A TOTAL ESTIMATOR OF THE HADAMARD FUNCTION USED FOR GPS OPERATIONS *

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Abstract

We describe a method based on the Total deviation approach whereby we improve the confidence of the estimation of the Hadamard deviation that is used primarily in GPS operations. The Hadamard-total deviation described in this paper provides a significant improvement in confidence indicated by

an increase of 1.3 to 3.4 times the one degree of freedom of the plain Hadamard deviation at the longest averaging time. The new Hadamard-total deviation is slightly negatively biased with respect to the usual Hadamard deviation, and τ values are restricted to less than or equal to $T/3$, to be consistent with the usual Hadamard's definition. We give a method of automatically removing bias by a power-law detection scheme. We review the relationship between Kalman filter parameters and the Hadamard and Allan variances, illustrate the operational problems associated with estimating these parameters, and discuss how the Hadamard-total variance can improve management of present and future GPS satellite clocks.

1 INTRODUCTION

Using a type of Hadamard variance, the goal of this paper is to reduce the uncertainty of long-term estimates of frequency stability without increasing the length of a data run. For measurements of frequency stability, the two-sample frequency variance known as the Allan variance was generalized to an N-sample variance weighted with binomial coefficients by R. A. Baugh [1]. The case of the three-sample frequency variance that is used here is the Picinbono variance [2] times $\frac{3}{2}$. However, in this paper, it will be called a Hadamard variance (following Baugh's work) that is defined as follows. Given a finite sequence of frequency deviates $\{y_n, n = 1, \ldots, N_{ymax}\}$, presumed to be the measured part of a longer noise sequence and with a sampling period between adjacent observations given by τ_0, define the $\tau = m\tau_0$-average frequency deviate as

$$\overline{y}_n(m) \equiv \frac{1}{m} \sum_{j=0}^{m-1} y_{n+j}.$$ (1)

Let $H_n(m) = \overline{y}_n(m) - 2\overline{y}_{n+m}(m) + \overline{y}_{n+2m}(m)$ be the second difference of the time-averaged frequencies over three successive and adjacent time intervals of length τ. Define the Hadamard variance as

$$_H\sigma_y{}^2(\tau) = \frac{1}{6} \left\langle H_n^2(m) \right\rangle,$$ (2)

where $< \cdot >$ denotes an infinite time average over n, and $_H\sigma_y{}^2$ depends on m.

The GPS program office uses this particular time-series statistic for estimating Kalman algorithm coefficients according to [3], which coefficients will be discussed in a later section. The Hadamard deviation $_H\sigma_y(\tau)$ is a function that can be interpreted like the more efficient Allan deviation as a frequency instability *vs.* averaging time τ for a range of frequency noises that cause different slopes on $_H\sigma_y(\tau)$. This is shown in figure 1. For estimating Kalman drift noise coefficients, $_H\sigma_y(\tau)$ is inherently insensitive to linear frequency drift and reports a residual "noise on drift" as a $\tau^{\frac{3}{2}}$ slope, or what is commonly called random run FM (RRFM). This is in contrast to the Allan deviation, which is sensitive to drift and causes a τ^{+1} slope. If the level of drift is relatively high, it masks the underlying random noise. It is customary to estimate and remove overall frequency drift. Depending on the method of drift removal, this procedure can significantly alter the Allan deviation in the longest term τ region of interest, so estimating underlying noise can be a formidable task for any given data span. On the other hand, the Hadamard deviation is unaffected by removing overall frequency drift. For this reason, it is the preferred statistic in situations in which the frequency drift may be above the random noise effects, which is the case with the use of Rb clocks in the GPS Block II satellite program. We do not imply that systematics such as frequency drift can be ignored. Indeed, satellite clocks are changed and these systematics must be learned as quickly as possible to ensure a smooth changeover.
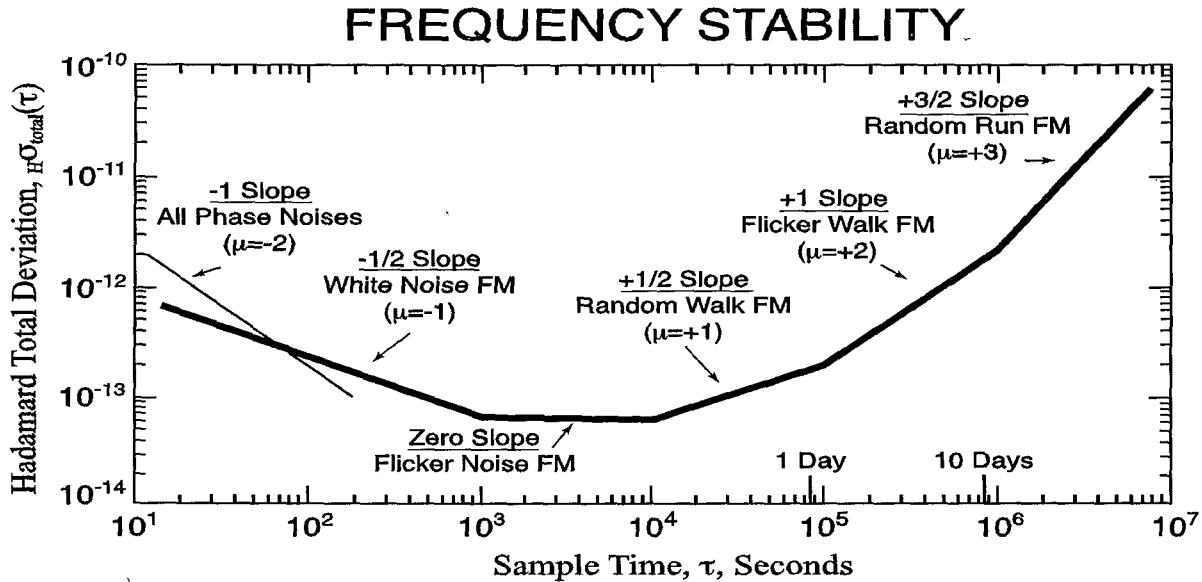
FREQUENCY STABILITY

Figure 1: The Hadamard deviation (root Hvar) shows FM power-law noises as straight lines in addition to PM sources of noise for τ-domain power-law exponent μ (that is, $_H\sigma_y^2(\tau) \propto \tau^\mu$) range of $-2 \leq \mu \leq 3$. We define a new estimator that can be interpreted identically called Hadamard-total deviation (root TotHvar) and that has significantly improved confidence at long term. The Hadamard-total deviation is insensitive to linear frequency drift that can mask characteristic random noise typically encountered here in the region where $\tau =$ one-week and longer. The goal is to identify μ even-integer power-law noises and accurately estimate their levels in order to set system parameters associated with the GPS Kalman filter.

Throughout this writing, we will make comparisons using the traditional best statistical estimators, denoted by "Hvar" and "Avar" referring to the maximum-overlap estimators of the Hadamard and Allan variances. Section 2 reviews the "total" approach to improving statistical estimation. Sections 3 and 4 give two methods of computing total Hadamard variance, designated as TotHvar, using measurements first of fractional frequency deviations and then of time deviations. Then we quantify the advantage of TotHvar over Hvar in Section 5, giving formulae for computing bias and equivalent degrees of freedom (edf) of TotHvar. Section 6 gives a method for efficiently determining the noise type at a given τ-value for automatically correcting the bias and determining confidence intervals for the range of noises considered by TotHvar. Section 7 reviews how an estimate of τ-domain frequency stability is used to set Kalman filter parameters (or q's) used in GPS operations, problems associated with the application of either the traditional Allan variance or Hvar to the Kalman filter, and how TotHvar serves as a unifying solution. Finally, Section 8 discusses a past scenario in GPS operations in which TotHvar is applied to real data showing the benefit of improved estimation of long-term frequency stability.

2 THE "TOTAL" APPROACH

The total estimator approach has been developed to improve confidence of major statistical tools used in analyzing and characterizing instabilities in phase and frequency of oscillators and synchronization systems [4-9]. Making a "total" estimator of eqn. (2) involves joining each real data

subsequence, namely the subsequence of y_i that goes into each $H_n(m)$ term, at both its endpoints by the same original data subsequence so that it repeats. This creates a new extended version of each y_i subsequence that may be extended by a forward or backward repetition, with or without sign inversion, thus with four possible ways to extend. From numerous simulation studies, we have determined that an extension by even (uninverted) mirror reflection of linear-frequency-detrended $H_n(m)$ subsequences yields the largest edf gain and least bias for the range of noise types identified by standard Hvar.

3 COMPUTATION USING y_n-SERIES

$H_n(m)$ is computed from a $3m$-point data segment or *subsequence* $\{y_i\}_n = \{y_i, i = n, \ldots, n+3m-1\}$. Before applying any data extensions, we must remove a linear frequency trend (drift) from each subsequence by making

$$^\circ y_i = y_i - c_1 i,$$

where c_1 is a frequency offset that is removed to minimize $\sum_{i=n}^{n+3m-1} (^\circ y_i - \overline{^\circ y_i})^2$, to satisfy a least-squared-error criterion for the subsequence. In practice, it is sufficient to compute this background linear frequency slope by averaging the first and last halves of the subsequence divided by half the interval and subsequently subtracting the value. Now extend the "drift-removed" subsequence $\{^\circ y_i\}_n$ at both ends by an uninverted, even reflection. Utility index l serves to construct the extensions as follows. For $1 \le l \le 3m$, let

$$^\circ y_{n-l}^{\#} = {}^\circ y_{n+l-1}, \quad ^\circ y_{n+3m+l-1}^{\#} = {}^\circ y_{n+3m-l},$$ (3)

to form a new data subsequence denoted as $\{^\circ y_i^{\#}\}_n$ consisting of the drift-removed data in its center portion, plus the two extensions, and thus having a tripled range of $n-3m \le i \le n+6m-1$ with $9m$ points. To be clear, we now have *extended subsequence* $\{^\circ y_i^{\#}\}_n = \{^\circ y_i^{\#}, i = n-3m, \ldots, n+6m-1\}$.

Define

$$\text{Total}_H \sigma_y^2(m, \tau_0, N_{ymax}) = \frac{1}{6(N_{ymax}-3m+1)} \sum_{n=1}^{N_{ymax}-3m+1} \left(\frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} \left(^\circ H_i^{\#}(m) \right)^2 \right),$$ (4)

for $1 \le m \le \left\lfloor \frac{N_{ymax}}{3} \right\rfloor$, where $\lfloor c \rfloor$ means the integer part of c and notation $^\circ H_i^{\#}(m)$ means that $H_n(m)$ above is derived from the new triply-extended subsequence $\{^\circ y_i^{\#}\}$. The symmetries of the extension and the Hvar filter allow the computational effort to be halved. Let $k = \lfloor 3m/2 \rfloor$. We need to calculate $^\#y_i^\circ$ only for $n-k \le i \le n+k+3m-1$, and $^\# H_i^\circ(m)$ only for $n-k \le i \le n+k$. Then

$$\sum_{i=n-3m}^{n+3m-1} \left(^\# H_i^\circ(m) \right)^2 = 2 \sum_{i=n-k+1}^{n+k-1} \left(^\# H_i^\circ(m) \right)^2 + \left(^\# H_{n-k}^\circ(m) \right)^2 + \left(^\# H_{n+k}^\circ(m) \right)^2, m \text{ even},$$

$$= 2 \sum_{i=n-k}^{n+k} \left(^\# H_i^\circ(m) \right)^2, m \text{ odd}.$$ (5)

4 COMPUTATION USING x_n-SERIES

The methodology described above can be written in terms of calculations on residual time differences between clocks, namely an x_i-series (to adhere to usual notation), recalling that

$$\bar{y}_i\,(m) = (x_{i+m} - x_i)\,/\,(m\tau_0).$$

Thus in the total approach applied to x_i-series, the data extensions on subsequences of x_i will be constructed in such a way that

$$^{\circ}y_i^{\#} = \left(^{\circ}x_{i+1}^{\#} - ^{\circ}x_i^{\#}\right)/\tau_0,$$

in agreement with section 3 above. This has the effect of requiring an *odd* mirror extension and a *third-difference* operator when considering subsequences of x_i. The Hadamard variance discussed in section 3 as a second-difference operator on τ-averaged y_n values can now be re-expressed in terms of a third-difference operator on time-error x_i-values. The sample variance (or mean square) of these third differences falls neatly into a class of structure functions, namely the variance produced by a difference operator of order three [10]. The modified Allan variance can also be treated as a third-difference variance [11].

The x_i-subsequence that corresponds to the y_i-subsequence starting at n is $\{x_i,\ n \le i \le n + 3m\}$, which has $3m + 1$ terms. Compute the detrended subsequence $^{\circ}x_i$ according to

$$k = \left\lfloor \frac{3m}{2} \right\rfloor, \quad c_2 = \frac{x_n - x_{n+k} - x_{n+3m-k} + x_{n+3m}}{k\,(3m - k)},$$

$$^{\circ}x_i = x_i - \tfrac{1}{2}c_2\,(i - n)\,(i - n - 3m), \quad n \le i \le n + 3m.$$

Define the extended subsequence $\left\{^{\circ}x_i^{\#},\ n - 3m \le i \le n + 6m\right\}$ by

$$
\begin{aligned}
^{\circ}x_i^{\#} &= ^{\circ}x_i, \quad n \le i \le n + 3m, \\
^{\circ}x_{n-l}^{\#} &= 2\,(^{\circ}x_n) - ^{\circ}x_{n+l}, \quad 1 \le l \le 3m, \\
^{\circ}x_{n+3m+l}^{\#} &= 2\,(^{\circ}x_{n+3m}) - ^{\circ}x_{n+3m-l}, \quad 1 \le l \le 3m.
\end{aligned}
$$

Then

$$m\tau_0\left(^{\circ}H_i^{\#}\,(m)\right) = -^{\circ}x_i^{\#} + 3\left(^{\circ}x_{i+m}^{\#}\right) - 3\left(^{\circ}x_{i+2m}^{\#}\right) + ^{\circ}x_{i+3m}^{\#}, \quad n - 3m \le i \le n + 3m - 1,$$

where $^{\circ}H_i^{\#}\,(m)$ has the same meaning as in Section 3. Now the Hadamard-total variance is computed from (4) as before with $N_{ymax} = N_{xmax} - 1$. Because of symmetry we need $^{\#}x_i^{\circ}$ only for $n - k \le i \le n + k + 3m$, and (5) applies.

5 BIAS AND EQUIVALENT DEGREES OF FREEDOM

We consider the random frequency-modulation (FM) noises since these dominate at long-term averaging times where we can capitalize on the improved confidence of using the total approach. To analyze phase-modulation (PM) noises, one would usually use Total TDEV [6] rather than the Hadamard deviation. For brevity, let $\text{Total}_{\text{H}}\sigma_y^2(m, \tau_0, N_{ymax})$ be $\text{TotHvar}\,(\tau, T)$, where $\tau = m\tau_0$, $T = N_{ymax}\tau_0$. The normalized bias and edf for TotHvar are given by

$$\text{nbias}\left(\tau\right) = \left[\frac{E\left\{\text{TotHvar}\left(\tau, T\right)\right\}}{E\left\{\text{Hvar}\left(\tau, T\right)\right\}} - 1\right] = a, \qquad (6)$$

$$\text{edf}\left(\tau\right) = \text{edf}[\text{TotHvar}\left(\tau, T\right)] = \frac{T/\tau}{b_0 + b_1 \tau/T}, \qquad (7)$$

where $E\{\cdot\}$ is expectation of $\{\cdot\}$, $0 < \tau \leq \frac{T}{3}$, $\tau \geq 16\tau_0$ (to be explained), and a, b_0, and b_1 are given in Table 1 for the five FM noise types considered by the Hadamard variance. α is the corresponding power-law exponent of the fractional-frequency noise spectrum $S_y(f) \propto f^\alpha$. In the context here, its valid range is $-4 \leq \alpha \leq 2$. $E\{\text{TotHvar}\left(\tau, T\right)\}$ relative to $E\{\text{Hvar}\left(\tau, T\right)\}$ in (6) is independent of τ and T, dependent on noise type, and biased low, giving a the negative sign in Table 1. The edf formula (7) is a convenient, empirical or "fitted" approximation with an observed error below 10% of numerically computed exact values derived from Monte-Carlo simulation method using the b_0 and b_1 coefficients of Table 1 and with the error decreasing with averaging factor $m = \tau/\tau_0$ increasing. In fact, (7) should be used only if data-sampling period τ_0 is sufficiently short compared to the averaging time τ by $\tau/\tau_0 \geq 16$. Otherwise, there are not enough points for the data-extension procedure in the total estimator to have significant advantage over the plain Hadamard estimator. In other words, the τ_0-dependence of the total estimator of (4) plays a significant role, whereas the weaker τ_0-dependence of the maximum-overlap estimator of plain $_H\sigma_y{}^2(\tau)$ given by (2) is generally suppressed as in (2). It is well known that maximum-overlap statistical estimators will increase edf, hence confidence, and the degree of data overlap is dependent on sampling interval τ_0 relative to τ [12, 13]. Real data should be sampled as fast as practical for a given averaging time. This is especially true in order for the data extension of each subsequence to be effective in the total approach.

Assuming chi-square distribution properties and edf computed by (7) and the values of Table 1, confidence intervals will be conservative since the distribution is actually narrower than chi-square. Although not quantitatively investigated, the narrowing of the distribution is proportional to increasing averaging factor $m = \tau/\tau_0$. Fortunately with real data runs, m is, of course, always largest at longest-term. Depending on the noise type, we have seen narrowing by as much as 15% for $m \approx 100,000$.

To show the improvement in estimating the Hadamard function, Table 2 lists the exact values of edf from theory for computations using TotHvar vs. plain Hvar for the longest averaging factor in which $\tau = T/3$. This point is the last point in the estimate, and the improvement in confidence using

Table 1: Coefficients for computing (6) and (7), normalized bias and edf of TotHvar.

Noise	Abbrev.	α	a	b_0	b_1
White FM	WHFM	0	-0.005	0.559	1.004
Flicker FM	FLFM	-1	-0.149	0.868	1.140
Random Walk FM	RWFM	-2	-0.229	0.938	1.696
Flicker Walk FM	FWFM	-3	-0.283	0.974	2.554
Random Run FM	RRFM	-4	-0.321	1.276	3.149

TotHvar is substantial, particularly for the general case of WHFM noise. TotHvar is a significantly improved estimator that offsets much of the criticized inefficiency in using the sample Hadamard deviation as opposed to the sample Allan deviation in the presence of common WHFM noise in frequency standards.

Table 2: Exact $\frac{edf\{\text{TotHvar}(T/3,T)\}}{edf\{\text{Hvar}(T/3,T)\}}$ gain for $\tau_{max} = T/3$.

Noise	edf gain of TotHvar $(T/3, T)$
WHFM	3.447
FLFM	2.448
RWFM	2.044
FWFM	1.676
RRFM	1.313

6 POWER LAW DETECTION

It is important to be able to determine which power-law noise type is present for a given τ-value in the range $-4 \le \alpha \le 0$ so that TotHvar's bias can be removed automatically. Similarly, before the edf can be determined to establish confidence intervals and set error bars for a stability measurement, it is necessary to identify the dominant noise process. This section describes a noise-identification (noise-ID) algorithm that has been found effective in actual practice, and that works for a single τ-point over the full range of $-4 \le \alpha \le 2$. It is based on the Barnes B1 function [14], which is the ratio of the N-sample (standard) variance to the two-sample (Allan) variance, supplemented by applying this function to frequency data, and the R(n) function [15], which is the ratio of the modified Allan to the normal Allan variances.

The B1 function has as arguments the number of frequency data points N, the dead time ratio r (which is set to 1), and the power-law τ-domain exponent μ. The B1 dependence on μ is used to determine the power-law noise type for $-2 \le \mu \le 2$ (WHPM and FLPM to FWFM). For a B1 that is consistent with a $\mu = -2$ result, the $\alpha = 1$ or 2 (FLPM or WHPM noise) ambiguity can be resolved with the R(n) ratio using the modified Allan variance.

For the Hadamard variance, the noise determination must be extended to $\mu = 3$ (or $\alpha = -4$, RRFM). This can be done by applying the B1 ratio to *frequency* (instead of the usual phase) data and *adding 2* to the resulting μ. This procedure is called "*B1" herein. Since the *B1 procedure simply applies the Barnes B1 ratio to frequency data instead of phase data, its use is as before, but now its range is effective from WHFM to RRFM noise. (This is analogous to simulation of RRFM data by treating RWFM phase data as frequency data.)

The overall noise identification process is as follows:

- calculate the standard and Allan variances for the applicable τ averaging factor,

- calculate B1, $B1(N, r = 1, \mu) = \frac{N(1-N^{\mu})}{2(N-1)(1-2^{\mu})}$,

- determine the expected B1 ratios for $\alpha = $ -3 through 1 or 2,

- set boundaries between them and find the best power-law noise match,

- resolve an $\alpha = 1$ or 2 ambiguity with the modified Allan variance and R(n), or

- resolve an $\alpha = $ -3 or -4 ambiguity with *B1.

Table 3: Formulae for $B1(N, r = 1, \mu)$. Substituting frequency data into the usual phase-data measurement of B1 ratio will shift these formulae to the $\mu + 2$ range, thus covering RRFM.

Noise	μ	$B1 =$
FWFM	2	$\frac{(N)(N+1)}{6}$
RWFM	1	$N/2$
FLFM	0	$\frac{N \, \ln N}{2(N-1) \, \ln 2}$
WHFM	-1	1
WH or FL PM	-2	$\frac{N^2-1}{1.5(N)(N-1)}$

For a data run of length N, Table 3 gives five specific formulae for B1 corresponding to $\mu =$ -2, -1, 0, 1, and 2. Table 4 summarizes the power-law detection scheme and gives the boundaries for demarcating each noise type. The boundaries between the B1, *B1, and R(n) functions are, in general, set as the geometric means of their expected values, and the actual measured ratio is tested against those values *downward* from the largest applicable μ. For example, if, during the testing, the measured B1 ratio is greater than the square root of the product of the expected B1 values for RWFM and FLFM noise, it is determined to be the former ($\alpha = -2$, RWFM).

High levels of frequency drift should be removed to best identify the underlying noise process by this method. Also, the R(n) ratio cannot, of course, be used for $\tau = \tau_0$ averaging factor (in which case it is 1 for all noise types). Finally, at the very longest averaging factor or last τ-point, it is better to use the previous or $\tau - \tau_0$ point to estimate the noise type. This algorithm has been used in commercial frequency-stability software [16] for the past decade with good success. It allows bias corrections and error bars to be calculated automatically during an analysis for all of the common time-domain stability statistics (including the new Hadamard total variance here) over the full range of noise types and for essentially all τ averaging times.

7 THE KALMAN NOISE MODEL AND THE GPS OPERATIONS PROBLEM

The time update of clock states in the Master Control Station (MCS) Kalman prediction algorithm is based on an average of the the most recent measurement of these states for each individual clock, modeled simply by random noise acting on phase $x(t)$, frequency $y(t)$, and frequency drift $z(t)$. With this model, the measured power-law α exponents of the frequency-fluctuation noise spectrum take on only the values 0,-2, and -4, corresponding to WHFM, RWFM, and RRFM, or $\mu = -1$, 1, and 3 in the τ-domain. Hence, we want to precisely extract the level of these noises for each clock using the most efficient method possible, which heretofore has been the sample Allan variance with drift removed from the data run, and more recently the sample Hadamard variance, because of its logical link to the model. If white PM (WHPM) is a significant noise component, and for completeness, the $\alpha = 2, \mu = -2$ case corresponding to WHPM is included as a separate error.

The parameters used by the MCS within GPS system operations are denoted as Kalman filter q's. By convention, each filter parameter $q_i, i = 0, 1, 2, 3$ corresponds respectively to τ-domain power law exponents $\mu = -2, -1, 1, 3$. For the Hadamard variance, the relationship is [3]

$$_H\sigma_y{}^2(\tau) = \sigma^2_{WHPM} + \sigma^2_{WHFM} + \sigma^2_{RWFM} + \sigma^2_{RRFM}$$

$$= \tfrac{10}{3}q_0\tau^{-2} + q_1\tau^{-1} + \tfrac{1}{6}q_2\tau + \tfrac{11}{120}q_3\tau^3. \tag{8}$$

For the Allan variance, the relationship is [17]

$$\sigma_y{}^2(\tau) = 3q_0\tau^{-2} + q_1\tau^{-1} + \tfrac{1}{3}q_2\tau \left[+\tfrac{1}{20}q_3\tau^3\right], \tag{9}$$

where the inclusion of the RRFM noise term as $\left[+\tfrac{1}{20}q_3\tau^3\right]$ is a point of contention for two reasons.

Table 4: Power-Law Noise Identification.

Noise	α	μ	ID by	Remarks
RRFM	-4	3	B1&*B1	Use *B1 to resolve α = -3 or -4 ambiguity
Decision boundary: {B1(FWFM) + B1(RWFM)} / 2				
FWFM	-3	2	B1&*B1	Use *B1 to resolve α = -3 or -4 ambiguity
Decision boundary: {B1(FWFM) + B1(RWFM)} / 2				
RWFM	-2	1	B1	
Decision boundary: sqrt {B1(RWFM) x B1(FLFM)}				
FLFM	-1	0	B1	
Decision boundary: sqrt {B1(FLFM) x B1(WHFM)}				
WHFM	0	-1	B1	
Decision boundary: sqrt {B1(WHFM) x B1(FLPM)}				
FLPM	1	-2	B1&R(n)	Use R(n) to resolve α = 1 or 2 ambiguity
Decision boundary: sqrt {B1(FLPM) x B1(WHPM)}				
WHPM	2	-2	B1&R(n)	
Noise ID Methods: B1 = Barnes B1(N, r, m) bias function with r = 1 [14]. *B1 = B1 applied to frequency data as phase data with $\mu = \mu + 2$. R(n) = ratio, mod Allan variance/Allan variance. [15].				

First, estimating q_3 by (9) using real data is unreliable because RRFM is inconsistent by the definition of the Allan variance. Second, ref. [17], from which the term derives, does not compute the Allan variance; instead, it computes the optimal mean-square prediction error variance of $\bar{y}(t_0, t_0 + \tau)$ based on $\{x(t), t \leq t_0\}$, for frequency noise spectra with $\alpha = 0$, -2, and -4. For these reasons, we advise omitting the RRFM term entirely from (9). The other terms of (9) happen to be correct for Allan variance.

The GPS Hadamard variance is defined to be equivalent to the Allan variance for WHFM, which is confirmed in comparing (8) and (9); however the variances differ by a factor of two for RWFM, therefore they cannot be used interchangeably under normal circumstances and involving drift-free stochastic processes.

Tuning the Kalman filter depends on the ability to "q" each individual clock according to estimates of its noise. The GPS Block IIR satellite program incorporates Rb atomic oscillators that are characterized by a mix of various levels and types of random noise and with frequency drift that may be significantly above noise. This kind of oscillator mix is difficult to manage using Avar and (9), which must be used based on drift-removed frequency residuals. However, reverting to using "frequency-drift insensitive" Hvar and using (8), the confidence becomes a factor of about $\frac{1}{3}$ less near the last and crucial long-term $\tau_{max} = T/3$ value owing to the plain sample Hadamard's edf of one less as compared to Allan's edf. For the proper perspective, note that we are in the one-week averaging τ-region with a last real-time data run of about one month, thus edf $\approx$ 1–2; so estimating filter q's is somewhat subjective. Figure 2 illustrates a summary of estimates of frequency stability for each GPS satellite clock as published in reports issued by the Naval Research Laboratory [18].

Table 2 shows that the new TotHvar $(T/3, T)$ edf is multiplied by a factor of 1.3 to 3.4 over

plain Hvar $(T/3, T)$. TotHvar can be applied directly and reliably, while retaining the efficiency of the sample Allan variance without the difficulty associated with real-time drift removal.

The work of this paper has impact on two GPS operational issues. The first is that the time needed to estimate the Hadamard variance is substantially reduced. For example, to obtain a $\tau =$ one-week estimate of the Hadamard variance with, say, the last 40 days of measured data, the Total approach using TotHvar obtains a one-week estimate with the same or better confidence in about 26 to 34 days of measured data. The second issue is that satellite data are obtained by the linked common-view method [19], and the delay in receiving the monitor station tracking data is currently at 2 to 3 days. Thus, it is important to extract maximum information from data at hand.
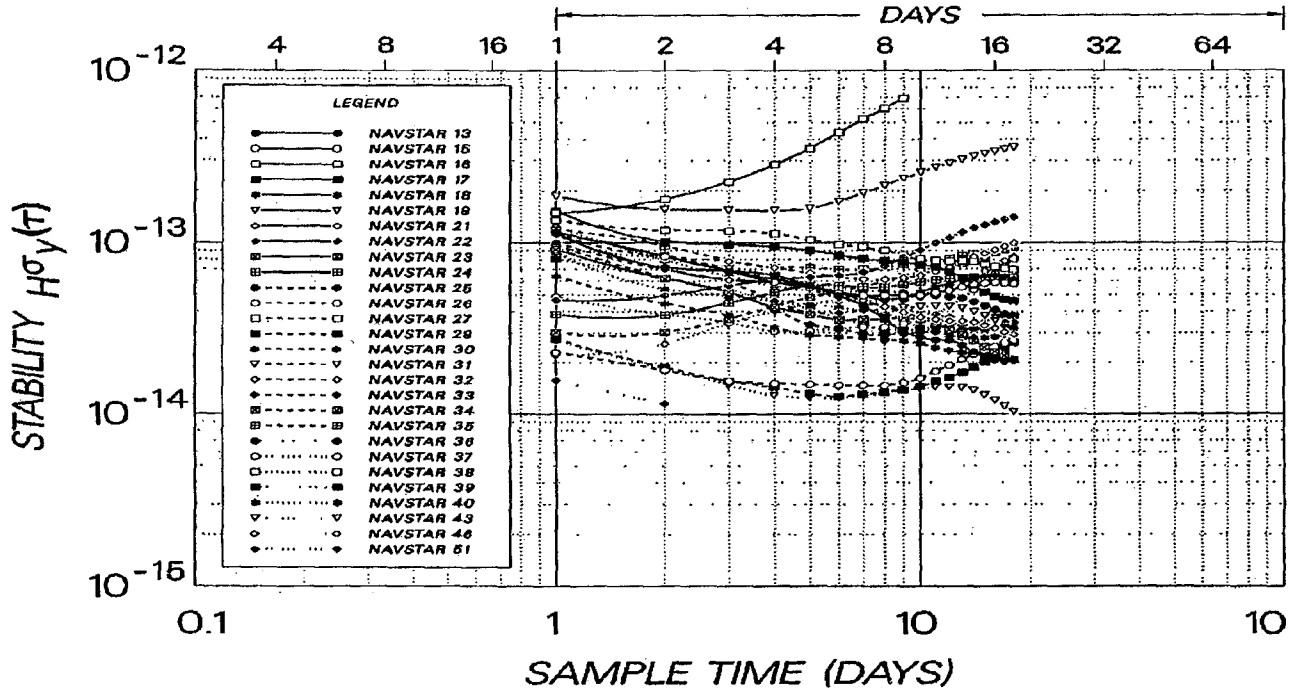

Figure 2: Hadamard-deviation frequency stability of individual GPS satellite clocks vs. USNO Master Clock for the period 1 January to 1 July, 2000 [18].

8 EXAMPLE

Figure 3 is data of SV24, a Block IIA GPS satellite. Total Hadamard deviation, plain Hadamard deviation, and Allan deviation are compared with increasing data spans starting at 7 days and extending to 28 days and shows how each of these statistics behaves as it evolves. As is generally the case, TotHdev performs better at estimating the longest-term noise level than plain Hdev for measured data spans as indicated by estimated levels from later (longer) data spans.

9 CONCLUSION

We have developed a significantly improved estimator of the three-sample Hadamard frequency variance based on the so-called "total" approach and denoted as TotHvar, for use in GPS operations and analysis. Practically speaking, we have reduced the long-term estimation uncertainty in terms of edf by a factor of 1.3 to 3.4, depending on the noise type, and we have presented a way to
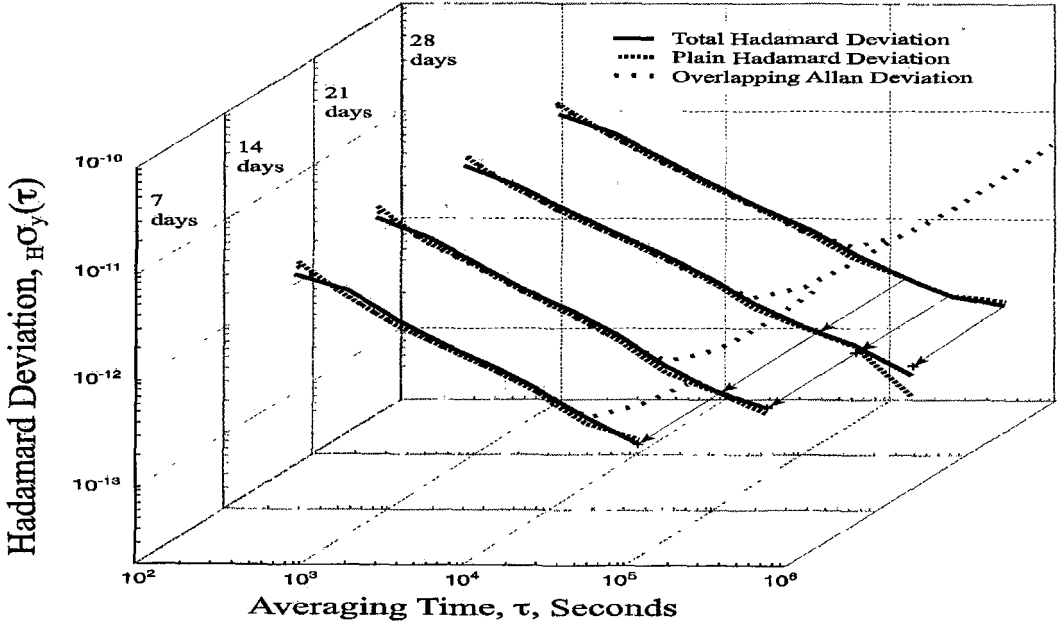
Figure 3: Total Hadamard deviation, plain Hadamard deviation, and Allan deviation for SV24 satellite clock data as the data run increases from 7 days (front plot) to 28 days (rear plot). The last (rightmost) values of TotHdev for shorter data runs anticipates the underlying noise level of longer runs compared to plain Hdev (arrowed lines are projected off 28-day data run). The Allan deviation's response to frequency drift masks the long-term noise level.

automatically remove the moderate negative bias of TotHvar by a power-law detection algorithm. Having confidence greater than plain Hvar and even equal to or greater than Avar, TotHvar is a statistic that permits tuning of the MCS Kalman filter with more accurately chosen clock-estimation parameters (or q's) that are linked to the most recent measurements of frequency stability of each clock. The increased confidence from TotHvar and shorter data processing delays will play significant roles in adequately managing future GPS system events.

ACKNOWLEDGMENTS

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Questions and Answers

MASSIMO TINTO (JPL): I have a quick question. The Hadamard function, in a sense, can be seen as the Allan variance of the Allan variance because you're taking the difference of the difference. From your definition, you have Y (T) minus Y (T) plus τ, minus the shift of 1 by τ. So have you thought about doing the Hadamard of the Hadamard, so going to the third order? So going a step further from what you have done and see if that will also give you some extra—

DAVID HOWE: I haven't. The problem with going to higher differences is that the efficiency goes down for the kinds of noise processes which are characteristic of the clocks, because you lose degrees of freedom as you have to use longer data lengths to estimate averaging times.

TINTO: Oh, I see.

STEVE HUTSELL (USNO AMC): Dave, I was wondering if you could comment on the increase of the effective degrees of freedom. How much of it was due to the overlapping technique and how much of it was due to the extension in the total technique?

HOWE: All of this was due to the extension. That gain was due to the extension entirely because the numbers were taken at the last point. There is not overlapping at the last point.

Now, that does raise the point that, as a practical matter, total and overlapping estimators rely heavily on sampling as fast as practical. You understand that you take estimates and you shift by τ_0, your sampling interval. Total takes advantage of that more fully than overlapping, but that has been true for quite a while.