Convexity and simplicity of polygonal curves via Fourier representation

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Abstract

Knowing what we know now maybe investigate other famous flows and how they look in terms of Fourier coefficients

A polygonal curve in the plane with n sides can be represented by an ordered list of n points $x_i \in \mathbb{R}^2$, alternatively in $\mathbb{C}^n$. [Some sentence connecting this to its "Fourier" representation, with coefficients α_i.] We consider the following question: is it possible to determine whether a polygonal curve is convex and simple by directly considering the coefficients α_i? To answer this question we look at the regions of $\mathbb{C}^n$ corresponding to convex and simple curves. Blah blah blah. This approach to thinking about convexity and simplicity can be useful in the evolution of polygonal curves under general non-linear flows.

Through these notes I hope to track my understanding of the discrete Fourier transform and how it relates to convexity. I hope to describe the relationship between the convexity of a shape in $\mathbb{C}$ and its Fourier coefficients.

1 Introduction

To describe a polygon in $\mathbb{R}^2$, Chow uses its equivalent representation in $\mathbb{C}$, where $x \in \mathbb{R}^2$ is mapped to the real component in $\mathbb{C}$ and $y \in \mathbb{R}^2$ is mapped to the imaginary component in $\mathbb{C}$. Each polygon of size n is then described with the following notation:

$$\vec{X} = (X_0, X_1, \ldots, X_{n-1}) = (x_0 + iy_0, x_1 + iy_1, \ldots, x_{n-1} + iy_{n-1})$$

Since each vertex is a complex number we can also describe $\vec{X}$ in terms of polar coordinates, which will be helpful later on: $\vec{X} = (r_1 e^{i\gamma_1}, r_2 e^{i\gamma_2}, \ldots, r_{n-1} e^{i\gamma_{n-1}})$.

Not only do these vectors represents a polygon in $\mathbb{C}$, but also a point in $\mathbb{C}^n$.

We can describe convex polygons as those which fulfill the following condition:

$$\forall X_i, X_j, s.t. i, j, < n, \overline{X_i X_j} \in \text{the open region } \mathcal{S} \text{ containing all the vertices } X_i$$

This is a discrete example of the generalized convexity for a curve. We only need to consider whether the line segments between the polygons vertices are within the polygon, rather than every pair of points, since the vertices are its outermost points. (can add proof) Intuitively this means that standing on any edge of the polygon as the outwards facing normal, the entire polygon should be behind you. For numerical analysis we will use an equivalent but more easily computable definition. Consider each difference vector of the form $Z_i = X_{i+1} - X_i$ (where the index is taken mod n). The sum of the difference vectors of a convex polygon will satisfy:

$$\sum_{i=0}^{m-1} \arccos \frac{\langle Z_i, Z_i + 1 \rangle}{\|Z_i\| \|Z_{i+1}\|} \leq 2\pi \tag{1}$$

$$\operatorname{sign}(Z_{i+1} \times Z_i) \equiv \text{const}$$

where m is the length of the ordered set of vertices of $\vec{X}$.

To make it easier to deal with the vectors in the discrete fourier matrix, we will define orientations of convexity. Since all convex shapes satisy $\operatorname{sign}(Z_{i+1} \times Z_i)$ is constant, we will define positively convex shapes as those with a positive sign (and counter-clockwise orientation), and negatively convex shapes as those with a negative sign (and clockwise orientation). This distinction will allow us to consider conventionally non-convex shapes such as a doubly oriented triangle as convex.

A polygons is considered simple if for every pair of distinct vertices X_i and X_j and for every pair of distinct edges $\overline{X_m X_k}$ and $\overline{X_i X_j}$, there is no intersection between them.

$$\forall X_i, X_j, X_m, X_k : i \neq j, m \neq k, (i,j) \neq (m,k), \overline{X_i X_j} \cap \overline{X_m X_k} = \varnothing$$

Through these definitions, the set of of convex and simple polygons are distinct and within their intersection is the set of regular polygons.

2 The Discrete Fourier Transform

Every polygon of the aforementioned form can be decomposed into constituent 'perfect' polygons with the aid of the discrete fourier transform. Similar to the fourier transform which decomposes a function, usually sinusodial waves, into sinusodial waves describing its frequencies, the discrete fourier transform breaks polygons into sums of 'perfect' polygons generated by visiting points on the complex unit circle at different frequencies.

The DFT is of the form:

$$\vec{\alpha_k} = \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \vec{X_k} e^{-\frac{i2\pi mk}{n}} \tag{2}$$

where each m, k is taken $\mod n$. We can also express it as a matrix $\mathbf{V}$, where $\mathbf{V}_{mk} = e^{-\frac{i2\pi mk}{n}}$.

While the IDFT, which maps the coefficients back to a polygonal curve is as follows:

$$\vec{X_k} = \frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} \vec{\alpha_k} e^{\frac{i2\pi mk}{n}} \tag{3}$$

In matrix form this is $\mathbf{V}^{-1}_{mk} = e^{\frac{i2\pi mk}{n}}$.

Lemma 2.1. *The sum of the mth row or kth column of the dft matrix $\forall m, k \neq 0$ is zero.*

Proof. To prove this we can show that $\forall m \neq 0$, the sum: $\sum_{k=0}^{n-1} e^{\frac{i2\pi mk}{n}}$, is a geometric series which equals zero. Expanding the series we see it is $1 + e^{\frac{i2\pi m}{n}} + e^{\frac{i2\pi 2m}{n}} + \cdots + e^{\frac{i2\pi(n-1)m}{n}}$. It is clear that this is a geometric series where our initial term is 1 and our ratio is $e^{\frac{i2\pi m}{n}}$. Indeed putting the series in terms of $\omega = e^{\frac{i2\pi}{n}}$, we see $\omega^{0m} + \omega^{m} + \omega^{2m} + \cdots + \omega^{(n-1)m}$. The sum is given as $1 * \frac{1-e^{\frac{i2\pi m}{n}n}}{1-e^{\frac{i2\pi m}{n}}} = \frac{1-e^{i2\pi m}}{1-e^{\frac{i2\pi m}{n}}} = \frac{1-1}{1-e^{\frac{i2\pi m}{n}}} = 0$. The same can be done to show that the sum of the kth column is also zero. $\square$

Also note that if m or k is zero, the sum is $\sum_{k=0}^{n-1} e^{\frac{i2\pi mk}{n}} = \sum_{k=0}^{n-1} 1 = n$.

Lemma 2.2. *V is symmetric, so that $V^T = V$ and $V = V^T$*

Proof. $V_{mk} = e^{-\frac{i2\pi mk}{n}} = e^{-\frac{i2\pi km}{n}} = V_{km}$ $\square$

Lemma 2.3. *Denoting V as $V = \frac{1}{\sqrt{n}}V$ is unitary so that $V^H V = V V^H = V V^{-1} = V^{-1} V = I$.*

Proof. Each entry is given by: $V_{mk} = \frac{1}{\sqrt{n}} e^{-\frac{i2\pi mk}{n}}$. The product $V^H V$ is then: $(V^H V)_{mk} = \sum_{j=0}^{n-1} V^H_{mj} V_{jk}$. Since $V = V^T$, $(V^H V)_{mk} = \sum_{j=0}^{n-1} \overline{V_{mj}} V_{jk}$. Notice that $\overline{V_{mj}}$ is simply $\frac{1}{\sqrt{n}} \overline{e^{-\frac{i2\pi mj}{n}}} = \frac{1}{\sqrt{n}} e^{\frac{i2\pi mj}{n}}$. So $\overline{V_{mj}} V_{jk} = \frac{1}{\sqrt{n}} e^{\frac{i2\pi mj}{n}} \frac{1}{\sqrt{n}} e^{-\frac{i2\pi jk}{n}} = \frac{1}{n} e^{\frac{i2\pi(mj-jk)}{n}}$. The sum is then $\sum_{j=0}^{n-1} \overline{V_{mj}} V_{jk} = \sum_{j=0}^{n-1} \frac{1}{n} e^{\frac{i2\pi(mj-jk)}{n}}$. This is also a geometric series where our initial term is $\frac{1}{n}$ and with the ratio $e^{\frac{i2\pi(m-k)}{n}}$. Therefore the sum is $\frac{1}{n} \frac{1-e^{\frac{i2\pi(m-k)}{n}n}}{1-e^{\frac{i2\pi(m-k)}{n}}}$. So $\forall m-k \neq 0$, the sum is $\frac{1}{n} \frac{1-1}{1-e^{\frac{i2\pi(m-k)}{n}}} = 0$, while if $m = k$ on the diagonal, the sum is $\sum_{j=0}^{n-1} \frac{1}{n} e^{\frac{i2\pi(mj-mj)}{n}} = \sum_{j=0}^{n-1} \frac{1}{n} = \frac{n}{n} = 1$. The same can be done for $V V^H$, to show that both equal I. $\square$

Putting the DFT and IDFT in matrix form we can see the consequences clearly:

The DFT becomes:

$$\vec{X_k} = \frac{1}{\sqrt{n}} V \vec{\alpha_k} \tag{4}$$

While the IDFT is:

$$\vec{\alpha_k} = \frac{1}{\sqrt{n}} V^{-1} \vec{X_k} = \frac{1}{\sqrt{n}} \overline{V} \vec{X_k} \tag{5}$$

Now since $V = V^T$ and V is unitary, the complex inner product of columns V_k have some interesting properties. As we have seen V is unitary so that it forms an orthonormal basis on $\mathbb{C}^n$. The V are orthogonal $\forall k, j$ such that $k \neq j$ so that $\langle V_k, V_j \rangle = \sum_{m=0}^{n-1} \frac{1}{n} e^{\frac{i2\pi(mj-mk)}{n}} = 0$ and for $k = j$, the sum is 1.

Lemma 2.4. *V^2 is a permutation matrix*

Proof. Each entry of the product V^2 is : $(V^2)_{mk} = \sum_{j=0}^{n-1} V_{mj} V_{jk}$ which becomes $\sum_{j=0}^{n-1} \frac{1}{n} e^{-\frac{i2\pi j(m+k)}{n}}$. As we have seen this the geometric series which sums to zero for all pairs of m, k except $m + k \mod n = 0$ in which case, since indices are always taken $\mod n$, then $e^{-\frac{i2\pi j(m+k)}{n}} = e^{-\frac{i2\pi j(0)}{n}} = 1$ so that the sum becomes $\sum_{j=0}^{n-1} \frac{1}{n} = 1$. $\square$

—-show that $V^4 = I = VV^H$, so that eigenvalues and V^2

3 Algebra of Fourier Coefficients

Further because the DFT is unitary the DFT of the dot product of two vectors is the dot product of the DFT of two vectors: $\langle u, v \rangle = \langle F(u), F(v) \rangle$

Due to the linearity of the discrete fourier transform operations can be performed easily. $F\{a\alpha + b\beta\} = aF\{\alpha\} + bF\{\beta\}$

Is the same true of convexity? Since the equation for convexity is quadratic the same is not true. The convexity and simpleness of the sum of two polygonal curves is

4 Numerical Properties of the DFT

As Chow explains, each column vector consists of the nth roots of unity: $\omega_k = e^{\frac{2\pi ijk}{n}}$. Considering

Every point in the column vectors can be understood as describing a different angle/point on the unit circle in $\mathbb{C}$. With this understanding it is easier to describe the properties of each vector. As you increase j you traverse the unit circle at different points. Increasing k affects the frequency at which you visit them.

To see this more explicitly we can consider purely the order at which each points traverses the unit circle by considering the transformation $f : \mathbb{C} \to \mathbb{R}$ which takes $\mathbf{V}_{mk} = e^{-\frac{i2\pi mk}{n}}$ to $mk \mod n$. This matrix

Note that this can also be seen by taking the sum of

The column vectors will represent regular shapes iff v_k is coprime with n. Intuitively this is because it will repeat if there is some $ks.t.mk = n$. If m, k is not coprime with n the entries of the vectors will repeat themselves.

5 Convexity with the DFT

Of course, when we graph the changing of a_2, we see some surface in $\mathbb{C}^n$, sometimes changing values of a_2 results in curl Changes in Perhaps can also measure divergence. What do critical points correspond to? Take vector field in $\mathbb{C}$. Stokes theorem relates integral of curl over surface to line integral around boundary which we have as well defined. Question is how does this help us?

[-maybe something about initial visualizations or research chronology is probably not important]

Rather than looking directly at the angle between each adjacent difference vector Z_i as in , we can return to treating Z_i as a point in $\mathbb{R}$ and consider what would be the 'cross product'. The complex cross product of $u, v \in \mathbb{C}$ where $u = u_1 + u_2 i$ and $v = v_1 + v_2 i$ is $u \times v = u_1 v_2 - u_2 v_1$ which can be seen as $\Im[u\overline{v}]$. This is equivalent to $|u||v|\sin(\theta_{uv})$ however the first definition makes it easier to convert Z_i into its fourier coefficients.

So we can express the region of simplicity as when the complex cross product between adjacent difference 'vectors' is less than zero:

[-something about being aware of convex and anti-convex since both allowed]

$$\Im[(X_{m+1} - X_m)\overline{(X_{m+2} - X_{m+1})}] \forall m \in [0, 1, \cdots, n-1] \tag{6}$$

Note that under this equation polygonal curves which maintain a negative cross product however still self-intersect, will be considered simple. For this reason we can look to the sum of 6 for all m.

5.1 Connection to winding number

For a continuous curve in $\mathbb{C}$ recall its winding number

Lemma 5.1. *The sum of the arguments of the cross products of difference vectors, or* $\frac{1}{2\pi}\sum_{m=0}^{n-1} \arg((X_{m+1} - X_m)\overline{(X_{m+2} - X_{m+1})})$ *is equivalent to the turning number of a polygon* $T = \frac{1}{2\pi}\sum_{m=0}^{n-1} \arg(\frac{X_{m+2} - X_{m+1}}{X_{m+1} - X_m})$.

Proof. This is straightforward as we only have to multiply the argument $\frac{X_{m+2} - X_{m+1}}{X_{m+1} - X_m}$ of the sum of the turning number by the conjugate of its denominator:

$$\begin{aligned}
&\arg(\frac{X_{i+2} - X_{i+1}}{X_{i+1} - X_i}) \\
&= \arg(\frac{X_{i+2} - X_{i+1}}{X_{i+1} - X_i} \cdot \frac{\overline{X_{i+1} - X_i}}{\overline{X_{i+1} - X_i}}) \\
&= \arg(\frac{(X_{i+2} - X_{i+1})\overline{(X_{i+1} - X_i)}}{|X_{i+1} - X_i|^2}) \\
&= \arg((X_{i+2} - X_{i+1})\overline{(X_{i+1} - X_i)}) - \arg(|X_{i+1} - X_i|^2) \\
&= \arg((X_{i+2} - X_{i+1})\overline{(X_{i+1} - X_i)})?
\end{aligned}$$

□

Figure 1: Im

Figure 2: Abs

We can also arrive at the angles between edges by using the definition of the cross product in place of taking their arguments. As above we can consider $u \times v = u_1 v_2 - u_2 v_1 = \Im[u\overline{v}] = |u||v|\sin(\theta_{uv})$. Solving for θ we have:

$$\theta_{uv} = \arcsin \frac{u_1 v_2 - u_2 v_1}{|u||v|}$$

Both of these approaches measure the angle of each edge in comparison to its predecessor. Accordingly we can express the transformation from a polygonal curve $N : \mathbb{C} \to \mathbb{C}$ as that which transforms edges to angles, a discrete analog to the Gaussian map. [Relate convexity with gaussian curvature. Can we bypass considering 2 conditions - the sum and inequality by relating sum of angles with sum of absolute value - what if anything would be improperly categorized?]

Alternatively by considering the dot products between the normal of every edge and all other edges we can tell if a shape is convex. Intuitively we can define a convex shape is that whose edge-normals are always fully outwards pointing.

First notice that the normal of an edge is the complex rotation by i, so Z_i $= iZ_i$.

$$\forall m \in [0, 1, \cdots, n-1] \forall_{j \neq m, m+1} j \in [0, 1, \cdots, n-1] i(X_{m+1} - X_m)(X_m - X_j) \tag{7}$$

Now we can take 6 and express it in term of its coefficients and the DFT matrix. To do this we will consider the complex inner product between the difference of the rows of the DFT and $\vec{\alpha}$, the vector of coefficients. Since this is an inner product, rather than matrix-vector multiplication, we will take the conjugate of the coefficients:

$$\Im[\langle \mathbf{V}_{m+1}^{-1} - \mathbf{V}_m^{-1}, \overline{\vec{\alpha}} \rangle \overline{\langle \mathbf{V}_{m+2}^{-1} - \mathbf{V}_{m+1}^{-1}, \overline{\vec{\alpha}} \rangle}] \forall m \in [0, 1, \cdots, n-1]$$

This simplifies to:

$$\Im[\langle \mathbf{V}_{m+1}^{-1} - \mathbf{V}_m^{-1}, \overline{\vec{\alpha}} \rangle \langle \mathbf{V}_{m+2} - \mathbf{V}_{m+1}, \vec{\alpha} \rangle \rangle] \forall m \in [0, 1, \cdots, n-1] \tag{8}$$

We can also express 7 in terms of its coefficients:

$$\Re[\langle \mathbf{V}_{m+1}^{-1} - \mathbf{V}_m^{-1}, \overline{\vec{\alpha}} \rangle \langle \mathbf{V}_m - \mathbf{V}_j, \vec{\alpha} \rangle \rangle] \forall m \in [0, 1, \cdots, n-1] \forall_{j \neq m, m+1} j \in [0, 1, \cdots, n-1] \tag{9}$$

We can also express this as a vector valued function using matrix multiplication. Define $f(\vec{\alpha}) : \mathbb{C}^n \to \mathbb{C}^n$:

$$f(\vec{\alpha}) = (\frac{1}{n})\Im[\begin{bmatrix} \omega^{10} - \omega^{00} & \omega^{11} - \omega^{01} & \cdots & \omega^{1(k)} - \omega^{0(k)} \\ \omega^{20} - \omega^{10} & \omega^{21} - \omega^{11} & \cdots & \omega^{2(k)} - \omega^{1(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{00} - \omega^{(m)0} & \omega^{01} - \omega^{(m)1} & \cdots & \omega^{0(k)} - \omega^{m(k)} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$$
$$\times \begin{bmatrix} \omega^{20} - \omega^{10} & \omega^{21} - \omega^{11} & \cdots & \omega^{2(k)} - \omega^{1(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{00} - \omega^{(m)0} & \omega^{01} - \omega^{(m)1} & \cdots & \omega^{0(k)} - \omega^{m(k)} \\ \omega^{10} - \omega^{00} & \omega^{11} - \omega^{01} & \cdots & \omega^{1(k)} - \omega^{0(k)} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}] \tag{10}$$

$$f(\vec{\alpha}) = \Im[(\mathbf{PV}^2 - \mathbf{I})\mathbf{V}^{-1}\vec{\alpha}\overline{(\mathbf{PV}^2\mathbf{PV}^2 - \mathbf{PV}^2)\mathbf{V}^{-1}\vec{\alpha}}] \tag{11}$$

As we saw V^2 is a permutation matrix, and $\mathbf{P}$ is the identity matrix where the diagonal goes over the opposite line so that it flips any matrix and indeed $\mathbf{P}^2 = I$ so that $\mathbf{P} = \mathbf{P}^{-1}$

or equivalently, since $\overline{\mathbf{V}^{-1}} = \mathbf{V}$:,

$$f(\vec{\alpha}) = \Im[(\mathbf{PV}^2 - \mathbf{I})\mathbf{V}^{-1}\vec{\alpha} \circ (\mathbf{PV}^2\mathbf{PV}^2 - \mathbf{PV}^2)\mathbf{V}\overline{\vec{\alpha}}] \tag{12}$$

Further simplifying:

$$f(\vec{\alpha}) = \Im[(\mathbf{PV}^2 - \mathbf{I})\mathbf{V}^{-1}\vec{\alpha} \circ (\mathbf{PV}^2 - I)\mathbf{PV}^{-1}\overline{\vec{\alpha}}] \tag{13}$$

Generalize to:

$$(\mathbf{PV}^2 - \mathbf{I})\mathbf{V}^{-1}\vec{\alpha} \circ (\mathbf{PV}^2 - \mathbf{I})\mathbf{PV}^{-1}\overline{\vec{\alpha}} \tag{14}$$

which looks extremely close to the -something- correlation of α. Or perhaps some extension of Parseval's theorem.

Where $\mathbf{P}$ is the permutation matrix constructed by flipping the identity matrix over a single axis. tentatively problem with 2d mobius strip. How to categorize? Perhpas sum of Im[] tells winding number. So only rely if sum is zero. As we had done with the difference vectors, we can notice that the sum of the arguments of the entries of this vector tell us the turning number. Namely $T = \sum_{k=0}^{n-1} \arg f(\vec{\alpha}_k)$

If we cant simplify this, we will need to go with the more complicated definition. Define $g(\vec{\alpha}) : \mathbb{C}^n \to \mathbb{C}^{n^2}$

$$g(\vec{\alpha}) = \langle\, (\mathbf{PV}^2 - \mathbf{I})\mathbf{V}, \vec{\alpha}\rangle \circ \langle\, (\mathbf{PV}^2)(I - (\mathbf{PV}^2)^{j-1})\mathbf{V}^{-1}, \overline{\vec{\alpha}}\rangle \forall_{j \neq m,m+1 j=0,1,\cdots,n-1} \tag{15}$$

Needs check and proof but maybe under the desirable conditions then $(\mathbf{PV}^2 - I)(\mathbf{V}\vec{\alpha} \circ \mathbf{PV}\vec{\alpha}$ whether thtough properties of Dft or under convexity and or simplicity is there relation to convolution?

To simplify this we can consider $(\mathbf{PV}^2 - I)$:

Lemma 5.2. $\mathbf{PV}^2 - I$ *is singular so that* $\det[\mathbf{PV}^2 - I] = 0$

As a consequence the columns of $\mathbf{PV}^2 - I$ are linearly dependent, and so are the columns of $(\mathbf{PV}^2 - \mathbf{I})\mathbf{V}^{-1}$ and $(\mathbf{PV}^2 - \mathbf{I})\mathbf{PV}^{-1}$. Particularly, both only form a basis for $\mathbf{C}^{n-1}$ which is of course because we do not care about alpha0 Now we can find non-trivial vectors $\vec{\alpha}$ such that either of these matrices equal zero

Further lets look at the eigenvalues of vectors of the circulant matrix $\mathbf{PV}^2 - \mathbf{I}$ and the anti-circulant matrix $(\mathbf{PV}^2 - \mathbf{I})\mathbf{P}$. First notice that Chow used the circulant matrix which is the product of ours: $(\mathbf{PV}^2 - \mathbf{I})^2\mathbf{P}$

The associated polynomial of $C_1 = \mathbf{PV}^2 - \mathbf{I}$ is $f(x) = -1 + x^{n-1}$ and its eigenvalues are given by $\lambda_k = -1 + \omega^j$. As it is a circulant matrix its eigenvectors are the columns of the dft.

For the anti-circulant matrix $C_2 = (\mathbf{PV}^2 - \mathbf{I})\mathbf{P}$, we can simplify it by considering the eigensystem of its corresponding circulant matrix. Denoting its eigenvalues as μ_k notice that $\mathbf{C_2}\mathbf{V}^{-1} = \vec{\mu}\mathbf{V} = \mathbf{VD}_2$, where $\mathbf{D}_2 = \operatorname{diag}\vec{\mu}$. Specifically its corresponding circulant matrix has the associated polynomial $f(x) = x^2 - x$, while its eigenvalues are given by $\mu_k = -\omega^{(n-1)k} + \omega^{(n-2)k}$, and of course its eigenvectors are given by the columns of the dft.

-something about noticing the two matrices for a dihedral group. Maybe more can be known about shapes based The associated polynomial of $(\mathbf{PV}^2 - \mathbf{I})\mathbf{P}$ is $f(x) = x^2 - x$, while its eigenvalues are given by $\mu_k = \pm|-\omega^{(n-1)k} + \omega^{(n-2)k}|$.

Now using the eigenvalues, which are generated from the DFT: -section on that- D = diag(eigenvals)

So that we can express their eigendecomposition $\mathbf{C}_1 = \mathbf{V}^{-1}\mathbf{D}_1\mathbf{V}$ and $\mathbf{C}_2 = \mathbf{VD}_2\mathbf{V}$.

This makes 1 become:

$$\mathbf{V}^{-1}\mathbf{D}_1\vec{\alpha} \circ \mathbf{VD_2}\overline{\vec{\alpha}} \tag{16}$$

-now from above that diag has zero element, rank of VD is n-1 where first column is zero, therefore theorem that convexity and simplicity independent of $\vec{\alpha}_0$