partial t^2} \\ \dfrac{\partial S_{13}}{\partial x_1} + \dfrac{\partial S_{23}}{\partial x_2} + \dfrac{\partial S_{33}}{\partial x_3} + \rho b_3 = \dfrac{\partial^2 (\rho u_3)}{\partial t^2} \end{cases}$$

And respect the boundary conditions:
- Displacement
- Boundary stresses
- Boundary Forces
- Body Forces

How do we relate stresses to displacements?
- Displacements → Strains (**Kinematic equations**)
- Strains → Stresses (**Constitutive equations**)


Vec Value
0
0.00248574
0.00497148
0.00745722
0.00994296
0.0124287
0.0149144
0.0174002
0.0198859
0.0223717
0.0248574
0.0273431
0.0298289
0.0323146
0.0348004
0.0372861
0.0397718
0.0422576
0.0447433
0.0472291

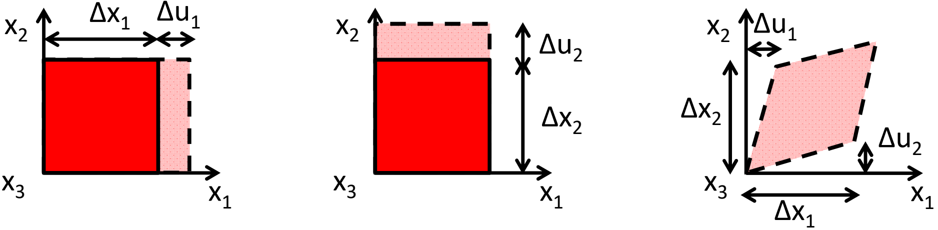
$$\varepsilon_{11} = \frac{\Delta u_1}{\Delta x_1}$$

$$\varepsilon_{22} = \frac{\Delta u_2}{\Delta x_2}$$

$$\pi - [\pi - \arctan(\Delta u_1/\Delta x_2) + \arctan(\Delta u_2/\Delta x_1)]$$

$$\arctan(x/y) \sim x/y$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\Delta u_1}{\Delta x_2} + \frac{\Delta u_2}{\Delta x_1}\right)$$



$$\varepsilon_{11} \approx \frac{\Delta u_1}{\Delta x_1}$$

$$\varepsilon_{22} \approx \frac{\Delta u_2}{\Delta x_2}$$

$$\varepsilon_{12} \approx \frac{1}{2}\left(\frac{\Delta u_1}{\Delta x_2} + \frac{\Delta u_2}{\Delta x_1}\right)$$

x_i

$$\varepsilon_{vol} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\Delta x_1 \Delta x_2 \Delta x_3$$

$$\Delta V$$

$$V_0$$

$$\varepsilon_{vol} = \frac{\Delta V}{V_0}$$

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

$$(\mathrm{d}x_1 + \mathrm{d}u_1)(\mathrm{d}u_2 + \mathrm{d}u_2)(\mathrm{d}x_3 + \mathrm{d}u_3)$$

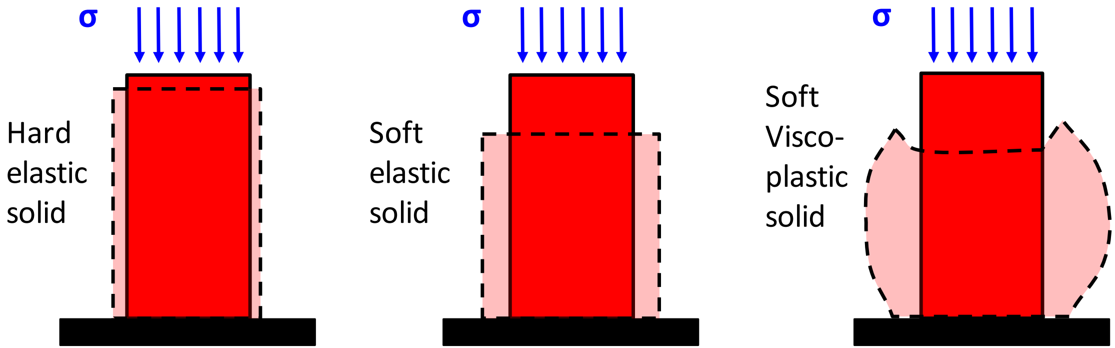
$$\varepsilon_{vol} = \frac{[(\mathrm{d}x_1 + \mathrm{d}u_1)(\mathrm{d}x_2 + \mathrm{d}u_2)(\mathrm{d}x_3 + \mathrm{d}u_3) - (\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3)]}{(\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3)}$$

$$\mathrm{d}u_i \mathrm{d}u_j$$

$$\mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3$$

$$\mathrm{d}u_i << \mathrm{d}x_j$$

$$\varepsilon_{vol} \sim \frac{(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}u_3 + \mathrm{d}x_1\mathrm{d}x_3\mathrm{d}u_2 + \mathrm{d}x_2\mathrm{d}x_3\mathrm{d}u_1)}{(\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3)} = \frac{\mathrm{d}u_1}{\mathrm{d}x_1} + \frac{\mathrm{d}u_2}{\mathrm{d}x_2} + \frac{\mathrm{d}u_3}{\mathrm{d}x_3} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \quad \blacksquare$$


σ
Hard
elastic
solid
σ
Soft
elastic
solid
σ
Soft
Visco-
plastic
solid

F

$$\Delta x$$

$$F = k\Delta x$$

E

$$\frac{F}{A} = E\frac{\Delta x}{L}$$

$\underline{\underline{\sigma}}$

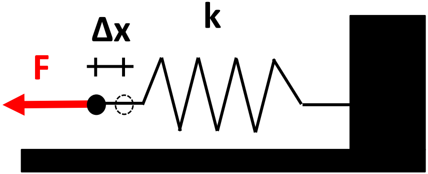
E

$$\underline{\underline{C}}$$

$$\underline{\underline{\sigma}} = \underline{\underline{C}}\,\underline{\underline{\varepsilon}}$$

Hooke's law

$$F = k\Delta x$$

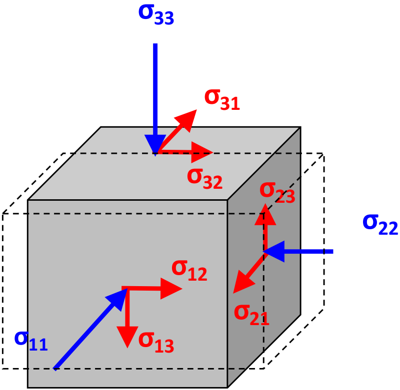


1-D (stress-strain) Hooke's law

$$\sigma = E\varepsilon$$

Generalized Hooke's law

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{\varepsilon}}$$



σ_{33}

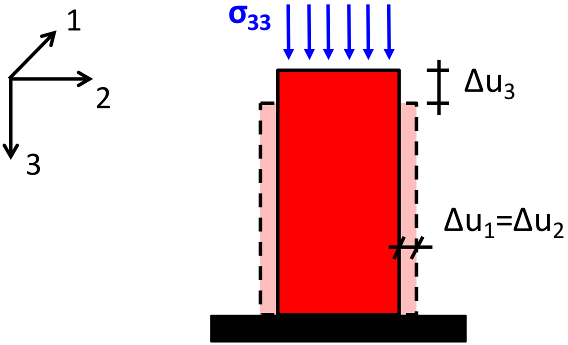
$$\varepsilon_{33}$$

$$E = \frac{\sigma_{33}}{\varepsilon_{33}}$$

$$\varepsilon_{11}$$

ε_{22}

$$\nu = -\frac{\varepsilon_{11}}{\varepsilon_{33}}$$



$$E = \frac{\sigma_{33}}{\varepsilon_{33}}$$ Young's Modulus

$$\nu = -\frac{\varepsilon_{11}}{\varepsilon_{33}}$$ Poisson's ratio (nu)

σ_a

ϵ_a

$$\sigma_a = 3000$$

$$= 3000\frac{1}{145}$$

$$= 20.7$$

$\varepsilon_a(1$

$$) = \frac{\sigma_a}{E} = \frac{20.7\ \text{MPa}}{1000\ \text{MPa}} = 0.0207 = 2.07\%$$

$\varepsilon_a(10$

$$) = \frac{\sigma_a}{E} = \frac{20.7 \text{ MPa}}{10000 \text{ MPa}} = 0.00207 = 0.207\%$$

$\varepsilon_a(50$

$$) = \frac{\sigma_a}{E} = \frac{20.7 \text{ MPa}}{50000 \text{ MPa}} = 0.00041 = 0.041\%$$

$$\begin{cases} \varepsilon_{11} & = & +\dfrac{1}{E}\,\sigma_{11} - \dfrac{\nu}{E}\,\sigma_{22} - \dfrac{\nu}{E}\,\sigma_{33} \\ \varepsilon_{22} & = & -\dfrac{\nu}{E}\,\sigma_{11} + \dfrac{1}{E}\,\sigma_{22} - \dfrac{\nu}{E}\,\sigma_{33} \\ \varepsilon_{33} & = & -\dfrac{\nu}{E}\,\sigma_{11} - \dfrac{\nu}{E}\,\sigma_{22} + \dfrac{1}{E}\,\sigma_{33} \end{cases}$$

$$\varepsilon_{ij}$$

$$G = E/[2(1 + \nu)]$$

$$\left\{ \begin{array}{lcl} 2\varepsilon_{12} & = & (1/G)\,\sigma_{12} \\ 2\varepsilon_{13} & = & (1/G)\,\sigma_{13} \\ 2\varepsilon_{23} & = & (1/G)\,\sigma_{23} \end{array} \right.$$

$$\underline{\underline{D}}$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{D}}\,\underline{\underline{\sigma}}$$

$$6 \times 1$$

6×6

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}
=
\begin{bmatrix}
+\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & +\frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & +\frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E}
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}
$$

$$\underline{\underline{\sigma}} = [0, 0, \sigma_{33}, 0, 0, 0]^T$$

$$\underline{\underline{D}}\ \underline{\underline{\sigma}}$$

$$\underline{\underline{\varepsilon}} = \left[-\frac{\nu}{E}\,\sigma_{33}, -\frac{\nu}{E}\,\sigma_{33}, \frac{1}{E}\,\sigma_{33}, 0, 0, 0 \right]^T$$

ν

$$\underline{\underline{C}} = \underline{\underline{D}}^{-1}$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}
$$

λ

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu\varepsilon_{22} + \nu\varepsilon_{33}\right]$$

$$\sigma_{11} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{\nu}\right)\varepsilon_{11}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$2\mu = \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{\nu}\right) = \frac{E}{(1+\nu)}$$

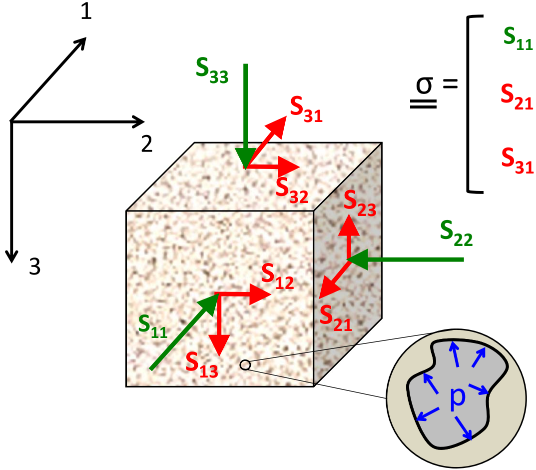
$$\mu = G$$

$$\left\{ \begin{array}{rcl} \sigma_{11} & = & (\lambda + 2\mu)\,\varepsilon_{11} + \lambda\,\varepsilon_{22} + \lambda\,\varepsilon_{33} \\ \sigma_{22} & = & \lambda\,\varepsilon_{11} + (\lambda + 2\mu)\,\varepsilon_{22} + \lambda\,\varepsilon_{33} \\ \sigma_{33} & = & \lambda\,\varepsilon_{11} + \lambda\,\varepsilon_{22} + (\lambda + 2\mu)\,\varepsilon_{33} \\ \sigma_{12} & = & 2\mu\,\varepsilon_{12} \\ \sigma_{13} & = & 2\mu\,\varepsilon_{13} \\ \sigma_{23} & = & 2\mu\,\varepsilon_{23} \end{array} \right. .$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} \quad \blacksquare$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{D}}\,\underline{\underline{S}}$$

$$\underline{\underline{\varepsilon}} = \underline{\underline{\underline{\underline{D}}}} \left(\underline{\underline{S}} - P_p \underline{\underline{I}} \right) = \underline{\underline{\underline{\underline{D}}}}\, \underline{\underline{\sigma}}$$



$$\underline{\underline{\sigma}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} - \begin{bmatrix} P_p & 0 & 0 \\ 0 & P_p & 0 \\ 0 & 0 & P_p \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} S_{11}-P_p & S_{12} & S_{13} \\ S_{21} & S_{22}-P_p & S_{23} \\ S_{31} & S_{32} & S_{33}-P_p \end{bmatrix}$$

α

$$\underline{\underline{\varepsilon}} = \underline{\underline{D}} \left(\underline{\underline{S}} - \alpha P_p \underline{\underline{I}} \right)$$

$$\alpha \sim 1$$

$\alpha \sim 0.5$

$$S_v = S_{33}$$

$$\sigma_v(z) = S_v(z) - P_p$$

$$\varepsilon_{11} = \varepsilon_{22}$$

$$\underline{\underline{\sigma}} = \underline{\underline{D}}\,\underline{\underline{\varepsilon}}$$

$$\underline{\underline{\varepsilon}} = [0, 0, \varepsilon_{33}, 0, 0, 0]^T$$

$$\begin{cases} \sigma_{11} = \sigma_{22} = \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{33} \\ \sigma_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \varepsilon_{33} \end{cases}$$

$$\sigma_{11} = \sigma_{22} = \frac{\nu}{1-\nu}\sigma_{33}$$

$$\sigma_h = \frac{\nu}{1-\nu}\sigma_v$$

$$\nu \sim 0.25$$

$$\nu/(1-\nu) \sim 1/3$$

$$\nu \to 0.5$$

$$\nu/(1-\nu) \to 1$$

$\nu \sim 0.5$

$$S_{11} = \sigma_{11} + P_p$$

$$S_{33} = \sigma_{33} + P_p$$

$$\varepsilon_{11} \neq \varepsilon_{22} \neq 0$$

$$\underline{\underline{\varepsilon}} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 0, 0, 0]^T$$

$$\left\{\begin{array}{l}\sigma_{11}=\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{11}+\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{22}+\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{33}\\ \sigma_{22}=\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{11}+\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{22}+\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{33}\\ \sigma_{33}=\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{11}+\frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{22}+\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\varepsilon_{33}\end{array}\right.$$

σ_{11}

σ_{22}

$$\left\{\begin{array}{l} \sigma_{11} = \frac{\nu}{1-\nu}\sigma_{33} + \frac{E}{1-\nu^2}\varepsilon_{11} + \frac{\nu E}{1-\nu^2}\varepsilon_{22} \\ \sigma_{22} = \frac{\nu}{1-\nu}\sigma_{33}\frac{\nu E}{1-\nu^2}\varepsilon_{11} + \frac{E}{1-\nu^2}\varepsilon_{22} \end{array}\right.$$

$$\varepsilon_{Hmax}$$

$$\varepsilon_{Hmin}$$

$$\begin{cases} \sigma_{Hmax} = \frac{\nu}{1-\nu}\sigma_v + E'\varepsilon_{Hmax} + \nu E'\varepsilon_{hmin} \\ \sigma_{hmin} = \frac{\nu}{1-\nu}\sigma_v + \nu E'\varepsilon_{Hmax} + E'\varepsilon_{hmin} \end{cases}$$

$$E' = \frac{E}{1-\nu^2}$$

$$S_{Hmax} = \sigma_{Hmax} + P_p$$

$$S_{hmin} = \sigma_{hmin} + P_p$$

$$\sigma_v = S_v - P_p$$

	(E,ν)	(K,G)
$G =$	$\dfrac{E}{2(1+\nu)}$	
$M =$	$\dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}$	$K+\dfrac{4}{3}G$
$\lambda =$	$\dfrac{\nu E}{(1+\nu)(1-2\nu)}$	
$K =$	$\dfrac{E}{3(1-2\nu)}$	

Shear modulus
(also noted as μ, S-wave)

Constrained modulus
(uniaxial compaction, P-wave)

Lamé first parameter
(volumetric strain component)

Bulk modulus
(relates volumetric strain and isotropic stress)

$$\mathrm{d}S_v/\mathrm{d}z = 23.8$$

$$\lambda_p = 0.7$$

$$E = 5$$

$$\nu = 0.22$$

$$\varepsilon_{hmin} = 0$$

$$\varepsilon_{Hmax} = 0.0002$$

$$S_v = 23.8\frac{\text{MPa}}{\text{km}}\frac{1\,\frac{\text{psi}}{\text{ft}}}{\frac{\text{MPa}}{\text{km}}} * 7950\ \text{ft} = 8227\text{psi}$$

$$P_p = \lambda_p S_v = 0.7 * 8227$$

$$\text{psi} = 5759 \text{ psi}$$

$$\sigma_v = S_v - P_p = 8227$$

$$\text{psi} - 5759\ \text{psi} = 2468\ \text{psi}$$

$$E' = \frac{E}{1-\nu^2} = \frac{5 \times 10^6 \text{ psi}}{1-0.22^2} = 5.25 \times 10^6 \text{ psi}$$

$$\begin{cases} \sigma_{Hmax} = \frac{\nu}{1-\nu}\sigma_v + E'\varepsilon_{Hmax} + \nu E'\varepsilon_{hmin} = \frac{0.22}{1-0.22}2468 \text{ psi} + 5.25 \times 10^6 \text{ psi} * 0.0002 = 1745 \text{ psi} \\ \sigma_{hmin} = \frac{\nu}{1-\nu}\sigma_v + \nu E'\varepsilon_{Hmax} + E'\varepsilon_{hmin} = \frac{0.22}{1-0.22}2468 \text{ psi} + 0.22 * 5.25 \times 10^6 \text{ psi} * 0.0002 = 927 \text{ psi} \end{cases}$$

$$\begin{cases} S_{Hmax} = \sigma_{Hmax} + P_p = 1745 \text{ psi} + 5759 \text{ psi} = 7504 \text{ psi} \\ S_{hmin} = \sigma_{hmin} + P_p = 927 \text{ psi} + 5759 \text{ psi} = 6686 \text{ psi} \end{cases} \quad \blacksquare$$

$$C_{pp}$$

V_p

$$C_{pp} = \frac{1}{V_p} \frac{\mathrm{d}V_p}{\mathrm{d}P_p}\bigg|_{S_v, \varepsilon_h}$$

ε_h

$$C_{pp} = \frac{C_{bp}}{\phi}$$

$$C_{bp} \sim 1/M$$

$$M = (1 - \nu)E/[(1 + \nu)(1 - 2\nu)]$$

$$C_{pp} = \frac{(1+\nu)(1-2\nu)}{(1-\nu)E\phi}$$

$$E = 10$$

$\nu = 0.20$

$$10^6$$

$$M = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} = \frac{(1-0.20)10\ \text{GPa}}{(1+0.20)(1-2\times 0.20)} = 11.11\ \text{GPa} = 1.6\times 10^{6}\ \text{psi}$$

$$C_{pp} = \frac{1}{M\phi} = \frac{1}{1.6 \times 10^6 psi \times 0.20} = 3.1 \; [10^6$$

$]^{-1} = 3.1\,\mu$

$\underline{u}$

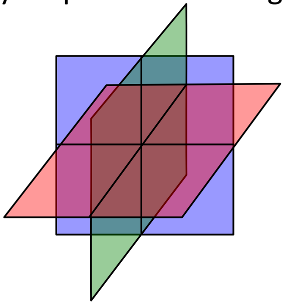
$$(\lambda + G)\nabla(\nabla \cdot \underline{u}) + G\nabla^2\underline{u} + \rho\underline{b} = 0$$

$$\lambda = (\nu E)/[(1+\nu)(1-2\nu)]$$

G

$\underline{b}$

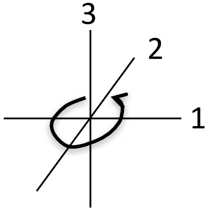
Orthorhombic symmetry
(symmetry respect to 3 orthogonal planes)



$$\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & 0 & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & 0 & 0 \\ & 0 & & 0 & C_{55} & 0 \\ & & & 0 & 0 & C_{66} \end{bmatrix}$$

9 independent parameters

Transverse Isotropy
(symmetry respect to 1 axis)



$$\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & 0 & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & 0 & 0 \\ & 0 & & 0 & C_{44} & 0 \\ & & & 0 & 0 & C_{66} \end{bmatrix}$$

5 independent parameters ($C_{12}=C_{11}-2C_{66}$)

$$E_h > E_v$$

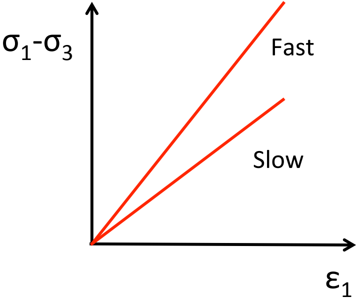
$\varepsilon >\sim 0.001$

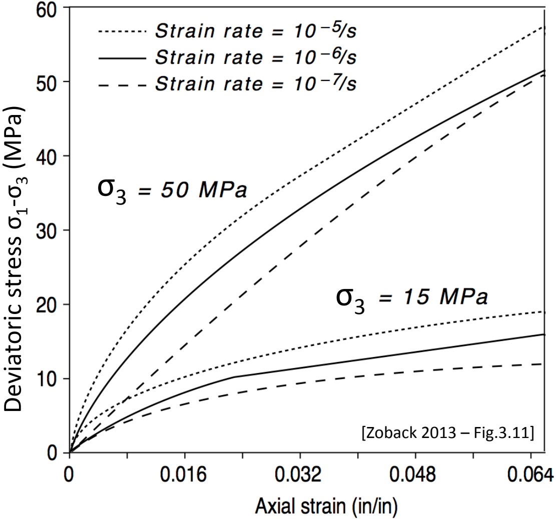
$$E_{load}$$

$$E_{unload}$$

$$E_{unload} \sim E_{unload}$$

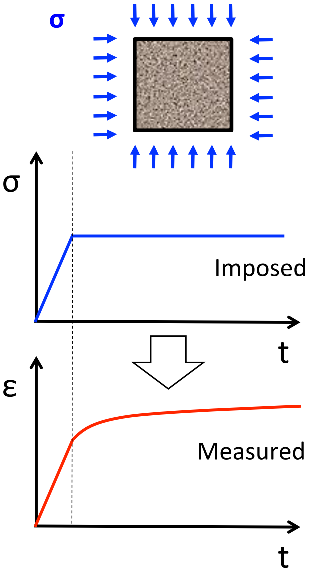
Strain rate hardening: The faster the loading, the stiffer the material

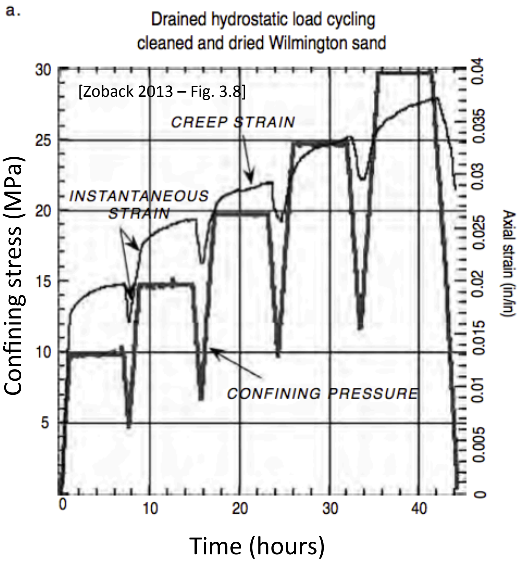




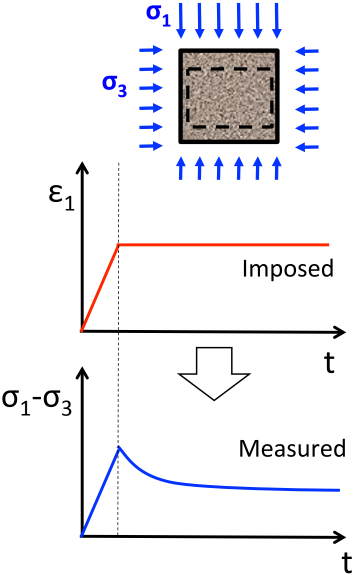
[Zoback 2013 – Fig.3.11]

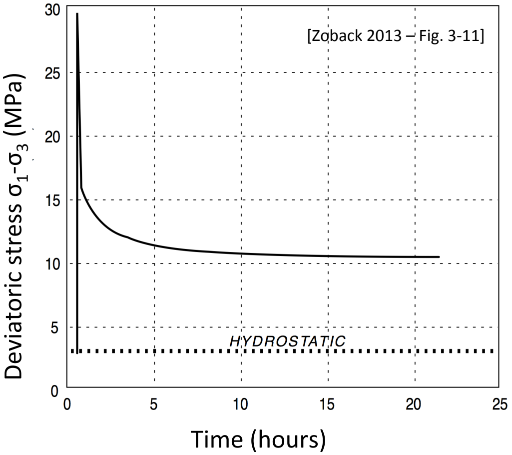
Creep strain at constant stress





(2) Stress relaxation at constant strain





$$\underline{\underline{\varepsilon}} = \underline{\underline{\underline{D}}}\,\underline{\underline{\sigma}}$$

$$\underline{\underline{\sigma}} = \underline{\underline{S}} - \alpha P_p \underline{\underline{I}}$$

$\underline{\underline{I}}$

$$\alpha = 1 - \frac{K_{drained}}{K_{unj}}$$

$$K_{drained}$$

$$K_{unj}$$

$$K_{unj} = K_m$$

$\underline{\underline{\alpha}}$

$$\alpha \neq 1$$

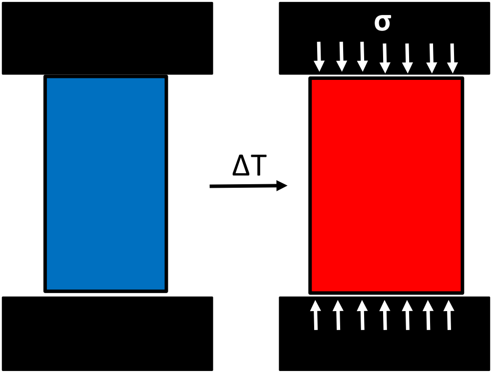
α_L

p

$$\alpha_L = \left.\frac{1}{L}\frac{dL}{DT}\right|_p$$

ΔT

$$
\left\{
\begin{array}{lcl}
\sigma_{11} & = & (\lambda + 2\mu)\, \varepsilon_{11} + \lambda\, \varepsilon_{22} + \lambda\, \varepsilon_{33} + 3K\alpha_L \Delta T \\
\sigma_{22} & = & \lambda\, \varepsilon_{11} + (\lambda + 2\mu)\, \varepsilon_{22} + \lambda\, \varepsilon_{33} + 3K\alpha_L \Delta T \\
\sigma_{33} & = & \lambda\, \varepsilon_{11} + \lambda\, \varepsilon_{22} + (\lambda + 2\mu)\, \varepsilon_{33} + 3K\alpha_L \Delta T \\
\sigma_{12} & = & 2\mu\, \varepsilon_{12} \\
\sigma_{13} & = & 2\mu\, \varepsilon_{13} \\
\sigma_{23} & = & 2\mu\, \varepsilon_{23}
\end{array}
\right.
$$


σ
ΔT

$$\sigma_{11} = \sigma_{22} = 0$$

$$\varepsilon_{11} = \varepsilon_{22} \neq 0$$

$$\varepsilon_{33} = 0$$

$$\Delta T \neq 0$$

$$\left\{ \begin{array}{rcl} 0 & = & (\lambda + 2\mu)\, \varepsilon_{11} + \lambda\, \varepsilon_{11} + 3K\alpha_L \Delta T \\ \sigma_{33} & = & \lambda\, \varepsilon_{11} + \lambda\, \varepsilon_{11} + 3K\alpha_L \Delta T \end{array} \right.$$

$$\sigma_{33} = \left(\frac{6 - \mu/K}{3 + \mu/K} \right) K \alpha_L \Delta T \quad \blacksquare$$

$$\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$$

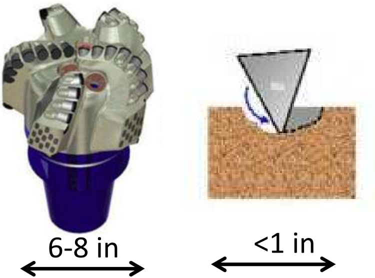
σ_{iso}

$$\varepsilon_{vol} = \frac{3(1-2\nu)}{E}\sigma_{iso}$$

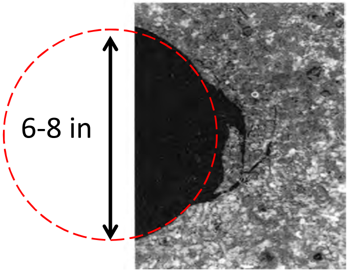
$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11}$$

(a) Uncemented sand (b) Cemented sandstone (c) Vuggy carbonate

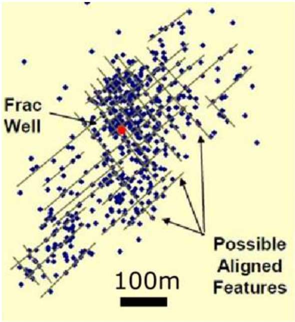
Rock cutting at the drill bit



Wellbore breakout

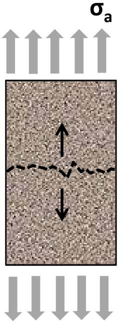


Shale hydraulic fracture

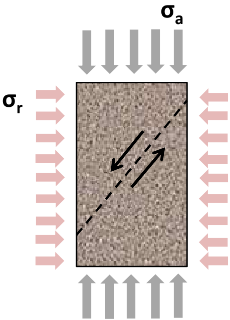


Reservoir depletion

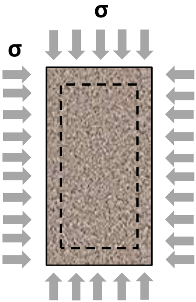
Images: Schlumberger/Terratek, Zoback 2013, Warpinski 2008, doe.gov



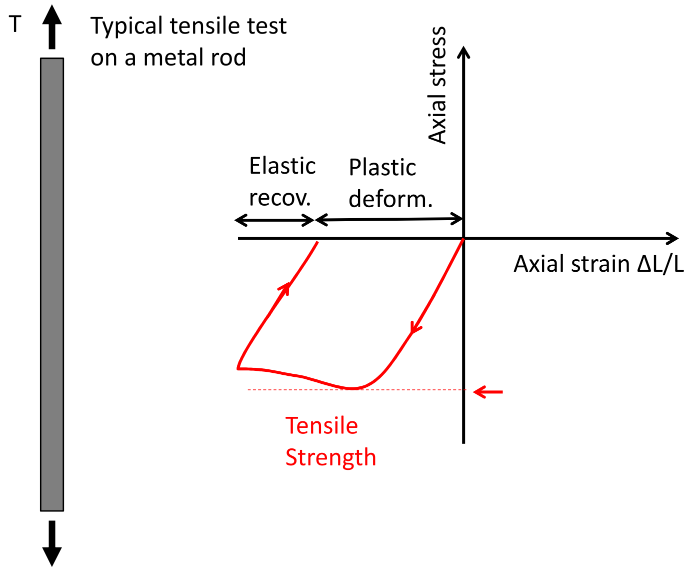
Tension
(bond breakage)
Ex: drilling-induced tens. fracs



Shear
(friction failure)
Ex: fault, breakout



Compression
(pore collapse)
Ex: reservoir compaction


T
Typical tensile test
on a metal rod
Axial stress
Elastic
recov.
Plastic
deform.
Axial strain ΔL/L
Tensile
Strength

Tensile test on a rock rod

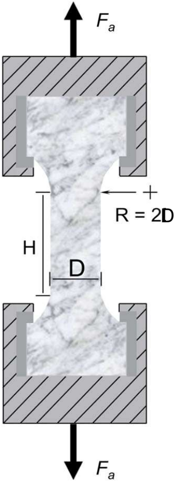
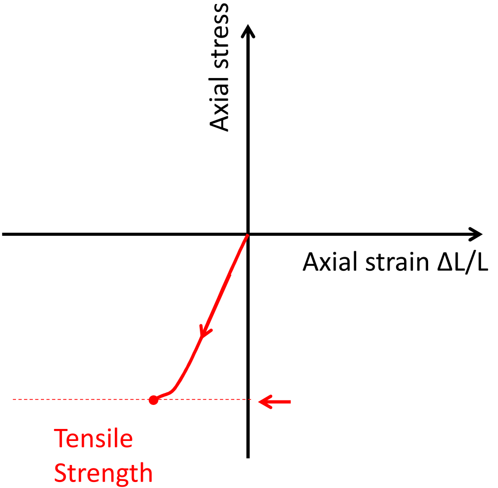


Figure direct tension: Perras and Diederichs (2014)

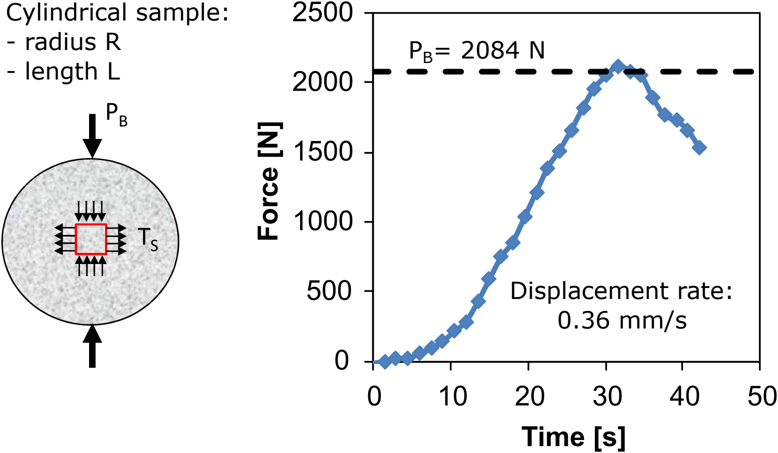


T_S

$$T_S = \frac{P_B}{\pi L R}$$

P_B

R



Cylindrical sample:
- radius R
- length L
P_B
T_S
P_B= 2084 N
Force [N]
Time [s]
Displacement rate:
0.36 mm/s
2500
2000
1500
1000
500
0
0
10
20
30
40
50

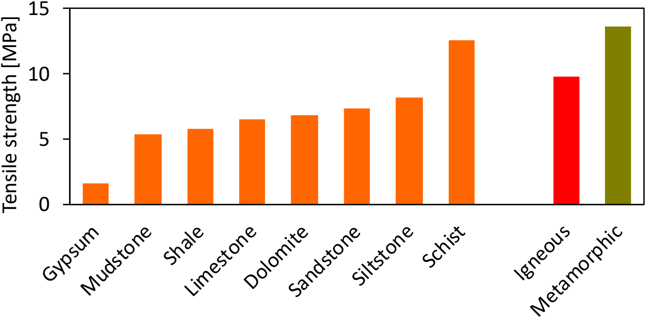
$$R = \frac{1}{2}$$

$$= 0.0127$$

$L = 1$

$$= 0.0254$$

$$T_S = \frac{2084 \text{ N}}{\pi(0.0254 \text{ m})(0.0127 \text{ m})} = 2.06 \times 10^6 \text{ Pa} = 2.06 \text{ MPa} \quad \blacksquare$$



Tensile strength [MPa]
15
10
5
0
Gypsum
Mudstone
Shale
Limestone
Dolomite
Sandstone
Siltstone
Schist
Igneous
Metamorphic

S_0

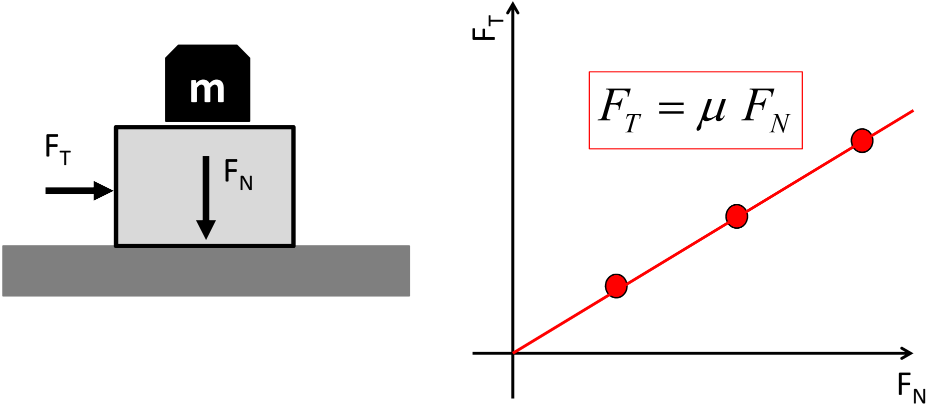
$$F_T$$

F_N

$$F_T = \mu F_N$$

$$F_N = 0$$

$$F_T = 0$$


m
F_T
F_N
F_T
F_T = μ F_N
F_N

τ

σ_n

μ_i

$$\tau = \mu_i \sigma_n$$

$$\sigma_n = 0$$

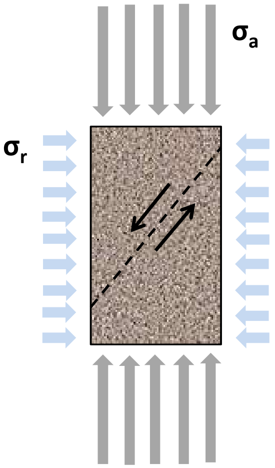
$$\tau = 0$$

φ

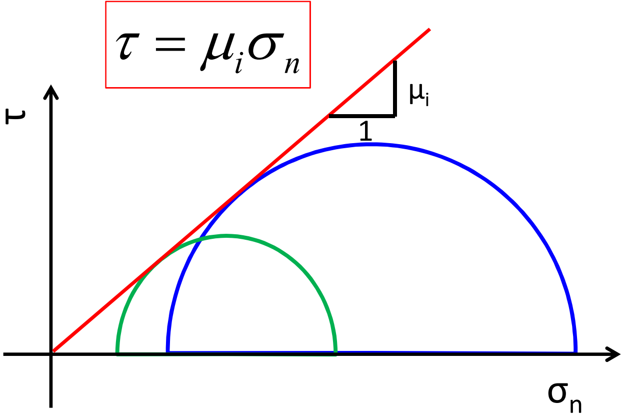
$$\tan(\varphi) = \mu_i$$

$\mu_i = 0.5$

$$\varphi \sim 30^\circ$$



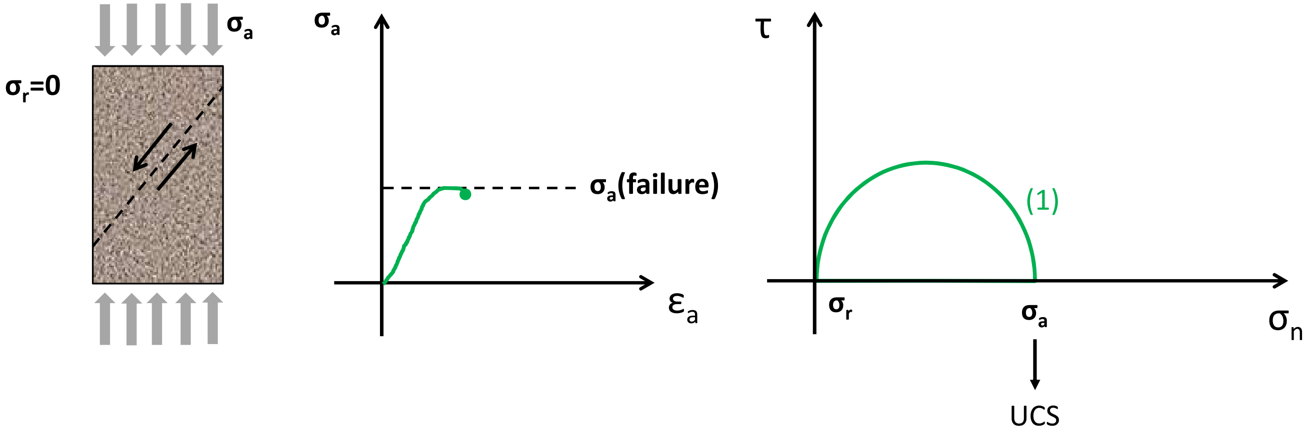
Unconsolidated Sand



$$\sigma_r = 0$$

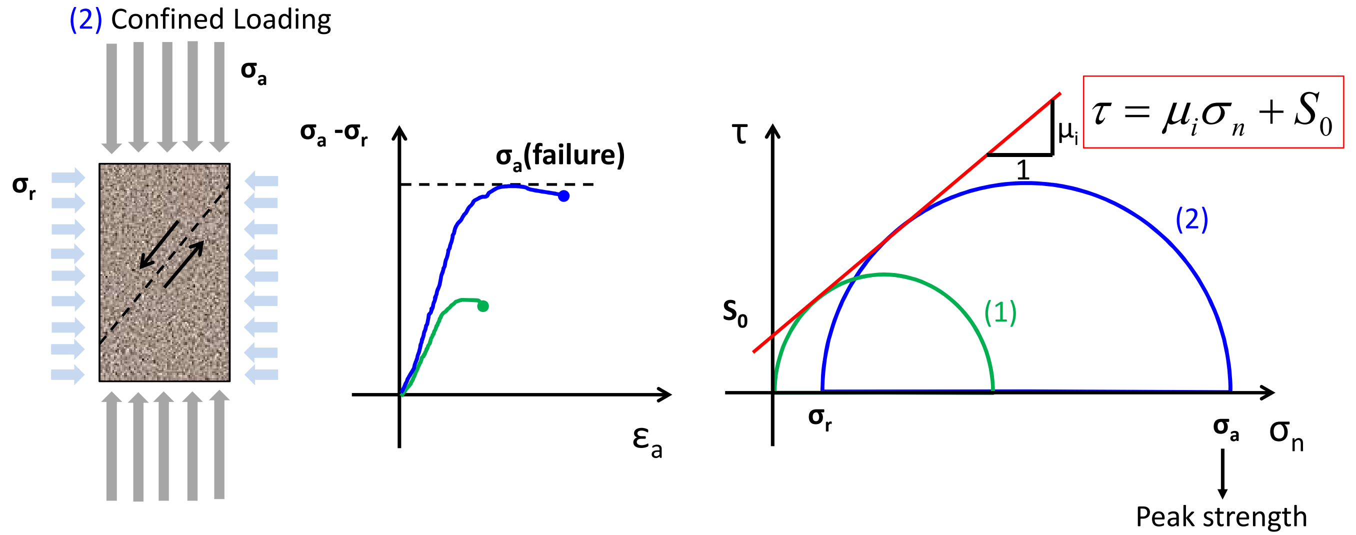
UCS

(1) Unconfined Loading



$\sigma_r \neq 0$

$$\tau = S_0 + \mu_i \sigma_n$$

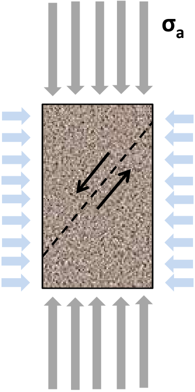

(2) Confined Loading
σa
σr
σa -σr
σa(failure)
εa
τ
μi
1
τ = μiσn + S0
(2)
(1)
S0
σr
σa
σn
Peak strength

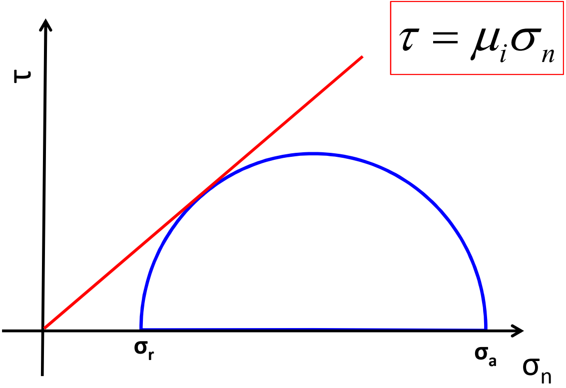
$$\pi/4 + \varphi/2$$

$$\pi/4 + \varphi/2 = 60^\circ$$

Which is point in the Mohr circle with maximum τ/σ_n?

What is the angle of the failure plane?





$$\sigma_1 = UCS + q\,\sigma_3$$

σ_1

σ_3

q

$$p' - q$$

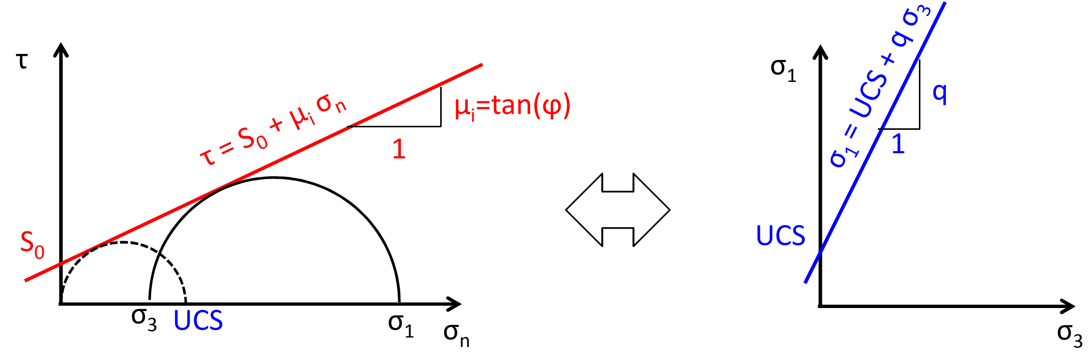
$$\sigma_m - q$$

$$q = \frac{1 + \sin \varphi}{1 - \sin \varphi}$$

$\varphi\ 30^\circ$

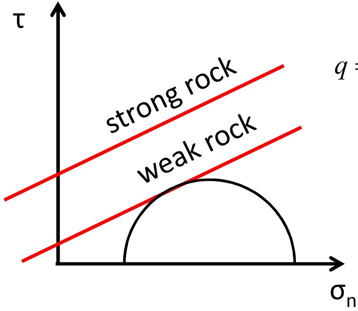
$$q = 3$$

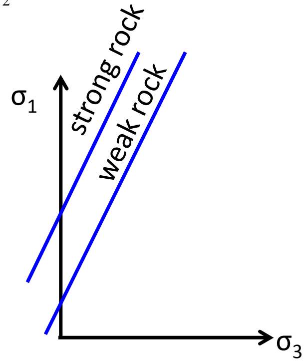
$$UCS = 2S_0 \left(\frac{1 + \sin \varphi}{1 - \sin \varphi} \right)^{1/2} = 2S_0 \sqrt{q}$$

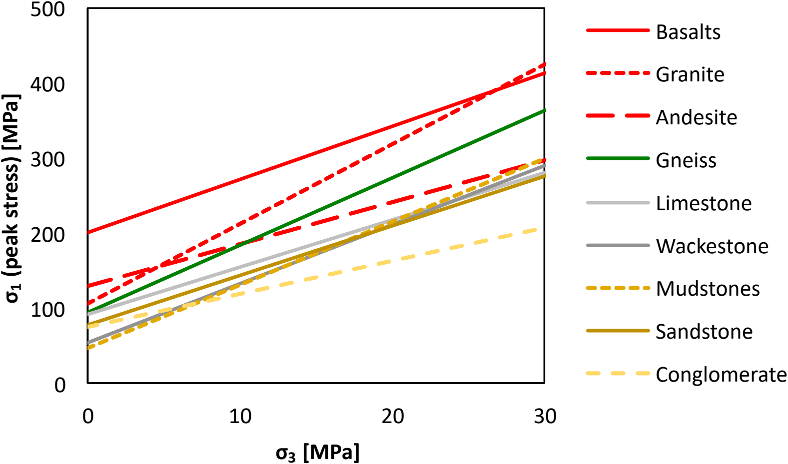


$$UCS = 2S_0\left(\sqrt{\mu_i^2+1}+\mu_i\right) = 2S_0\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^{1/2}$$

$$q = \left(\sqrt{\mu_i^2+1}+\mu_i\right)^2 = \frac{1+\sin\varphi}{1-\sin\varphi}$$







500
400
300
200
100
0
σ1 (peak stress) [MPa]
0
10
20
30
σ3 [MPa]
Basalts
Granite
Andesite
Gneiss
Limestone
Wackestone
Mudstones
Sandstone
Conglomerate

$$(S_1, S_2, S_3)$$

$(\sigma_1, \sigma_2, \sigma_3)$

P_c

σ_D

$$P_p < P_c$$

$$\sigma_D = S_1 - S_3$$

$$S_2 = S_3$$

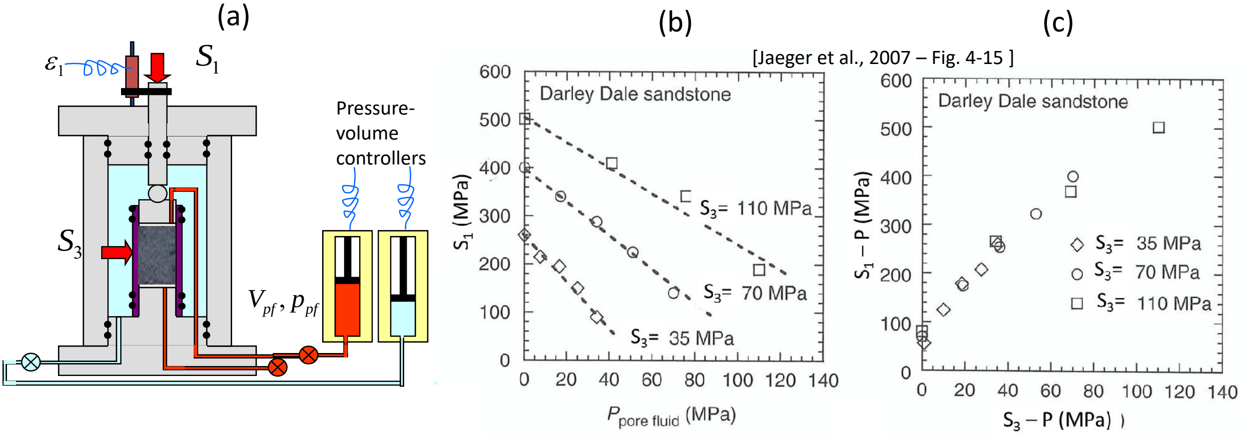
$$S_1 - S_3 = \sigma_1 - \sigma_3$$

$$d\varepsilon_a/dt$$

$$\varphi = \arctan[(q-1)/(2\sqrt{q})],$$

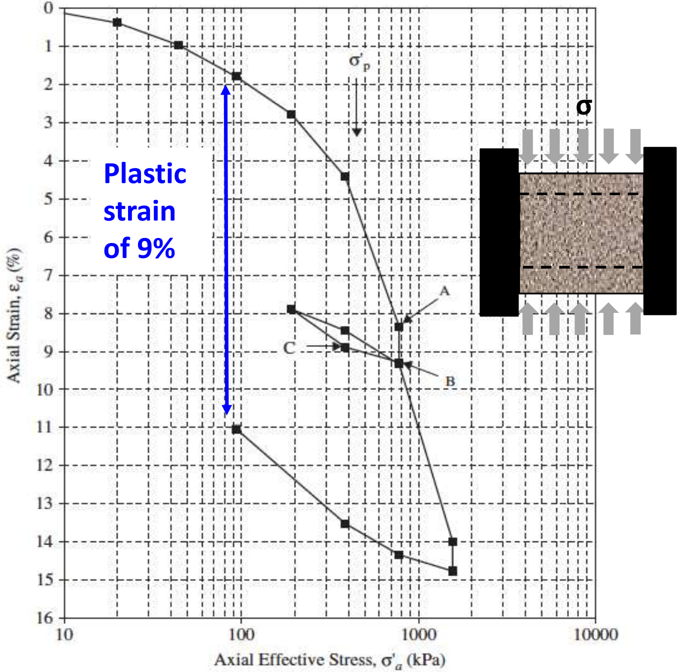
$$\varepsilon_v = \varepsilon_a + 2\varepsilon_r$$

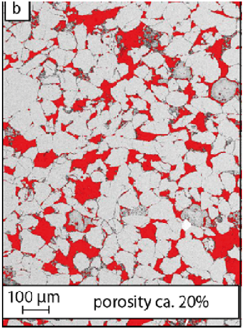
$$d\varepsilon_v/dt < 0$$

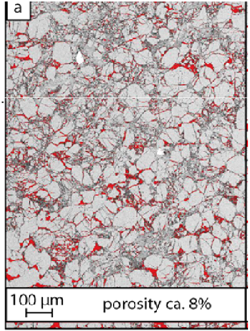


(a)
ε_1
S_1
S_3
Pressure-volume controllers
V_{pf}, p_{pf}
(b)
(c)
[Jaeger et al., 2007 – Fig. 4-15]
Darley Dale sandstone
S_1 (MPa)
S_3= 110 MPa
S_3= 70 MPa
S_3= 35 MPa
$P_{pore\ fluid}$ (MPa)
Darley Dale sandstone
$S_1 - P$ (MPa)
$S_3 - P$ (MPa)

Germaine and Germaine, 2009



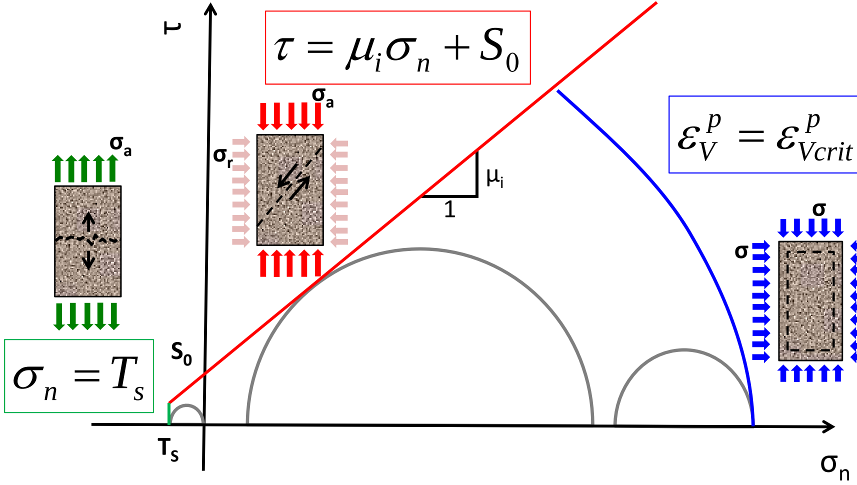




www.memin.de/results-tp-2.html

$$(\sigma_n, \tau)$$

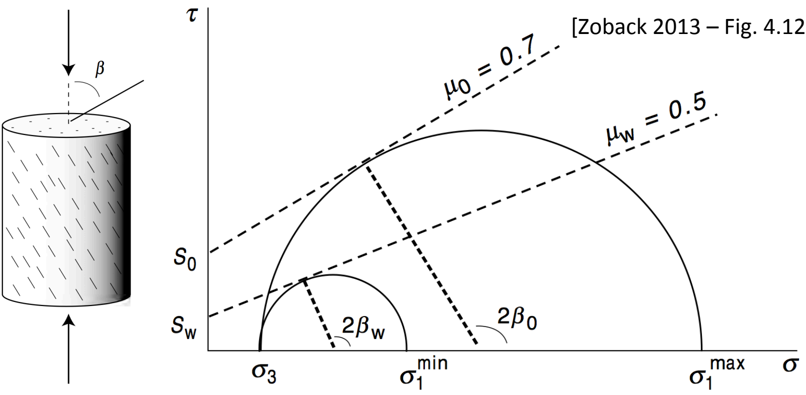
$$\tau = S_0 + \mu_i \sigma_n$$



τ
$\tau = \mu_i \sigma_n + S_0$
σ_a
σ_r
σ_a
μ_i
1
$\varepsilon_V^p = \varepsilon_{Vcrit}^p$
σ
σ
S_0
$\sigma_n = T_s$
T_s
σ_n

$$\beta_w$$

$$S_w + \mu_w \sigma_n$$



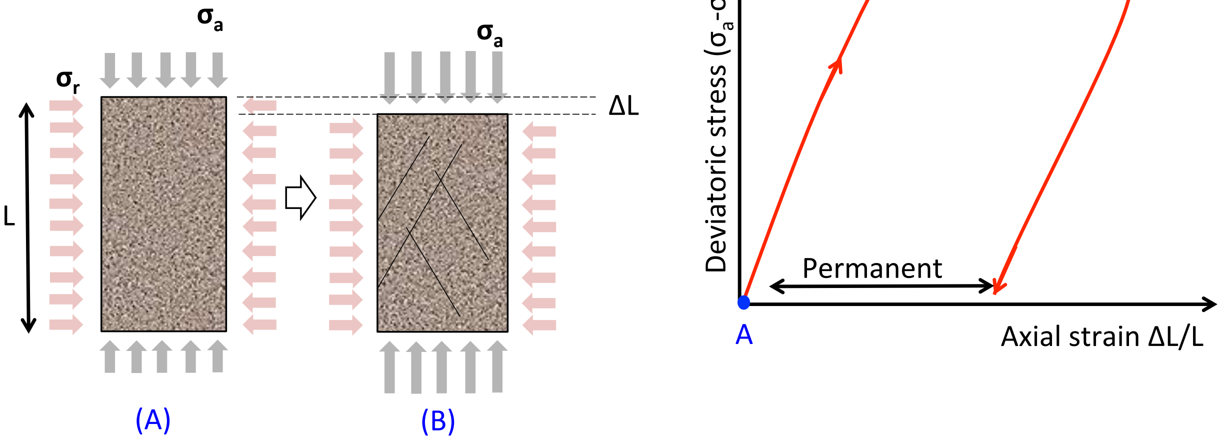
β
τ
[Zoback 2013 – Fig. 4.12]
μ0 = 0.7
μW = 0.5
S0
SW
2βW
2β0
σ3
σ1min
σ1max
σ

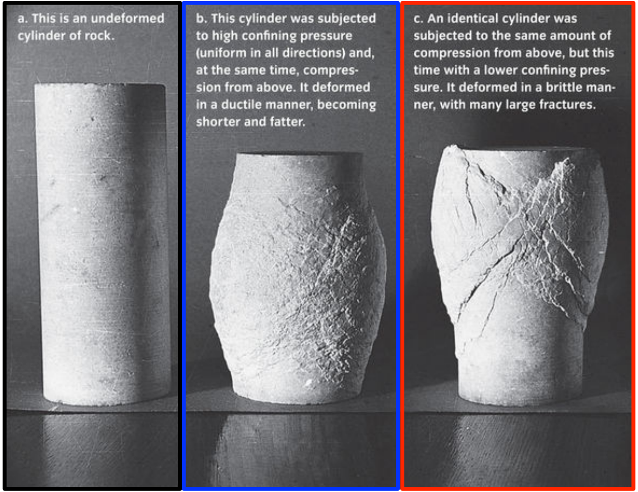
Rock deformation

Relation strain V.S. stress

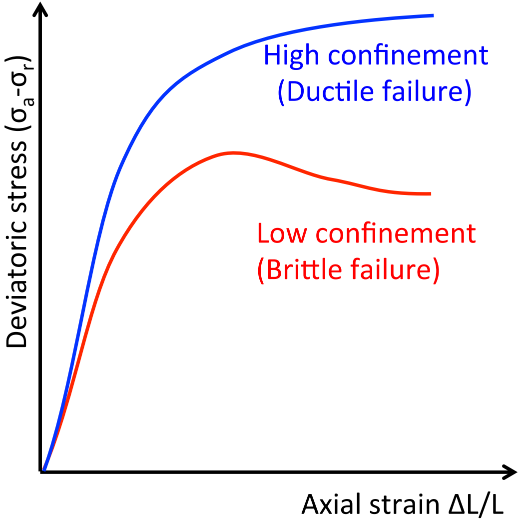
Elastic (Young modulus)

Plastic (~Viscosity)





Visualizing Geology, WileyPlus



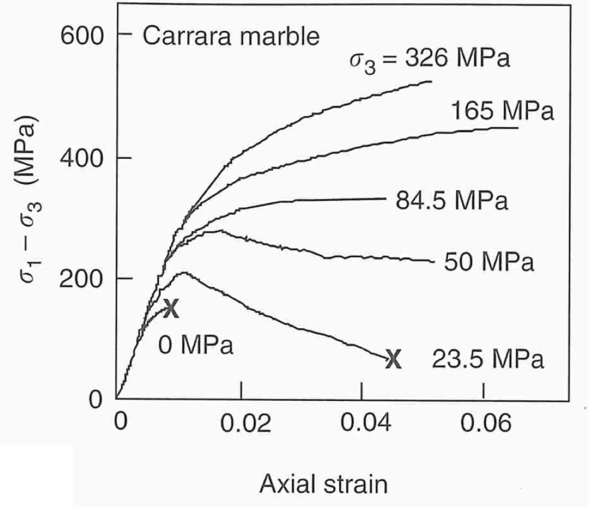
$$\sigma_d = (\sigma_1 - \sigma_3)$$

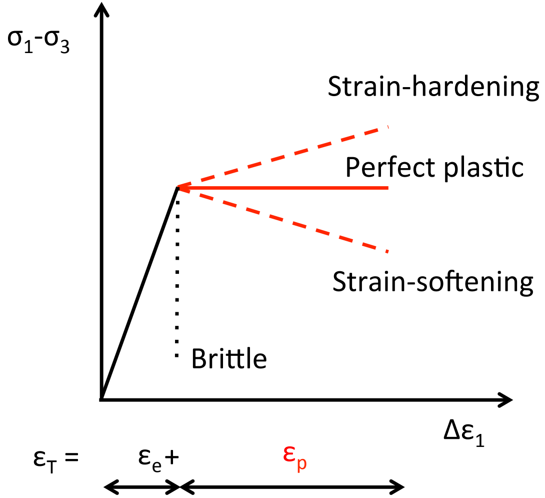
$$\underline{\underline{\varepsilon_p}}$$

$$\underline{\underline{C_p}}$$

$$\Delta\underline{\underline{\sigma}} = \underline{\underline{C_p}}\Delta\underline{\underline{\varepsilon}}$$

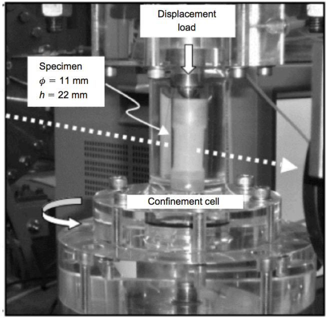
[Jaeger et al. 2007 – Fig. 4.5]

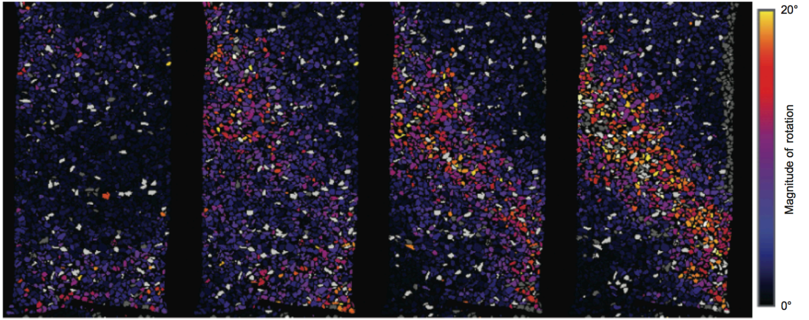




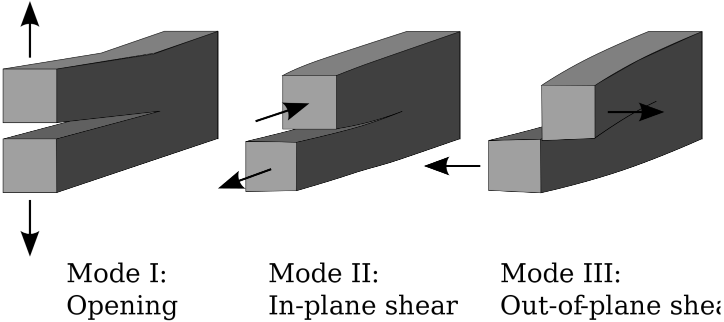
$$\boxed{\underline{\underline{\Delta\sigma}} = \underline{\underline{\underline{\underline{C}}}}\;\underline{\underline{\Delta\varepsilon}}} \qquad \boxed{\underline{\underline{\Delta\sigma}} = \underline{\underline{\underline{\underline{C_p}}}}\;\underline{\underline{\Delta\varepsilon_p}}}$$

(a) Uncemented or poorly cemented rock → grain friction, dilation, grain crushing/rotation





(b) Cemented rock → Propagation of microfractures, grain friction/crushing

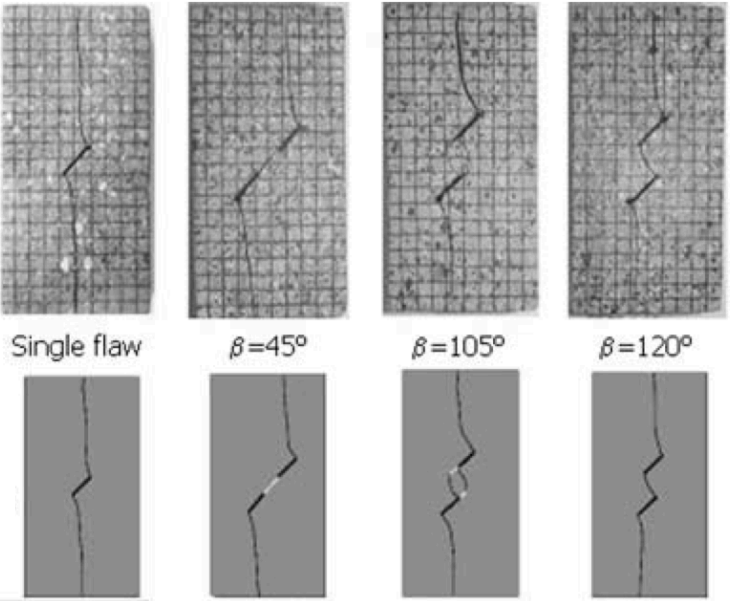


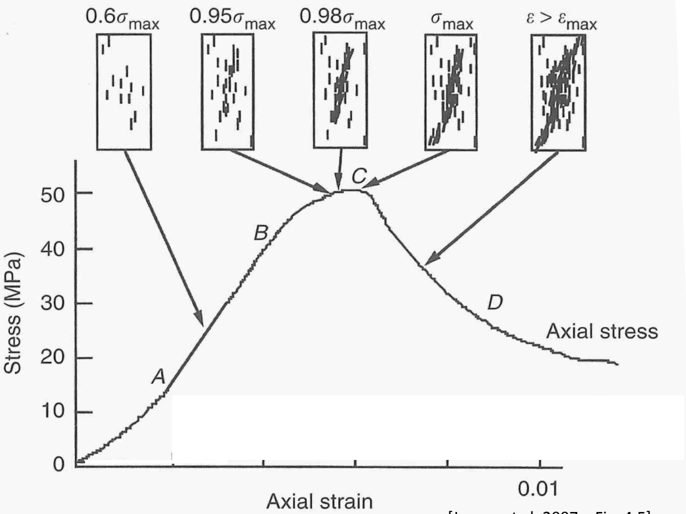
Stress intensification at the tip of fractures

Propagation starts at fracture tips

Napolitan Tuffo

[Hall et al. 2006 – Pure Appl. Geophys.]





[Jaeger et al. 2007 – Fig. 4.5]