TIME DOMAIN MEASUREMENT OF FREQUENCY STABILITY

A tutorial introduction

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ABSTRACT

The paper outlines the theoretical basis behind the definition of frequency stability in the time domain. Various types of variances are examined. Their differences and interrelation are pointed out. Systems that are generally used in the measurement of these variances are described.

INTRODUCTION

Radio frequency sources give an output signal which, in general, is affected by small fluctuations in their amplitude, phase or frequency. The nature of these fluctuations may be random or deterministic. Due to the large number of users of these sources and the variety of fields in which they are applied, a problem has arisen in the method of characterization of the frequency fluctuations. In the case of the deterministic fluctuations such as linear frequency drift, it has been found, in most cases, that a specification of the fractional frequency deviation on a per day or per month basis is satisfactory. However, in the case of non deterministic or random fluctuations, a mathematical treatment based on probability concepts is necessary. In the past, depending on the field of interest, several methods of characterization have been used and sometimes have led to confusion.

More recently a proposal has been made by the IEEE Subcommittee on Frequency Stability that the spectral density $S_y(f)$ of the fractional frequency fluctuation, y, and the two sample variance, $\sigma_y(\tau)$, be used to characterize frequency stability respectively in the frequency domain and in the time domain [1]. These two parameters have resulted as a logical conclusion from a rather large amount of theoretical and experimental work on the subject [2], [3]. However, other parameters, specially in the time domain, have been studied and have been found most interesting in specific cases [4], [5], [6], [7]. The subject still raises great interest.

In the present paper we give a brief description of the main theoretical concepts involved in the definition of the frequency stability in the time domain. The interrelation through the spectral density $S_y(f)$ of some of the various variances that have been studied up to now is made explicit. Finally several systems used for the measurement of stability in the time domain are described.

This paper was prepared at the request of the Program Committee of this Meeting. Several excellent reviews and tutorial papers on this subject have been published in recent years [8], [9], [10], [11], [12]. Consequently it appears difficult in the writing of such a tutorial introduction, to avoid repetitions or to improve on all these papers. The most one can do at this stage, is to present the material in a slightly different manner. In particular the present text, specially on the theoretical section, owes a great deal to the recent review by Dr. Rutman [8]; the reader is strongly encouraged to consult that excellent article.

A - THEORY

Definitions

According to a well accepted notation, the instantaneous output voltage of a signal generator can be written as [1]:

$$V(t) = \left[V_0 + \varepsilon(t) \right] \sin \left[2\pi \nu_0 t + \varphi(t) \right] , \tag{1}$$

where V_0 and ν_0 are the nominal amplitude and frequency and $\varepsilon(t)$ and $\varphi(t)$ are amplitude and phase fluctuations. It is assumed that $\varepsilon(t)$ is very small with respect to V_0 and can be entirely neglected. The instantaneous frequency of the oscillator is defined as:

$$\nu(t) = \nu_0 \left(1 + \frac{\dot{\varphi}(t)}{2\pi\nu_0} \right) , \tag{2}$$

where $\dot{\varphi}(t)$ stands for $d\varphi(t) / dt$. We also define the fractional frequency fluctuation as:

$$y(t) \equiv \frac{\dot{\varphi}(t)}{2\pi\nu_0} , \tag{3}$$

and we assume that

$$\left| \frac{\dot{\varphi}(t)}{2\pi\nu_0} \right| << 1 . \tag{4}$$

A proposed means to characterize the frequency stability of an oscillator is the spectral density of y denoted by $S_y(f)$. Its dimensions are Hz^{-1}. A measurement giving an estimate of $S_y(f)$ would then characterize the stability of the oscillator in the frequency domain.

This can be done in practice with a spectrum analyzer. However, one must be aware that only estimates of $S_y(f)$ can be obtained because of frequency range limitations and finite observation times.

Experimental studies of various frequency sources have shown that for all practical purposes, the frequency fluctuations spectrum of the most common oscillators can be represented by a truncated polynomial in the Fourier frequency domain:

$$S_y(f) = h^{\alpha} f^{\alpha} , \tag{5}$$

where α is an integer, ranging from -2 to +2. The frequency fluctuations spectral density can be related to the phase fluctuations spectral density through the relation

$$S_y(f) = \frac{f^2}{\nu_0^2} S_{\varphi}(f) . \tag{6}$$

The α has been associated with various types of fluctuations either in the phase or frequency representation.

α	Type of fluctuations
2	White Phase
1	Flicker of Phase
0	White Frequency
-1	Flicker of Frequency
-2	Random Walk of Frequency

However, another characterization of the frequency stability can be made by considering that $y(t)$ is a random function of time. Then, a statistical parameter measuring in some sense the excursions of the values y of the random function $y(t)$ around its mean value should characterize the frequency stability of the oscillator. In statistics, the standard deviation σ or the variance σ^2 is often used as statistical parameter. We could define a variance for the instantaneous frequency. In practice the frequency is measured over a time interval ,τ, called the averaging time and the variance is calculated through the relation:

$$\sigma^2(\tau) = < \left((\bar{y}_i) - <\bar{y}> \right)^2 > , \tag{7}$$

where $< >$ means an average over an infinite number of samples. Without going too deeply into questions of statistical concern, we make a few assumptions. First, we assume that stationnarity applies to our model. By this we mean that a displacement of the time coordinates does not change the statistics of our ensemble. Secondly we assume ergodicity, that is, averages over the ensemble can be replaced by time averages on one of the samples. The sign $< >$ in equation (7) then becomes a time average. Furthermore, we can assume that $<\bar{y}> = 0$ and $\sigma^2(\tau)$

becomes simply $< (\bar{y})^2 >$. Since $< (\bar{y})^2 >$ implies an infinite time average this variance is an ideal theoretical concept which is commonly called the true variance $I^2(\tau)$. In practice it is clear that one can only do measurements either for a finite time or on a finite number of samples and thus obtain an estimate of this ideal statistical parameter. Furthermore it is found that $I^2(\tau)$ diverges for certain types of noise such as flicker frequency fluctuations. In order to avoid these problems, various scientists have proposed several types of variances obtained from limited amount of samples. As will be explained below, one of them, the "two - adjacent - sample variance" studied by Barnes and Allan, has been proposed as a time domain measurement of frequency stability [1].

An attempt at measuring time domain frequency stability

Before defining the two sample variance let us examine the process of frequency measurement itself. We assume that a digital frequency counter is used to measure the frequency. The measurement is then made over a finite time τ and one obtains an average of the frequency over this time interval τ. In other words, the counter gives the number of cycles n_k during the time interval τ. Figure 1 is an experimental arrangement by which the frequency of oscillator (1) is measured, oscillator (2) being used as the time base of the counter. For the purpose of simplifying the picture let us assume that oscillator (2) is perfect in the sense that its frequency is free of fluctuations. All fluctuations in the measurements would then come from frequency fluctuations of oscillator (1). The counter takes measurements in the sequence shown in figure 2(a). The result for N measurements may be as shown in figure 2(b). Here $\tau = t_{k+1} - t_k$ and

$$\bar{\nu}_{k\tau} = \nu_0 \, (1 + \bar{y}_{k\tau}) \quad , \tag{8}$$

$$< \bar{\nu} > \simeq \frac{1}{N} \sum_{i}^{N} \bar{\nu}_{i\tau} \quad . \tag{9}$$

The average value of the random variable $\bar{\nu}$ is only an estimate of the actual average frequency, average being done on N samples. One may then calculate for the N samples the variance:

$$\sigma_{\bar{\nu}}^2 = \frac{1}{N-1} \sum_{j=1}^{N} (\bar{\nu}_j - < \bar{\nu} >)^2 \quad . \tag{10}$$

In order to continue the analysis it is assumed that the oscillator does not show systematic drifts with time. If such drifts are present, they are removed from the data and the following analysis applies.

The variance of the random variable $\bar{y}$ may be readily written as:

$$\sigma_y^2(N,T,\tau) = \frac{1}{N-1}\sum_{k=1}^{N}\left(\bar{y}_k - \frac{1}{N}\sum_{j=1}^{N}\bar{y}_j\right)^2 \quad . \tag{11}$$

1. First we note that $\sigma_y^2(N,T,\tau)$ is itself a random variable. It is an estimate of the true variance $I^2(\tau)$ made on N samples. Its average value made on several sets on N samples, $\sigma_y^2(N,T,\tau)$, should be close to the true variance. At the limit where N tends to infinity it should be equal to $I^2(\tau)$.

2. The results of the experiments discussed above can be used to form an histogram as illustrated in figure 3. If the number N of samples is large enough the figure may be rather smooth and a good estimate of $\sigma_y^2(N,T,\tau)$ can be obtained from this curve through a measurement of the half width at half the height. (For a normal distribution $\sigma = 1.17\ (\Delta\bar{y})_{\frac{1}{2}}$).

3. If the experiment is repeated for other averaging times τ, the results obtained with the histogram technique may behave as shown in figure 4. The value of $\sqrt{\sigma_y^2(N,T,\tau)}$ for each of the histograms may then be plotted as a function of τ. The results are shown in figure 5. These results appear very interesting, and give in a sense an indication of the frequency stability of the oscillator in the time domain. However, several difficulties arise when the technique of measurements is changed. For example it is observed that in the region where the variance varies as $1/\tau^2$ an increase in the number of samples N does not alter its value, providing this number of samples is large enough. This is not the case in the region where $\sigma_y^2(N,T,\tau)$ is independent of τ. In that region an increase in the number of samples N shows up by an increase of the variance. One is then faced with a problem of a variance whose values depend on the number of samples. Furthermore, in the region where σ^2 varies as $1/\tau^2$ the value of σ^2 depends on the frequency bandwidth of the measurement system.

The two-sample variance

In order to avoid these problems, and to facilitate intercomparison between the results reported by workers in various fields, one has then to make a choice on N and preferably the ratio T/τ. It has been proposed, following the work of Allan [13]:

1) that the following weighed sample variance be used:

$$\sigma(N,T,\tau) = \frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{y}_i - \frac{1}{N}\sum_{j=1}^{N}(\bar{y}_j)\right)^2 \quad ; \tag{12}$$

2) that $N = 2 \rightarrow$ two-sample variance ;

3) that $\frac{T}{\tau} = 1$, or no dead time between measurements [1]. This variance is a random variable and its average, abreviated

$\sigma_y^2(\tau)$, is given by:

$$\sigma_y^2(\tau) = \langle \sigma_y^2 (2, T, \tau) \rangle = \frac{1}{2} \langle (\overline{y}_{k+1} - \overline{y}_k)^2 \rangle \quad . \tag{13}$$

It is proposed as a characterization of frequency stability in the time domain.

This variance has very interesting properties:

1. It is standardized in the sense that N and T are fixed.

2. It is equal to the true variance $I^2(\tau)$ for white frequency noise. It is equal to $3/2\, I^2(\tau)$ for white phase noise thus close to it.

3. It is convergent for all types of frequency noise normally encountered in oscillators, that is the five power laws mentioned earlier.

4. Although by definition, one is still faced with an average on an infinite number of samples (in this sense $\sigma_y^2(\tau)$ is still an idealization), good estimates of it can be obtained by a relatively limited number of measurements, m , of the pairs:

$$\sigma_y^2(\tau , m) \approx \frac{1}{m} \sum_{j=1}^{m} \frac{1}{2} (\overline{y}_{k+1} - \overline{y}_k)_j^2 \quad . \tag{14}$$

For $n > 10$ it has been shown that the confidence interval to be associated to $\sigma_y(\tau)$ in such an estimate if of the order of [14].

Confidence interval = $K_\alpha \sigma_y(\tau) / \sqrt{m}$,

where K_α is a constant depending on the power law predominent, but is not far from unity.

5. Finally, tables have been compiled which translates from one type of variance to another in relation to variations in N and the ratio T/τ , and this, for the five power laws commonly encountered in oscillators [15]. Of particular interest is the bias function

$$B_2 (\frac{T}{\tau} , u) = \frac{\langle \sigma_y^2 (2, T, \tau) \rangle}{\langle \sigma_y^2 (2, \tau, \tau) \rangle} \quad ,$$

since in general a simple counter will be characterized by a dead time ($T/\tau \neq 1$) between successive measurements.

The main disadvantages of $\sigma_y^2(\tau)$ are:

1) it diverges for power law spectral densities greater than -2;

2) it does not discriminate between white phase noise and flicker of phase noise.

Relation between the time domain and the frequency domain: other types of variance

The time domain frequency stability, as characterized earlier either through the true variance or the two-sample variance, can be interpreted in a different way. The operation of the counter, averaging the frequency for a time τ may be thought of as a filtering operation. The transfer function , $H(f)$, of this equivalent filter is then the Fourier transform of the impulse response $h(t)$. It can be shown that the time domain frequency stability is then given by [16], [17].

$$\langle \sigma^2 (N, T, \tau) \rangle = \int_0^\infty S_y(f) \, | H(f) |^2 \, df \quad , \tag{15}$$

where $S_y(f)$ is the spectral density of frequency fluctuations. In the case of the true variance and of the Allan variance, we have:

$$h_I(t) = \begin{cases} 0 & t < -\tau \\ \frac{1}{\tau} & -\tau < t < 0 \\ 0 & t > 0 \end{cases} \quad , \qquad h_A(t) = \begin{cases} 0 & t < -\tau \\ \frac{1}{\sqrt{2}\,\tau} & -\tau < t < 0 \\ -\frac{1}{\sqrt{2}\,\tau} & 0 < t < \tau \\ 0 & t > \tau \end{cases} \quad , \tag{16}$$

$$H_I(f) = \frac{\sin \frac{\omega\tau}{2}}{\frac{\omega\tau}{2}} \quad , \qquad H_A(f) = \frac{\sin \frac{2\omega\tau}{2}}{\left(\frac{\omega\tau}{2}\right)^2} \quad . \tag{17}$$

These relations are illustrated in figure 6.

This "transfer function approach" has been exploited by several authors to elaborate new types of variances for characterizing oscillator frequency stability in the time domain. It is the equivalent of digital filtering used in data processing.

Hadamard Variance

The sequence of measurements shown in figure 6 for the Allan variance, which consists of two samples ($N = 2$) can be expanded to a sequence of a greater number of samples. A sequence for the case where $N = 10$ is shown in figure 6(e), where the impulse response of the equivalent filter is plotted as a function of t . The variance for this sequence is then [18]:

$$\langle \sigma_H^2 (N, T_D, \tau) \rangle = \langle (\bar{y}_1 - \bar{y}_2 + \bar{y}_3 - \ldots - y_N)^2 \rangle \quad , \tag{18}$$

where T_D is the dead time between measurements. The square modulus of the transfer function of this filter is [20]:

$$|H_H(f)|^2 = \left(\frac{\sin \pi \tau f}{\pi \tau f}\right)^2 \left(\frac{\sin N \pi T f}{\cos \pi T f}\right)^2 \quad . \tag{19}$$

It is illustrated in figure 6(f) for no dead time and $N = 10$. The characteristics of this variance, interesting for the topic of frequency stability characterization are as follows:

a) For the case of no dead time ($T_D = 0$), the transfer function of the equivalent filter has a main lobe, centered at $f_1 = 1/2\tau$, and whose width is equal to:

$$f = \frac{\pi^2}{16 \tau N} \text{ (equivalent rectangular filter)}$$

It may thus be made very narrow by increasing N. In this sense this filtering process is well suited for spectral analysis and this property has been exploited to obtain the spectral density of frequency fluctuations using time domain measurements [20]. This is easily seen from relation (15) by realizing that $|H(f)|^2$ can be approximated by a narrow square window over which $S_y(f)$ does not vary much. The spectral density is then given by:

$$S_y(f_1) \simeq \frac{\tau}{N} < \sigma_H^2 (N, \tau, \tau) > \tag{20}$$

This is seen to lend itself to a straightforward computation, in order to obtain an estimate of the spectral density without the use of a spectrum analyser.

b) However, one should be aware of severe limitations in this technique. There exist secondary side lobes in $H_H(f)$, which appear at harmonics of $f_1 = 1/2\tau$, for no dead time between measurements. These can be minimized by proper adjustment of the dead time between measurements or by proper weighing of the samples of the measurements. The properties of the transfer function have been well studied in the case of a weighing by the binomial coefficients and by a pseudo-sinusoïdal function [18], [19], [20].

Modified sample variance

Boileau and Picinbono have introduced a variance which can be interpreted in the case of its digital realization as a modified sample variance. [6] Their relations, when translated into the notation generally adopted in the field of frequency stability gives a variance as follows:

$$\sigma^2_{mod}(N, T, \tau) = \bar{y}_{(N+1)/2} - \frac{1}{N} \sum_{i=1}^{N} \bar{y}_i \Big)^2 \quad , \tag{21}$$

where N is odd. Thus $y_{(N+1)/2}$ is the central sample of the set of N

samples. Rutman has examined the case when $N = 3$ and $T = \tau$ [8]. The measurement sequence and the square modulus of the transfer function are shown in figure 6(g) and 6(h) respectively. The variance is given by:

$$< \sigma^2_{mod} (3, \tau, \tau) > = \int_0^\infty S_y(f) \frac{16}{9} \frac{\sin^6 \pi \tau f}{(\pi \tau f)^2} df \quad . \tag{22}$$

The modified sample variance has the advantage of being convergent for all five power law spectral densities examined up to now, plus two others, where $\alpha = -3$ and $\alpha = -4$. However, it does not discriminate between white phase noise and flicker phase noise better than $\sigma_y(\tau)$.

A special case of the sample variance

It is general practice to study the frequency stability or measure the variance as a function of averaging time τ. This is what is done in the sample variance described above. In opposition to this practice, De Prins and Cornelissen [7] have studied frequency fluctuations over intervals T for fixed averaging times τ. One can then in principle study the variance $\sigma_y^2(T)$ of the instantaneous frequency ($\tau \to 0$) as a function of the time interval T. This variance is very different from the one described previously. In the present case all values of $y(t)$ are scanned as T is varied in opposition to the averaging over τ considered up to now.

In the case where τ does not tend to zero but is fixed at a given value, the variance then becomes a special case of the sample variance and the behaviour of $\sigma_y^2(T)$ can be obtained from the bias function, B_2.

The high pass variance

A close look at equation (15) suggests that $\sigma^2(\tau)$ can actually be defined through the transfer function $H(f)$ of the equivalent filter corresponding to the measurement sequence. Rutman has suggested that this approach could be taken even if the actual measurement sequence was not existing [5]. Then, $H(f)$ could be given the shape desired. Of course the inverse Fourier transform of $H(f)$ is not necessarily a step wise function that could be implemented in a straightforward manner by a counting technique. Other measurement techniques have then to be implemented.

In this approach the variance is written:

$$\sigma^2(\tau) = \frac{1}{\pi^2 \nu_0^2 \tau^2} \int_0^\infty S_\varphi(f) \, | H_\varphi(f) |^2 \, df \quad , \tag{23}$$

where $S_\varphi(f)$ is the phase spectral density and is related to $S_y(f)$ through relation (6). The variance is then defined in terms of $H_\varphi(f)$ and not in terms of the measurement sequence.

From equation (23) one sees that for the Allan variance, the square modulus of the phase transfer function of the equivalent filter is

$$| H_{\varphi_A}(f) |^2 = \sin^4 \pi f \tau \ . \quad \text{(See figure (7))} \tag{24}$$

This is essentially a high pass filter, having an oscillating nature with a period $1/\tau$. Low frequency components $f < (\pi\tau)^{-1}$ are filtered out, this being an essential character of the Allan variance. Consequently it appears that $| H_{\varphi}(f) |^2$ could be essentially a high pass filter and essentially the same character for the variance would be obtained. In fact, calculations show that, when a second order high pass filter, with cut off frequency $f_c = (\pi\tau)^{-1}$, is used to calculate a so called "high pass variance", the general behaviour with the power law spectral density is essentially the same as the behaviour of the Allan variance. Both variances have the same asymptotic slopes with τ and both variances cannot differentiate between white phase noise and flicker phase noise.

Band pass variance

Following this line of thought and recognizing the nature of the limitations of the high pass variance, Rutman [8] has suggested that a bandpass filter be used for $| H_{\varphi}(f) |^2$ with a center frequency equal to $(1/2\tau)$ and a constant Q factor, say equal to 1. In that case the behaviour of $\sigma_{BP}(\tau)$ is quite different from that of $\sigma_y(\tau)$ or $\sigma_{HP}(\tau)$; it shows complete discrimination between the five power law models in its asymptotic behaviour as a function of τ.

Of course, the method for measuring $\sigma_{BP}(\tau)$ is not a conventional one incorporating a frequency counter. One uses a phase comparator (loose-phase-lock technique), a bandpass filter and a r.m.s. voltmeter. In this sense, it is the same type of system as the one used in reference [17] and essentially falls in the class of systems used for studying frequency stability in the frequency domain. It appears natural to think of frequency stability measurements in the time domain as being done through a time sequential technique and a statistical analysis of the resulting data. This should be kept in mind in the practical implementation of systems designed for the measurement of frequency stability in the time domain.

An unified approach

In the previous sections, various types of variance were examined. The approach taken has been one in which the measurement sequence was identified; the transfer function of the equivalent filter implementing the impulse response for the sequence in question was established, and the variances could be calculated through relation (15). This method is very useful in pointing out the limit of utilization of a particular variance in respect to the power law frequency model and also in understanding the reason of these limits.

These variances however have all been introduced as particular cases for special needs. Recently, Lindsey, Chie, Leavitt and Lewis [21], [22],[23] have introduced in the picture a method of analysis called the "structure function approach", which emphasizes the fundamental ties between these variances rather than their differences.

The k^{th} average frequency fluctuation over time τ can be written as a difference of phase:

$$\overline{y}_k = \frac{1}{2\pi \nu_0 \tau} \left[\varphi(t_k + \tau) - \varphi(t_k) \right] \quad . \tag{25}$$

We may define the first difference or first increment in phase as:

$$\Delta^{(1)} \varphi(t_k, \tau) = \varphi(t_k + \tau) - \varphi(t_k) \quad . \tag{26}$$

The frequency difference $(\overline{y}_{k+1} - \overline{y}_k)$ appears as a second difference in phase:

$$(\overline{y}_{k+1} - \overline{y}_k) = \frac{1}{2\pi \nu_0 \tau} \left[\varphi(t_k + \tau) - 2\varphi(t_k) + \varphi(t_k - \tau) \right] \quad , \tag{27}$$

and the second phase increment is defined as:

$$\Delta^{(2)} \varphi(t_k, \tau) = \varphi(t_k + \tau) - 2\varphi(t_k) + \varphi(t_k - \tau) \quad . \tag{28}$$

It is readily realized that the first difference in phase is used in the definition of the true variance, while the second difference in phase is used in the definition of the Allan variance (two samples). Lesage and Audoin [20] have proposed to continue the process further and have obtained an expression for the n^{th} difference in phase, which includes the binomial coefficient as a weighing factor. This analysis has led them to the implementation of the Hadamard variance in which the measurement sequence is weighed by the binomial coefficients.

From this it appears that a common basis may be expected under the definition of the various variances examined above. The rank of the phase increment appears to play a major role. In the approach of Lindsey and Chie this point is stressed. The N^{th} phase increment is defined as:

$$\Delta^N \varphi(f, \tau) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \varphi(t + (N-k)\tau) \quad , \tag{29}$$

where $$\binom{N}{k} = \frac{N!}{k!\,(N-k)!} \quad \text{(Binomial Coefficient)} \quad . \tag{30}$$

The structure function of phase is then defined as:

$$D^{(N)}(\tau) = < (\Delta^{(N)} \varphi(t, \tau) > \quad . \tag{31}$$

Stationnarity of the N^{th} difference is assumed in the wide sense. It is then shown that the variances defined earlier can all be expressed in terms of these structure functions:

True variance

$$I^2(\tau) = \frac{1}{(2\pi \nu_0 \tau)^2} D_{\varphi}^{(1)}(\tau) \quad ; \tag{32}$$

Allan variance

$$\sigma_y^{\,2}(\tau) = \frac{1}{2(2\pi \nu_0 \tau)^2} \quad D_{\varphi}^{(2)}(\tau) \quad ; \tag{33}$$

Modified three-sample variance

$$< \sigma^2_{y\,\mathrm{mod}}(3, \tau, \tau) > = \frac{1}{9(2\pi \nu_0 \tau)^2} D_{\varphi}^{(3)}(\tau) \quad ; \tag{34}$$

Hadamard variance (weighed by the binomial coefficients)

$$< \sigma^2_{\mathrm{H.B.C.}}(N, \tau, \tau) > = \frac{1}{(2\pi \nu_0 \tau)^2} D_{\varphi}^{(N)}(\tau) \quad ; \tag{35}$$

This approach thus clearly shows that the variances utilized up to now by various authors have a common basis, in occurence, a structure function of phase. On the other hand, this structure function is related to the spectral density through the relation:

$$D_{\varphi}^{N}(\tau) = 2^{2(N-1)} (2\pi \nu_0 \tau)^2 \int_0^{\infty} S_y(f) \frac{\sin^{2N} \pi \tau f}{(\pi \tau f)^2} \, df \quad , \tag{36}$$

which effectively, as stressed earlier, provides means for evaluating the spectral density $S_y(f)$ as filtered with a transfer function

$$H_{sf}(f) = \frac{\sin^{2N} \pi \tau f}{(\pi \tau f)^2} \, x, \; 2^{\,2(N-1)} \tag{37}$$

through sequential measurements in the time domain.

In the previous paragraphs a structure function of phase was introduced as a means for describing frequency stability in the time domain. A structure function of frequency, however, can also be introduced to describe frequency stability; it is written $D_y^{(N)}(\tau)$. Lindsey and Chie [22] have shown, by studying the mathematical differences between these structure functions, that the true variance is essentially a measure of phase instability while the Allan variance is a measure of frequency instability. This type of reasoning has led them to suggest that a function of the product of the two types of variance could be a parameter by which frequency stability in the time domain could be characterized.

It should be pointed out that the characterization of the stability of an oscillator could in principle be made through tables of the structure functions $D^{(N)}(\tau)$. The user could calculate from these tables the type of variance that is best suited for his particular application. In a sense, these structure functions can be thought of, as characterizing completely the frequency stability of the oscillator in the time domain in the same sense as $S_y(f)$, does it in the frequency domain.

Long time frequency fluctuations

In real oscillators it is possible to observe very long term frequency fluctuations, that is, very slow fluctuations which may appear over periods of days, months and years. These may originate either from slow random fluctuations or from deterministic drifts in the behaviour of the oscillator.

Slow random fluctuations

The above analysis was limited to five power laws of the spectral density ($-2 < \alpha < +2$). Very slow frequency fluctuations predominate at very low frequencies and are thus represented by more negative slope power laws such as $\alpha = -3$ or -4. The greatest negative slope that the Allan variance can handle is $\alpha = -2$, a random walk of frequency type of noise; for more negative slopes it diverges. Since direct spectral analysis of these slow fluctuations is not experimentally feasible, it appears that the other types of variance mentioned earlier may be useful. In fact the modified three sample variance $\sigma^2_{y\,mod}(3, \tau, \tau)$ converges for f^{-3} and f^{-4} types of noise with respective slope τ^2 and τ^3. Consequently for very slow frequency fluctuations one may then have to use a variance different from the two sample variance in order to have meaningfull interpretation of time domain data.

Deterministic drifts

Systematic drifts are generally observed in oscillators. These drifts may be represented by a polynomial [9]. A model for fractional frequency drifts is:

$$y(t) = d_1 t^1 + d_2 t^2 + \ldots d_n t^n \quad . \tag{38}$$

For the first term, representing a linear frequency drift, the Allan variance varies as τ^2. It is time dependent for higher order drifts. The other variances mentioned above consequently are found very useful in characterizing these higher order polynomial drifts. In particular the modified three sample variance varies as τ^2 for quadratic frequency drifts while the Hadamard variance weighed by binomial coefficients varies as τ^3 for cubic frequency drifts. The behaviour of the asymptotic value of $\sigma(\tau)$ versus τ for various power laws of spectral density and various orders of frequency drifts are summarized in table 1. It should be under-

stood that the structure function approach can also be applied to analyze these long term frequency fluctuations, although emphasis has not been placed on this point in this paper.

In this section we have examined the properties of several types of variances that can be used for characterizing frequency stability of oscillators in the time domain. Differences between these variances have been shown; their interrelation through the fractional frequency fluctuations spectral density has also been emphasized. In fact, it has been shown that these variances, in the case of random fluctuations, are essentially elegant means of representing the spectral density $S_y(f)$ through parameters which can be measured with simple systems implemented with a frequency counter and a calculator for doing statistical analysis. Such systems will now be described.

B - TIME DOMAIN MEASUREMENT SYSTEMS

Ideally, frequency stability measurements require a frequency reference much more stable than the oscillator to be studied. A system realizing this condition can be implemented easily for the measurement of the frequency stability of most common oscillators. However, for stable, state-of-the art, oscillators, it is necessary that the reference oscillator be at least as stable as the oscillator studied. Frequency stability measurements in the time domain can be done by two different methods which are related to the detection of two different parameters: frequency and phase.

One can determine the mean frequency over finite observation times and calculate, for a given number of samples, a certain variance as described in the previous section. Although these measurements involve the well developed technology of frequency or period counting, their use is limited by the fact that they give information only on the mean frequency and the frequency fluctuations. Furthermore, in many systems, the measurement samples are not adjacent in time which, in some cases, alter the value of the statistical parameter calculated.

Many experimental set up's have been considered during the last several years, as the technology evolved (see for example reference [24], [25] and [26]. We shall limit our discussion to a few of them illustrating their main principles.

The most simple set up is the "Direct Frequency Counting System" shown in figure 8. In this case the reference oscillator is the frequency counter time base. It is suitable for the study of low performance oscillators. The output of the counter gives the mean frequency over a preselected time interval. Its recording on a trip chart, magnetic tape or in a digital memory allows one to use best estimate curve fitting methods, in order to find any systematic frequency trends or drifts. These drifts are removed prior to statistical analysis. The two sample variance (Allan

variance) or any other desired variances are calculated from the set of corrected data. The main limitation of this set up is the ± one count and the accuracy of the time base reference.

The "Frequency Heterodyne Technique", shown in figure 9, is another important set up used when the oscillator to be studied is more stable than the frequency counter time base. One must use another oscillator as the reference. First, the oscillator frequency is translated down to a value which can be conveniently measured by the counter. The translation is realized by mixing the oscillator signal with the reference signal set at a convenient different frequency and by detecting only the difference frequency (beat frequency). A synthesizer can be very helpful in many cases. The optional link between the reference and the counter allows an increase in the counter performance and a precise measurement of the absolute frequency. Statistical analysis is done on the beat frequency as described for the former set up. One must refer the fluctuations to the nominal frequency of the oscillator.

If the oscillator and the reference frequencies are very close, one may multiply each one by a different factor, creating a more sizeable beat frequency. Such a system is shown in figure 10. Frequency counting and statistical analysis are achieved as in the previous set up. Unfortunately, noise may be introduced in the multiplier chains which set limits to this technique.

Measurements of stability in the frequency domain require the detection of phase or frequency fluctuations. While equipped to do so, the same set up can be used to do measurements in the time domain. These experimental systems are called "Phase Locked Reference Systems". A voltage controllable reference oscillator is phase-locked to the oscillator signal. When loosely locked, as indicated in figure 11, the phase detector delivers a signal which is proportional to the phase difference between the two oscillators. If the reference is considered more stable than the oscillator, the fluctuations of this signal is attributed to the random changes of the oscillator phase. When this signal is processed through a differentiator, a low frequency signal, proportional to the frequency fluctuations, is obtained. It is then possible to use a voltage to frequency converter driven by this signal, to generate a low frequency oscillation, fluctuating in the same manner as the frequency of the original oscillator. Average frequencies, over time intervals τ, are determined by a counter and statistical analysis is performed as in the previous systems.

If the reference oscillator is tightly locked in phase to the oscillator studied, as shown in figure 12, the command signal, applied to the reference is proportional to the frequency changes between the two oscillators. Again if the reference oscillator is much more stable than the other oscillator, this command signal fluctuates in the same way as does the frequency of the oscillator. Statistical analysis of

this signal is performed with the set up described in the loose phase locked reference case.

The two systems act simply as frequency translators with non unity conversion factor. The sensitivity of these systems is enhanced by a factor proportional to the ratio of the nominal oscillator frequency to the nominal frequency of the converter. Time domain regions, where the results are significant, are determined by the servo loop characteristics. Usually a second order loop is used in order to achieve optimum performance; the parameters to adjust are the natural frequency and the damping factor [27],[28],[29]. The loose Phase Locked Reference System is normally used when frequency stability for averaging times below 1 sec is needed, while the tight Phase Locked Reference System is preferred for longer averaging times.

Obviously these systems are well suited for the measurement of phase or frequency fluctuations in the frequency domain. For this type of measurement, it is necessary to measure the spectral density at the phase detector in the first system or the spectral density of the command signal in the second system. Since for highly stable oscillators the information lies in a spectrum containing frequencies much lower than one hertz, a very low frequency spectrum analyser is required. Digital real time spectrum analysers are indicated for such measurement but they are expensive. The method of bandpass filtering proposed by Rutman can be applied to give measurements either in the time domain or in the frequency domain with these systems [5].

All the experimental systems described previously deal with the measurement of frequencies or periods averaged over a finite observation time, τ. Characterization of frequency stability is done through statistical analysis on an ensemble of these averaged frequencies. It was pointed out in the theoretical section that the frequency stability of an oscillator can also be characterized by a measurement of phase differences. To do so, one has to measure the phase of an oscillator and to calculate phase differences, spaced in time by an interval, τ. Measurement techniques of phase are well developed in the field of time scale implementation since a time scale can be graduated in terms of phase with 2π radians as a unit of time (one period). Thus, measurements of phase differences correspond to time difference measurements.

We will describe two systems for measuring frequency stability by time differences. They are the Dual Mixer Time Difference System [30] and the Phase Modulated Phasemeter [31]. Both systems give about the performance of ±1 picosecond time interval resolution and a limit of resolution of roughly $1 \times 10^{-13}\,\tau^{-1}$ when used to measure fractional frequency stability. Since these systems measure the phase of oscillators, the evaluation of the two-sample variance can be done without dead time and no correction factors are needed. Furthermore the systems can be used to compare oscillators of exactly the same nominal frequency (clock oscil-

lators).

The Dual Mixer Time Difference System is illustrated in figure 13. The two oscillators are of the same type and have the same nominal frequency, ν_0. A common oscillator translates down to ν_b each oscillator frequency in two identical channels. Phase comparison is done between the two beat signals. Zero crossing detection on each signal is achieved and serves to trigger pulse generators. The time interval between each pair of pulses is a measure of the relative phase of the two oscillators; the system acts as a sophisticated phase detector. Any fluctuations of this time interval can be considered as fluctuations of the phase of one or both oscillators. A time interval counter measures these phase (time) differences and statistical analysis is done according to prescribed theoretical calculations.

The translation has the effect of increasing the system resolution by a factor which is approximately ν_0/ν_b. When used to characterize the frequency stability in the time domain, the sampling time is the period or a multiple integer of the beat signal. Consequently a synthesizer used as common oscillator becomes a very convenient tool. Noise contribution from the common oscillator is greatly reduced when the nominal phase shift between the two beat signals is small. An adjustable phase shifter is then placed in series with one oscillator in order to satisfy this condition. (See the Appendix for a calculation on that system.)

The Phase Modulated Phasemeter System is illustrated in figure 14. Again two identical oscillators are compared in phase. Each signal frequency is multiplied, then mixed and filtered to get a beat signal. A non zero frequency beat is generated by modulating the phase of one oscillator with a low frequency signal. Zero crossing detection of this signal gives pulses which can be located in time when compared to a reference signal triggered by the modulating signal. This time interval is a measurement of the phase of one oscillator compared to the other. If one oscillator is considered as a reference, the time interval will be a measurement of the phase of the other oscillator. As in the case of the Dual Mixer Time Difference System, the frequency stability is calculated from this time interval recording for different observation times by simple phase differences and statistical weighing. With this system, the measurement resolution is increased by the up conversion factor.

The various measurement systems described show a certain hierarchy in the parameters evaluated. When one has access to a signal proportional to the phase, he is allowed to calculate any combination of phase differences, then, any variances. In fact, it is possible to calculate all the structure functions of the phase fluctuations which gives a complete measurement of the frequency stability in the time domain. This is possible

with the two last systems and the loose Phase Locked Reference System when the phase is recorded instead of being differentiated. All the other systems described give access only to frequency and its fluctuations, and this limits the amount of statistical information which can be obtained.

APPENDIX: EXPRESSION OF THE FREQUENCY STABILITY IN THE TIME DOMAIN FOR THE DUAL MIXER TIME DIFFERENCE SYSTEM

The phases at the output of oscillator 1 and oscillator 2 are :

$$\Phi_1(t) = \omega_1 t + \theta_1 + \varphi_1(t) \quad ,$$

and $$\Phi_2(t) = \omega_2 t + \theta_2 + \varphi_2(t) \quad ,$$

where ω_1 and ω_2 are the angular frequencies, θ_1 and θ_2 are the initial phase offsets and $\varphi_1(t)$ and $\varphi_2(t)$ are the phase fluctuations. The phase at the output of the common oscillator is :

$$\Phi_R(t) = \omega_R t + \theta_R + \varphi_R(t)$$

with the corresponding meaning for each parameter or variable.

At the input of each mixer, the phase of the oscillator signals can be represented by the equations already given where θ includes the phase shift added by the variable shifter and $\varphi_1(t)$ and $\varphi_2(t)$ include any phase fluctuations added by the transmission links. The phase of each reference signal can be written as :

$$\Phi_{R,1}(t) = \omega_R t + \theta_{R,1} + \varphi_{R,1}(t)$$

$$\Phi_{R,2}(t) = \omega_R t + \theta_{R,2} + \varphi_{R,2}(t)$$

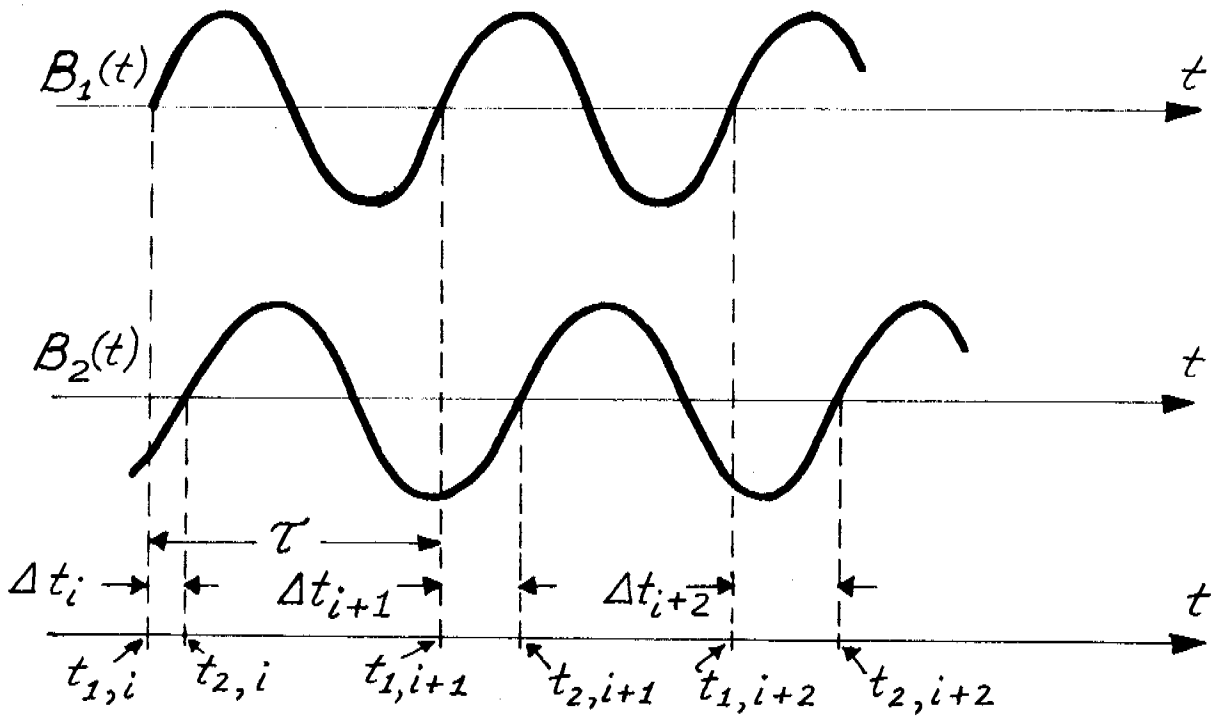
in order to account for any phase shifts and phase fluctuations introduced by the isolation amplifiers and the transmission links. We will see below how these phase perturbations can be made negligeable in the measurement system.

The output of each mixer delivers low frequency beats whose phase are :

$$\Phi_{B,1}(t) = (\omega_R - \omega_1)t + (\theta_{R,1} - \theta_1) + \varphi_{R,1}(t) - \varphi_1(t)$$

$$\Phi_{B,2}(t) = (\omega_R - \omega_2)t + (\theta_{R,2} - \theta_2) + \varphi_{R,2}(t) - \varphi_2(t)$$

These two signals have a time evolution represented as follow :



The positive zero crossings give time events that are a measurement of the relative phase between the two oscillators.

At a certain time, $t_{1,i}$, the phase of the first beat signal is such that the signal is zero; then

$$(\omega_R - \omega_1)t_{1,i} + \theta_{R,1} - \theta_1 + \varphi_{R,1}(t_{1,i}) - \varphi_1(t_{1,i}) = m\,2\pi$$

At a certain time $t_{2,i}$, the phase of the second beat signal corresponds to the same criterion :

$$(\omega_R - \omega_2)t_{2,i} + \theta_{R,2} - \theta_2 + \varphi_{R,2}(t_{2,i}) - \varphi_2(t_{2,i}) = (m+n)2\pi$$

where m and n are integers.

Now if we impose that: $\omega_2 \equiv \omega_1$ and define $\omega_B = \omega_R - \omega_2 = \omega_R - \omega_1$, we obtain the identities

$$\omega_B t_{2,i} + \theta_{R,2} - \theta_2 + \varphi_{R,2}(t_{2,i}) - \varphi_2(t_{2,i}) = (m+n)\,2\pi$$

$$\omega_B t_{1,i} + \theta_{R,1} - \theta_1 + \varphi_{R,1}(t_{1,i}) - \varphi_1(t_{1,i}) = m\,2\pi$$

and by subtraction :

$$\omega_B(t_{2,i} - t_{1,i}) + \theta_{R,2} - \theta_{R,1} - \theta_2 + \theta_1 + \varphi_{R,2}(t_{2,i}) - \varphi_{R,1}(t_{1,i})$$

$$- \varphi_2(t_{2,i}) - \varphi_1(t_{1,i}) = n\,2\pi$$

In this expression, $\theta_{R,2}$ and $\theta_{R,1}$ are constants different by a value introduced by the isolation amplifiers and cable lengths. It is possible to define a constant phase offset,

$$\Delta\theta = \theta_{R,2} - \theta_{R,1} - \theta_2 + \theta_1$$

which can be adjusted by the phase shifter. By doing so, we also set the nominal value of $t_{2,i} - t_{1,i}$; for small time offsets, the two phase fluctuation terms coming from the reference oscillator are correlated and their difference is negligeable when compared to the phase fluctuation difference of the two oscillators. Within this approximation, the time difference becomes :

$$\Delta t_i = t_{2,i} - t_{1,i} = \frac{\varphi_2(t_{2,i}) - \varphi_1(t_{1,i})}{\omega_B} - \frac{\Delta\theta}{\omega_B} + \frac{n\,2\pi}{\omega_B}$$

If we look at the next pair of zero crossings we obtain, in a similar way, the time difference :

$$\Delta t_{i+1} = t_{2,i+1} - t_{1,i+1} = \frac{\varphi_2(t_{2,i+1}) - \varphi_1(t_{1,i})}{\omega_B} - \frac{\Delta\theta}{\omega_B} + \frac{n2\pi}{\omega_B}$$

and for the following pair

$$\Delta t_{i+2} = t_{2,i+2} - t_{1,i+2} = \frac{\varphi_2(t_{2,i+2}) - \varphi_1(t_{1,i+2})}{\omega_B} - \frac{\Delta\theta}{\omega_B} + \frac{n2\pi}{\omega_B}$$

The definition of the two sample variance in term of the second difference of phase is :

$$\sigma^2(\tau) = \frac{1}{2}\left\langle \frac{\varphi(t_{i+2}) - 2\varphi(t_{i+1}) + \varphi(t_i)}{2\pi\nu_0\tau} \right\rangle^2$$

where the averaging time, τ, is the time interval between t_i and t_{i+1} or the time interval between two successive phase measurements of each oscillator. Such a linear combination of phase fluctuations can be obtained by grouping the time differences just derived. Then

$$\Delta t_{i+2} - 2\Delta t_{i+1} + \Delta t_i = \frac{\varphi_2(t_{2,i+2}) - 2\varphi_2(t_{2,i+1}) + \varphi_2(t_{2,i})}{\omega_B} - \frac{\varphi_1(t_{1,i+2}) - 2\varphi_1(t_{1,i+1}) + \varphi_1(t_{1,i})}{\omega_B}$$

If the two oscillators are statistically independent, we can write :

$$(\Delta t_{i+2} - 2\Delta t_{i+1} + \Delta t_i)^2 = \left\langle \frac{\varphi_2(t_{2,i+2}) - 2\varphi_2(t_{2,i+1}) + \varphi_2(t_{2,i})}{\omega_B} \right\rangle^2 + \left\langle \frac{\varphi_1(t_{1,i+2}) - 2\varphi_1(t_{1,i+1}) + \varphi_1(t_{1,i})}{\omega_B} \right\rangle^2$$

and in terms of the two sample variance of each oscillator :

$$(\Delta t_{i+2} - 2\Delta t_{i+1} + \Delta t_i) = 2\left(\frac{\nu_0}{\nu_B}\right)^2 \tau^2 \left[\sigma_2^2(\tau) + \sigma_1^2(\tau)\right]$$

This relation is the corner stone of the Dual Mixer Time Difference System. It shows that a linear combination of the time interval measurements give a value proportional to the two sample variance of each oscillator. When the two oscillators are identical, $\sigma_2^2(\tau) = \sigma_1^2(\tau)$, and the calculated value is twice the value for each oscillator. If one oscillator is much more stable than the other oscillator, $\sigma_1^2(\tau) << \sigma_2^2(\tau)$, the calculated value is directly the value for the instable oscillator. In these calculations, we consider three successive pairs of time event; they are then spaced in time by an interval: $\tau = (\nu_B)^{-1}$, the beat period. It is also possible to skip a determined number of zero crossings between each pair of time events. In this case, the observation time is a multiple integer of $(\nu_B)^{-1}$.

A similar type of calculations applies to the phase modulated phase-meter.

REFERENCES

1. J.A. Barnes, A.R. Chi, S.L. Cutler, D.J. Healey, D.B. Leeson, T.E. McGunigal, J.A. Mullen, W.L. Smith, R.L. Sydnor, R. Vessot and G.M.R. Winkler, "Characterization of Frequency Stability", IEEE Trans. Instrum. Meas. IM 20, pp. 105-120, 1971.

2. IEEE-NASA Symposium on Short Term Frequency Stability, Goddard Space Flight Center, Greenbelt MD, 1964 (Document NASA SP-80).

3. Special Issue on Frequency Stability, Proc. IEEE 54, 1966.

4. R.A. Baugh, "Frequency Modulation Analysis with the Hadamard Variance", Proc. 25th Annual Frequency Control Symposium, Atlantic City, N.J., pp. 222-225, 1971.

5. J. Rutman, "Characterization of Frequency Stability: A Transfer Function Approach and its Application to Measurement via Filtering of Phase Noise", IEEE Trans. Instrum. Meas. IM 23, pp. 40-48, 1974.

6. E. Boileau and B. Picinbono, "Statistical Study of Phase Fluctuations and Oscillator Stability", IEEE Trans. Instrum. Meas. IM 25, pp. 66-75, 1976.

7. J. De Prins and G. Cornelissen, "Power Spectrum, Frequency Stability and Flicker Noise", Proc. Frequency Standards and Metrology Seminar, Laval University, Quebec, Canada, pp. 368-387, 1971 (unpublished).

8. J. Rutman, "Characterization of Phase and Frequency Instabilities in Precision Frequency Sources: Fifteen Years of Progress", Proc. IEEE 66, pp. 1048-1075, 1978.

9. G.M.R. Winkler, "A Brief Review of Frequency Stability Measures", Proc. 8th Annual PTTI Applications and Planning Meeting U.S. Naval Research Laboratory, Washington, D.C., pp. 489-527, 1976.

10. J.A. Barnes, "A Review of Methods of Analyzing Frequency Stability", Proc. 9th Annual PTTI Applications and Planning Meeting (NASA Goddard Space Flight Center Greenbelt MD), 1977.

11. D.A. Howe, "Frequency Domain Stability Measurements: A tutorial Introduction", NBS Technical Note 579, 1976.

12. G. Rovera, "Stabilita di Frequenza", Lezioni tenuto presso la Scuola di Perfezionamento in Fisica dell'Universita di Pisa, 1974 (Pisa Editrice Tecnica Scientifica).

13. D.W. Allan, "Statistics of Atomic Frequency Standards", Proc. IEEE 54, pp. 221-230, 1966.

14. P. Lesage and C. Audoin, "Characterization of Frequency Stability: Uncertainty due to the Finite Number of Measurements", IEEE Trans. Instrum. Meas. IM 22, pp. 157-161, 1973 (Corrections on this paper have been published later in the issues March 1974 and September 1976 of the same Journal).

15. J.A. Barnes, "Tables of Bias Functions B_1 and B_2 for Variances Based on Finite Samples of Processes with Power Law Spectral Densities", NBS Technical Note 375, 1969.

16. L.S. Cutler and C.L. Searle, "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards", Proc. IEEE 54, pp. 136-154, 1966.

17. R. Vessot, L. Mueller and J. Vanier, "The Specification of Oscillator Characteristics from Measurements Made in the Frequency Domain", Proc. IEEE 54, pp 199-207, 1966.

18. R.A. Baugh, "Frequency Modulation Analysis with the Hadamard Variance", Proc. 25th Annual Frequency Control Symp., Atlantic City, N.J., pp. 222-225, 1971.

19. G. Sauvage et J. Rutman, "Analyse Spectrale du Bruit de Fréquence des Oscillateurs par la Variance Hadamard", Ann. Télécom. 28, pp. 301-314, 1973.

20. P. Lesage et C. Audoin, "Instabilité de Fréquence des Oscillateurs - Mesure dans le Domaine Fréquence par une Méthode Temporelle", L'Onde Electrique 55, pp. 82-89, 1975.

21. W.C. Lindsey and J.L. Lewis, "Modeling, Characterization and Measurement of Oscillator Frequency Instability", Report CDRL item No B020, Lincoln Corporation Pasadena, Cal. 91104, 1974.

22. W.C. Lindsey, C.M. Chie and W.E. Leavitt, "Interpretation and Application of Oscillator Instability Measures", Proc. 8th Annual Precise Time and Time Interval Applications and Planning Meeting, U.S. Naval Research Laboratory, Washington, D.C., pp. 535-560, 1976.

23. W.C. Lindsey and C.M. Chie, "Theory of Oscillator Instability Based Upon Structure Functions", Proc. IEEE 64, pp. 1652-1666, 1976.

24. L.S. Cutler, C.L. Searle, "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards", Proc. IEEE, Vol 54, No 2, p. fig. , February 1966.

25. "Time and Frequency: Theory and Fundamentals", B.E. Blair, editor, NBS Monograph 140, U.S. Dept. of Commerce, Washington, D.C., May 1974.

26. "Frequency and Time", P. Kartaschoff, Academic Press, Monograph in Physical Measurement, London, England, 1978.

27. F.L. Walls, S.R. Stein, "Design Considerations in State-of-the-Art Signal Processing and Phase Noise Measurement Systems", Proc. 30th Annual Symposium on Frequency Control, Electronic Industries Ass., pp. 269-274, June 1976.

28. J. Vanier, M. Têtu, L.G. Bernier, "Transfer of Frequency Stability from an Atomic Frequency Reference to a Quartz Crystal Oscillator", Proc. 32nd Annual Symposium on Frequency Control, Electronic Industries Ass., pp. 520-526, June 1978.

29. "Phase-Locked Loops: Application to Coherent Receiver Design", A. Blanchard, Wiley-Interscience publication, 1976.

30. D.W. Allan, H. Daams, "Picosecond Time Difference Measurement System", Proc. 29th Annual Symposium on Frequency Control, Electronic Industries Ass., pp. 404-411, May 1975.

31. A. Gabry, B. Dubouis, D. Landréa, G. Faucheron, "Dispositif de mesure de la stabilité des horloges étalons, utilisant un phasemètre à modulation de phase", Note Technique NT/TCR/DEF/1606, Centre National des Etudes en Télécommunications, 92131 Issy-les-Moulineaux, Mai 1978.

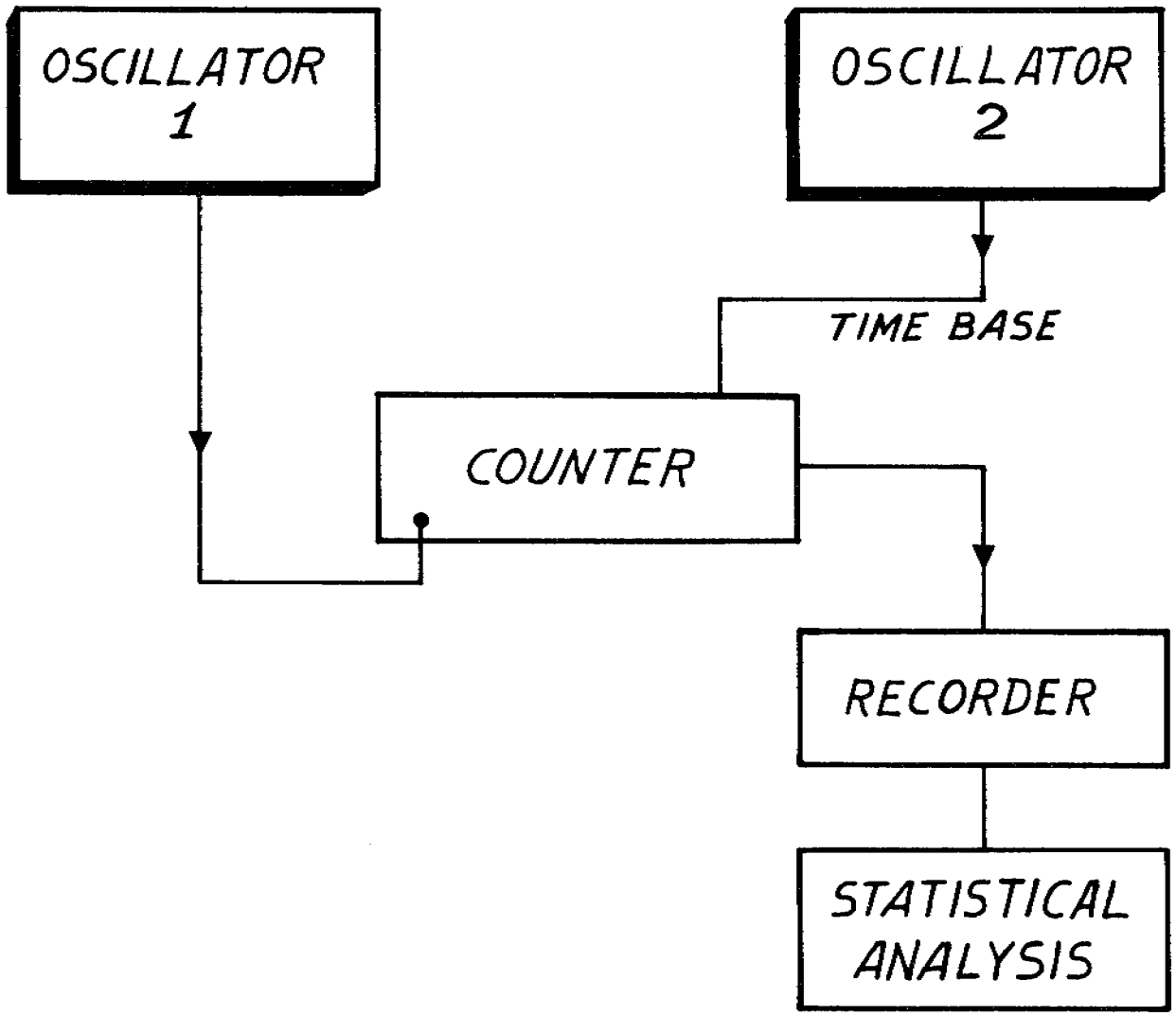


Figure 1. Block diagram of the ideal experimental set up used to measure the frequency of oscillator (1) over an averaging time τ. The frequency of oscillator (2) is assumed to be free of fluctuations.

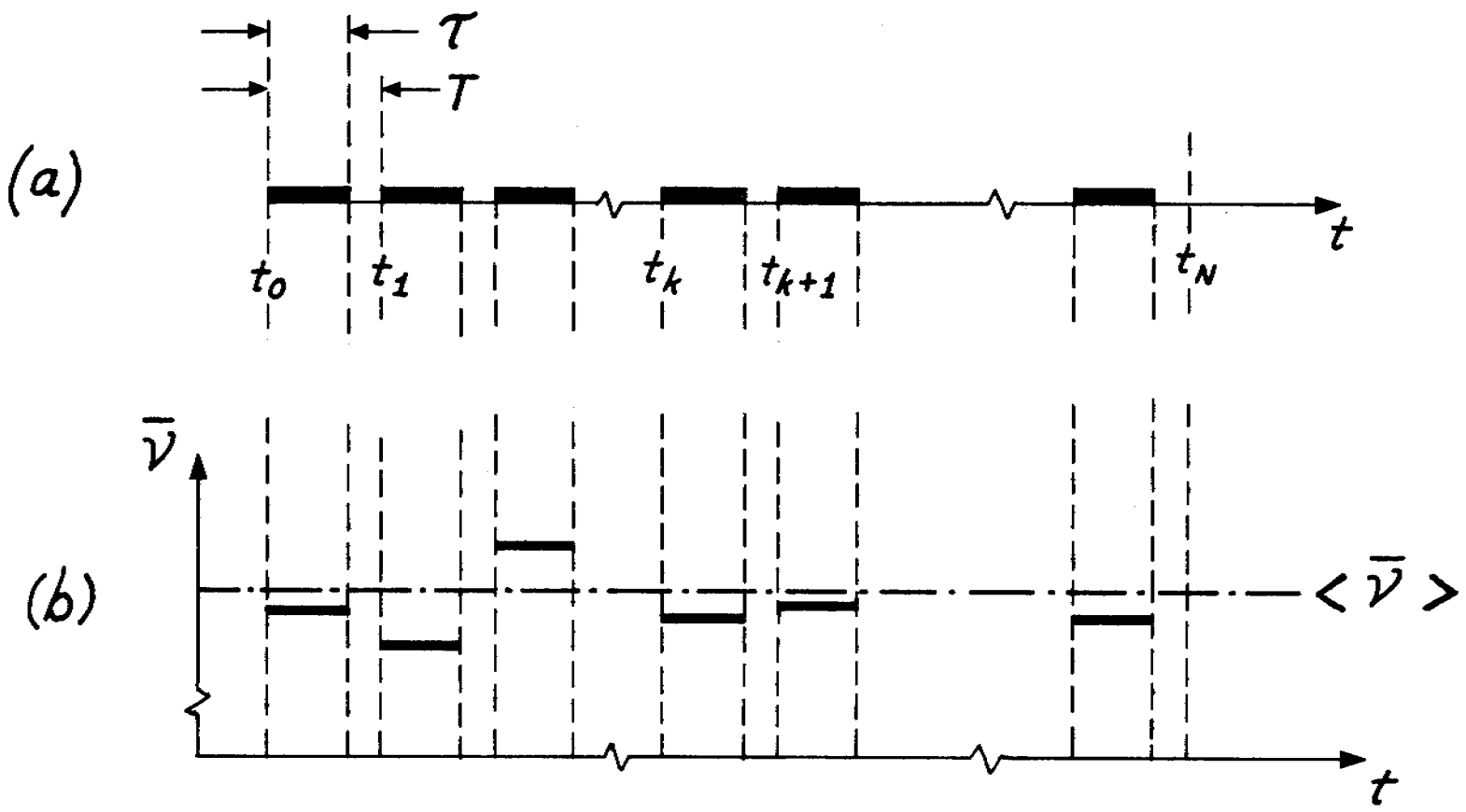


Figure 2. (a) Time sequence used in the measurement of the frequency of an oscillator.
(b) Hypothetical result of the measurements of the frequency of oscillator (1) in the set up of figure (1).

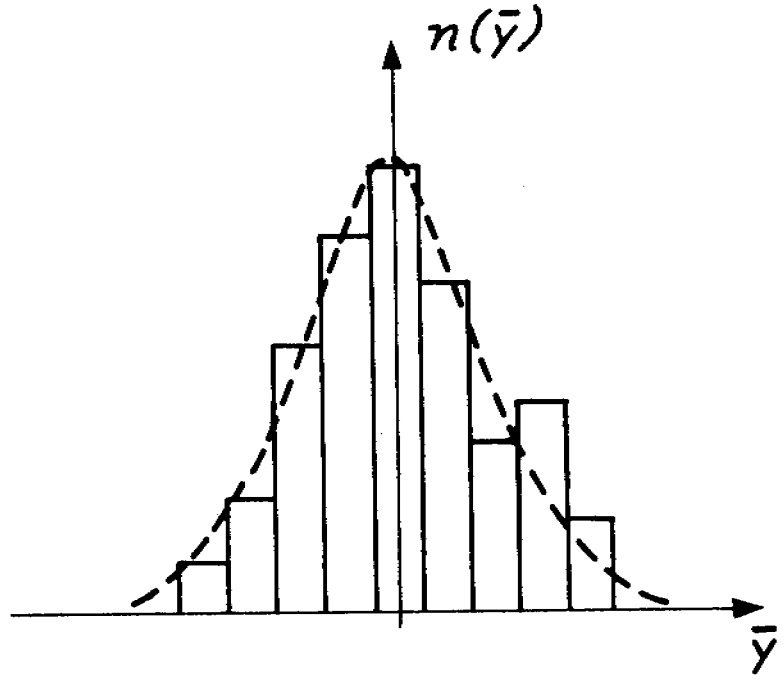


Figure 3. Histogram of the hypothetical results of the frequency fluctuations shown in figure 2.

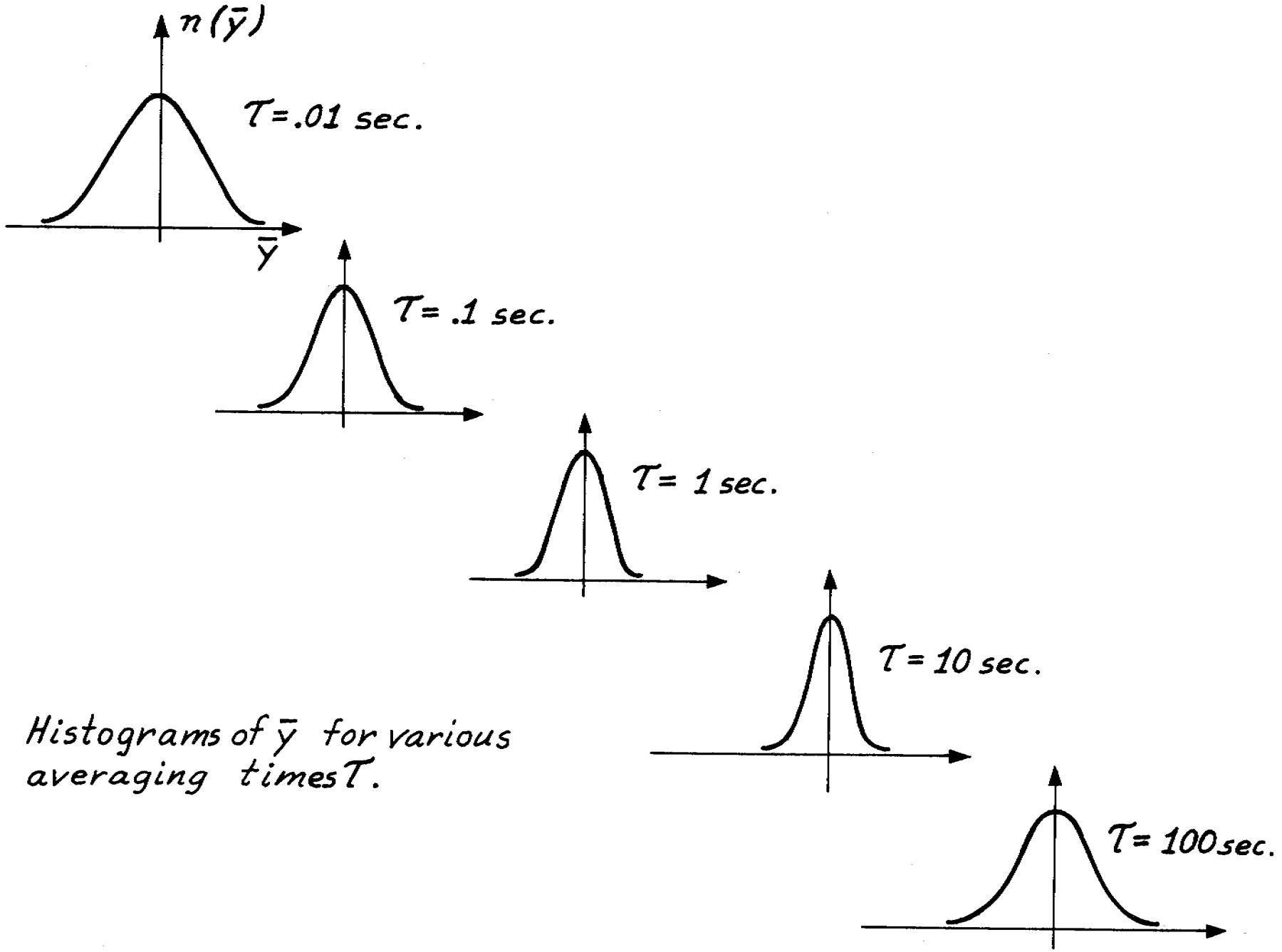


Figure 4. Hypothetical histograms obtained by repeating the measurements of frequency on N samples for various

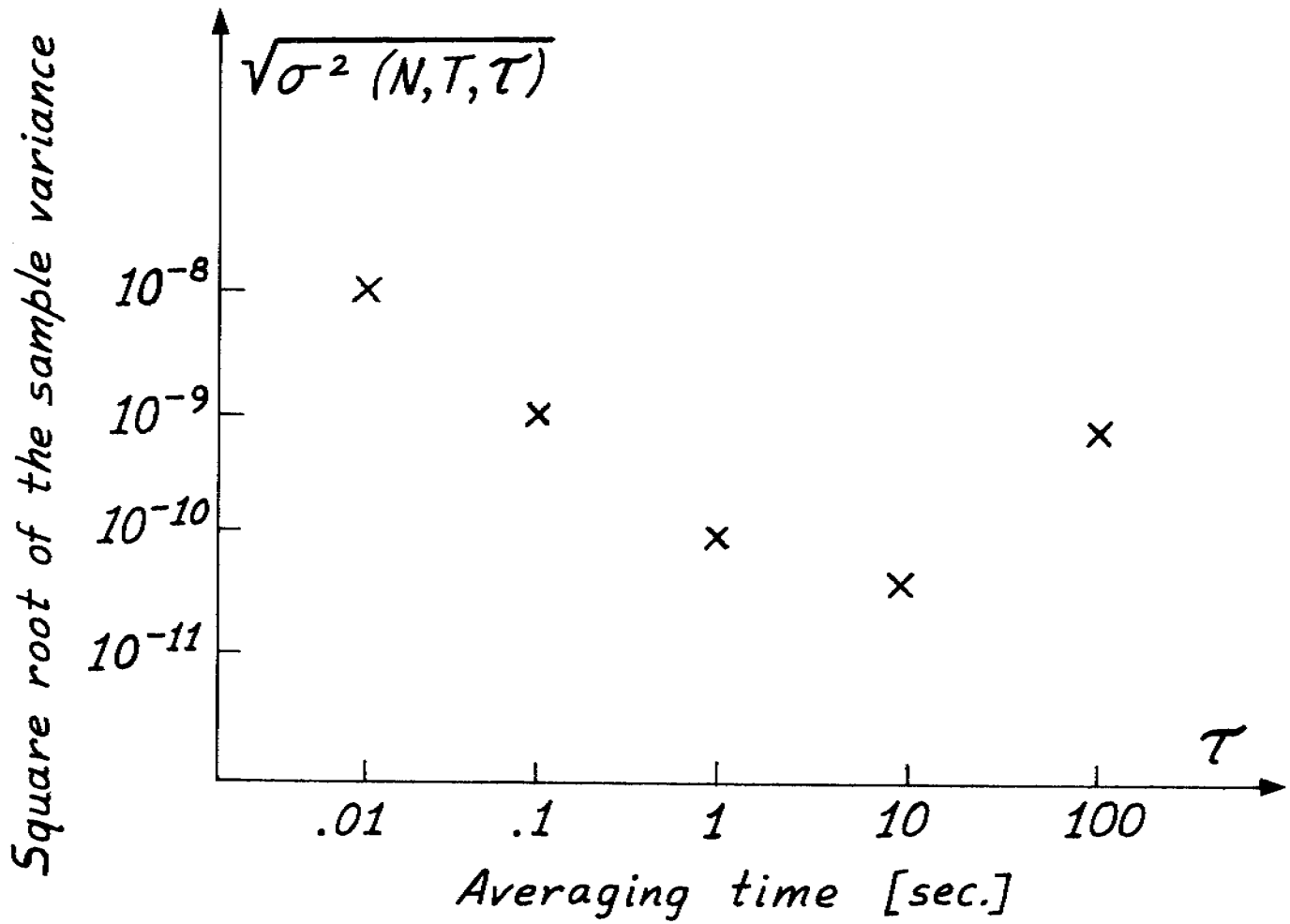


Figure 5. Plot of the square root of the sample variance $\sigma^2(N, T, \tau)$ (standard deviation) obtained for the hypothetical results of figure 4, as a function of the averaging time.

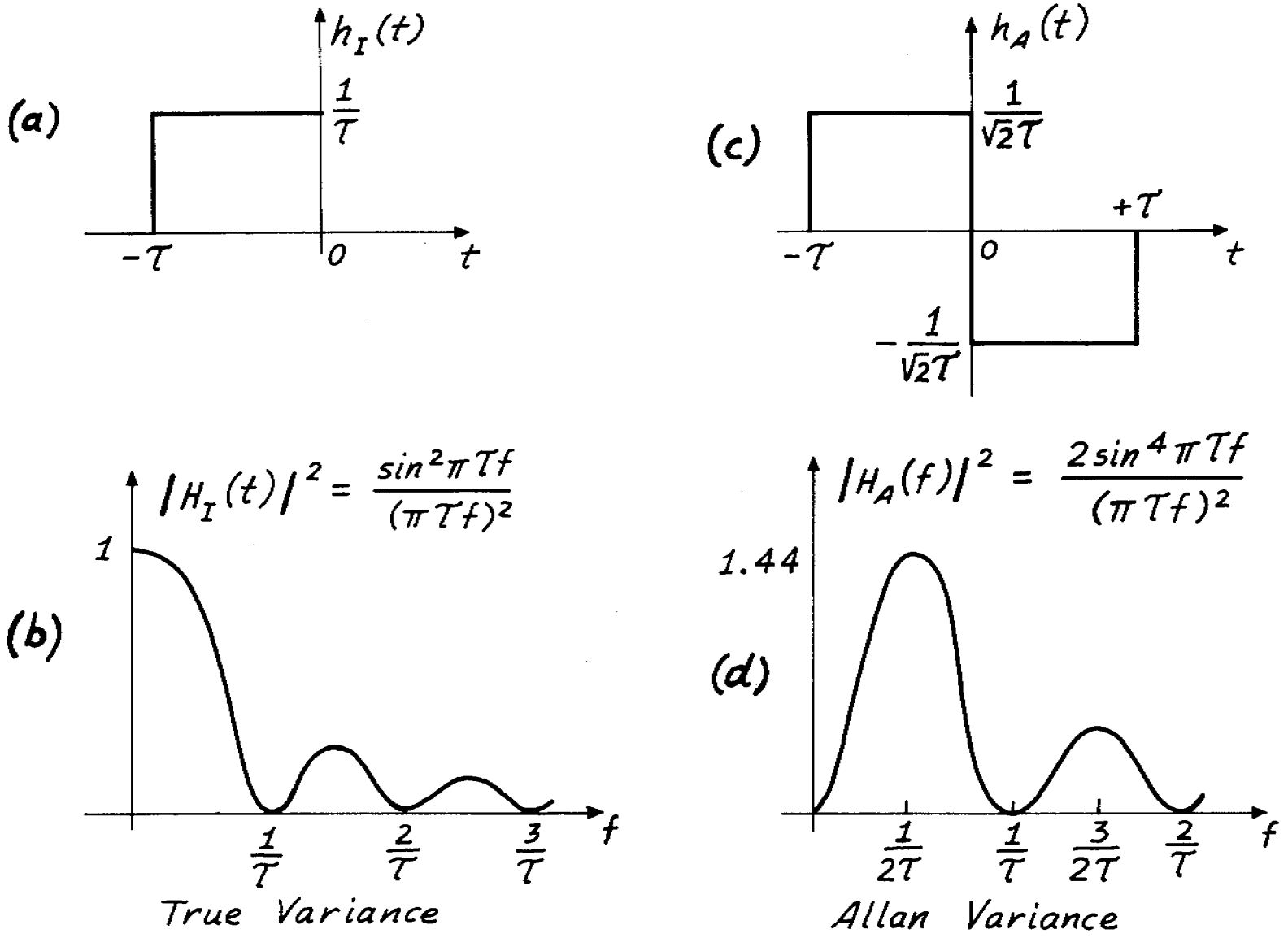
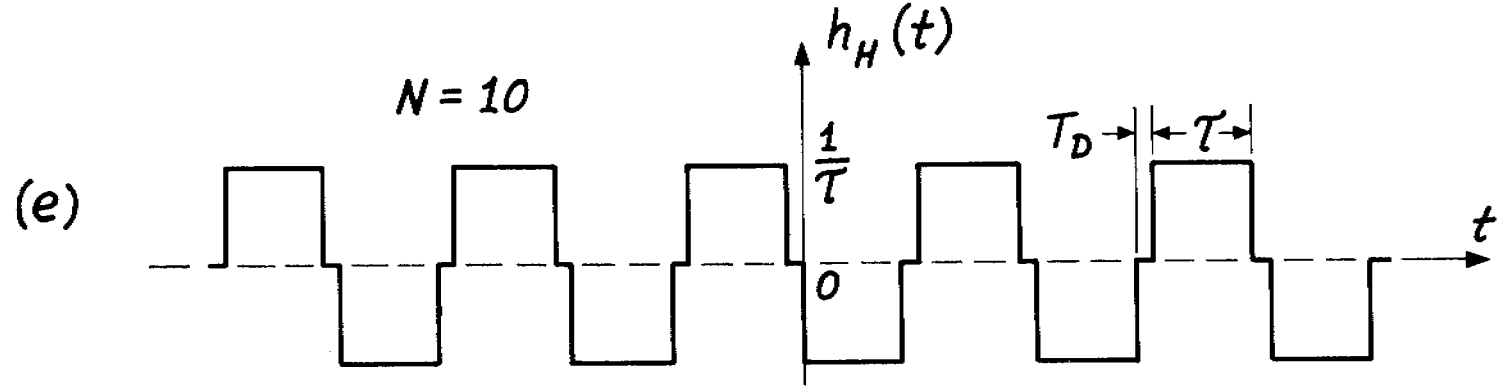


Figure 6. Impulse response h(t) and square modulus of the transfer function H(f) of the equivalent filters used in the calculations of variances (a) and (b): True variance; (c) and (d): Allan variance; (e) and (f): Hadamard variance; (g) and (h): Modified sample variance.



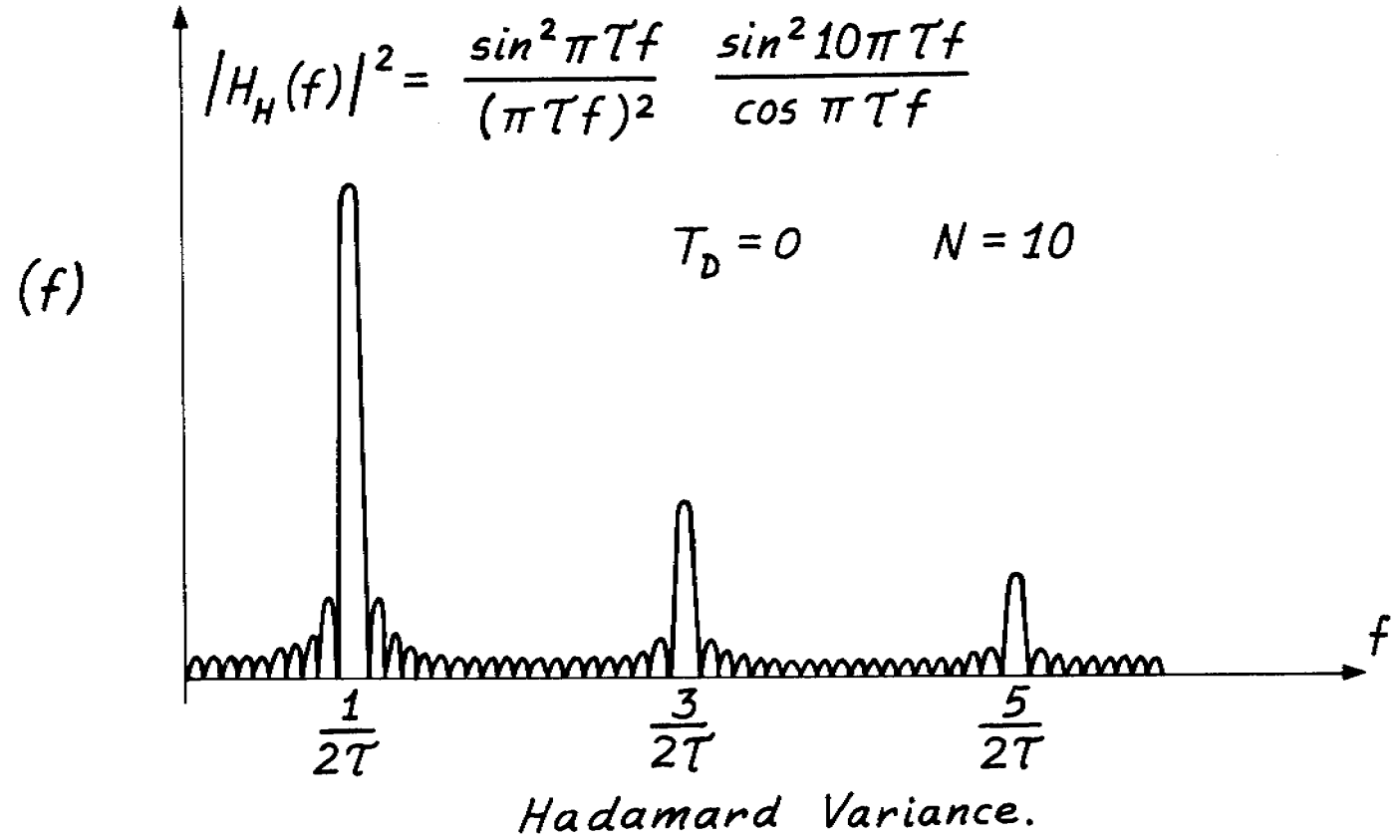
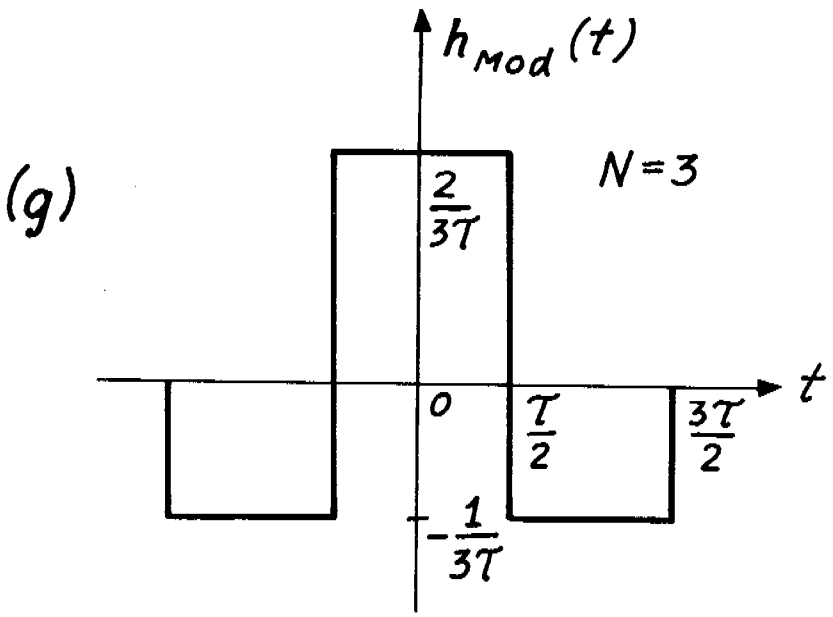
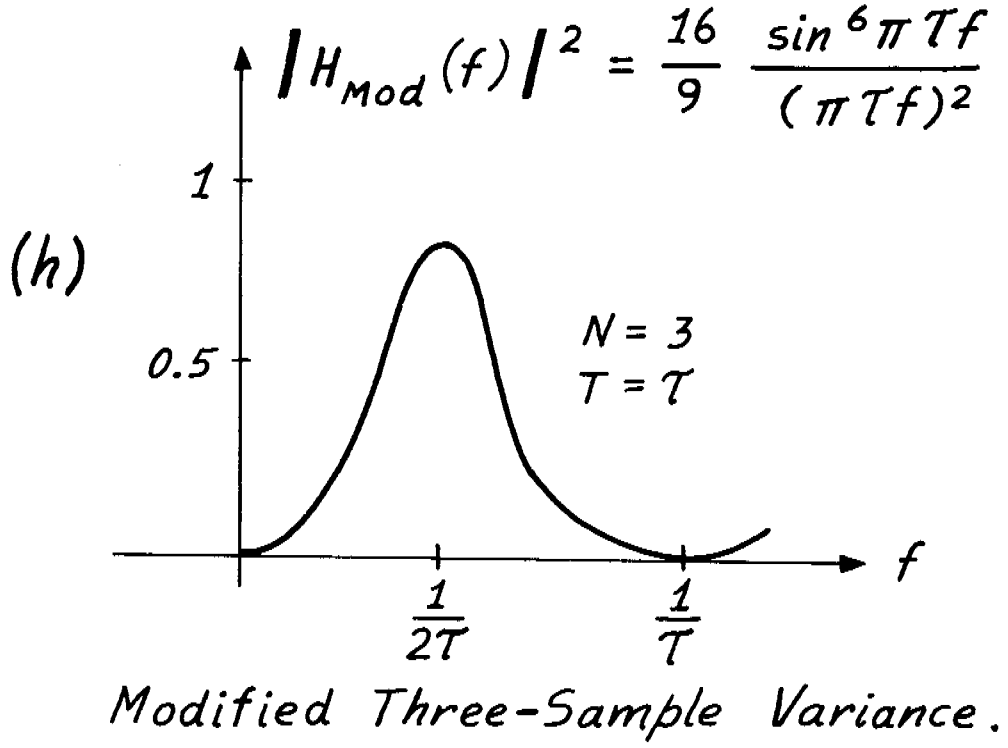


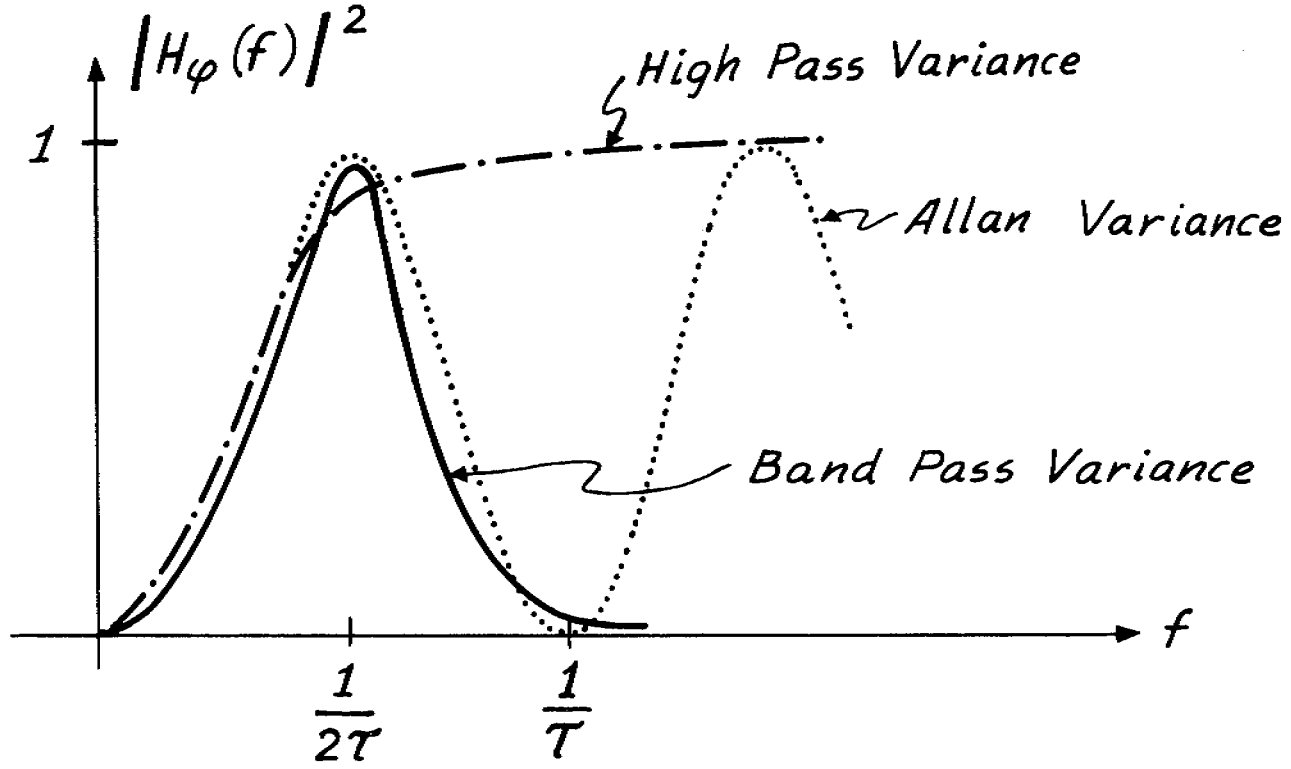
Figure 6 (Cont.)





Modified Three-Sample Variance.

Figure 6 (Cont.)



$$\sigma^2(\tau) = \int_0^\infty S_y(f)\,|H(f)|^2\,df$$

$$\sigma^2(\tau) = \frac{2}{(\pi \nu_0 \tau)^2} \int_0^\infty S_\varphi(f)\,|H_\varphi(f)|^2\,df$$

High Pass and Band Pass Variances.

Figure 7. Square modulus of the phase transfer function $H_\varphi(f)$, for the high pass, low pass and Allan variances.

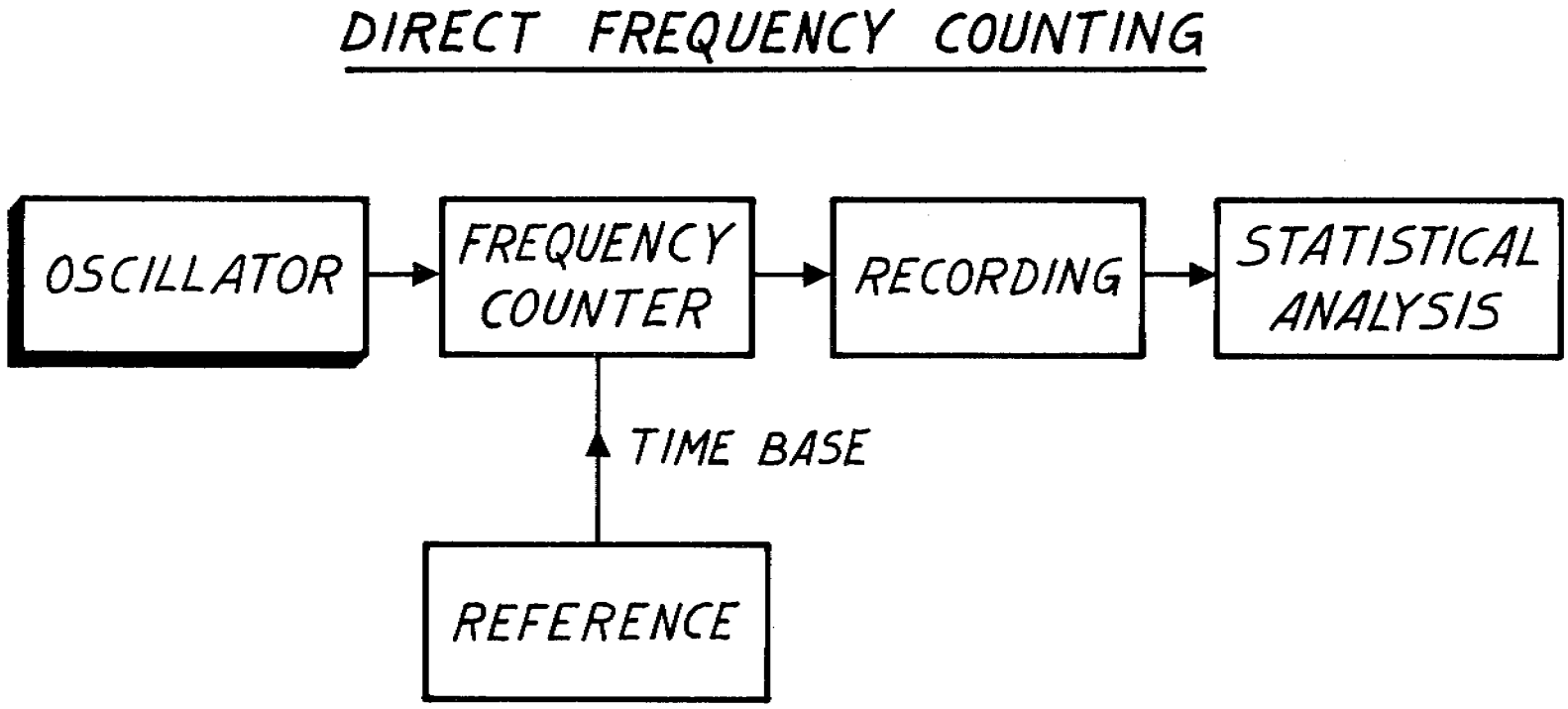


Figure 8. Block diagram illustrating the Direct Frequency Counting method.

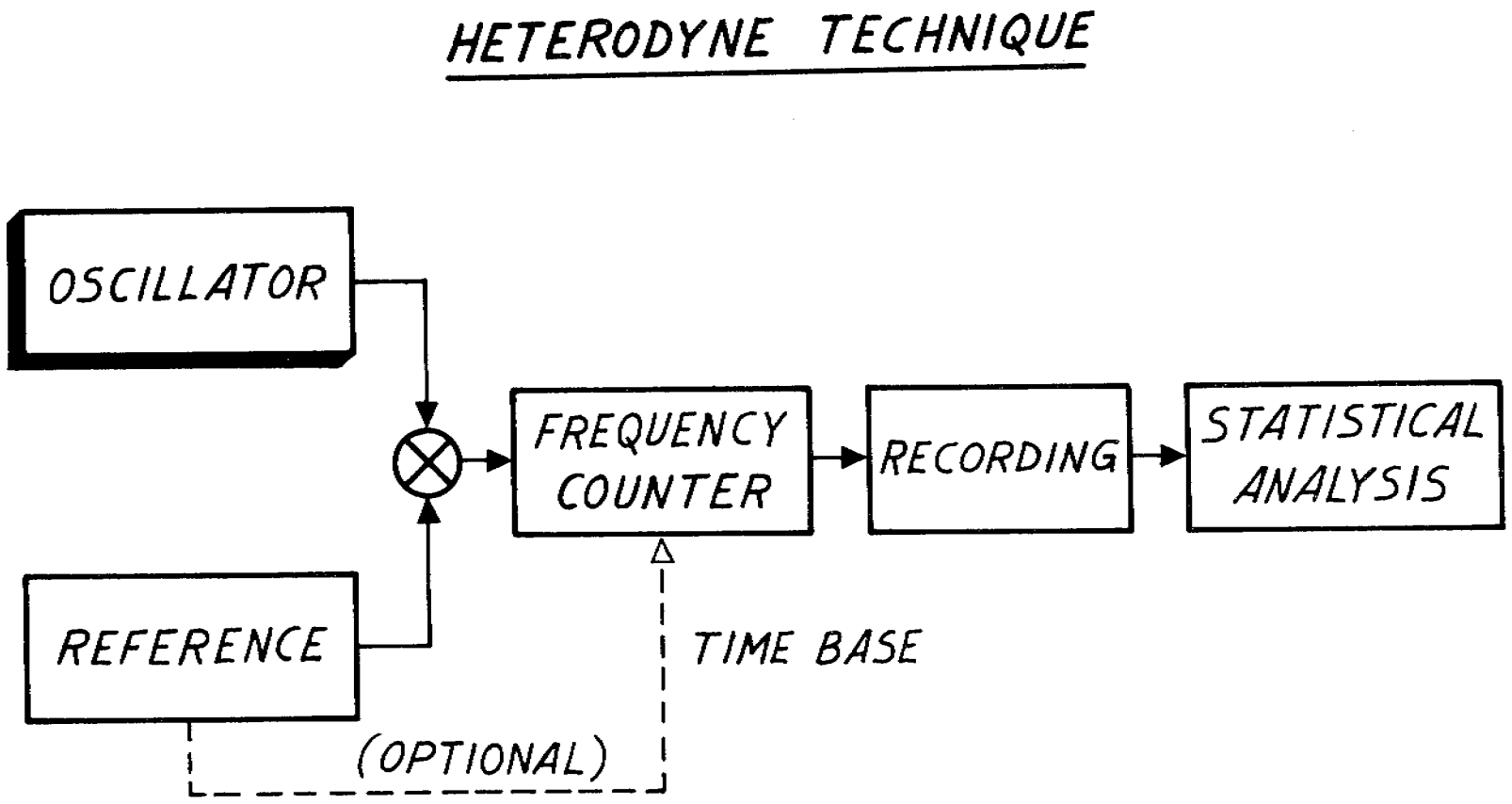


Figure 9. Block diagram illustrating the Heterodyne Technique.

OFFSET FREQUENCY COMPARATOR

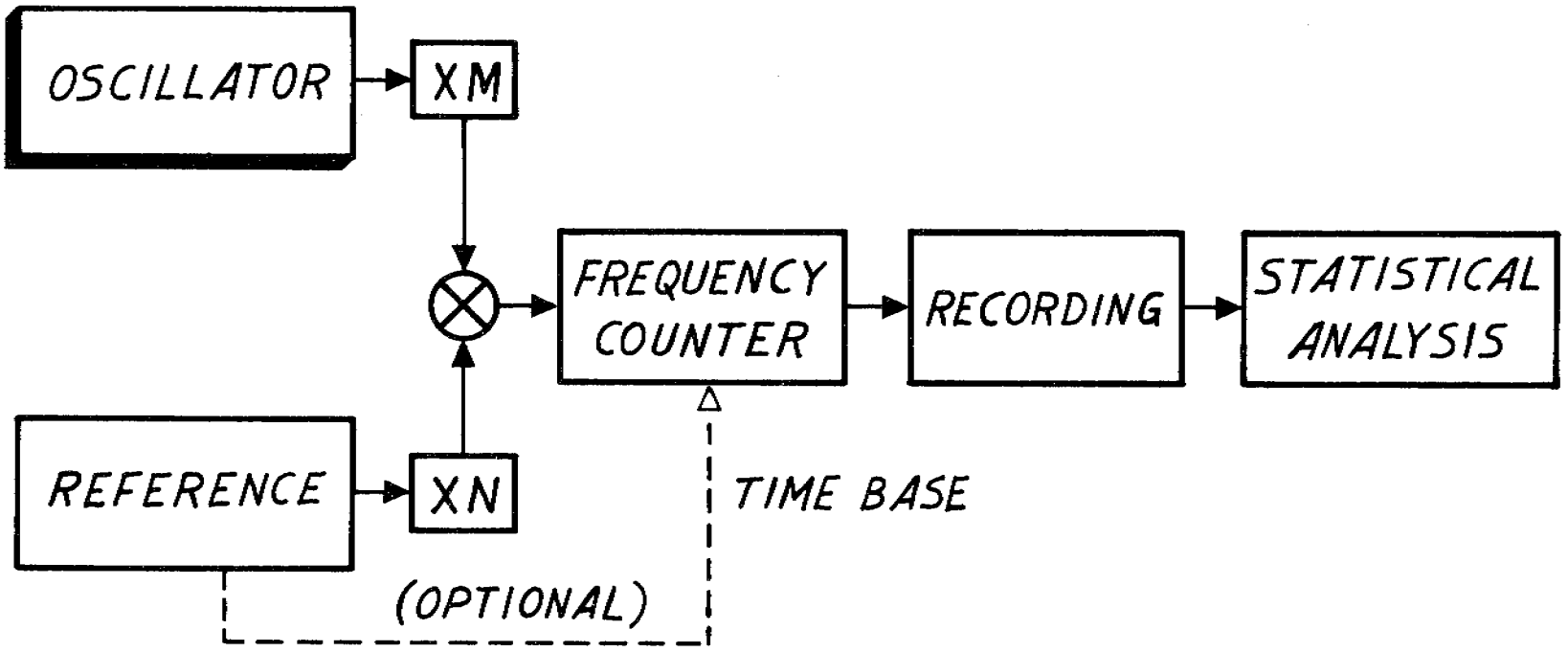


Figure 10. Block diagram illustrating the Offset Frequency Comparator method.

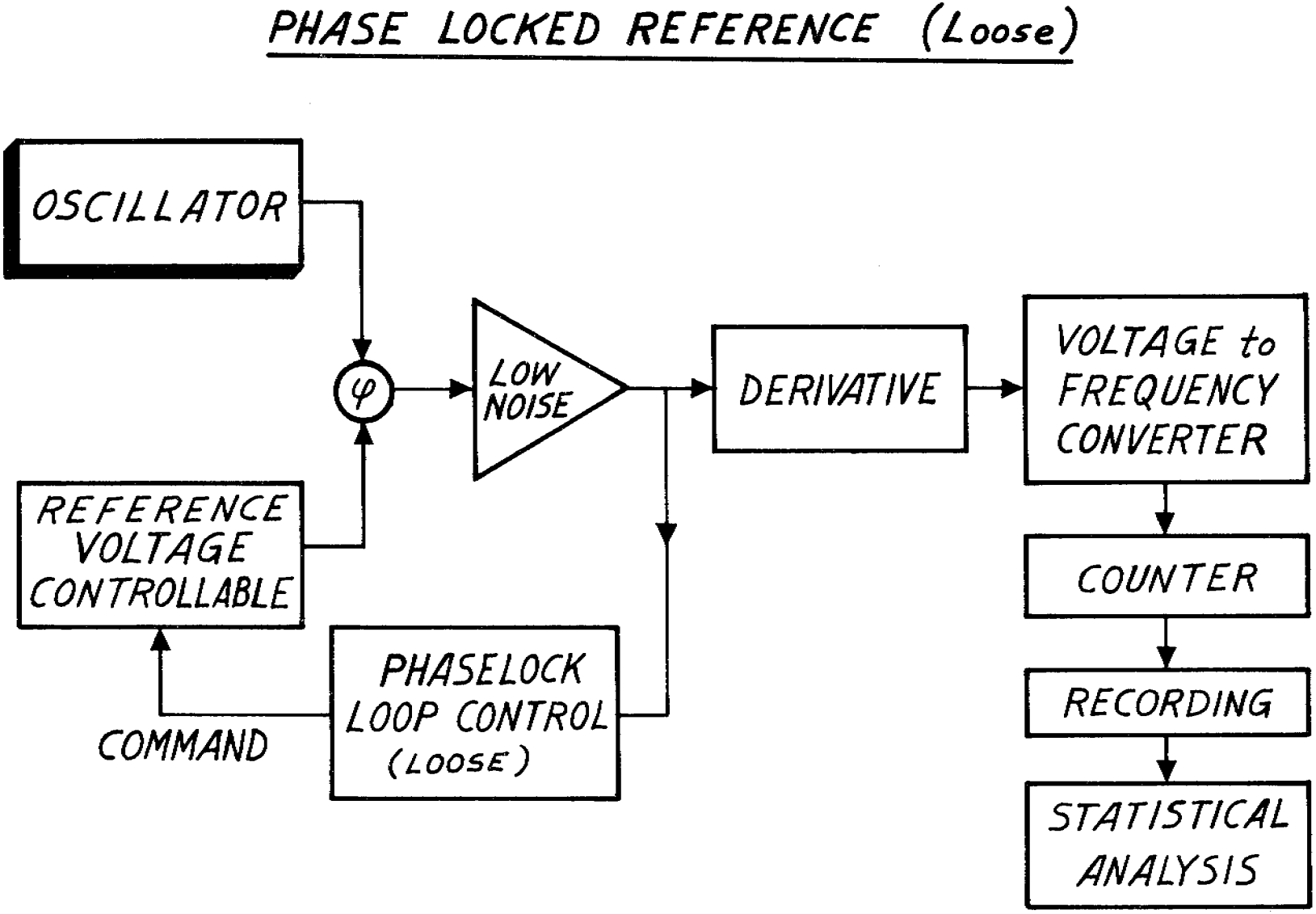


Figure 11. Block diagram illustrating the Phase Locked Reference System. In this system the reference oscillator is loosely locked.

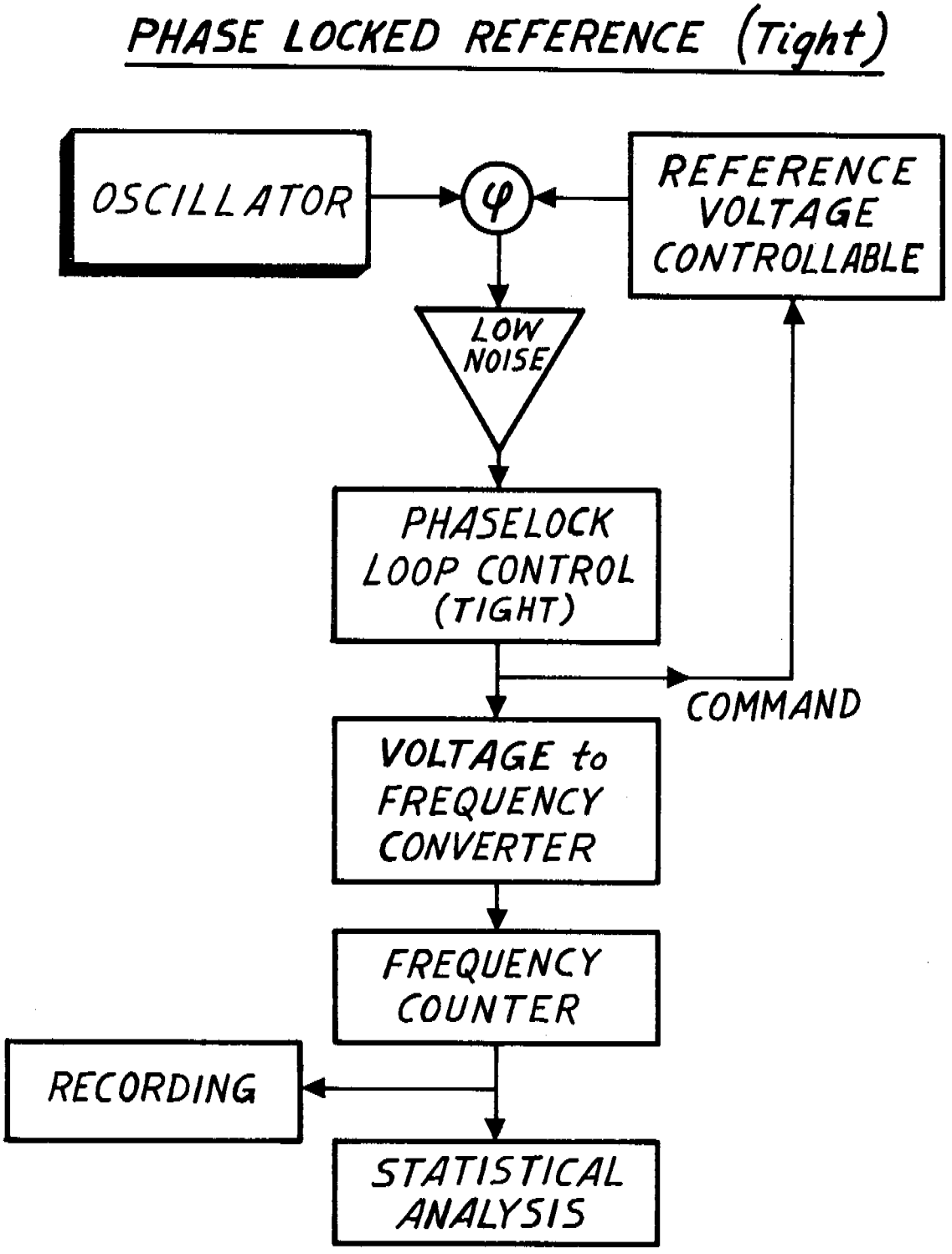


Figure 12. Block diagram illustrating the Phase Locked Reference oscillator in the tightly locked mode.

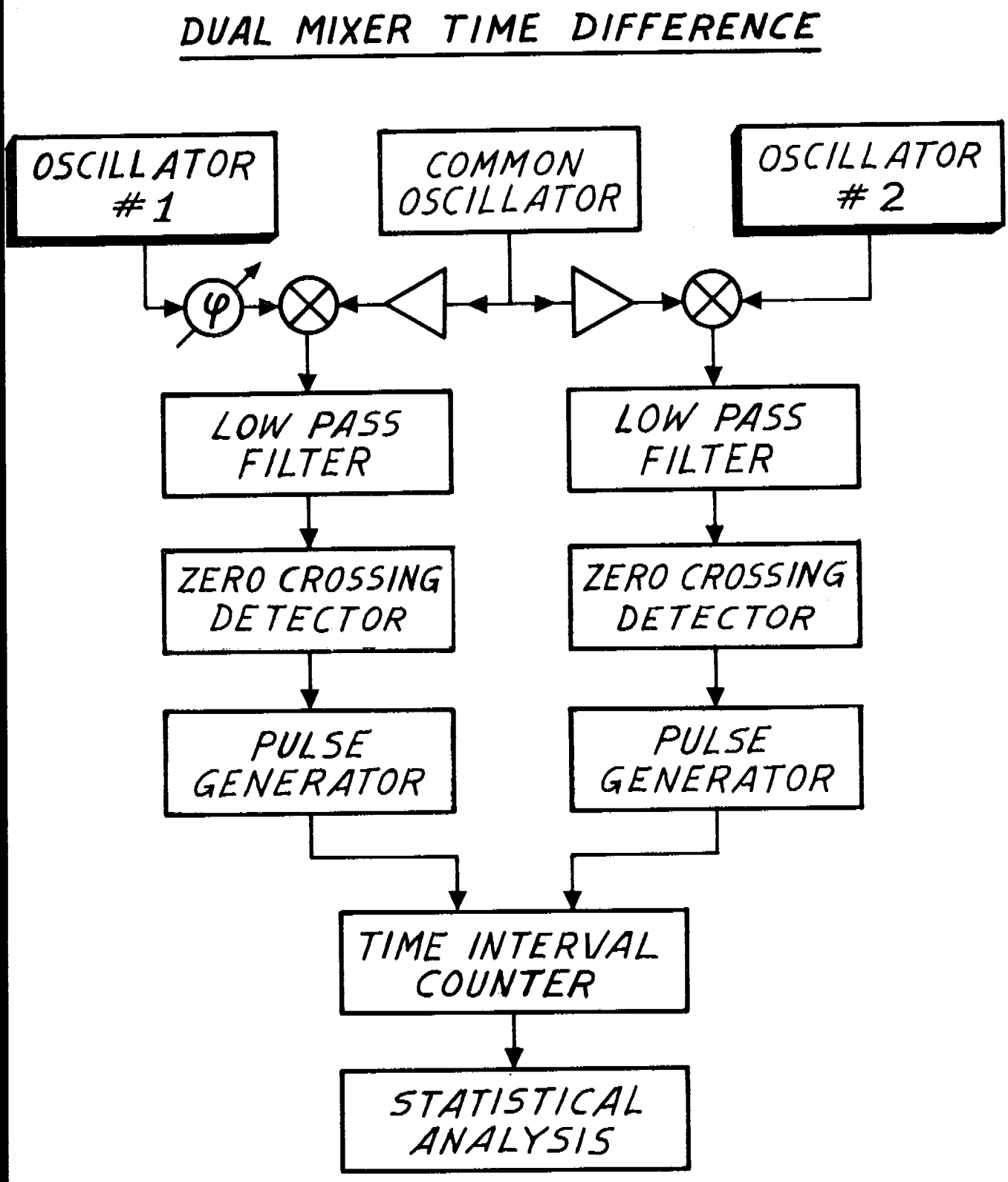


Figure 13. Block diagram illustrating the Dual Mixer Time Difference System. When the two oscillators have tendency to move off frequency they must be phase locked loosely together. This loop is not shown on this figure.

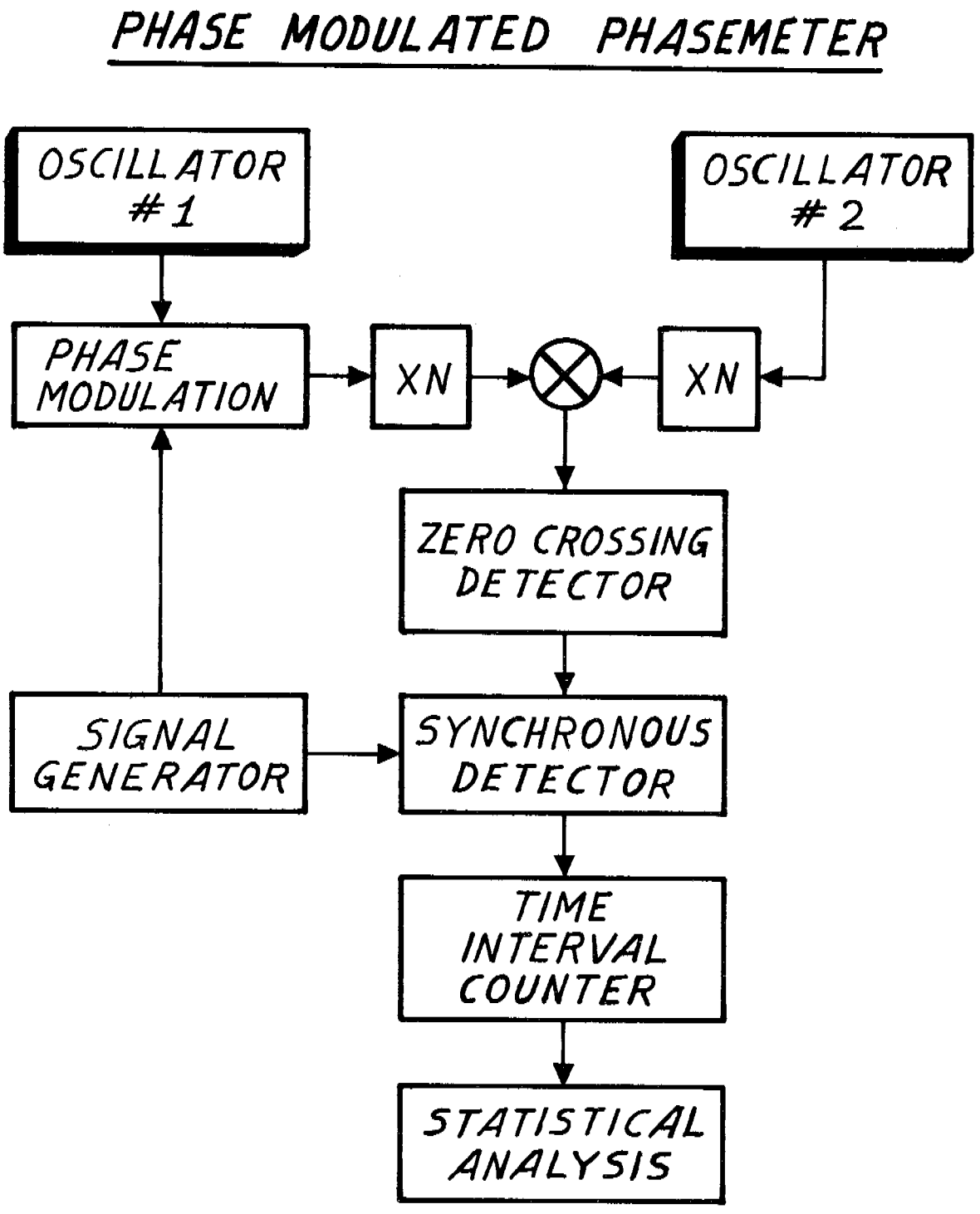


Figure 14. Block diagram illustrating the Phase Modulated Phasemeter System.

Table 1

Asymptotic Behaviour of Various Variances for Power Law Spectral Models and Polynomial Drifts

ASSYMPTOTIC BEHAVIOUR of VARIANCES

$$S_y(f) = h_\alpha f^\alpha$$

$S_y(f)$	$I^2(\tau)$	$\sigma_y^2(\tau)$	$\sigma_m^2(3,\tau,\tau)$
$h_2 f^2$	τ^{-2}	τ^{-2}	τ^{-2}
$h_1 f$	$\sim\tau^{-2}$	$\sim\tau^{-2}$	$\sim\tau^{-2}$
h_0	τ^{-1}	τ^{-1}	τ^{-1}
$h_{-1} f^{-1}$	∞	τ^0	τ^0
$h_{-2} f^{-2}$	∞	τ^1	τ^1
$h_{-3} f^{-3}$	∞	∞	τ^2
$h_{-4} f^{-4}$	∞	∞	τ^3

$$y(t) = d_n t^n$$

$d_n t^n$	$I^2(\tau)$	$\sigma_y^2(\tau)$	$\sigma_m^2(3,\tau,\tau)$
$d_1 t^1$	—	τ^2	0
$d_2 t^2$	—	—	τ^4

QUESTIONS AND ANSWERS

DR. GERNOT M. R. WINKLER, U. S. Naval Observatory:

I have nothing to criticize. I think it was a very clear explanation. However, I think there is still a problem if you want to introduce a generally educated person to the subject because of the terminology, which was, of course, adopted a long time ago; it is misleading. And I would like to suggest an additional approach to such a generally educated person.

There are two ways, two main distinctions, in which we can measure and/or characterize frequency instability. The first one, called time domain measurement, essentially measures and/or characterizes the instabilities by measuring phase differences. And we obtain a statistical distribution of the carrier frequency, and we characterize that statistical distribution of the carrier frequency.

In the frequency domain, we interpret the variations and measure them as variations of the modulation frequency or of a modulation frequency around a fixed carrier.

Now, I think that this is the first thing. Now in the time domain, there are again two essentially different methods to do it. One is to obtain samples of the time readings which are equally spaced, and then you look at the statistics and have various ways to characterize it.

The other way essentially is counting phase differences between zero crossings, and you obtain your desired statistics this way. Now by doing that you inevitably have the question of whether you have dead time or not dead time. And you have the various variations of your methods.

But I believe the essential point which we tried to get across is that the distinction of time domain or frequency domain is not in frequency, but it is the distinction of frequency measurements of a carrier or frequency measurements of a modulation frequency around a fixed carrier.

In both cases, we really measure frequency. But they mean two different things. Thank you.

DR. HARRY PETERS, Sigma Tau:

I also thought that was one of the best summaries that I have ever heard on the subject. If you will allow me, I would like to make one comment regarding the use of such systems.

In the use of any of these systems, you need a frequency referance. And preferably the reference should be much better than the things you wish to measure. If you wish to measure a crystal in a rubidium in 1 to 100 seconds averaging time, of course you have no standard which is significantly better. And for such systems, the

use of a cesium as a reference in all of these systems is particularly inappropriate because you need a subsidiary standard as sort of a flywheel for all measuring times out to on the order of a thousand seconds. And this is why many people want hydrogen masers in their systems. For any of these systems, you would like a standard which is superior to all the other standards for all the measuring times in which you are interested. I had many other extensions of these comments, but I think I will stop here. Thank you.

MR. DAVE ALLAN, National Bureau of Standards:

Let me clarify one thing the novice to the community. I think they have done an outstanding job in showing how you can characterize an oscillator as to the random fluctuations basically. One must be very careful; to really characterize an oscillator, there are all kinds of systematic effects that must be included as far as the manufacturer and the user are concerned. The dependence upon temperature, pressure, humidity, whatever you have, that the oscillator may depend upon, is a whole set that must be included in a proper characterization of an oscillator.