

Variable Structure Control: A Survey

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Abstract—A tutorial account of variable structure control with sliding mode is presented in this paper. The purpose is to introduce in a concise manner the fundamental theory, main results, and practical applications of this powerful control system design approach. This approach is particularly attractive for the control of nonlinear systems. Prominent characteristics such as invariance, robustness, order reduction, and control chattering are discussed in detail. Methods for coping with chattering are presented. Both linear and nonlinear systems are considered. Future research areas are suggested and an extensive list of references is included.

CONVENTIONS FOR SYMBOLS

Capital, italic letters represent matrices, e.g., A and K .
 Boldface, roman lower-case letters represent vectors, e.g., \mathbf{a} and \mathbf{k} .
 Lower-case, italic letters represent scalars, e.g., a and k .
 $\dim\mathbf{x}$ and $\dim B$ stand for dimensions of \mathbf{x} and B , respectively.

I. INTRODUCTION

VARIABLE structure control (VSC) with sliding mode control was first proposed and elaborated in the early 1950's in the Soviet Union by Emelyanov and several coresearchers [43], [79], [154]. In their pioneer works, the plant considered was a linear second-order system modeled in phase variable form. Since then, VSC has developed into a general design method being examined for a wide spectrum of system types including nonlinear systems, multi-input/multi-output systems, discrete-time models, large-scale and infinite-dimensional systems, and stochastic systems. In addition, the objectives of VSC has been greatly extended from stabilization to other control functions. The most distinguished feature of VSC is its ability to result in very robust control systems; in many cases invariant control systems result. Loosely speaking, the term "invariant" means that the system is completely insensitive to parametric uncertainty and external disturbances. Today, research and development continue to apply VSC to a wide variety of engineering systems. In this introductory section, the basic notion of VSC and

brief discussions about its historical development are presented.

A. The Basic Notion of VSC

The basic idea of VSC was originally illustrated by a second-order system [43], [154] similar to the following

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 2y - x + u \\ u &= -\psi x\end{aligned}\quad (1a)$$

where

$$\begin{aligned}\psi &= 4 \quad \text{when } s(x, y) > 0 \\ &= -4 \quad \text{when } s(x, y) < 0\end{aligned}\quad (1b)$$

and

$$s(x, y) = x\sigma, \quad \sigma = 0.5x + y. \quad (1c)$$

A block diagram of the system is shown in Fig. 1(a). The variable $s(x, y)$ in (1.1c) is the product of two functions

$$x = 0 \quad \text{and} \quad \sigma = 0.5x + y = 0. \quad (2)$$

The functions describe lines dividing the phase plane (xy plane) into regions where $s(x, y)$ has different sign as shown in Fig. 1(b). As such, the lines (2) are often called *switching lines* and $s(x, y)$ is called a *switching function*. The lines also define the set of points in the phase plane where $s(x, y) = 0$. This set of points is known as the *switching surface*, despite the fact that the set composed of two lines is not a surface in the strict sense. All of these terms are more carefully defined and used later.

The feedback gain ψ is switched according to (1b), i.e., the sign of $s(x, y)$. Therefore, the system (1a, b) is analytically defined in two regions of the phase plane by two different mathematical models:

In region I where $s(x, y) = x\sigma > 0$, model is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 2y - x - 4x = 2y - 5x\end{aligned}\quad (3)$$

In region II where $s(x, y) = x\sigma < 0$, model is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 2y - x + 4x = 2y + 3x.\end{aligned}\quad (4)$$

The phase plane trajectories for (3) and (4) are shown as portraits in Fig. 2(a) and (b). The equilibrium point of (3) is an unstable focus at the origin. The equilibrium point of (4) is a saddle at the origin; the saddle point is also unstable.

The phase portrait for the system (1a, b) is formed by drawing the portrait for (2) in region I of the phase plane

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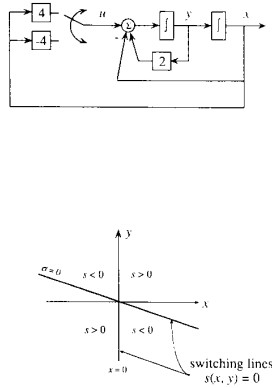


Fig. 1. A simple VSC example. (a) System model. (b) Regions defined by the switching logic.

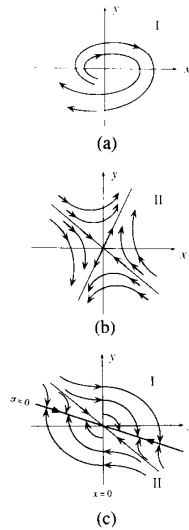


Fig. 2. Phase portraits. (a) For system (1.3). (b) For system (1.4). (c) For system (1.1a, b).

and drawing the portrait for (3) in region II. The resultant portrait is shown in Fig. 2(c). To obtain the complete phase portrait, the trajectory of the system on the set $s(x, y) = 0$ must be described. On the line $x = 0$, the phase trajectories of regions I and II are just joined together without any ambiguity. On the line

$$\sigma = 0.5x + y = 0.5x + \dot{x} = 0 \quad (5)$$

which itself is a dynamical equation, the phase portrait is a trajectory along the switching line $\sigma = 0$ as shown in Fig. 2(c).

The complete phase portrait of the system shows that there are no unusual motion characteristics on the line $x = 0$ other than possible discontinuities on motion direction. However, the line $\sigma = 0$ contains only endpoints of those trajectories coming from both sides of the line. These points constitute a special trajectory along the

$\sigma = 0$ line, representing motion called a *sliding mode*. Thus, a phase trajectory of this system generally consists of two parts, representing two modes of the system. The first part is the *reaching mode*, also called *nonsliding mode*, in which the trajectory starting from anywhere on the phase plane moves toward a switching line and reaches the line in finite time. The second part is the sliding mode in which the trajectory asymptotically tends to the origin of the phase plane, as defined by the differential equation (5). Four basic notions of this example VSC system should be observed:

- 1) Since the origin of the phase plane represents the equilibrium state of the system, the sliding mode represents the behavior of the system during the transient period. In other words, the line that describes $\sigma = 0$ defines the transient response of the system during the sliding mode.
- 2) During the sliding mode, trajectory dynamics (5) are of a lower order than the original model (1).
- 3) During the sliding mode, system dynamics are solely governed by the parameters that describe the line $\sigma = 0$.
- 4) The sliding mode is a trajectory that is not inherent in either of the two structures defined by (2) or (3).

During the control process, the structure of the control system (1a, b) varies from one structure (2) to another (3), thus earning the name *variable structure control*. To emphasize the important role of the sliding mode, the control is also often called *sliding mode control*. It should be noted that a variable structure control system can be devised without a sliding mode, but such a system does not possess the associated merits. In the next section, the general VSC problem is stated.

B. Statement of the VSC Problem

For a given control system represented by the state equation

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\mathbf{u} \quad (6)$$

where $\dim \mathbf{x} = n$ and $\dim \mathbf{u} = m$, find:

- 1) m switching functions, represented in vector form as $\mathbf{s}(\mathbf{x})$, and
- 2) a variable structure control

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}^+(\mathbf{x}, t) \quad \text{when } \mathbf{s}(\mathbf{x}) > 0 \\ &= \mathbf{u}^-(\mathbf{x}, t) \quad \text{when } \mathbf{s}(\mathbf{x}) < 0 \end{aligned}$$

such that the reaching modes satisfy the reaching condition, namely, reach the set $\mathbf{s}(\mathbf{x}) = 0$ (switching surface) in finite time.

The physical meaning of above statement is as follows:

- 1) Design a switching surface $\mathbf{s}(\mathbf{x}) = 0$ to represent a desired system dynamics, which is of lower order than the given plant.
- 2) Design a variable structure control $\mathbf{u}(\mathbf{x}, t)$ such that any state \mathbf{x} outside the switching surface is driven to

reach the surface in finite time. On the switching surface, the sliding mode takes place, following the desired system dynamics. In this way, the overall VSC system is globally asymptotically stable.

C. Brief Theoretical Background

Before the emergence of the early stages of VSC development, its foundation had already been laid. Elements of the foundation consist of the theory of oscillation and the qualitative theory of differential equations. Brief discussions of these elements are given below.

(a) *Phase Plane Method*: As a powerful graphical tool for studying second-order dynamic systems, the phase plane method was well established in the realm encompassing the qualitative (geometric) theory of differential equations and oscillation theory. The classical literature of Andronov *et al.* [6] and Flugge-Lotz [52] cited many early works in these areas. In their outstanding works, two contributions provided the foundation for the emergence of VSC:

- 1) Regionwise linearization of nonlinear dynamic systems in which linearization of nonlinear systems was applied in partitioned regions of the phase plane. This gave the initial prototype VSC systems.
- 2) The sliding mode motion, a term first used by Nikol'ski [111]. This was the first concept of sliding mode control.

(b) *Theory of Differential Equations with a Nonanalytic Right-Hand Side*: Two kinds of nonanalyticity are of importance with respect to VSC:

- 1) Finite discontinuous right-hand side, which is the relay type discontinuity, and
- 2) Double-valued right-hand side, which is the relay type discontinuity with hysteresis.

The problem is that a differential equation is not defined at the point where the right-hand side of the equation is not analytic because the existence and uniqueness of the solutions at these points are not guaranteed. Hence, the phase plane method cannot give a complete solution without defining an auxiliary equation at these points. The auxiliary equation is the model of switching that occurs in VSC systems with discontinuous control. Five methods have been suggested to define the differential equation for the system at points of discontinuous dynamics. They are summarized as follows.

Method 1: First, transform the system model to controllable canonic form. For example, the controllable canonic form for a time-invariant, single-input linear system is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \sum_{i=1}^n -a_i x_i + bu\end{aligned}\quad (7)$$

Then, a function $s(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \cdots + x_n$ defines a surface

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots + x_n = 0 \quad (8)$$

in the n -dimensional phase space. If switching is constrained to occur on this $(n-1)$ -dimensional surface, then all points of discontinuity lie on the surface. By solving (8) for x_n and then substituting the result in (7), the differential equation of the sliding mode is found:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ x_{n-1} &= -c_1 x_1 - c_2 x_2 - \cdots - c_{n-1} x_{n-1}\end{aligned}\quad (9)$$

In summary, the coefficients in the switching function (8) define the characteristic equation of the sliding mode if the system model is described in controllable canonic form.

Method 2: This method obtains the differential equation of sliding mode by means of two coordinate transformations. Consider a linear, time-invariant plant and a switching function described by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ s(\mathbf{x}) &= \mathbf{C}\mathbf{x}\end{aligned}\quad (10)$$

where $\dim \mathbf{x} = n$ and $\dim \mathbf{u} = \dim \mathbf{s} = m$. The dynamics of the sliding mode could be more easily described if the state vector was composed with \mathbf{s} as m of the state variables. Therefore, the objective is to transform the model (10) into a form that contains \mathbf{s} as m of the state variables. The plant is first transformed by $\mathbf{y} = \mathbf{T}_1 \mathbf{x}$ to the form [158]

$$\begin{aligned}\dot{\mathbf{y}}_1 &= \bar{\mathbf{A}}_{11} \mathbf{y}_1 + \bar{\mathbf{A}}_{12} \mathbf{y}_2 \\ \dot{\mathbf{y}}_2 &= \bar{\mathbf{A}}_{21} \mathbf{y}_1 + \bar{\mathbf{A}}_{22} \mathbf{y}_2 + \bar{\mathbf{B}} \mathbf{u}\end{aligned}\quad (11)$$

where $\dim \mathbf{y}_1 = n - m$, $\dim \mathbf{y}_2 = m$, and $\det \bar{\mathbf{B}} \neq 0$. A second transformation

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{s} \end{bmatrix} = \mathbf{T}_2 \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$$

brings (11) to the form [60]

$$\begin{aligned}\dot{\mathbf{y}}_1 &= \tilde{\mathbf{A}}_{11} \mathbf{y}_1 + \tilde{\mathbf{A}}_{12} \mathbf{s} \\ \dot{\mathbf{s}} &= \tilde{\mathbf{A}}_{21} \mathbf{y}_1 + \tilde{\mathbf{A}}_{22} \mathbf{s} + \tilde{\mathbf{B}} \mathbf{u}\end{aligned}\quad (12)$$

When the system is in the sliding mode, the dynamics satisfy $\mathbf{s} = 0$ and $\dot{\mathbf{s}} = 0$. From (12), the differential equation for the sliding mode is easily solved by setting $\mathbf{s} = 0$:

$$\dot{\mathbf{y}}_1 = \tilde{\mathbf{A}}_{11} \mathbf{y}_1. \quad (13)$$

Method 3: Fillipov established a systematic mathematical theory for differential equations with discontinuities [50], [51], [110]. Consider a general plant

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \quad (14)$$

and switching function $s(\mathbf{x})$ where $\dim \mathbf{x} = n$, $\dim \mathbf{u} = m$.

Given that the system is of variable structure, the system dynamics can be described by two structures

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}^+(\mathbf{x}, \mathbf{u}) \quad \text{when } s(\mathbf{x}) > 0 \\ &= \mathbf{f}^-(\mathbf{x}, \mathbf{u}) \quad \text{when } s(\mathbf{x}) < 0. \end{aligned} \quad (15)$$

The system dynamics are not directly defined on $s(\mathbf{x}) = 0$ by (1.15). Instead, Fillipov describes the dynamics on $s(\mathbf{x}) = 0$ as a type of “average” of the two structures in (15)

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) = \mu \mathbf{f}_0^+ + (1 - \mu) \mathbf{f}_0^-, \quad 0 \leq \mu \leq 1 \quad (16)$$

where

$$\mathbf{f}_0^+ = \lim_{s \rightarrow 0} \mathbf{f}^+(\mathbf{x}, \mathbf{u}) \quad \text{and} \quad \mathbf{f}_0^- = \lim_{s \rightarrow 0} \mathbf{f}^-(\mathbf{x}, \mathbf{u})$$

are functions of \mathbf{x} . The term μ is also a function of \mathbf{x} and can be specified in such a way that the “average” dynamic $\mathbf{f}_0(\mathbf{x})$ is tangent to the surface $s(\mathbf{x}) = 0$. The geometric concept is illustrated in Fig. 3.

Method 4: A fourth method for describing the dynamics of the sliding mode is called the *equivalent control approach* [153]. Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \quad (17)$$

with a switching function $s(\mathbf{x})$. The first step of the equivalent control approach is to find the input $\mathbf{u}_e(\mathbf{x})$ such that the state trajectory stays on the switching surface $s(\mathbf{x}) = 0$. Once the equivalent control input is known, the sliding mode dynamics can be described by substituting $\mathbf{u}_e(\mathbf{x})$ in (17).

The equivalent control is found by recognizing that $\dot{s}(\mathbf{x}) = 0$ is a necessary condition for the state trajectory to stay on the switching surface $s(\mathbf{x}) = 0$. Differentiating $s(\mathbf{x})$ with respect to time along the trajectory of (17) gives

$$\dot{s}(\mathbf{x}) = \frac{\partial s}{\partial \mathbf{x}} \mathbf{A}(\mathbf{x}) + \frac{\partial s}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x})\mathbf{u} = 0. \quad (18)$$

Solving (18) for \mathbf{u} yields the equivalent control

$$\mathbf{u}_e(\mathbf{x}) = - \left(\frac{\partial s}{\partial \mathbf{x}} \mathbf{B}(\mathbf{x}) \right)^{-1} \frac{\partial s}{\partial \mathbf{x}} \mathbf{A}(\mathbf{x}) \quad (19)$$

where the existence of the matrix inverse is a necessary condition.

Method 5: The theory of general dynamical system concerns a differential equation with its right-hand side being a set-mapping function; the theory is naturally applicable for describing the dynamics of a VSC system on its switching surface [121]. It has been applied to min-max control, which can be considered a special case of VSC [64], [66].

In all five methods just described, the objective is to find the differential equation of the sliding mode, which is defined only on the switching surface. Methods 1 and 2 are often more simple to use than the latter three, even though their results do not describe the sliding motion in the original state space because some state transformation is used. Methods 3 and 4 establish the sliding mode

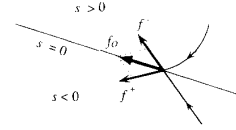


Fig. 3. Fillipov's construction of the equivalent control.

differential equations in the original state space, but they are seldom used in practice. Method 5 is devoted to establishing a more general theoretical foundation.

D. Brief History of VSC

The history of VSC development is marked by three stages of development—the early stage of VSC, the stage of VSC for multi-input linear systems, and the stage of complete VSC development. Each of them is briefly described below.

(a) *The Early Stage of VSC (1957–1970):* VSC systems studied during this period possessed three characteristics. The first characteristic is that each system was modeled by either a high-order, linear differential equation with a single input:

$$x^{(n)}(t) + a_n x^{(n-1)}(t) + \cdots + a_2 \dot{x}(t) + a_1 = bu(t) \quad (20)$$

or by its equivalent state space model in controllable canonic form (7). The second characteristic is that the switching surface was defined in a special quadratic form:

$$s(\mathbf{x}) = x_1(c_1 x_1 + c_2 x_2 + \cdots + c_n x_n). \quad (21)$$

The third characteristic is that the structure of the control was described by

$$\begin{aligned} u &= \psi x_1 \quad \text{where } \psi = \alpha \quad \text{when } s(\mathbf{x}) > 0 \\ &= \beta \quad \text{when } s(\mathbf{x}) < 0. \end{aligned} \quad (22)$$

Equations (20)–(22) describe a single-input, linear system with switched feedback gain. Numerous VSC papers were published in this area and their results were well documented [6], [43], [79], [154]. The issues studied in detail included:

- 1) Existence of the sliding mode
- 2) Stability of the sliding mode
- 3) Systems with time-varying coefficients
- 4) Effects of system parameter perturbations and outside disturbances
- 5) Systems having unmeasurable state variables

The type of systems with the aforementioned three characteristics is quite restrictive. The quadratic nature of the switching function (21) and the structure of the control (22) are sufficient to guarantee a sliding mode response, but other structures exist that are easier to implement and are more flexible in design. In addition, the controllable canonic representation cannot always reveal the general nature the VSC systems.

(b) *The Stage of VSC for Multi-Input Linear Systems (1970–1980):* During this period, the theory of VSC for

general linear systems was more firmly established; the general system is of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (23)$$

where $\dim \mathbf{x} = n$ and $\dim \mathbf{u} = m$. an m -dimensional linear vector switching function $\mathbf{s}(\mathbf{x})$ was postulated for the VSC. The control structure for each of the m inputs was described as

$$\begin{aligned} u_i(\mathbf{x}) &= \psi_i^+(\mathbf{x}) \quad \text{when } s_i(\mathbf{x}) > 0 \\ &= \psi_i^-(\mathbf{x}) \quad \text{when } s_i(\mathbf{x}) < 0 \quad i = 1, 2, \dots, m \end{aligned} \quad (24)$$

and every scalar switching function $s_i(\mathbf{x})$ was linear in the state variables rather than quadratic. However, the established VSC theory did not attract much attention for many practical applications. The reason seems to be twofold. First, VSC theory was overshadowed by the popular linear control system design techniques. Second, the important robustness properties of the VSC system were not yet fully recognized or appreciated.

(b) *The Present Stage of Advancing Development (1980–Present)*: Since 1980, two developments have greatly enhanced the attention given to VSC systems. The first is the existence of a general VSC design method for complex systems. The second is a full recognition of the property of perfect robustness of a VSC system with respect to system perturbation and disturbances. As a result, research and development of VSC methods have been greatly accelerated, both in theory and in applications. The R & D work may be classified into five categories:

- 1) Development for different system models. This includes the development of VSC theory for nonlinear systems, discrete-time systems, systems with time delay, stochastic systems, large-scale systems and infinite-dimensional systems.
- 2) Extension of the objectives of control. The functions of VSC have been extended beyond system stabilization to include motion following or tracking, model following, model reaching, adaptive and optimal control, and state observation.
- 3) Exploration of additional properties of VSC. Such properties include invariance of the sliding mode to system perturbations, robustness of the reaching or nonsliding mode, and the elimination or reduction of control chatter.
- 4) Establishment of VSC laws that possess certain characteristics.
- 5) Applications in various engineering problems.

The purpose of this tutorial paper is to present the fundamental theory and main results for the design of VSC systems. The basic notions and a brief history of VSC development have already been presented. More precise definitions and deeper concepts are presented next by considering VSC for linear plants. Concepts discussed are the design of switching surfaces, characterization of the sliding mode and the reaching or nonsliding mode, control law design, and basic properties of a vari-

able structure system response. The extensions of these concepts to nonlinear systems are then presented. Further extensions of variable structure concepts to other control objectives besides stabilization and regulation are briefly described. Known studies of variable structure control for some special systems are noted. Since application of VSC has been widespread in recent years, some notable contributions are listed. Finally, a very extensive list of references is included. Other survey papers by Utkin [155]–[157], DeCarlo *et al.* [35], and a monograph by Gao [60] are also recommended.

II. VSC FOR LINEAR SYSTEMS

The discussion in this section concerns the general linear time-invariant system represented by the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (25)$$

The state vector \mathbf{x} is n -dimensional and the input \mathbf{u} is m -dimensional. The m column vectors of the \mathbf{B} matrix are designated as \mathbf{b}_i , for $i = 1$ to m .

A. Basic Definitions

Basic terminology of variable structure systems with sliding mode are more carefully defined in this section.

Definition 1: The structure in a VSC system is governed by the sign of a vector-valued function $\mathbf{s}(\mathbf{x})$, which is defined to be the *switching function*. A switching function is generally assumed to be m dimensional and linear, i.e.,

$$\mathbf{s}(\mathbf{x}) = \mathbf{C}\mathbf{x} \quad (26)$$

where

$$\mathbf{s}(\mathbf{x}) = [s_1(\mathbf{x}) \quad s_2(\mathbf{x}) \quad \dots \quad s_m(\mathbf{x})]^T \quad (27)$$

and

$$\mathbf{C} = [c_1^T \quad c_2^T \quad \dots \quad c_m^T]^T \quad (28)$$

Thus

$$s_i(\mathbf{x}) = \mathbf{c}_i^T \mathbf{x}. \quad (29)$$

Each scalar switching function $s_i(\mathbf{x})$ describes a linear surface $s_i(\mathbf{x}) = 0$, which is defined to be a *switching surface*. The term *switching manifold* is often used. In addition, the surface can be called a *switching hyperplane* because the switching function is linear. Notice in the introductory example, however, that the set $s(x, y) = 0$ consisted of two intersecting lines. Such a set is not a manifold in the mathematical sense. Hence, the terms *manifold* and *hyperplane* are avoided in the remainder of this paper.

Let \mathbf{x}_0 be the initial state of the system at the initial time t_0 , $\mathbf{x}(t)$ be the state at any time t , and S be a switching surface that includes the origin $\mathbf{x} = 0$.

Definition 2: If, for any \mathbf{x}_0 in S , we have $\mathbf{x}(t)$ in S for all $t > t_0$, then $\mathbf{x}(t)$ is a *sliding motion* or *sliding mode* of the system.

Definition 3: If every point in S is an end point, that is, for every point in S there are trajectories reaching it from

both sides of S , then switching surface S is called a *sliding surface*.

Definition 4: The condition under which the state will move toward and reach a sliding surface is called a *reaching condition*.

From the definitions above, it is shown next that an n th-order system with m inputs will have $2^m - 1$ switching surfaces.

- 1) Let \mathbf{c}_i^T be the i th row vector of the matrix C . Then $\mathbf{c}_i^T \mathbf{x} = 0$ defines a surface S_i of dimension $(n - 1)$. There are m such surfaces:

$$S_i = \{\mathbf{x} | s_i(\mathbf{x}) = \mathbf{c}_i^T \mathbf{x} = 0\} \quad i = 1, 2, \dots, m. \quad (30)$$

- 2) Consider the intersection of two surfaces S_i and S_j , $i \neq j$. Their intersection is an $(n - 2)$ -dimensional switching surface. The total number of such intersections equals the number of combinations of m surfaces S_i taken two at a time

$$\binom{m}{2} = \frac{m!}{(m-2)!2!} = \frac{m(m-1)}{2}.$$

These switching surfaces are mathematically described by

$$S_{ij} = \{\mathbf{x} | s_i(\mathbf{x}) = \mathbf{c}_i^T \mathbf{x} = 0 \quad \text{and} \quad s_j(\mathbf{x}) = \mathbf{c}_j^T \mathbf{x} = 0\} \\ = S_i \cap S_j \quad i, j = 1, 2, \dots, m, i < j. \quad (31)$$

A geometric interpretation is shown in Fig. 4. The intersection of the two planes, S_1 and S_2 , is the switching surface S_{12} , which is a line.

- 3) Further intersections involving multiple surfaces S_i can be described in the same manner as in (31). For example, the intersection of three surfaces S_i , S_j , and S_k is a switching surface S_{ijk} of dimension $(n - 3)$. There are $\binom{m}{3}$ such surfaces.
- 4) Finally, there is a single switching surface S_E of dimension $(n - m)$, which is the intersection of all surfaces S_i , $i = 1, 2, \dots, m$ taken together. S_E may be called the *eventual switching surface* to which all trajectories must eventually reach. The surface is given by

$$S_E = \{\mathbf{x} | s(\mathbf{x}) = C\mathbf{x} = 0\} = S_1 \cap S_2 \cap \dots \cap S_m. \quad (32)$$

The total number of switching described by 30–32 is $2^m - 1$. On each of the switching surfaces, there may be a sliding mode that is described by a differential equation of the same dimension as the switching surface. Therefore, it is possible to have a total of $2^m - 1$ different sliding modes. The sliding mode associated with S_E may be called the *eventual sliding mode*. This way of defining sliding modes was implied in the hierarchical control scheme of Utkin [154]. However, many authors consider only the motion on S_E as the sliding motion. Different definitions described here are all useful because there are actually many ways in which a sliding motion can begin.

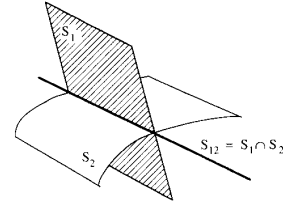


Fig. 4. Geometric interpretation of various switching surfaces.

These are called *switching schemes* and are described in more detail in the next section.

B. Switching Schemes

As discussed earlier, a system having m inputs can have m switching functions and up to $2^m - 1$ sliding surfaces. Although some authors consider dynamics constrained to the eventual sliding surface S_E to be the sliding mode, the sliding mode can actually begin in a number of different ways, hereafter referred to as a *switching scheme*. The number of switching schemes that exist depends on the order of entering different sliding modes.

(a) *Fixed-Order Switching Scheme:* In this scheme, sliding modes take place in a preassigned order as the system state traverses the state space. For example, the state can first move from the initial state \mathbf{x}_0 onto the switching surface S_1 , which has dimension $n - 1$. The sliding mode can then move to the surface $S_{12} = (S_1 \cap S_2)$, which has dimension $n - 2$. Sliding moves to progressively lower dimensional sliding surfaces and eventually reaches the surface S_E , which has dimension $n - m$:

$$\mathbf{x}_0 \rightarrow S_1 \rightarrow (S_1 \cap S_2) \rightarrow (S_1 \cap S_2 \cap S_3) \rightarrow \dots \rightarrow S_E.$$

This early scheme has been called the hierarchical VSC scheme [154]. The scheme has several weaknesses, i.e., giving slow and poor transient response in general, resulting in large magnitude for control effort, and great difficulty in its solution.

(b) *Free-Order Switching Scheme:* Here, the order of sliding modes is not preassigned but follows the natural trajectory on a first-reach-first-switch scheme. The switching takes place depending on the location of the initial state in the state space. This scheme [31], [61] is more plausible than the fixed-order scheme for three reasons. First, using this the solution of VSC is easy to determine. Second, the reaching mode possesses better dynamic characteristics. Finally, the resulting control effort is smaller in magnitude so saturation is less likely to occur.

(c) *Eventual Sliding Mode Switching Scheme:* In this scheme the state is driven from any initial state to the eventual sliding surface S_E on which the sliding mode control takes place. There may or may not be sliding modes on other switching surfaces. This scheme is simple in implementation and its control is easy to be made smooth [35], [155]. However, the scheme does not guarantee good transient characteristics.

(d) *Decentralized Switching Scheme:* The system is treated as m single-input subsystems, each having a scalar

switching function and its associated sliding mode. The subsystems are coupled in general. The combined vector switching function has the form

$$\mathbf{s}(\mathbf{x}) = [s_1(\mathbf{x}_1), \dots, s_m(\mathbf{x}_m)]$$

$$s_i(\mathbf{x}_i) = \mathbf{c}_i^T \mathbf{x}_i, \quad i = 1 \text{ to } m$$

where \mathbf{x}_i and \mathbf{c}_i are n_i -dimensional vectors with

$$\sum_{i=1}^m n_i = n.$$

The scheme is intended for large-scale systems and gives good results [88], [143].

In general, the free-order switching scheme is the best among all schemes. But, for large scale systems, the decentralized switching scheme is preferred.

C. Reaching Conditions and the Reaching Mode

The condition under which the state will move toward and reach a sliding surface is called a *reaching condition*. The system trajectory under the reaching condition is called the *reaching mode* or reaching phase. Three approaches for specifying the reaching condition have been proposed.

(a) *The Direct Switching Function Approach*: The earliest reaching condition proposed [43], [154] was

$$\dot{s}_i > 0, \quad \text{when } s_i < 0$$

$$\dot{s}_i < 0, \quad \text{when } s_i > 0 \quad i = 1, \dots, m$$

or, equivalently,

$$s_i \dot{s}_i < 0 \quad i = 1, \dots, m. \quad (33)$$

This reaching condition is global but does not guarantee a finite reaching time. A similar sufficient condition that is local in nature was proposed by Utkin [155]

$$\lim_{s_i \rightarrow 0^+} \dot{s}_i < 0 \quad \text{and} \quad \lim_{s_i \rightarrow 0^-} \dot{s}_i > 0. \quad (34)$$

It will be shown in Section II-D-(a) that (33) is very difficult to use for the multiple-input VSC. Even with a simplifying assumption, such as adopting fixed-order switching, the approach remains difficult.

(b) *The Lyapunov Function Approach*: By choosing the Lyapunov function candidate

$$V(\mathbf{x}, t) = \mathbf{s}^T \mathbf{s} \quad (35)$$

a global reaching condition is given by [79]

$$\dot{V}(\mathbf{x}, t) < 0 \quad \text{when } \mathbf{s} \neq 0. \quad (36)$$

Finite reaching time is guaranteed by modifying (36) to

$$\dot{V}(\mathbf{x}, t) < -\epsilon \quad \text{when } \mathbf{s} \neq 0, \quad \text{where } \epsilon \text{ is positive.} \quad (37)$$

Clearly, this approach leads to the eventual sliding mode switch scheme.

(c) *The Reaching Law Approach*: The crux of the reaching law approach is a new method called the *reaching law*

method. It directly specifies the dynamics of the switching function. Let the dynamics of the switching function be specified by the differential equation

$$\dot{\mathbf{s}} = -Q \mathbf{sgn}(\mathbf{s}) - K \mathbf{f}(\mathbf{s}) \quad (38)$$

where gains Q and K are diagonal matrices with positive elements, and

$$\mathbf{sgn}(\mathbf{s}) = [\text{sgn}(s_1) \cdots \text{sgn}(s_m)]^T$$

$$\mathbf{f}(\mathbf{s}) = [f_1(s_1) \cdots f_m(s_m)]^T.$$

The scalar functions f_i satisfy the condition

$$s_i f_i(s_i) > 0 \quad \text{when} \quad s_i \neq 0, \quad i = 1 \text{ to } m.$$

Equation (38) is called the *reaching law*. Various choices of Q and K specify different rates for \mathbf{s} and yield different structures in the reaching law. Three examples are

1) The constant rate reaching law

$$\dot{\mathbf{s}} = -Q \mathbf{sgn}(\mathbf{s}) \quad (39)$$

2) The constant plus proportional rate reaching law

$$\dot{\mathbf{s}} = -Q \mathbf{sgn}(\mathbf{s}) - K \mathbf{s} \quad (40)$$

3) The power rate reaching law

$$\dot{s}_i = -k_i |s_i|^\alpha \text{sgn}(s_i) \quad 0 < \alpha < 1 \quad i = 1 \text{ to } m. \quad (41)$$

The reaching law approach not only establishes the reaching condition but also specifies the dynamic characteristics of the system during the reaching phase [61]. Additional merits of this approach include simplification of the solution for VSC and providing a measure for the reduction of chattering [63]. This approach results in the free-order switching scheme.

D. The Control Law

Design of the VSC control law is affected by two factors: 1) the choice of a sliding mode entering scheme, and 2) whether or not the *structure* of the control law has been prespecified. In this section, various design approaches are presented. The influence of the sliding mode entering scheme is discussed first. Consider the model represented by (25)–(29) for the following discussion.

(a) *Effect of the Switching Scheme*: Solutions for the VSC law are straightforward in general, except for the fixed-order sliding mode scheme.

(i) *Fixed-order switching scheme*: For the fixed-order scheme, the determination of the control u involves the solution of m pairs of inequalities [154, 174].

$$\dot{s}_i = \frac{\partial s_i}{\partial \mathbf{x}} (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}) = \begin{cases} > 0, & \text{when } s_i < 0 \\ < 0, & \text{when } s_i > 0 \end{cases}$$

for $i = 1, \dots, m$. (42)

Let \mathbf{b}_i be the i th column vector of the \mathbf{B} matrix. Develop-

ing (2.18) and using the notations of (2.1) to (2.5), gives

$$\left. \begin{aligned} \mathbf{c}_1^T \mathbf{A} \mathbf{x} + \mathbf{c}_1^T \mathbf{b}_1 u_1 + \cdots + \mathbf{c}_1^T \mathbf{b}_m u_m & \begin{cases} < 0 & \text{when } s_1 > 0 \\ > 0 & \text{when } s_1 < 0 \end{cases} \\ \vdots \\ \mathbf{c}_m^T \mathbf{A} \mathbf{x} + \mathbf{c}_m^T \mathbf{b}_1 u_1 + \cdots + \mathbf{c}_m^T \mathbf{b}_m u_m & \begin{cases} < 0 & \text{when } s_m > 0 \\ > 0 & \text{when } s_m < 0 \end{cases} \end{aligned} \right\} \quad (43)$$

Thus (42) consists of $2m$ conditional inequalities for $2m$ unknowns. The $2m$ unknowns are control signals $u_i(\mathbf{x})$, $i = 1$ to m , and each has two values for two different conditions. Solving (42) for them is a very difficult task. The fixed-order switching is helpful in concept but difficult in realization. In addition, its solution tends to result in a very conservative VSC, meaning that the magnitudes of $u_i(\mathbf{x})$ are often unnecessarily large. As a result, the scheme is seldom used.

(ii) *Free-order switching scheme*: For the free-order switching scheme, the control is solved directly from the reaching law specification described by (38)

$$\dot{\mathbf{s}} = \frac{\partial \mathbf{s}}{\partial \mathbf{x}} (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}) = -\mathbf{Q} \operatorname{sgn}(\mathbf{s}) - \mathbf{K} \mathbf{f}(\mathbf{s}). \quad (44)$$

The VSC is easily solved from (44) as

$$\mathbf{u}(\mathbf{x}) = -\left(\frac{\partial \mathbf{s}}{\partial \mathbf{x}} \mathbf{B}\right)^{-1} \left[\frac{\partial \mathbf{s}}{\partial \mathbf{x}} \mathbf{A} \mathbf{x} + \mathbf{Q} \operatorname{sgn}(\mathbf{s}) + \mathbf{K} \mathbf{f}(\mathbf{s}) \right] \quad (45)$$

(iii) *Eventual sliding mode switching scheme*: For the eventual sliding mode switching scheme, the control is designed to satisfy a Lyapunov stability criteria

$$\dot{V} = \frac{d}{dt} (\mathbf{s}^T \mathbf{s}) = 2\mathbf{s}^T \dot{\mathbf{s}} = 2\mathbf{s}^T \frac{\partial \mathbf{s}}{\partial \mathbf{x}} (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}) > 0. \quad (46)$$

The solution for VSC is more involved than that for the free-order switching scheme but is probably easier than that for the fixed-order switching.

(iv) *Decentralized switching scheme*: For the decentralized switching scheme, the VSC controls of all the subsystems are separately designed. Each subsystem control can be obtained independently using any one of the above three approaches. Consider an n th-order large-scale system with m single-input subsystems

$$\dot{\mathbf{x}}_i = \mathbf{A}_i \mathbf{x}_i + \mathbf{B}_i u_i + \sum_{\substack{j=1 \\ j \neq i}}^m \mathbf{A}_{ij} \mathbf{x}_j \quad i = 1, \dots, m. \quad (47)$$

Here $\dim \mathbf{x}_i = n_i$, and $n_1 + n_2 + \cdots + n_m = n$. For the i th subsystem, \mathbf{A}_i represents dynamics internal to the subsystem, \mathbf{B}_i is the input vector, and \mathbf{A}_{ij} represents coupling to other subsystems. Let there be m switching functions (one associated with each subsystem)

$$s_i(\mathbf{x}_i) = \mathbf{C}_i^T \mathbf{x}_i.$$

The overall system sliding mode consists of m independent sliding modes, each moving on its own sliding surface

$s_i(\mathbf{x}_i) = 0$ such that $\mathbf{x}_i \rightarrow 0$. This scheme is intended for large-scale systems, for which the result is generally good.

(b) *Properties of VSC Structures*: The design of VSC can proceed with the structure of the control $\mathbf{u}(\mathbf{x})$ free or preassigned at the outset. In either case, the objective is to satisfy a reaching condition. In the free structure approach, generally, the control $\mathbf{u}(\mathbf{x})$ can be solved by constraining the switching function to any one of the following conditions.

- (i) $s_i \dot{s}_i < 0$ (Direct switching approach)
- (ii) $\dot{V} = \frac{d}{dt} (\mathbf{s}^T \mathbf{s}) < 0$ (Lyapunov function approach)
- (iii) $\dot{s}_i = -q_i \operatorname{sgn}(s_i) - k_i f_i(s_i)$ (Reaching law approach).

Additional forms of constraint exist, including

- (iv) $\dot{V} = -q - kV \quad V = \mathbf{s}^T \mathbf{s}$
- (v) $\frac{d}{dt} s_i^3 \leq -\eta |s_i|$ [156]
- (vi) $\dot{s} = -\mathbf{F}(\mathbf{s})$ [49]

The latter forms are seldom used in practice.

In some cases, it is convenient to preassign the structure of the VSC and then determine the values of the controller gains such that the desired reaching law is satisfied. Three popular types of preassigned structures are given here.

1) *Relay control*: Here, the VSC for each element of the control vector \mathbf{u} takes the form of a relay. The relay gain may be either fixed or state dependent

$$\begin{aligned} u_i(\mathbf{x}) &= k_i^+, \quad \text{when } s_i(\mathbf{x}) > 0 \\ &= k_i^-, \quad \text{when } s_i(\mathbf{x}) < 0 \quad i = 1, \dots, m. \end{aligned} \quad (48)$$

The values of k_i^+ and k_i^- are chosen to satisfy the desired reaching condition.

2) *Linear feedback with switched gains*: The preassigned structure is

$$\mathbf{u}(\mathbf{x}) = \psi(\mathbf{x}) \mathbf{x} \quad (49)$$

where $\psi = [\psi_{ij}(\mathbf{x})]$ is an $m \times n$ matrix of state-dependent gains. A popular structure for the gains is

$$\begin{aligned} \psi_{ij}(\mathbf{x}) &= \alpha_{ij}, \quad \text{when } s_i(\mathbf{x}) x_j > 0 \\ &= \beta_{ij}, \quad \text{when } s_i(\mathbf{x}) x_j < 0 \quad \text{for } \begin{cases} i = 1, \dots, m \\ j = 1, \dots, n. \end{cases} \end{aligned} \quad (50)$$

Parameters α_{ij} and β_{ij} are chosen to satisfy the desired reaching condition. The details of (50) can be varied to suit the problem at hand. For example, in the VSC of model following systems, it is easier to guarantee the reaching condition $s_i s_i < 0$, $i = 1$ to m , by giving \mathbf{u} a

preassigned structure such as

$$\mathbf{u} = \Psi \mathbf{x} + \Phi \mathbf{e} + \theta \quad (51)$$

where $\mathbf{e} = \mathbf{x} - \mathbf{x}^d$ is the tracking error, \mathbf{x}^d is the reference state vector, and θ is a vector. In fact, this approach can be included in the more general technique of the *reaching law method* and be handled in a much simpler way.

3) *Augmenting the equivalent control* [35]: Here, the VSC control takes the form

$$\mathbf{u} = \mathbf{u}_e + \Delta \mathbf{u} \quad (52)$$

where \mathbf{u}_e is the equivalent sliding mode control (1.27), whereas $\Delta \mathbf{u}$ is added to satisfy the reaching condition. A commonly used form for $\Delta \mathbf{u}$ is a relay control (48).

III. VSC FOR NONLINEAR SYSTEMS

The most general state equation for nonlinear systems is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \quad (53)$$

where $\dim \mathbf{x} = n$ and $\dim \mathbf{u} = m$. The most commonly used special form of (53) describes systems that are linear in the input \mathbf{u} . That is,

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\mathbf{u}. \quad (54)$$

Note that some systems that are not linear in the input \mathbf{u} can still be put in the form of (54) by using an invertible input transformation. An input-output model of the system is represented by appending an m -dimensional output function

$$\mathbf{y} = \mathbf{c}(\mathbf{x}, \mathbf{u}, t) \quad (55)$$

to (54). The nonlinear systems considered in this paper will have the same numbers of outputs and inputs. Time-invariant systems are modeled by omitting the variable t in (54)–(55).

Three points are stressed at the outset for nonlinear systems:

- 1) Basic concepts and the fundamental theory of VSC are similar to those for linear systems.
- 2) The derivation of a control $\mathbf{u}(\mathbf{x})$ is similarly simple and straightforward.
- 3) But the analysis of the sliding mode and the search for the corresponding switching function become a more difficult problem.

To study the stability of sliding modes occurring in nonlinear systems, various state transformations are used to put the differential equations of the system in one of several possible canonical forms. Reaching laws are generally tailored to take advantage of canonical form characteristics.

A. Canonical Forms

1) *Reduced Form*: The state vector \mathbf{x} is partitioned into \mathbf{x}_1 and \mathbf{x}_2 , where $\dim \mathbf{x}_2 = \dim \mathbf{u} = m$ and $\dim \mathbf{x}_1 = n - m$. In addition, the input matrix takes the special form

$\mathbf{B} = [0 \ B^*]^T$, where B^* is an invertible $m \times m$ matrix. The reduced form of the system model takes the form

$$\dot{\mathbf{x}}_1 = \mathbf{A}_1(\mathbf{x}) \quad \dim \mathbf{x}_1 = n - m \quad (56a)$$

$$\dot{\mathbf{x}}_2 = \mathbf{A}_2(\mathbf{x}) + B^*(\mathbf{x})\mathbf{u} \quad \dim \mathbf{x}_2 = \dim \mathbf{u} = m \quad (56b)$$

Necessary and sufficient conditions for the existence of the reduced form have been established in [10].

2) *Controllability Form*: Given m inputs, the system model is partitioned into m subsystems, each of which is represented in controllable canonic form [78]. More explicitly, the state vector is in the form

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T \\ \vdots \\ \mathbf{x}_m^T \end{bmatrix}, \quad \dim \mathbf{x}_i = n_i \quad \text{and} \quad \sum_{i=1}^m n_i = n. \quad (57a)$$

Dynamics of each subsystem are represented in controllable canonic form

$$\dot{\mathbf{x}}_i = \mathbf{A}_i \mathbf{x}_i + \alpha_i(\mathbf{x}) + \beta_i(\mathbf{x})\mathbf{u} \quad i = 1, \dots, m. \quad (57b)$$

where

$$\mathbf{A}_i = \begin{bmatrix} 0 & \mathbf{I}_{n_i-1} \\ 0 & 0 \end{bmatrix} \quad \dim \mathbf{A}_i = n_i \times n_i$$

$$\alpha_i(\mathbf{x}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_{i0}(\mathbf{x}) \end{bmatrix} \quad \dim \alpha_i = n_i$$

$$\beta_i(\mathbf{x}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta_{i0}(\mathbf{x}) \end{bmatrix} \quad \dim \beta_i = n_i \times m.$$

Finally, the overall system dynamics are given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \alpha(\mathbf{x}) + \beta(\mathbf{x})\mathbf{u} \quad (58)$$

where

$$\mathbf{A} = \text{diag}[\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m]$$

$$\alpha(\mathbf{x}) = \begin{bmatrix} \alpha_1(\mathbf{x}) \\ \vdots \\ \alpha_m(\mathbf{x}) \end{bmatrix} \quad \beta(\mathbf{x}) = \begin{bmatrix} \beta_1(\mathbf{x}) \\ \vdots \\ \beta_m(\mathbf{x}) \end{bmatrix}.$$

Note that there can be coupling between the m subsystems (57b).

3) *The Input-Output Decoupled Form*: Consider a single input/single output (SISO) nonlinear system with output $y = c(\mathbf{x}, u)$. Let $y^{(r)}$ denote the r th differentiation of y with respect to time. The number of differentiations of the output function $y = c(\mathbf{x}, u)$ required for the input u to appear explicitly is called the *relative degree* r of the system. Note that the relative degree of a system cannot be greater than the system order n . This concept of relative degree for a SISO nonlinear system is compatible with the definition of relative degree for linear systems, i.e., the excess of poles over zeros.

Relative degree in a MIMO system is defined in a

similar fashion. For a given output y_i , define the relative degree r_i to be the smallest integer such that at *least one* of the inputs appears explicitly in a differentiation. Since there are m outputs in the system, there are m such integers r_i . The relative degree is denoted by the set (r_1, \dots, r_m) and the *total relative degree* is defined by $r = r_1 + \dots + r_m$.

If the total relative degree r equals the system order n , then there are conditions such that the system

$$\begin{aligned}\dot{\mathbf{x}} &= A(\mathbf{x}) + B(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{c}(\mathbf{x})\end{aligned}\quad (59)$$

can be represented as a set of m decoupled higher-order differential equations of the form

$$\begin{aligned}y_i^{(r_i)} &= f_i(y_1, \dots, y_1^{(r_1-1)}; \dots; y_n, \dots, y_m^{(r_m-1)}) \\ &\quad + g_i(y_1, \dots, y_1^{(r_1-1)}; \dots; y_m, \dots, y_m^{(r_m-1)})u_i \\ i &= 1, \dots, m.\end{aligned}\quad (60)$$

Necessary and sufficient conditions for this transformation have been established in [54], [78]. When the total relative degree $r < n$, then the system can be represented in the *normal form*, which is presented next.

4) *Normal Form*: Under certain mild restrictions, a system model (54) that has total relative degree $r < n$ can be transformed into the normal form using differential geometry methods [24], [78]. The state transformation is divided into two groups of scalar functions. The first group of functions $z_{i,j}$ are simply the m output functions $y_i = c_i(\mathbf{x}, \mathbf{u})$ and their derivatives up through order $j = r_i - 1$, where r_i is the relative degree of the i th output. The total number of state transformation functions in this first group equals $r = r_1 + \dots + r_m$. The new state vector is completed by finding $n - r$ additional state transformations η_k , $k = 1, \dots, n - r$, which are independent of each other and the functions $z_{i,j}$. Since the variables $z_{i,j}$ are associated with the outputs, they can be considered the “external” dynamic variable. The functions η_k are interpreted as being “internal” variables. The system dynamics are represented in terms of $z_{i,j}$ and $\eta = [\eta_1, \dots, \eta_{n-r}]^T$ as follows.

$$\begin{aligned}\dot{z}_{i,j} &= z_{i,j+1}, \quad j = 1, \dots, r_i - 1 \\ &\vdots \\ \dot{z}_{i,r_i} &= \alpha_i(\mathbf{z}, \eta) + \sum_{k=1}^m \beta_{i,k}(\mathbf{z}, \eta)u_k \quad i = 1, \dots, m \\ \dot{\eta} &= \gamma(\mathbf{z}, \eta) \quad \dim \eta = n - r.\end{aligned}\quad (61)$$

In (61), α_i and $\beta_{i,k}$ are scalar-valued functions, and γ is a vector-valued nonlinear function. The reduced form, controllability canonic form, and the input-output decoupled form are all special cases of the normal form.

B. VSC for Canonical Forms

1) *VSC for the Reduced Form*: Let the state vector be partitioned according to (56). Consider a general m -di-

mensional switching surface equation

$$\mathbf{s}(\mathbf{x}) = \mathbf{s}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{0} \quad (62)$$

Theoretically, this equation can be solved for \mathbf{x}_2 in terms of \mathbf{x}_1 in the form of $\mathbf{x}_2 = \mathbf{w}(\mathbf{x}_1)$. This is equivalent to expressing the switching function as

$$\mathbf{s}(\mathbf{x}) = \mathbf{x}_2 - \mathbf{w}(\mathbf{x}_1). \quad (63)$$

Thus the problem of determining the switch function $\mathbf{s}(\mathbf{x})$ is to find $\mathbf{w}(\mathbf{x}_1)$ such that the sliding mode is asymptotically stable. That is, the system (56a), with $\mathbf{x}_2 = \mathbf{w}(\mathbf{x}_1)$ is asymptotically stable. In general, it is difficult to find a desired $\mathbf{w}(\mathbf{x}_1)$ unless matrix A_1 in (56) is of a certain special structure. At any rate, the dimension of \mathbf{x}_1 is $n - m$, less than that of the original system. Once $\mathbf{w}(\mathbf{x}_1)$ is found, so is $\mathbf{s}(\mathbf{x})$. Then the VSC $\mathbf{u}(\mathbf{x})$ may be designed using the *reaching law method* in a way similar to that for linear systems.

Using the reaching law (38) in the system (56) yields the VSC signal

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= - \left(\frac{\partial \mathbf{s}}{\partial \mathbf{x}_2} B^*(\mathbf{x}) \right)^{-1} \left(Q \mathbf{sgn}(\mathbf{s}) + k \mathbf{f}(\mathbf{s}) \right. \\ &\quad \left. + \frac{\partial \mathbf{s}}{\partial \mathbf{x}_1} A_1(\mathbf{x}) + \frac{\partial \mathbf{s}}{\partial \mathbf{x}_2} A_2(\mathbf{x}) \right).\end{aligned}\quad (64)$$

2) *VSC for the Controllability Form*: In this form, the entire system is decomposed into m subsystems, so it is convenient to use a decentralized sliding mode scheme with decoupled switching functions, such as

$$s_i = \mathbf{c}_i^T \mathbf{x}_i \quad i = 1, \dots, m. \quad (65)$$

The stability of the sliding mode of each subsystem is guaranteed if \mathbf{c}_i^T is appropriately chosen. From (57), one sees that \mathbf{x}_i is a substate vector in the phase variable form and the equations of the subsystem are

$$\begin{aligned}\dot{x}_{i1} &= x_{i2} \\ \dot{x}_{i2} &= x_{i3} \\ &\vdots \\ \dot{x}_i(n_i + 1) &= x_{in_i} \\ \dot{x}_{in_i} &= \alpha_i(\mathbf{x}) + \beta_i u_i \quad i = 1, \dots, m.\end{aligned}\quad (66)$$

Therefore the switch plane $\mathbf{c}_i^T \mathbf{x}_i = 0$ becomes

$$x_{i1}^{(n_i)} + c_{i1} x_{i1}^{(n_i-1)} + \dots + c_{i(n_i-1)} \dot{x}_i + c_{in_i} x_i = 0. \quad (67)$$

The VSC can be obtained from (57) and (65) using the reaching law method.

Since each subsystem is in controllable canonic form, the dynamics of the sliding modes are easily determined. Recall that $\dim \mathbf{x}_i = n_i$. On the sliding surface, the n_i th component of the state vector \mathbf{x}_i can be represented in terms of the remaining components by solving (63), $s_i = 0$. Substituting the result in (57b) give the dynamics of the sliding mode s_i . The model is in controllable canonic form

with order $n_i - 1$, and the characteristic polynomial is specified by the coefficients of the vector \mathbf{c}_i . Therefore, stability of each sliding mode is guaranteed by choosing the elements of \mathbf{c}_i to match a desired characteristic equation.

3) *VSC for the Input-Output Decoupled Form*: This case is very similar to the controllability form case, with the only difference in the meaning of the transformed state variables. Variable structure control issues remain unchanged. The case was studied in [143], [148], where the switching function used took a special form

$$s_i(\mathbf{x}_i) = (p + \lambda)^{n_i-1} x_i \quad p \equiv \frac{d}{dt}(\cdot), \quad \lambda > 0.$$

It should be pointed out that it is very advisable to apply the centralized VSC where a set of subsystem switching function is given by

$$s_i(\mathbf{x}_i) = \mathbf{c}_i^T \mathbf{x}_i \quad i = 1 \text{ to } m.$$

4) *VSC for the Normal Form*: The known results for variable structure control of a system in normal form (61) assume stability of the zero dynamics. Zero dynamics are dynamics of the system under the condition that outputs and their derivatives equal zero. Therefore, the zero dynamics are described by the internal dynamic variables with $\mathbf{z} = 0$:

$$\dot{\eta} = \gamma(0, \eta).$$

Then, the switching function for a system in the normal form can be chosen as

$$s_i = \mathbf{c}_i^T \mathbf{z}_i \quad i = 1, \dots, m.$$

Among the literature treating the normal form can be found the case of $r = n$ (total relative degree = system order) [49], the single-input case [138], and the multiple-input case [55], [90], [98]. The main issue in this approach is how to guarantee the stability of the zero dynamics $\dot{\eta} = \gamma(0, \eta)$ and how to relate to the formulation of the output vector function $\mathbf{y} = \mathbf{c}(\mathbf{x})$.

5) *VSC for More General Models*: As stated earlier, the most general nonlinear model takes the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$. Works in this area exist [13], [14], [135] and deal with some basic concepts, such as the definition of the sliding mode and the possibility of certain approximations in VSC systems. However, many issues remain unexplored.

In summary, the design of a VSC system requires the determination of appropriate switching functions and ensuring stability of the sliding modes. Up to now, these steps are reasonable only when the system is in one of the canonical forms discussed above. This often means that only special cases have been treated. Many problems still remain open and several important issues will be raised in the following discussion about the basic properties of variable structure control.

IV. PROPERTIES OF VSC

The complete response of a VSC system consists of three phases of different modes: 1) the reaching mode, 2)

the sliding mode, and 3) the steady-state mode. The "steady state" of a system is here defined to be the final periodic state of the system, which includes the constant state as a special case. Different forms of steady state include zero-error state, constant offset state, and limit-cycle state.

Since finite delays are inherent in all physical actuators, the sliding and steady-state modes are always accompanied by objectionable chattering. The performance of a VSC system is therefore measured by its response characteristics, robustness, and severity of chattering.

A. Characteristics of the Reaching Mode

To better understand the reaching mode dynamics in a VSC system, a reaching space representation has been proposed [61]. Consider an n th-order system with m inputs and a switching function \mathbf{s} , where $\dim \mathbf{s} = m$. Define an m -dimensional *reaching space* R , whose coordinates are the m scalar functions s_i . Thus, a reaching law differential equation such as

$$\dot{\mathbf{s}} = -Q \operatorname{sgn}(\mathbf{s}) - K \mathbf{f}(\mathbf{s}) \quad (68)$$

represents the dynamics of the reaching mode in the reaching space rather than in the original state space coordinates. Given $\mathbf{f}(\mathbf{s})$, (68) can be integrated to yield a solution $\mathbf{s}(t)$, which describes a unique trajectory in R . This trajectory is completely determined by the reaching law's design and it yields certain important information about the reaching mode. For example, the reaching time T from the initial state \mathbf{s}_0 to the sliding surface $\mathbf{s} = 0$, is more easily determined in reaching space coordinates. Further discussion on reaching space can be found in Gao and Hung in this issue [63].

B. Characteristics of the Sliding Mode

The behavior of the sliding mode for linear plants has been discussed in Section II. One of the main results is that a desired sliding mode dynamics can be achieved by an appropriate design of the switching function. In this section, another characteristic of the sliding mode is considered, namely, robustness or insensitivity to certain modeling errors and disturbances.

Robustness is one of the most distinguishing properties of VSC systems. For a plant represented by either a linear or nonlinear high-order differential equation, the differential equation of the sliding mode can be entirely independent of effects due to modeling error and external disturbances. Thus, the sliding mode is said to be *invariant* (better than just robust) to modeling error and disturbances. The invariance property requires that certain *matching conditions* be satisfied; these are examined in more detail next.

Consider a general linear system of the form

$$\dot{\mathbf{x}} = (\mathbf{A} + \Delta \mathbf{A})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{f}(t) \quad (69)$$

where $\Delta \mathbf{A}$ and $\mathbf{f}(t)$ represent the modeling error and external disturbance, respectively. If there exist $\Delta \tilde{\mathbf{A}}$ and

$\tilde{\mathbf{f}}(t)$ such that matching conditions [28], [39], [42]

$$\Delta A = B \Delta \tilde{A} \quad \text{and} \quad \mathbf{f}(t) = B \Delta \tilde{\mathbf{f}} \quad (70)$$

are satisfied, then the sliding mode is invariant. The physical meaning of (70) is that all modeling uncertainties and disturbances enter the system through the control channel.

The same result has been extended to nonlinear systems [133]

$$\dot{\mathbf{x}} = A(\mathbf{x}) + \Delta A(\mathbf{x}) + B(\mathbf{x})\mathbf{u} + \mathbf{f}(\mathbf{x}, t). \quad (71a)$$

In fact, for the more general case where $B(\mathbf{x})$ is also perturbed

$$\dot{\mathbf{x}} = A(\mathbf{x}, t) + \Delta A(\mathbf{x}, \mathbf{p}, t) + B(\mathbf{x})\mathbf{u} + \Delta B(\mathbf{x}, \mathbf{p}, t)\mathbf{u} + \mathbf{f}(\mathbf{x}, \mathbf{p}, t) \quad (71b)$$

where \mathbf{p} is an uncertain parameter vector, it has been shown [63] that invariance hold true if the following matching conditions are satisfied

$$\begin{aligned} \Delta A(\mathbf{x}, \mathbf{p}, t) &= B(\mathbf{x}, t) \Delta \tilde{A}(\mathbf{x}, \mathbf{p}, t) \\ \Delta B(\mathbf{x}, \mathbf{p}, t) &= B(\mathbf{x}, t) \Delta \tilde{B}(\mathbf{x}, \mathbf{p}, t) \\ \mathbf{f}(\mathbf{x}, \mathbf{p}, t) &= B(\mathbf{x}, t) \Delta \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{p}, t) \end{aligned} \quad (72)$$

for certain $\Delta \tilde{A}$, $\Delta \tilde{B}$, and $\tilde{\mathbf{f}}$.

C. Chattering in Sliding and Steady-State Modes

One of the underlying assumptions in the design and analysis of VSC systems is that the control can be switched from one value to another at will, infinitely fast. In practical systems, however, it is impossible to achieve the high switching control that is necessary to most VSC designs. There are several reasons for this. One cause is the presence of finite time delays for control computation. The second cause is the limitations of physical actuators. An example can be found in dc servomotor control design, where it is often assumed that the plant input is current. Because of winding inductance, it is impossible to switch current at an infinitely fast rate. Since it is impossible to switch the control at infinite rate, chattering always occurs in the sliding and steady-state modes of a VSC system. In the steady state, chattering appears as a high-frequency oscillation about the desired equilibrium point and may also serve as a source to excite the unmodeled high-frequency dynamics of the system [6], [52], [91], [147]. Since chattering is almost always objectionable, significant research effort has been directed at eliminating or reducing its effects. Several methods are described below.

1) *The Continuation Approach*: In many VSC designs, the control contains terms that are relaylike in nature, as shown in Fig. 6(a). The ideal relay characteristic is practically impossible to implement, so one approach to reducing the chatter is to replace relay control by a saturating, continuous approximation as shown in Fig. 6(b) [143], [148]. In state space, a *boundary layer* around the switching surface is introduced. Within this boundary layer, the control is chosen to be a continuous approximation of the

switching function. An interpretation is that a high-gain control is used near the surface. A consequence of the continuation method is that invariance is lost. The system possesses robustness that is a function of the boundary layer width. Another continuation method [3], [22] replaces a min-max type control

$$u = \frac{\mathbf{c}^T \mathbf{x}}{|\mathbf{c}^T \mathbf{x}|} \quad \text{by} \quad u = \frac{\mathbf{c}^T \mathbf{x}}{|\mathbf{c}^T \mathbf{x}| + \delta}$$

where the positive constant δ makes the control u continuous.

The general effects of continuation and the effects of delays in actual implementation of a variable structure control are now described for the four common types of switching characteristics shown in Fig. 5. The reader may find it helpful to consider a simple second-order system and switching surface such as

$$\begin{aligned} \ddot{x} + \alpha \dot{x} + \beta x &= -ku(s) \\ s &= \dot{x} + cx \end{aligned} \quad (73)$$

In the phase plane, the sliding surface equation describes a straight line passing through the origin with slope $-c$.

The ideal relay control (Fig. 5(a)): Here the control takes the form

$$\begin{aligned} u(s) = \text{sgn}(s) &= +1 \quad \text{when } s > 0 \\ &= -1 \quad \text{when } s < 0 \end{aligned}$$

The control is ideal in the sense that it switches instantly at the value $s = 0$. For this case it is easy to get the exact analytical solution and to sketch the entire phase portrait as shown in Fig. 6(a), from which the following conclusions can be illustrated:

- An ideal sliding mode exists on the line $s = 0$, meaning there is no chattering. This is because the control can switch infinitely close of the line $s = 0$.
- There is no steady-state error.
- The invariance property holds because the control can keep the system on the line $s = 0$.

The ideal saturation control (Fig. 5(b)): The most simple saturating control is described by

$$\begin{aligned} u(s) = \text{sat}(s) &= +1 \quad \text{when } s > L \\ &= \frac{s}{L} \quad \text{when } |s| \leq L \\ &= -1 \quad \text{when } s < -L \end{aligned}$$

where $L > 0$ and $\pm L$ defines the thresholds for entering the boundary layer. Outside the boundary layer, the control is identical to the ideal relay characteristic. Within the boundary layer, however, the control is a high-gain, linear control. As a result, the system will be driven to the boundary layer, but the trajectory cannot not be forced to follow the line $s = 0$. In the present case, the exact phase portrait of the system can still be obtained as shown in Fig. 6(b), illustrating the following conclusions:

- The sliding mode does not exist because the trajectory is not forced to stay on the $s = 0$ surface.

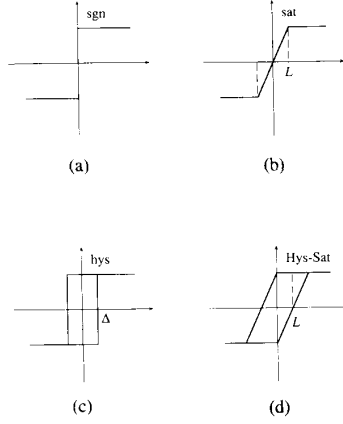


Fig. 5. Switching functions.

- No chattering can occur, since a continuous control is used within the boundary layer.
- In the absence of disturbances, there is no steady-state error.
- Invariance property does not exist.

Another type of boundary layer modification that has been proposed [143], [147] involves adaptively adjusting the boundary layer width to match the degree of uncertainty in the dynamics of the system.

The Practical relay control (Fig. 5c): A practical relay always exhibits hysteresis, modeled by

$$u(s) = \text{hys}(s) = \begin{cases} +1 & \text{when } s > \Delta, \\ -1 & \text{when } s < -\Delta, \end{cases} \quad \text{or} \quad \begin{cases} \text{when } \dot{s} < 0 \text{ and } |s| < \Delta \\ -1 & \text{when } \dot{s} > 0 \text{ and } |s| < \Delta \end{cases}$$

where $2\Delta > 0$ is the amount of the hysteresis in s . The hysteresis characteristic makes it impossible to switch the control on the surface $s = 0$. Instead, switching occurs on the lines $s = \pm\Delta$. The exact solution of the system can be determined with some effort and the phase portrait of the response, Fig. 6(c), illustrates the following general characteristics:

- A nonideal sliding mode exists, meaning there always is chattering in the sliding mode.
- The system has limit cycle behavior in "steady state." It is impossible to eliminate the chatter.
- The origin is not an equilibrium point.

The practical saturation control (Fig. 5(d)): A practical saturation element also exhibits hysteresis. Analysis of a system with such a control is complicated, especially in the case of a nonlinear system. Describing function techniques can be useful. Fig. 6(d) shows the phase portrait of the VSC system. Generally speaking:

- The sliding mode does not exist at all.
- The VSC system is asymptotically stable in the large,

but has two equilibrium points (points P and P' in Fig. 6(d)). There is steady-state error.

- There is no chattering phenomenon.
- When the slope of the linear part of the saturation function is not sufficiently large, the system ceases to possess any VSC system properties.
- The invariance characteristics do not exist.

In conclusion, continuation approaches eliminate the high-frequency chattering at the price of losing invariance. A high degree of robustness can still be maintained with a small boundary layer width, but significant delays in the control actuator may dictate the need for a "thicker" boundary layer. In the extreme case, large amplitude low-frequency oscillation may result and the system may cease to possess any variable structure behavior. The invariance and robustness properties of the system no longer exist in the latter case. It is interesting to note, in Fig. 6, that in all four cases the VSC systems are stable and their state trajectories are bounded in a strip in the state space. If the width of the strip is sufficiently narrow, the dynamics within the strip may be called the "quasi-sliding mode." Thus one can consider that the systems shown in b, c, and d of Fig. 7 have quasi-sliding modes.

2) *Tuning the Reaching Law Approach*: Chattering can be reduced by tuning parameters q_i and k_i in the reaching law

$$\dot{s}_i = -q_i \text{sgn}(s_i) - k_i s_i \quad i = 1, \dots, m.$$

Near the switching surface, $s_i \approx 0$, so $|\dot{s}_i| \approx q_i$. By choosing the gain q_i small, the momentum of the motion will be reduced as the system trajectory approaches the switching surface. As a result, the amplitude of the chatter will be reduced. However, q_i cannot be chosen equal to zero because the reaching time would become infinite; the system fails to be a sliding mode control system. A large value for k_i increases the reaching rate when the state is not near the switching surface. Finite reaching time with a zero reaching rate at the switching surface can be achieved using the power rate reaching law (41). By this method, the chattering can practically be suppressed altogether. Simulation results for such control can be found in [63]. Other works devoted to the elimination or reduction of chattering exist and can be found in the literature [12], [29], [47], [102], [181].

In summary, chattering is a hindrance to the widespread use of VSC in many practical control systems. Elimination or reduction of chattering remains an important and challenging problem.

V. EXTENSION TO OTHER CONTROL OBJECTIVES

Variable structure control concepts, and more specifically the sliding mode approach, were initially applied to system stabilization or regulator-type problems. The techniques have since been extended to many other control issues. The following discussion considers a general system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\mathbf{u} \quad (74)$$

where $\dim \mathbf{x} = n$ and $\dim \mathbf{u} = m$.

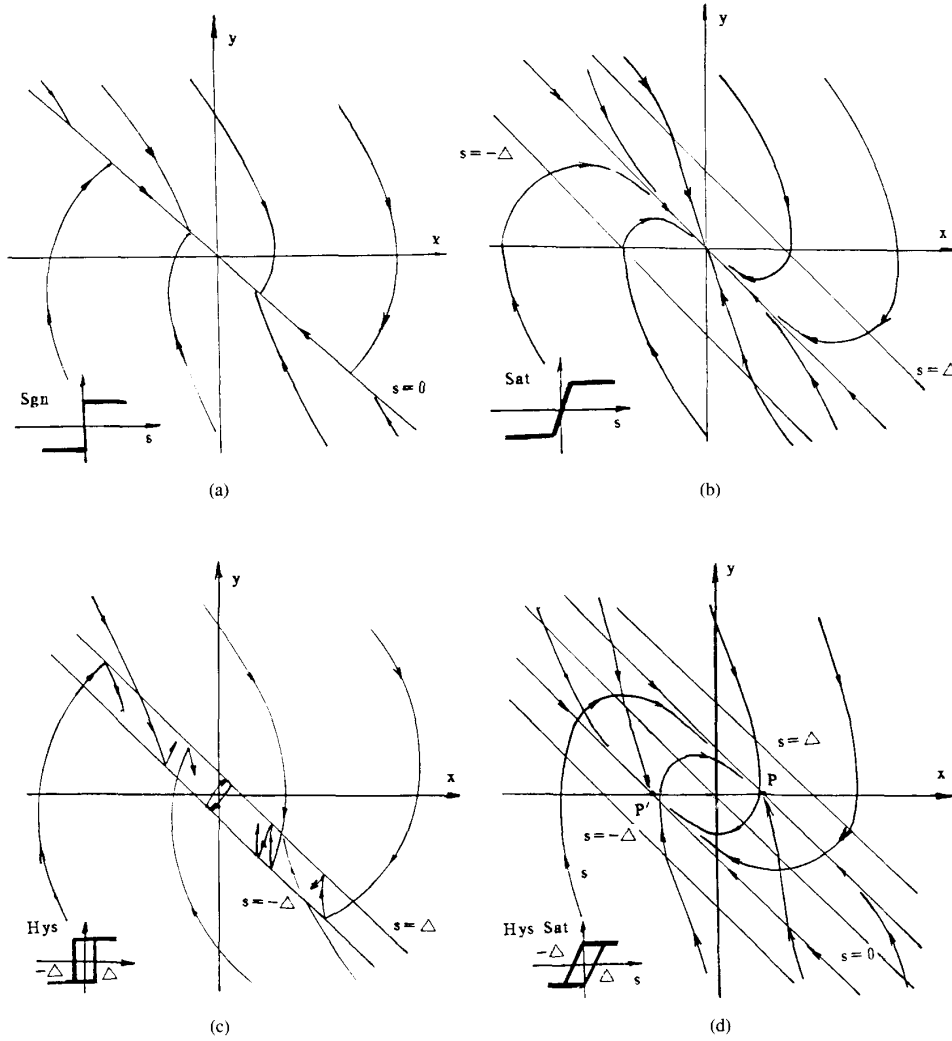


Fig. 6. Sliding modes and quasi-sliding modes.

The extension of VSC concepts to *tracking* problems can be found in several references [25], [60], [160]. Here, the objective is to cause the motion $\mathbf{x}(t)$ of the system (74) to follow a desired motion $\mathbf{x}_d(t)$ asymptotically.

A related, but different interpretation of the tracking problem is that of *model following*. That is, the desired motion is that of a reference model (n th order, p inputs):

$$\dot{\mathbf{x}}_m = \mathbf{A}_m \mathbf{x}_m + \mathbf{B}_m \mathbf{r} \quad \mathbf{x}_m(0) = \mathbf{x}_o \quad (75)$$

where \mathbf{r} is the command or reference input and \mathbf{x}_o is the initial condition of the system (74). Results in this area can be found in [15], [28], [33], [36], [67], [130], [173], [175]. There are also many papers investigating model following systems where some parameters or part of the model is not exactly known. Variable structure control has been used with a variety of other measures to produce *adaptive model following* systems [3], [5], [9], [10], [68], [75], [100], [145].

In addition to meeting the requirements of following a model output, one may desire the system (74) to behave like the model after finite time. This is called *model reaching control*. In fact, the most basic VSC can be considered a model reaching control, where the model reached in finite time is the dynamic equation of the sliding mode. This model has dimension $p = n - m$.

Variable structure control with dynamic feedback compensation has recently been proposed [62]. An m th-order state equation that represents the compensator is appended to the original system (74) to give an $(n + m)$ th-order system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\mathbf{u} \\ \dot{\mathbf{z}} &= \mathbf{Q}(\mathbf{x}, t) + \mathbf{W}(\mathbf{x}, t)\mathbf{u}. \end{aligned} \quad (76)$$

One chooses $\mathbf{s} = \mathbf{s}(\mathbf{x}, \mathbf{z}, t)$ to be an m -dimensional switch-

ing function whose eventual sliding mode represents the desired n th-order model. In the eventual sliding mode, the VSC system behaves as the model does.

A large body of work has contributed to control of *uncertain dynamic systems*. A general model is

$$\dot{\mathbf{x}} = A_o \mathbf{x} + \Delta A(\mathbf{x}, \mathbf{p}, t) + (B_o + \Delta B(\mathbf{x}, \mathbf{p}, t))\mathbf{u} + \mathbf{f}(\mathbf{x}, \mathbf{p}, t) \quad (77)$$

where \mathbf{p} is a uncertain parameter vector, ΔA and ΔB are uncertain perturbations, and \mathbf{f} is the unknown external disturbance vector. The VSC may be structured the same way as that for systems without uncertainties by taking into account the bounds on ΔA , ΔB , and \mathbf{f} [32], [63]. A contribution by Leitmann [94], which is often called Lyapunov's min-max method, has inspired another series of works [11], [30], [34], [64]–[66], [95], [96], [99], [150]. These works are closely related to variable structure control results.

When not all state variables of a system

$$\begin{aligned} \dot{\mathbf{x}} &= A(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} \end{aligned} \quad (78)$$

are measurable, a state observer is often proposed. The state estimation problem for nonlinear systems is, however, complicated and generally difficult to solve. The problem is often more tractable using VSC design techniques. Because the objectives of state estimation are quite similar to that of model following, one can design a sliding mode for the state observation error. Works on VSC observers have been reported for linear systems [71], for single-output nonlinear systems [145], for uncertain dynamic systems [161], [162], and for applications to motors [56], [93], [126]. The implementation of VSC systems using linear state observers has also been reported [13], [19], [161], [162].

VI. EXTENSION TO OTHER TYPES OF DYNAMIC SYSTEMS

A. Large-Scale Systems

Variable structure control for large-scale systems was briefly discussed in Section II-D-(a). Consider the large-scale system that may be represented by N coupled subsystems

$$\dot{\mathbf{x}}_i = A_i \mathbf{x}_i + B_i \mathbf{u}_i + \sum_{\substack{j=1 \\ j \neq i}}^N A_{ij} \mathbf{x}_j \quad i = 1, \dots, N \quad (79)$$

where $\dim \mathbf{x}_i = n_i$, $\dim \mathbf{u}_i = m_i$. In (79), (A_i, B_i) is a controllable pair. If all A_{ij} 's satisfy the matching condition

$$\text{rank}(A_{ij}, B_i) = \text{rank } B_i$$

then N -vector switching functions

$$\mathbf{s}_i(\mathbf{x}_i) = \mathbf{c}_i^T \mathbf{x}_i \quad \dim \mathbf{s}_i = m_i \quad i = 1, \dots, N$$

can be chosen such that the sliding modes of the overall

system describe N decoupled subsystems. The corresponding VSC law can be obtained by methods presented in Section II. In general, the obtained VSC laws are of two levels, the global control and the local control [60], [88]. The reader is referred to literature for both the linear [4], [87], [88] and nonlinear cases [103], [104], [120]. An application to power systems has also been reported in [89].

B. Discrete-Time Systems

For the VSC of discrete-time systems, the structures of the control are similar to that for continuous-time systems [57], [59], [106], [127]. However, the characteristics of discrete-time VSC systems differ from those of continuous-time VSC systems in two respects. First, discrete-time VSC systems can only undergo quasi-sliding modes [106], i.e., the state of the system can approach the switching surface but cannot stay on it, in general. This is due to the fact that control action can only be activated at sampling instants and the control effort is constant over each sampling period. Second, when the state does reach the switching surface, the subsequent discrete-time switching cannot generate the equivalent control to keep the state on the surface. Thus, discrete-time VSC systems do not possess the invariance properties found in continuous-time systems; the robustness issues are still under investigation.

C. Distributed-Parameter Systems

The study of the VSC for distributed-parameter systems is still in its infancy. The available results show that simple extensions of theory for lumped-parameter systems are not valid. The difficulties are threefold. First, there is no general theory on partial differential equations with a discontinuous right-hand side. Second, although the equivalent control approach for lumped-parameters systems could be applied to distributed-parameter cases, it is difficult to find and validate approximation conditions that are used [77], [113], [114]. Finally, the VSC law would be related to the characteristics for the system model in a very complex way. For example, application of VSC techniques to a heat flow process may exploit boundary conditions in order to develop the control law. Because boundary conditions vary, the designed sliding mode may not always exist.

An approximate (not precise) condition for a particular parabolic distributed parameter has been reported [77], [114]. For systems described by first-order differential equations, the problem can be transformed into that of finite-dimensional systems by using the concept of a characteristic direction field. Then the sliding mode conditions can be expressed in terms of a finite-dimensional sliding mode condition using the so-called "controlled characteristic equation" [136]. Another approach to the VSC of distributed-parameter systems is to approximate them by finite-dimensional lumped-parameters. Such an approach is widely used in dealing with VSC for flexible spacecraft structures [115], [116], [119].

D. Time-Delay Systems

To illustrate several features of VSC for systems with time delay, consider

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^N \mathbf{A}_i \mathbf{x}(t - t_i) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{s}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}\quad (80)$$

where $0 < t_1 < t_2 < \dots < t_N$. Note that the trajectory of \mathbf{x} from an initial time t_o can be determined only if the history of $\mathbf{x}(t)$ for $t_o - t_N < t < t_o$ is known. So the state space for (80) becomes an n -dimensional functional space whose elements are functions defined on $[-t_N, 0]$. The sliding modes should also be established in the functional space instead of the usual state space. Therefore, two important problems arise. First, one must find a good method to describe the sliding motion, i.e., define the equation of the sliding mode. The second challenge is the design of the matrix \mathbf{C} in (80) to yield a stable sliding motion. To the authors' knowledge, these two problems have not been fully solved, but several studies have been reported [76], [77], [83].

E. Other Systems

It is well known that in stochastic control the controllers are usually complex and hard to implement in real time, and the robustness of the control is not clear. Notable exceptions are linear quadratic Gaussian (LQG) systems and self-tuning regulators. Therefore, it is tempting to apply the merits of VSC to stochastic systems. Some result in this area have been reported in [181]. Other control systems in which VSC has been studied include learning control [1], neuromorphic networks [149], and fuzzy systems [38], [149]. It has already been pointed out that some VSC results have incorporated state estimators in the control. A related problem is to design VSC systems using only output feedback [75]. It is clear that many interesting problems remain to be studied.

VII. APPLICATIONS

A. Robot Control

The dynamics of an n -link manipulator are usually modeled by n coupled second-order nonlinear differential equations. Present-day control methods are often based on nonlinear compensations, which require an accurate manipulator model and, hence, load forecasting. Generally these methods are complex and costly in implementation. Fortunately, the dynamic equations can easily be transformed into a nonlinear canonical form and the matching conditions are always satisfied, so a VSC approach to manipulator control appears to be very promising. Studies using two-link manipulators have been reported in [173], [174], where a hierarchical VSC law was used. The simulation results showed the existence of chattering, a problem that has been studied in greater detail [107], [148], [169]. Applications to parameter identification of robotic systems have been reported [69], [145].

Other design methods for variable structure control of manipulators can be found in [1], [7], [26], [38], [70], [74], [97], [117], [163], [164], [170], [171], [177].

B. Motor Control

Control of electrical motors has been a popular application of VSC. The technique has been applied to the control of dc motors, synchronous motors, and induction motors. The following are just a few of many references in the literature [8], [18], [48], [56], [58], [68], [72], [82], [86], [93], [123]–[125]. Design of a sliding mode observer for an induction motor has also been done [126].

C. Aircraft Control

Variable structure control has been applied to the a variety of flight problems, including (1) control of lateral motion of an aircraft [166], (2) realization of asymptotically decoupled control of roll angle, angle of attack, and sideslip in the presence of rapid maneuvering [131], and (3) flight control [73], [108], [182].

D. Spacecraft Control

The VSC of multiaxial spacecraft for large-angle rotational maneuvering has been studied [40], [41], [159]. Application to spacecraft rotation damping [140] and orientation control [132] has also been reported.

E. Flexible Space Structure Control

Studies have been made on the use of VSC for stabilization, regulation, and maneuvering of large flexible space structures [115], [116], [130]. One interesting aspect is that the design of the control laws for the rigid mode control and for the elastic mode stabilization can be treated independently [130]. Because large flexible structures are distributed parameter systems, known approaches to study the control involve the use of finite-dimensional truncated models. Hence, many state variables are not measurable; a state observer has been used to implement the VSC law [119].

F. Other Applications

Listed as follows are some other known applications of variable structure control.

- 1) Load frequency control of power systems [17], [27], [89], [141], [142]
- 2) Servomechanisms [80], [92], [109], [176], [179]
- 3) Pulse-width modulation control [137], [139]
- 4) Guidance [22]
- 5) Process control [84], [85], [172]
- 6) Phase-locked loop control [128]
- 7) Power converters [134]
- 8) Digital implementation [2]
- 9) Remote vehicle control [118]

Undoubtedly, quite a few other published works of great interest have been missed.

VIII. CONCLUSION

Having traveled a long history of research and development, VSC is well established as a general control method. Its feasibility is increasingly recognized by control professionals, although there are still problems to be investigated. VSC is naturally attractive to control engineers because its basic concepts are rather easy to understand and has given satisfactory performance in many practical areas of industrial electronics. More importantly, VSC is applicable to many control systems where there are no other design methods are well-developed. Although contributions in the theoretical analysis and design of VSC systems will continue to improve, it is of major significance to also conduct experimental research for all kinds of industrial problems. The results of such studies will undoubtedly enhance the confidence of practicing engineers considering the use of VSC.

This tutorial attempts to introduce the fundamental theory, main results, and practical applications of VSC to instrumentation and control professionals of all engineering disciplines. It is hoped that this account will be of help to those who are interested in understanding the powerful capability of VSC and in learning how to design VSC systems. The reference list is intended to contain all significant papers published to date. The authors apologize for any papers missed.

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