Calculating Analytic Light Curves of Supernovae, Kilonovae, and Other Transients: A worksheet

D. Kasen (kasen@berkeley.edu)

Introduction and Approximations

Here we develop a simple, one-zone (semi-)analytic model for the light curves from the expanding debris of a supernova, kilonova or related sort of explosive transient. After completing the worksheet, one should (hopefully) be able to understand and calculate models that can be applied to theoretical and observational research studies.

Our discussion follows the analytic methods for supernova light curves presented in the papers of Arnett (1980, 1982). Consider a cloud of gas with mass M ejected from some explosion with a kinetic energy E_K. The fundamental equation describing the evolution of the expanding cloud of ejecta is the first law of thermodynamics, which expresses energy conservation

$$\frac{dE_{\text{int}}(t)}{dt} = -p\frac{dV}{dt} + \dot{Q}(t) - L(t) \qquad (1)$$

where E_{int} is the total internal energy of the ejecta of volume V. The pdV/dt term is the rate of work done by the ejecta pressure p, $\dot{Q}$ is the total heating input rate (erg s^{-1}) due to radioactivity (or any other energy input sources) and L is the luminosity (erg s^{-1}) escaping the system (which is light curve we would like to calculate).

We can solve Eq. 1 and so calculate the light curve $L(t)$ provide we make some simplifying assumptions.

Assumption 1: We consider a spherical, one-zone model of the ejecta, i.e., a spherical uniform density cloud of radius R. This is obviously a gross approximation of the actual remnant structure, but still turns out to give insightful scaling relations.

Assumption 2: The ejecta expands freely as $R = vt + R_0$ where $v = \sqrt{2E_K/M}$ is the characteristic velocity, R_0 is the radius of the system at the time of ejection[1], and t is the time since ejection. The volume of the cloud is

$$V(t) = \frac{4\pi}{3}(vt + R_0)^3 \qquad (2)$$

and the density is $\rho(t) = M/V$.

Assumption 3: Radiation energy dominates over gas energy. The internal energy density, u, and pressure are then

$$u = aT^4 \qquad p = \frac{1}{3}aT^4 \qquad (3)$$

[1] For example, R_0 is the radius of the progenitor star. In most cases we are interested at times when $vt \gg R_0$ and can simply take $R = vt$. Still, we retain R_0 for now for completeness.

In our one zone model, $u = E_{\text{int}}/V$. We will be able to confirm that the assumption of radiation domination is well motivated given the temperatures and densities we find for the ejecta.

Assumption 4: The radiation leaking from the cloud can be described by the standard diffusion equation in spherical coordinates[2]

$$L(r) = -4\pi r^2 \frac{c}{3\kappa\rho} \frac{\partial u}{\partial r} \qquad (4)$$

where κ is the opacity. To properly calculate the derivative of $u = aT^4$ we would need to know the radial run of temperature. However, since we are using a one-zone model we can approximate[3] the spatial derivative as

$$\frac{\partial u}{\partial r} \sim -\frac{E_{\text{int}}/V}{R} \qquad (6)$$

Assumption 5: The opacity, κ, is a constant. Of course, in reality the opacity is a complicated function of time, radius, and wavelength; nevertheless, we can try to identify an average effective grey opacity. For ordinary supernova material, a reasonable value is $\kappa = 0.1 \text{ cm}^2 \text{ g}^{-1}$. For the much more opaque heavy r-process ejecta from neutron star mergers, the value is more like $\kappa = 10 \text{ cm}^2 \text{ g}^{-1}$.

[2] Diffusion only applies at epochs when the ejecta is still reasonably optically thick.

[3] This simple approximation is equivalent to assuming u declines as

$$u(r) = \frac{E_{\text{int}}}{V}\left(1 - \frac{r}{R}\right) \qquad (5)$$

i.e., the energy density drops linearly from E_{int}/V at the center to 0 at the outer edge.

Problems:

A supernova light curve represents thermal radiation diffusing out of an expanding, hot cloud of ejecta. The diffusion time[4] through a static medium of radius of R is $t_{\text{d}} = \tau R/c$ where the optical depth $\tau = \kappa\rho R$. In a moving medium, photon escape is modified by the fact that cloud radius is growing and the density declining over time. The timescale at which photons effectively escape is when the diffusion time, t_{d}, becomes comparable to the current expansion time $t_{\text{exp}} = R/v$ of the cloud.

Problem 1: Setting $t_{\text{d}} \approx t_{\text{exp}}$, show that the timescale, t_{lc}, at which photons escape is

$$t_{\text{lc}} = \left[\frac{3}{4\pi}\frac{M\kappa}{vc}\right]^{1/2} \propto M^{3/4}\kappa^{1/2}E_k^{-1/4} \qquad (8)$$

where we assumed $vt \gg R_0$.

comment: The timescale t_{lc} provides an estimate of the peak time of the transient light curve. While the scaling with physical parameters is reasonable, the numerical constant out front would need to be calibrated to achieve some quantitative accuracy.

[4] In an optically thick medium, a photon steps (on average) one mean free path $l = 1/\kappa\rho$ before scattering. In a random walk diffusion process, after N steps the photon has only moved an average distance $R = \sqrt{N}l$. Thus, $N = R^2/l^2$ steps are needed for the photon to escape a region of radius R, and the diffusion time is

$$t_{\text{d}} = N\frac{l}{c} = \frac{R^2}{lc} = \frac{R^2\kappa\rho}{c} = \tau\frac{R}{c}. \qquad (7)$$

So the diffusion time is simply a factor of $\tau = R\kappa\rho$ longer than the free-streaming timescale R/c.

Problem 2: What is the optical depth of the ejecta at the time $t_{\rm lc}$ when photons can effectively escape?

comment: You see that photons begin to effectively escape for $\tau > 1$. The condition for escape is not that the ejecta be optically thin, but that photons can diffuse out faster than the ejecta itself grows.

Problem 3: Show that the luminosity from the diffusion equation (Eq. 4) can be written

$$L(t) \approx \frac{E_{\rm int}(t)}{t_{\rm lc}^2}(t + t_0) \qquad (9)$$

where $t_0 = R_0/v$ is the timescale[5] it took the ejecta to expand to double its initial size R_0. To calculate the light curve we just need to solve for how $E_{\rm int}(t)$ evolves with time. Let's do it step by step.

Problem 4: Consider first the evolution of $E_{\rm int}$ in the case of adiabatic expansion where $L = \dot{Q} = 0$ (i.e., no heat is entering or leaving the system). Show that

$$E_{\rm int}(t) = E_0 \left(\frac{t_0}{t + t_0} \right) \qquad (10)$$

where E_0 is the initial internal energy and time $t = 0$ (corresponding to the end of the explosion and the start of the ejecta expansion). Thus, the internal energy cools adiabaticaly like $E_{\rm int} \propto R_0/R \propto 1/t$.

comment: In a stellar explosion powered by a strong shock, the initial internal energy E_0 is typically comparable to the kinetic energy E_K. However, over time the internal energy turns into kinetic energy as pressure does pdV work on the ejecta. At times $t \gg t_0$, the internal energy has dropped to but a small fraction[6] of the kinetic energy, $E_{\rm int} \ll E_K$, and there is no longer enough internal energy to significantly accelerate the ejecta[7]. Thus, free homologous expansion becomes an increasingly excellent assumption at times $t \gg t_0$.

Problem 5: Next, consider the case where there is no heating ($\dot{Q} = 0$) but radiation can escape the remnant ($L \neq 0$). Solve equation 1 for $E_{\rm int}(t)$ and derive the analytic formula for the light curve

$$L(t) = \frac{E_0}{t_{\rm lc}} \frac{R_0}{vt_{\rm lc}} \exp\left[-\frac{(t + t_0)^2}{2t_{\rm lc}^2} \right] \quad \text{(no heating)} \qquad (11)$$

comment: As expected, the quantity $t_{\rm lc}$ gives the characteristic duration of the lightcurve of Eq. 11. The coefficient in front of the exponential is an estimate of the characteristic light curve luminosity. It is simply the total initial energy E_0 divided by the timescale for this

energy to leak out, t_{lc}, but *modified* by a factor R_0/vt_{lc} that accounts for the loss of internal energy due to adiabatic expansion.

Problem 6: Write the coefficient in front of Eq. 11 in terms of R_0, M and E_K to see how the (no-heating) luminosity scales with physical parameters. Plug in default values $M = M_\odot, R = R_\odot, E_0 = 10^{51}$ erg, and scale to discuss what luminosity is expected for a Type IIP supernova (massive red giant explosion), a Type Ia supernova (white dwarf) and a kilonova (neutron star merger ejecta). You'll see that a light curve powered by the diffusion of energy E_0 deposited in the explosion is very dim except for stars with large R_0. In other cases, continuing energy input is needed.

Problem 7: Finally, consider the general case where there is both radioactive or other heating ($\dot{Q} \neq 0$) and radiation ($L \neq 0$). Rearrange the equations to show that

$$\frac{dL(t)}{dt} = \frac{(t + t_0)}{t_{lc}^2}[\dot{Q}(t) - L(t)] \tag{12}$$

comment: Notice that according to Eq. 12, at the maximum of the light curve (where $dL/dt = 0$) we must have $L = \dot{Q}$. In other words, the luminosity at the peak of the light curve is equal to the instantaneous rate of heating at that time. You have just proven what is known as "Arnett's Law". It is only roughly true given the approximations we have made, but a useful rule of thumb.

Problem 8: There is no simple, general solution for Eq. 12, but you can write the light curve as an integral over $\dot{Q}(t)$. Consider attacking the differential equation using an [integrating factor](link) and show that[8]

[8] We now assume $t \gg t_0$, and so drop t_0, but it is straightforward to include it if you wish.

$$L(t) = \exp\left[-\frac{t^2}{2t_{lc}^2}\right]\left(\frac{E_0 R_0}{vt_{lc}^2} + \int_0^t \dot{Q}(t')\left(\frac{t'}{t_{lc}^2}\right)\exp\left[\frac{t'^2}{2t_{lc}^2}\right]dt'\right) \tag{13}$$

Which uses the initial condition for $L(0)$ given by Eq. 11. In the limit of no heating ($\dot{Q}(t) = 0$) we get only the first term in parenthesis, which reduces to the no-heating result Eq. 11. In the other limit that the initial energy contribution $E_0 t_0/t_{lc}^2$ is negligible the above simplifies to

$$L(t) = e^{-(t^2/2t_{lc}^2)}\int_0^t \frac{\dot{Q}(t')t'}{t_{lc}^2}e^{(t'^2/2t_{lc}^2)}dt' \quad \text{(with heating)} \tag{14}$$

This integral is relatively easy to calculate numerically, given some heating function $\dot{Q}(t)$.

Problem 9: Consider the earliest epochs of the light curve, when $t \ll t_{\text{lc}}$ and the heating rate can be assumed to be roughly constant $\dot{Q} = \dot{Q}(0)$. Expand the exponentials in Eq. 14 to show that

$$L(t \ll t_{\text{lc}}) \approx Q_0 \left(\frac{t}{t_{\text{lc}}} \right)^2 + \mathcal{O}(t/t_{\text{lc}})^4 \qquad (15)$$

So to the leading term we have a light curve rise of $L(t) \propto t^2$ at early times. You could keep higher order terms if you care to.

comment: Sometimes the scaling $L \propto t^2$ is explained by appealing to a "fireball" model, where at early times the photospheric radius grows as $R_{\text{ph}} = v_{\text{ph}}t$ while the photospheric temperature T_{ph} remains constant – hence the luminosity $L = 4\pi R_{\text{ph}}^2 \sigma_{\text{sb}} T_{\text{ph}}^4$ scales as t^2. The above derivation shows that the t^2 scaling can be derived from more physical considerations.

The $L \propto t^2$ behavior often gives a reasonable match to the early rise of observed supernova light curves, perhaps fortuitously given the many approximations we have made. More meaningful estimates of the light curve rise require specifying the radial profile of density and heating input.

Problem 10: Write a simple code to integrate Eq. 14 and so calculate a light curve. For example, the heating due to radioactive ^{56}Ni heating is

$$\dot{q}_{\text{ni}}(t) = 3.9 \times 10^{10} e^{-t/\tau_{\text{ni}}} + 6.78 \times 10^9 \left[e^{-t/t_{\text{co}}} - e^{-t/t_{\text{ni}}} \right] \ \text{erg g s}^{-1} \qquad (16)$$

where $t_{\text{ni}} = 8.8$ days and $t_{\text{co}} = 113.6$ days. The heating for radioactive r-process decay is

$$\dot{q}_{\text{rp}}(t) \approx 2 \times 10^{10} \left(\frac{t}{1 \text{ day}} \right)^{-1.3} \text{erg g s}^{-1} \qquad (17)$$

The heating from central magnetar spindown is

$$\dot{q}_{\text{mag}}(t) = \frac{E_m/t_m}{(1+t/t_m)^2} \text{erg s}^{-1} \qquad (18)$$

where E_m is the total rotational energy of the magnetar and t_m the spindown timescale.

Problem 11: The above model only allows us to calculate a bolometric light curve. In order to estimate the colors, we can make the (very rough) approximation that the spectrum is a blackbody[9] at the temperature at the photosphere, T_{ph}, which can be determined by

$$T_{\text{ph}}(t) = \left[\frac{L}{4\pi R_{\text{ph}} \sigma_{\text{sb}}} \right]^{1/4} \qquad (19)$$

[9] The opacity in supernova ejecta is typical strongly wavelength dependent, so the photospheric radius is a function of wavelength, hence blackbody emission at a single temperature is not a great assumption. But at least it is something we can do easily.

To derive the photospheric radius, $R_{\mathrm{ph}}(t)$, we must assume something about the density structure of the ejecta. In the absence of a detailed explosion model, we can use simple analytic profiles, two of which are given below.

Exponential Density Profile: Consider the density profile

$$\rho(r,t) = \rho_0(t)e^{-v/v_e} \tag{20}$$

where $\rho_0(t)$ and v_e are parameters and we assume $r = vt$. In homologous expansion, the shape of the density profile does not change with time, it simple scales outward and drops as $\rho \propto t^{-3}$.

11a) Integrate the density profile over radius[10] (at any given time) and determine $\rho_0(t)$ and v_e in terms of M, E_k, and t.

[10] The mass in an infinitesimal shell is $dm = 4\pi r^2 \rho(r)dr$ and the kinetic energy is $dE = \frac{1}{2}4\pi r^2\rho(r)v^2 dr$

The continuum optical depth integrated outward from some radius r to infinity is

$$\tau(r) = \int_r^\infty \rho(r)\kappa dr \tag{21}$$

where κ is the continuum opacity. The photosphere is defined[11] as the radius where $\tau(r) = 1$.

[11] Sometimes the photosphere is defined as $\tau = 2/3$ but the difference is not of great importance here.

11b) Assuming a constant opacity κ, determine the photospheric radius and photospheric velocity in terms of M, E_k, and t.

11c) Determine the time after which $\tau < 1$ throughout the entire ejecta. This marks the onset of the "nebular" phase, where the assumption of blackbody emission becomes severely bad.

Broken Power Law Density Profile: An alternative simple density structure is the broken power-law, given by

$$\rho(r,t) = \rho_0(t)(v/v_t)^{-\delta} \quad \text{for } v < v_t \tag{22}$$
$$\rho(r,t) = \rho_0(t)(v/v_t)^{-n} \quad \text{for } v \geq v_t \tag{23}$$

Supernova simulations show that the ejecta is often described by a shallow exponent δ in the interior layers and a steep exponent n in the outer layers.

11d) What are the allowed values of n and δ such that the total mass and energy converge?

11e) Integrate the density profile over radius to determine ρ_0 and v_t in terms of M, E, n and δ.

11f) Calculate the photospheric radius and velocity over time.

In the blackbody model, the specific luminosity L_ν (units ergs s^{-1} Hz^{-1}) at some frequency ν is[12]

[12] It is common to forget the factor of π in this expression, but it necessary since

$$\int_0^\infty B_\nu(T)d\nu = \frac{\sigma_{\mathrm{sb}}T^4}{\pi} \tag{24}$$

and so is needed to assure that $\int L_\nu d\nu = L$.

$$L_\nu(\nu) = B_\nu(T_{\text{ph}}) \frac{\pi L}{\sigma_{\text{sb}} T_{\text{ph}}^4} \tag{25}$$

11g) Generalize your code for the bolometric light curve $L(t)$ so that it also calculates the photospheric radius, photospheric temperature, and specific luminosity L_ν as a function of time.

comment: If you want to convert your specific luminosity into an apparent AB-magnitude, use the astronomer's relation

$$M_{\text{AB}} = -2.5\log_{10}\left[\frac{L_\nu}{4\pi D^2}\right] - 48.6 \tag{26}$$

where D is the distance to the source in cm. If you want to calculate an absolute AB magnitude, the convention is to use $D = 10$ parsecs $= 3.086 \times 10^{19}$ cm.

comment: The semi-analytic light curve code you have developed here is capable of predicting approximate light curves for theoretical models, or for fitting observational data. In fact, such models have been used in many, many research papers. However, before over-interpreting your results, keep in mind the limitations imposed by the several simplifying assumptions we have made.