MEM637 Theory of Nonlinear Control-II
Winter Final Project

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1 Introduction

In approximately early 1970's the control community witnessed the birth of a sustained interest in the applications of differential geometry concepts to the formulation and solution of a variety of long-standing non-linear control problems. The understanding of theoretical and practical implications of control theories and the issues associated with the design of non-linear control systems can be seen in the work of Hermann and Krener (1970), Sussman and Jurdjevic (1972) and Works by Brockett (1976)[1]. Variable structure systems and its sliding mode behavior has also seen extensive research in last fifty years. Scientist from the United States have contributed their expertise in this research sector from wide range of applications ranging from gas turbines to aerospace design problems. This report will focus on the problem of formulating a variable structure switching system based on the Feedback linearization (FBL) reduction for a wheelset through numerical analysis and simulations.

2 System Model

The wheelset used to model a simple wheeled robot as shown in Figure 1. The system is modeled by the differential equations (1). In this problem, the driving force, F and the steering torque, T are created by coordinating the wheel torques independently. For similicity, we consider M=1 and J=1, and we assume that the control inputs have magnitude limits given by $|F| \leq 1$, and $|T| \leq 4$.

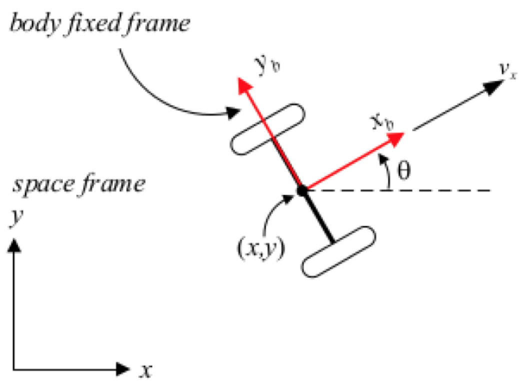


Figure 1: Simple wheelset

$$
\begin{aligned}
\dot{x} =& \quad v_x \cos\theta \\
\dot{y} =& \quad v_y \sin\theta \\
\dot{\theta} =& \quad \omega \\
M\dot{v_x} =& \quad F\omega \\
J\dot{\omega} =& \quad T
\end{aligned}
\tag{1}
$$

This systems can be feed back linearized with respect to the outputs (x, y) using dynamic extension. The extended equations are,

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v}_x \\ \dot{\omega} \\ \dot{F} \end{bmatrix} = \begin{bmatrix} v_x \cos\theta \\ v_x \sin\theta \\ \omega \\ F \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ T \end{bmatrix}
\tag{2}
$$

where u is the dynamically extended input.

The resulting FBL form of the dynamics are given by,

$$
\dot{\mathbf{z}} = A\mathbf{z} + E(\alpha + \rho\mathbf{u}),
\tag{3}
$$

where,

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\alpha &= \begin{bmatrix} -v_x\omega^2\cos\theta - 2\omega F\sin\theta \\ -v_x\omega^2\sin\theta + 2\omega F\cos\theta \end{bmatrix} \\
\rho &= \begin{bmatrix} \cos\theta & -v_x\sin\theta \\ \sin\theta & v_x\cos\theta \end{bmatrix} \\
\mathbf{u} &= \begin{bmatrix} u \\ T \end{bmatrix}
\end{aligned}
$$

In this project, the objective is to design a variable structure switching system based on the FBL form given in (3). The sliding surface is selected to have the form $s(\mathbf{x}) = K\mathbf{z} = 0$, and K is selected so that all the sliding eigenvalues are at -1.

3 Design of the VSC

Variable structure control systems are switching controllers that illustrate certain robustness properties. The VSC is designed in two stages,

1) design the switching surface $K\mathbf{z}$ using equivalent control, so that sliding eigenvalues are at -1,

2) use the Lyapunov approach to design the control law $\mathbf{u} = \mathbf{u}(\mathbf{x})$ so that the sliding surface is reached in finite time.

3.1 Designing the sliding surface

On the sliding surface, $s = K\mathbf{z} = 0$, and as a result $\dot{s} = K\dot{\mathbf{z}} \equiv 0$ on the surface. Therefore using the FBL dynamics given in (3), we can write,

$$K[A\mathbf{z} + E(\alpha + \rho\mathbf{u})] = 0$$

Solving for u, we get the equivalent control on the sliding surface to be,

$$\mathbf{u_{eq}} = -(KE\rho)^{-1}(KA\mathbf{z} + KE\alpha) \tag{4}$$

Substituting (4) back in the system dynamics (3), gives the following equivalent dynamics for the FBL system on the sliding surface,

$$\dot{z} = Az + E(\alpha - \rho(kE\rho)^{-1}(kAZ + kE\alpha)) \tag{5}$$

Selecting K such that $KE = I$, the equivalent dynamics can be simplified to

$$\dot{z} = [I - EK]Az \tag{6}$$

Now the objective is to design K such that,

1) $KE = I$,
2) eigenvalues of $H = [I - EK]A$ are at -1.

In order for $KE = I$, K should have the following form

$$K = \begin{bmatrix} k_{11} & k_{12} & 1 & k_{14} & k_{15} & 0 \\ k_{21} & k_{22} & 0 & k_{24} & k_{25} & 1 \end{bmatrix} \tag{7}$$

Then:

$$I - EK = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -k_{11} & -k_{12} & 0 & -k_{14} & -k_{15} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -k_{21} & -k_{22} & 0 & -k_{24} & -k_{25} & 0 \end{bmatrix} \tag{8}$$

and

$$H = (I - EK)A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -k_{11} & -k_{12} & 0 & -k_{14} & -k_{15} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -k_{21} & -k_{22} & 0 & -k_{24} & -k_{25} \end{bmatrix}. \tag{9}$$

The eigenvalues of the sliding dynamics are given by $|I\lambda - H| = 0$ where,

$$I\lambda - H = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 & 0 \\ 0 & k_{11} & \lambda + k_{12} & 0 & k_{14} & k_{15} \\ 0 & 0 & 0 & \lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & -1 \\ 0 & k_{21} & k_{22} & 0 & k_{24} & \lambda + k_{25} \end{bmatrix}. \tag{10}$$

Solving $|I\lambda - H| = 0$ further we get,

$$\lambda[\lambda[\lambda(\lambda + k_{25}) + k_{24}][\lambda(\lambda + k_{12}) + k_{11}] - [k_{14} + k_{15}\lambda][k_{22}\lambda^2 + k_{21}]] = 0 \tag{11}$$

Assuming that $k_{14} = k_{15} = k_{21} = k_{22} = 0$ the equation becomes,

$$\lambda^2[\lambda^2 + k_{25}\lambda + k_{24}][\lambda^2 + k_{12}\lambda + k_{11}] = 0 \tag{12}$$

Thus, two eqigenvalues are identically zero. If we set the remaining eigenvalues to -1, we obtain the following equations:

$$\begin{aligned} (\lambda + 1)(\lambda + 1)(\lambda + 1)(\lambda + 1) &= 0 \\ (\lambda^2 + 2\lambda + 1)(\lambda^2 + 2\lambda + 1) &= 0 \end{aligned} \tag{13}$$

Finally comparing (13) with (12), we obtain the remaining values of K.

$$K = \begin{bmatrix} k_{11} & k_{12} & 1 & k_{14} & k_{15} & 0 \\ k_{21} & k_{22} & 0 & k_{24} & k_{25} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \tag{14}$$

There are a total of six sliding eigenvalues which are $[0, 0, -1, -1, -1, -1]$.

3.2 Desgining control laws to reach the sliding surface

Once K has been defined, the next step in the variable structure control system design is the specification of the control function $u_i^{\pm}$ such that the manifold $s(x) = 0$ is reached in finite time so that sliding occurs. We use a Lypaunov control design approach to obtain this control law. We consider a Lyapunov function

$$V(x) = \frac{1}{2} s^T s \tag{15}$$

Thus,

$$V \begin{cases} = 0, & s(\mathbf{x}) = 0 \\ > 0, & otherwise \end{cases} . \tag{16}$$

Thus ensuring $\dot{V} < 0$ would result in the system exponentially approaching $s = 0$. By ensuring that $\dot{V} \leq -\epsilon < 0$, we can ensure that $s = 0$ is reached in finite time.
Upon differentiation of $V(x)$ we obtain,

$$\dot{V} = \dot{s}^T s = [KA\mathbf{z} + \alpha + \rho\mathbf{u}]^T K\mathbf{z}. \tag{17}$$

Let's consider the control input to be composed of two parts,

$$\mathbf{u} = \mathbf{u_{sm}} + \mathbf{u_{ds}},$$

and set

$$\mathbf{u_{sm}} = \rho^{-1}(-KA\mathbf{z} - \alpha). \tag{18}$$

Thus, (17) becomes,

$$\dot{V} = \mathbf{u_{ds}}^T \rho^T K\mathbf{z}. \tag{19}$$

In order to ensure that $\dot{V} \leq -\epsilon < 0$, we set components of $\mathbf{u_{ds}}$ as

$$u_{ds_i} = \begin{cases} -U_{lim,i} & s_i(x)^* = (\rho^T K\mathbf{z})_i > 0 \\ U_{lim,i} & s_i(x)^* = (\rho^T K\mathbf{z})_i < 0 \end{cases} \tag{20}$$

for $i = 1, 2, ..., m$, where $s^* = [s_1^*, s_2^*, \cdots, s_m^*] = \rho^T K\mathbf{z}$, and $U_{lim,i} > 0$ is an appropriately selected switching limit. The value $U_{lim,i}$ will decide the time taken to converge to the switching surface. A large $U_{lim,i}$ make the system converge on to the switching surface faster, however, it will also result in large magnitude switching of the control inputs across the sliding surface, which leads to pronounced chattering.

From (18) and (20) the final control law will be,

$$\mathbf{u_{fin}} = \rho^{-1}(-KA\mathbf{z} - \alpha) + \begin{cases} -U_{lim,i} & s_i(x)^* = (\rho^T K\mathbf{z})_i > 0 \\ U_{lim,i} & s_i(x)^* = (\rho^T K\mathbf{z})_i < 0 \end{cases} \tag{21}$$

If, as in our case, the control inputs have limited magnitude, $|u_i| < u_{i,max}$, we limit the result of (21) at these limits. That is,

$$u_i = \begin{cases} u_{fin_i} & , |u_{fin_i}| \leq u_{i,max} \\ sign(u_{fin_i})u_{i,max} & , otherwise \end{cases} \tag{22}$$

4 Results and Simulations

Simulations were conducted on MATLAB using the control proposed in (22). In the simulations, we would expect the value of the Lyapunov function V to stabilize to 0 in finite time and the control inputs to be contrained within their respective limits.

In the first simulations, we used the following initial conditions,

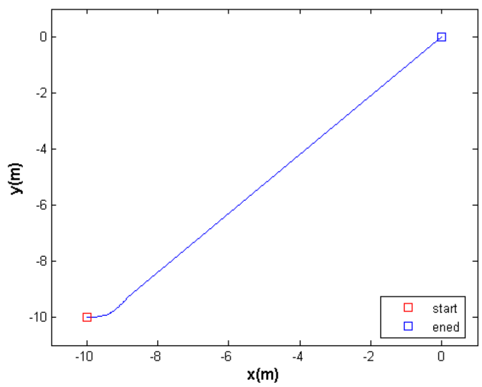
$$\begin{bmatrix} x_0 \\ y_0 \\ \theta_0 \\ v_{x_0} \\ \omega_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} -10 \\ -10 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

where we have selected $v_{x_0} = 1$ to ensure that ρ has an inverse. Figure 2(a) shows the path taken by the vehicle. Figures 2(b) and 2(c) show the inputs to the system and the evolution of the Lyapunov function V. As expected, the Lyapunov function goes to 0 in finite time and the inputs are constrained withing their respective limits. It can also be seen that once V stabilizes to 0, i.e., when the sliding mode is reached, the two controls u and T keeps switching to keep the system on the sliding surface. While on the sliding surface the sliding eigenvalues (set at -1) tends some of the systems states ($\mathbf{z}$) to 0 as well. Some states (corresponding to the zero eigenvalues) will tend to a a constant value (in this case the heading direction θ approaches a constant value).

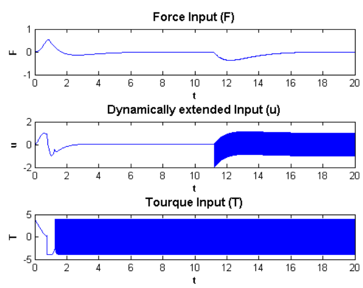
In the second simulation we use the initial conditions,

$$\begin{bmatrix} x_0 \\ y_0 \\ \theta_0 \\ v_{x_0} \\ \omega_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

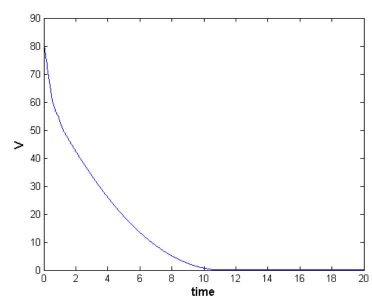
and the corresponding results are shown in Figures 3(a), 3(b) and 3(c). Similar to simulation 1, V tends to zero in finite time and the inputs are constrained



(a)

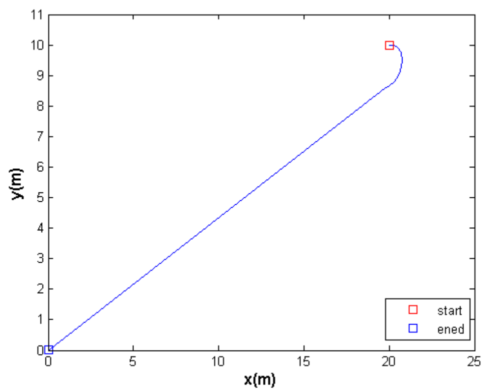


(b)

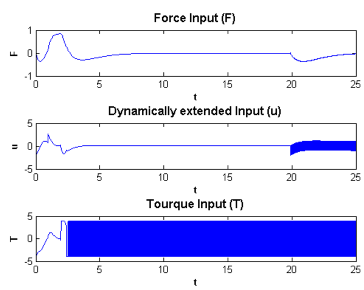


(c)

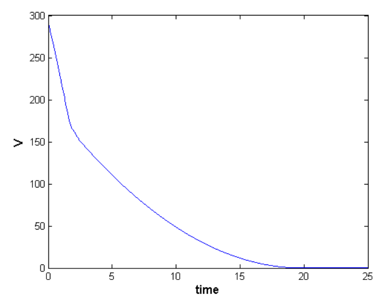
Figure 2: (a) Path taken by the vehicle, (b) time evolution of the inputs, (c) time evolution of the Lyapunov function, for simulation 1, for initial conditions given in (23). The inputs are well within their limits and the Lyapunov function stabilizes to 0 in finite time.



(a)



(b)



(c)

Figure 3: (a) Path taken by the vehicle, (b) time evolution of the inputs, (c) time evolution of the Lyapunov function, for simulation 2, for initial conditions given in (24). The inputs are well within their limits and the Lyapunov function stabilizes to 0 in finite time.

to be withing the corresponding limits. Furthermore, as before, the inputs keep 'chattering' once the sliding surface is reached. As before, the x, y states approaches 0, while the heading direction θ approaches a constant on the sliding surface.

5 Reference

1. Sira-Ramirez, H. (1987). Variable structure control for non-linear systems (Vol 18 No. 9). INT J. SYSTEMS SCI.
2. Harry Kwatny, Variable Structure Control
3. Nonlinear Control and Analytical Mechanics: a Computational Approach - Harry Kwatny