

# On systematic uncertainties in Coordinated Universal Time (UTC)

## Biography

Dr. Demetrios Matsakis is Chief Scientist for Time Services at the US Naval Observatory (USNO). He received his undergraduate degree in Physics from MIT. His PhD was from U.C. Berkeley, and his thesis, under Charles Townes, involved building masers and using them for molecular radio astronomy and interferometry. Hired at the USNO in 1979, he measured Earth rotation and orientation using Connected Element Interferometry and Very Long Baseline Interferometry (VLBI). Beginning in the early 90's, he started working on atomic clocks and in 1997 was appointed Head of the USNO's Time Service Department. He holds one patent, has over 140 publications, has served on many international commissions, and for three years was President of the International Astronomical Union's Commission on Time. Email: [demetrios.matsakis@usno.navy.mil](mailto:demetrios.matsakis@usno.navy.mil);

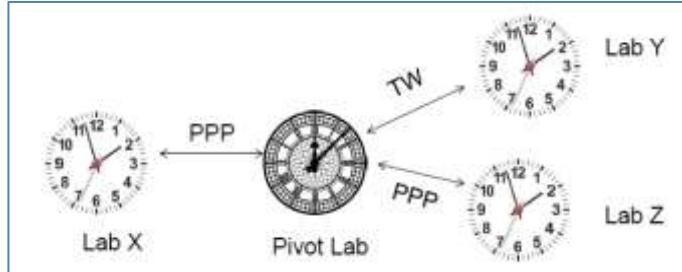
## Abstract

Lewandowski, Matsakis, Panfilo, and Tavella published a series of papers computing the statistical (Type A), and systematic (Type B) uncertainties in UTC [1-2]. The formalism did not include a method for incorporating redundant links, but an outline for this was later published by Petit and Jiang [3]. This paper explores the consequences of the topological location of labs, and of using links with large uncertainties and labs with large weights on the systematic uncertainties of the UTC realizations of the labs, which are known as UTC(k). We present a possible way to address the effects of uncalibrated laboratories.

## Introduction

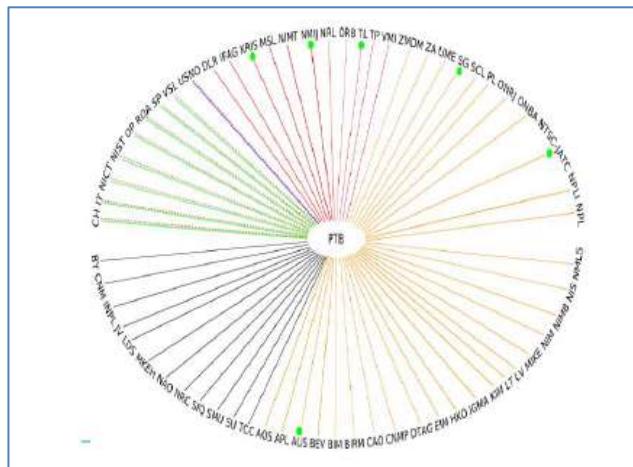
UTC/EAL is optimized for frequency stability, and calculated at epochs spaced far enough apart that time transfer noise is small or negligible. A rigorous analysis of the uncertainties in UTC was published in 2006 [1, 2; hereafter LMPF]; see also Appendix I. LMPF found that one could remove the deterministic contributions to UTC, such as leap seconds, clock models, and steering to the primary frequency standards; the uncertainties in UTC could then be computed using only the uncertainties of the links. To compute the uncertainties (but not the values) in UTC -UTC(k), where UTC(k) is lab k's realization of UTC, UTC could be considered a weighted average of the UTC(k), where the weight equals the sum of the weights of the individual clocks of each lab k. With this understanding, standard statistical techniques could be applied to compute the uncertainty of any lab's clocks with respect to the weighted average of clocks that make up UTC, always making due allowance for the correlations between the links (Appendix I).

The correlations between links depends on the equipment utilized in the measurements. Links involving GNSS receivers are examples of links with site-based uncertainties; in these cases each lab's system has its own uncertainty, and their link's uncertainty is the square root of the sum of the squares (RSS) of the uncertainties of the labs. With reference to Figure 1, this is the situation between labs X and Z; lab P's receiver is used as a pivot but its contribution cancels. In contrast, the uncertainties of links involving Two Way Satellite Time Transfer (TW, also known as TWSTT and TWSTFT) are a property of the link and not the individual labs. In Figure 1, the uncertainty of X-Y (the weighted average of the clock readings in lab X minus the weighted average of the clock readings in lab Y) would be the RSS of the two links X-P and P-Y.



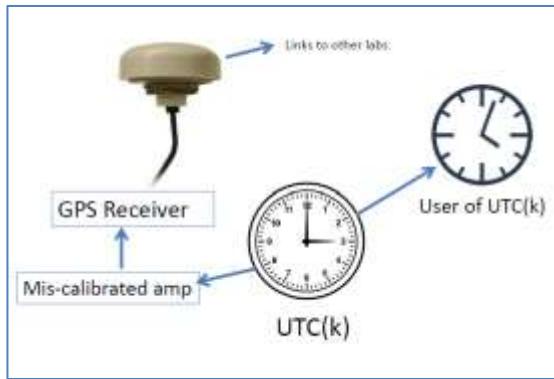
**Figure 1** Lab X and the Pivot lab P are linked by PPP, which has site-based uncertainties due to its associated GNSS receiver's calibration uncertainties, as are labs Z and P, but labs P and Y are linked by TW, with baseline-based uncertainties.

We shall address the management of complex chains of links below and in Appendix II. The current topography used by the BIPM has no redundant links and employs a star-pattern in which there is a single pivot, the PTB (Figure 2).



**Figure 2** Topology used to compute UTC in October, 2016. Only the links with the PTB are shown. Colors and dots indicate the type of time transfer system employed.

The uncertainty analysis treats Type A and Type B uncertainties separately, but with the same formalism. It makes no allowance for the fact that the clock models (predictions of each clock's individual time, frequency, and frequency drift deviations from UTC) can absorb systematic uncertainties. The models in essence cause UTC to be an integrated frequency scale, so that each clock's contribution is to change UTC by its weight times the difference between its reading and the modeled prediction of its reading. With reference to Figure 3, the effect of the link uncertainties can be understood by taking into account the fact that  $\text{UTC}(k)$  is defined by the voltage of a specially designated time tick (or its equivalent) reaching a specific value or zero-crossing at the laboratory. If a  $\text{UTC}(k)$  were defined by a sole global time-transfer mechanism, consistency would be satisfied only until that system failed. If all labs were linked in a non-redundant single-pivot system, then a bias common to every link would not be noticed by any user who checked his/her system against more than one lab. The user's system would simply incorporate the bias. If the labs had different biases discrepancies would be noticed, and this is why the uncertainties of the biases cannot be ignored.

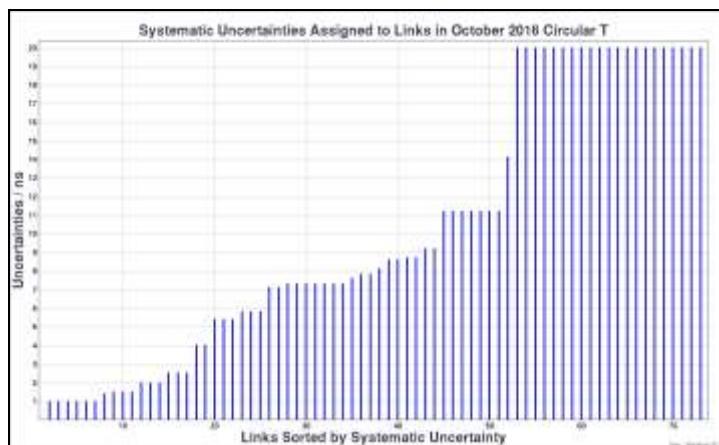


**Figure 3 A bias in the GNSS system of a laboratory will not be experienced by a user deriving time by an alternate means.**

To illustrate the effect of clock models in UTC, consider a new laboratory, with an ensemble of very good clocks but a totally uncalibrated time transfer system. It is unweighted until a clock model and precision-determined weight can be determined. With the model in use and the appropriate weight, UTC becomes more stable because it has more clocks to average. Because UTC is computed using the variations of the clocks of the new lab (call it lab X) from its initial model, all the numerical computations of UTC will be completely insensitive to the link's miscalibration. However the systematic uncertainties UTC-UTC(X) will reflect it, and they will add to the uncertainties of all the other laboratories in proportion to the weight assigned to the clocks of lab X.

#### Issues with Systematic Uncertainties

Issues arise because UTC is not designed to reduce the impact of link uncertainties. It is designed with a view towards optimizing frequency stability and frequency accuracy. The statistical uncertainties of the links become negligible on monthly scales, and the systematic uncertainties by definition do not depend on the averaging time. For statistical uncertainties this situation is strongly mitigated by the fact that the link jitters are easily measured and similar in magnitude, but this is not the case for systematic uncertainties (Figure 4). However, the differences in the link uncertainties are partially masked by the current practice of assigning a nominal uncertainty of 20 ns to all uncalibrated links. This is an overestimate for some receiver models because the manufacturers pre-calibrate the systems before shipment. However 20 ns is often an underestimate, as shown by the fact that six of the receivers calibrated for the first time by the BIPM had their values changed by 20 ns, 51 ns, 256 ns, 328 ns, 1000 ns, and 1600 ns. Fortunately most receivers are not so badly miscalibrated; in fact 24 other labs that were calibrated for the first time by the BIPM required no annotation of a change in the Circular T.

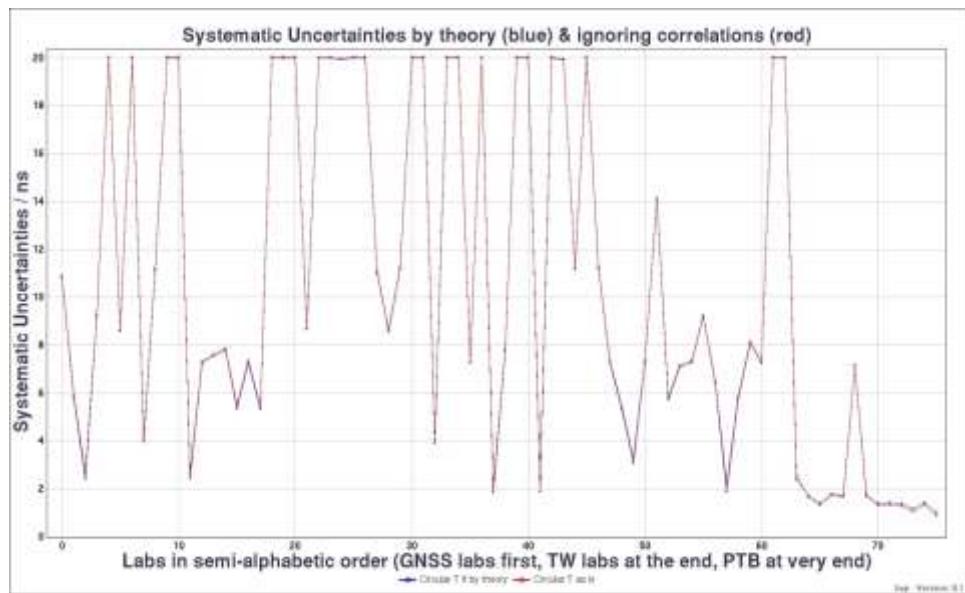


**Figure 4 Uncertainties assigned to the GNSS links in October 2016. TW-calibrated links were <=1.5 ns. The lowest GPS-calibrated links were 1.5 ns. Each of these links is with PTB, the pivot lab.**

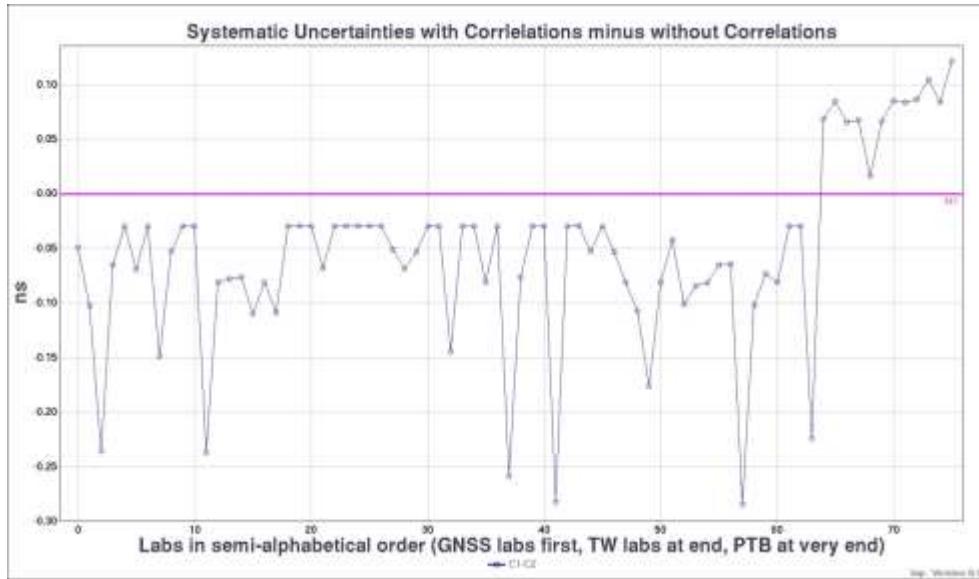
When the uncertainties of Figure 4 are inserted into the algorithm, the UTC(k) of the pivot lab (PTB) has the lowest uncertainty. Since the Circular T is computed assuming all links are TW, this can be explained as an inevitable outcome of the topology shown in Figure 2. For the single-pivot situation, the uncertainty of the PTB's link to each other lab is just the uncertainty of its direct link to the lab, and the average of all those links averages down towards zero. The uncertainty of every non-pivot TW lab's link to any other lab is the RSS of the uncertainty of its link to the PTB (first part) plus the uncertainty of the link from the PTB to the other lab (second part). Since the uncertainty of a lab k's direct link to the PTB is common to all links between lab k and other labs, the contribution of the "first part" to the RSS is common to all the links and does not average down. Appendix III shows that if every one of N equally weighted links was TW with uncertainty T, the pivot lab's uncertainty with UTC would approach  $T/\sqrt{N}$  (zero) as N increases, whereas the non-pivot labs' uncertainties would approach T.

The extra contribution would not be present if all non-pivot labs were linked via the same GNSS receiver and were treated as having site-based uncertainties. In this topology every lab is equally a pivot lab, and Appendix III shows that each lab's uncertainty with UTC would be  $G\sqrt{[(N-1)/N]}$ , which is essentially G for large N, where G is the calibration uncertainty of each GNSS receiver and where the uncertainty of the link between each individual pair of labs is  $\sqrt{2}G$ .

There are at least two justifications for the BIPM's computing the Circular T as if all links were baseline-based (TW). One reason for this is because GNSS receivers are not absolutely calibrated in practice. Calibrations are made relative to an uncalibrated receiver, so it is not obvious exactly how to divide the Type A or Type B uncertainties between the two labs. Also, any long-term drift of the receivers would effectively de-correlate the calibrations, rendering them baseline-based. In this work, we apply a simple algorithm to derive the individual calibration uncertainties for each GNSS receiver in the Circular T by finding the PTB-Lab(k) link that has the smallest uncertainty, assigning that uncertainty (divided by  $\sqrt{2}$ ) to the GNSS receivers of the PTB and Lab k, and determining all the other GNSS receiver's uncertainties by applying the RSS formula to their links with the PTB's receiver. The Circular T can then be computed treating GNSS links as site-based. The difference was found to be small (Figures 5 and 6). Note that the current link topology uses just one pivot, to which most of the weighted clocks link via TW. Experimentation with other topologies, and situations in which the TW links do not carry a large number of weighted clocks, show that in such cases it is important to take the site-based characteristic of GNSS links into account.



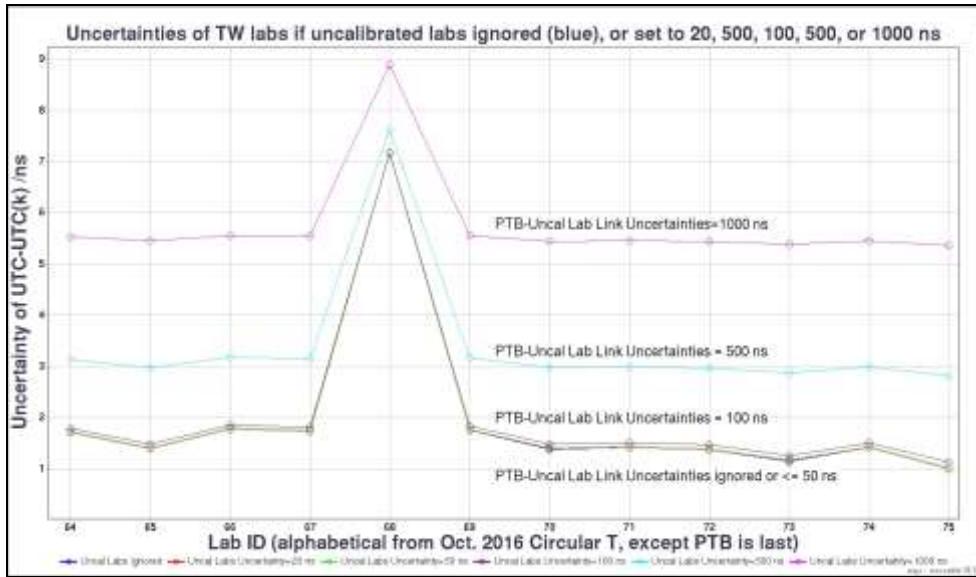
**Figure 5.** Systematic uncertainties computed treating all links as baseline-based (blue) and by treating the GNSS links as site-based (red). There is very little difference between the two curves.



**Figure 6 Difference between the systematic uncertainties as computed by the BIPM and how they would be computed if the site-based characteristics of GNSS noise were incorporated.**

In the figures, the PTB (last point) has the lowest uncertainty both ways, because the links with the most clocks remain computed via TW. The net effect of ignoring the site-based nature of the GNSS links is very subnanosecond. Also, while the GNSS labs would have a lower uncertainty if their link's site-based nature were incorporated it is found that the TW lab's uncertainties would increase. The effect is small because the GNSS receivers have large uncertainties, and the uncertainty calculations involve the RSS. The TW lab's uncertainties slightly increase because the correlation between a TW lab "T" and GNSS labs X and Y, would be the RSS of the T-PTB link plus the uncertainty of the GNSS receiver at the PTB (which is common to both the PTB-X and PTB-Y links); if all links were TW, then the GNSS receiver would not be contributing.

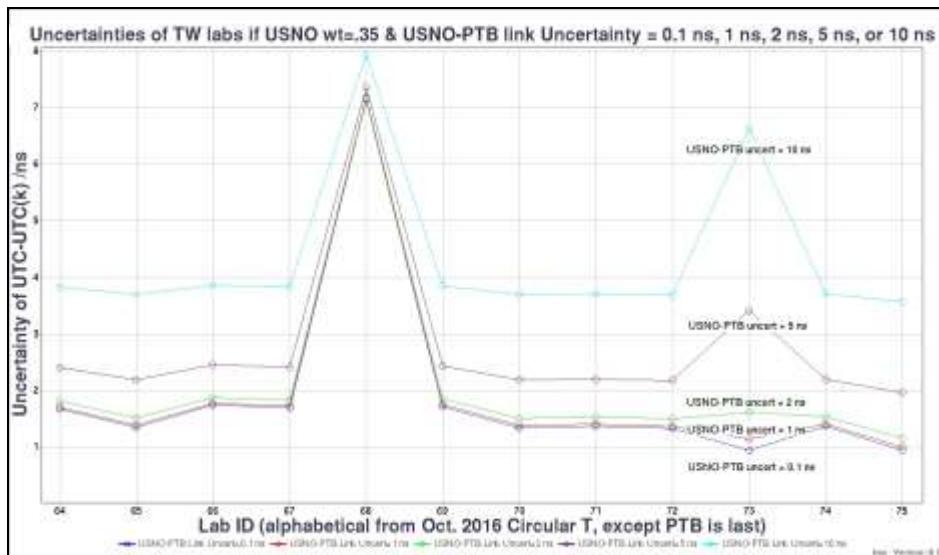
Issues have also arisen concerning the systematic (Type B) errors associated with uncalibrated labs and redundant links. Since UTC is not optimized with respect to link uncertainties, the uncertainty of each link adds to the variance of UTC-UTC(k), in rough proportion to its squared weight. In October 2016 there were 21 uncalibrated labs, with total weight of approximately 0.73%, and these contributed only about .3 ns to the overall UTC uncertainties at the current default uncertainty of 20 ns. Note that the effect of the uncalibrated labs is mostly seen in the TW labs, because the uncalibrated lab's contribution enters via RSS of the uncertainties. In order to study the dependence of the Circular T uncertainties on the nominal value assigned to uncalibrated links, the Circular T uncertainties were recomputed changing the uncertainties assigned to the uncalibrated labs (Figure 7).



**Figure 7 Uncertainties of the TW labs, each curve represents a different nominal uncertainty assigned to the uncalibrated GNSS labs. The lab at position 68 is NPL.**

In Figure 7, only the TW labs are shown because their low Type B uncertainties are most affected by the RSS of their links's variances with the uncalibrated labs. This is illustrated by the TW lab at position 68 (NPL), whose large uncertainty is due to having recently come on-line without a recent calibration. Note that its uncertainty is significantly less sensitive to the assigned value for the uncalibrated labs. (If the systematic uncertainty of NPL were based on a GNSS calibration it should properly have been treated as site-based, if it were a “bridge” calibration it should be treated as baseline-based). Figure 7 shows that assigning uncertainties of even 100 ns to all the uncalibrated links hardly affects even the TW lab uncertainties in the current situation. In the future we expect these links to become calibrated, although the problem may reoccur every time a new lab starts contributing to UTC with a uncalibrated system.

A third issue is that the large weight of the USNO causes any change in the uncertainty of its link to the PTB to negatively affect the uncertainties of UTC-UTC( $k$ ) for all other labs  $k$ , as in Figure 8.



**Figure 8 Effect on TW labs of changing the uncertainty of the highest-web lab (USNO, shown here as #73). NPL is identified as lab #68.**

Figure 8 indicates that the uncertainties of the TW labs can be noticeably affected if the USNO-PTB Type B uncertainty rises above 2 ns. This is a necessary consequence of the USNO's many clocks, and four rubidium fountains, which give frequency stability to UTC. The USNO takes its commitment to calibrate its link seriously, as well as to provide multiple time transfer modes so as to verify the calibration.

### A plausible but incorrect alternative technique

It is plausible to compute the Type B errors through an optimal weighting of weighted laboratories according to their links' Type B uncertainties; the resulting uncertainties would not be with respect to UTC but with respect to a UTC-like scale ( $UTC'$ ) that weighs clocks differently. In that case, as in Matsakis (2016) [4], the defining equation for each laboratory  $k$  would be:

$$B = \begin{pmatrix} G_{1k} \\ G_{2k} \\ G_{3k} \\ G_{4k} \\ G'_{xk} \\ \dots \\ T_{yk} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} (UTC'' - UTC(k)). \quad (1)$$

The solution is given by equations (4) and (5) and the measurement covariance matrix is given by the particulars of the links. In this case all possible chains of calibrated links would be taken into consideration, as the  $(n,m)^{\text{th}}$  element of the measurement covariance matrix is given by the covariance of the  $n^{\text{th}}$  link calibrated links emanating from  $k$  with the  $m$ -th calibrated link chain. If the two links terminate at the same lab, then the uncertainty of that lab must be included in the covariance. This leads to an iterative approach in which all the uncertainties are initially zero and each iteration uses the uncertainties derived in the previous iteration. Numerical simulations with 75 labs and actual Circular T data have shown that this procedure rapidly converges.

This method is not recommended because links to a laboratory with low weight in UTC but extremely low link uncertainties would dominate the computations, while an unweighted lab would decrease the computed uncertainty without adding new information (G. Petit, private communication). A further argument against this approach and for defining the uncertainties against UTC is that the derivative of the UTC time scale is its frequency scale, and for this the laboratory weights in UTC are well-defined and independent of the clock model.

### Discussion

Although assignment of 20 ns uncertainty to the uncalibrated labs is not strictly correct, the quantitative uncertainties of the uncertainties are not seriously affected. For the sake of metrological purity, one could compute the Type B errors using only calibrated links. The resulting uncertainties would not be with respect to UTC, but rather with a  $UTC'$  based upon the large subset of laboratories whose links are calibrated, and which contains over 99% of the weight. In the Circular T, the Type B and combined entries for the uncalibrated laboratories could be left blank or set to a higher nominal value such as 50 ns, and a clarifying note could be included. A variant of this would raise the uncertainty of the uncalibrated labs to 50 ns and use them in the computations; either method would have a political effect of stimulating calibrations. We note also that the BIPM's increased emphasis in calibrations will shortly reduce the total number of uncalibrated labs, perhaps to zero.

### Conclusions

The current computations of the systematic uncertainties in the Circular T could be modified to acknowledge the site-based characteristics of GNSS links. The quantitative difference would be small, although the symbolic

effect might be to emphasize the importance of calibration as well as the responsibilities of the pivot labs and highly-weighted labs to provide redundant, robust, and accurate time transfer systems.

### Acknowledgements

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### Appendix I. The statistical equations behind LMPT

EAL is the weighted sum of the deviations of clock readings from their predictions. Since a bias in a time transfer link is absorbed into the clock predictions, the numerical computation of EAL (and with it TAI and UTC) is insensitive to the bias. Because of this bias, an external observer using a very low-noise time transfer system to lab k (such as by optical fibers or carrier phase TWSTFT) would observe the UTC(k) as realized by the time tick, not the UTC(k) computed in the Circular T. This disagreement would be verified if the observer also linked to a different lab k', whose GPS and other links had different biases.

Since UTC for our purpose can be considered the weighted sum of the true difference between each laboratory's realization after correction by the Circular T, the systematic uncertainty of any lab k can be expanded as follows:

$$\begin{aligned} \langle(\text{UTC}-U_k)^2\rangle &= \langle(\sum(w_n U_n)-U_k)^2\rangle = \langle\sum w_n(U_n-U_k)^2\rangle \\ &= \langle[\sum w_n*(U_n-U_k)][\sum w_m*(U_m-U_k)]\rangle = \sum \sum w_m * w_n * \langle(U_n-U_k)*(U_m-U_k)\rangle, \end{aligned} \quad (\text{A1})$$

where UTC(k) minus its prediction is written  $U_k$ , the sum of the weights is 1. The summations are over all N laboratories, although the terms are zero when  $n=k$  or  $m=k$ . As in LMPT, the predictions are considered deterministic parameters and ignored.

In Appendix III are two simple geometries for which it is possible to derive a result analytically. One is GPS-only and the other is single-pivot TW-only. This is used as a test of computer programs implementing the models.

### Appendix II. A Straightforward Way to Compute the Uncertainties

The analysis shown here follows Petit and Jiang [3], in which the biases enter into the measurement covariance matrix and is assumed that the calibration measurements themselves have been incorporated into the time

transfer, and therefore there are no bias measurements to apply. An innovative approach was explored by Panfilo, Petit, and Harmegnies [5], who added explicit references to IGS time and added the calibrations as separate measurements.

In this formulation there are N labs, L GNSS measurements (which have site-based uncertainties), and M TW measurements (which are link-based). Only independently-calibrated links are applied. TW links calibrated via the triangle method [5] provide no new information, and using them would cause the determinant of the measurement matrix to be zero. Uncalibrated links would make it infinite.

In the first step, N-1 values of UTC (k)-UTC (1) are found with the L+M measurements in a least-squared solution. Uncalibrated links are ignored, and redundant measurements exist if L+M>N-1. Following the style of Petit and Jiang, 2005, a typical equation for a 4-lab solution takes the form B=AX.

$$B = \begin{pmatrix} G_{21} \\ G_{31} \\ G_{43} \\ G'_{21} \\ G'_{32} \\ T_{31} \\ T_{43} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} UTC(2) - UTC(1) \\ UTC(3) - UTC(1) \\ UTC(4) - UTC(1) \end{pmatrix} = AX. \quad (B1)$$

Here the measurement vector B's components are denoted G for GPS, G' for other GNSS, and TW for TW.  $G_{nm}$  is a GPS measurement between lab n and lab m, the non-GPS but GNSS measurement  $G'_{nm}$  is between labs n and m, and  $T_{nm}$  is a TW measurement between labs n and m. We later will denote UTC(k)-UTC(1) as UTC<sub>k</sub>, and note that UTC<sub>1</sub>=0. The site-based uncertainty of a GNSS lab n is written  $G_n$ , so that the uncertainty between labs n and m is  $\sqrt{(G_n^2 + G_m^2)}$ .

The measurement vector has L+M components, the parameter vector has N-1 components, and the design matrix has L+M rows and N-1 columns.

The solution is found using the symmetric covariance matrix S of the measurements, which is L+M square. If we denote the squared uncertainty by  $u^2$ , S can be written:

$$S = \begin{pmatrix} u_{G1}^2 + u_{G2}^2 & u_{G1}^2 & 0 & 0 & 0 & 0 & 0 \\ u_{G1}^2 & u_{G1}^2 + u_{G3}^2 & -u_{G3}^2 & 0 & 0 & 0 & 0 \\ 0 & -u_{G3}^2 & u_{G3}^2 + u_{G4}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{G',1}^2 + u_{G',2}^2 & -u_{G',2}^2 & 0 & 0 \\ 0 & 0 & 0 & -u_{G',2}^2 & u_{G',2}^2 + u_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_{T31}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{T32}^2 \end{pmatrix} \quad (B2)$$

As in Petit and Jiang, the solution can be written:

$$X = (A^T S^{-1} A)^{-1} (A^T S^{-1}) B. \quad (B3)$$

And the solution parameters' covariance matrix is given by:

$$(A^T S^{-1} A)^{-1}. \quad (B4)$$

The second step is to find UTC-UTC(1), which is possible because UTC is the weighted sum of the UTC(k), and therefore

$$UTC - UTC(1) = [\sum (w_n * UTC(n))] - UTC(1) = \sum w_n * [UTC(n) - UTC(1)]. \quad (B5)$$

The values within the summation brackets the parameters solved for in the first step. Since labs with no calibrated links are not used in the solution, we must normalize the weights so their sum remains one. Note that not normalizing the weights is equivalent to assuming the uncalibrated labs are realizing UTC exactly.

$u_k^2$ , the squared uncertainties of UTC-UTC(k) can be found as follows:

$$u_k^2 = \langle [UTC - UTC(k)]^2 \rangle = \langle [\sum w_n * UTC(n) - \sum w_n * UTC(k)]^2 \rangle = \langle \{\sum w_n * [UTC(n) - UTC(k)]\}^2 \rangle. \quad (B6)$$

Where the summations are over all labs. If we let  $UTC_n = UTC - UTC(1)$ ,

$$u_k^2 = \langle \{\sum w_n * [UTC(n) - UTC(1) + UTC(1) - UTC(k)]\}^2 \rangle = \langle \{\sum w_n * (UTC_n - UTC_k)\}^2 \rangle. \quad (B7)$$

$$= \langle (\sum w_n * [UTC_m - UTC_k]) * \sum w_m * [UTC_m - UTC_k] \rangle. \quad (B8)$$

$$= \sum \sum w_n * w_m * [\langle (UTC_n * UTC_m) \rangle - 2\langle (UTC_m - UTC_k) \rangle + \langle (UTC_k * UTC_k) \rangle]. \quad (B9)$$

$$= \sum \sum w_n * w_m * \langle (UTC_n * UTC_m) \rangle - 2 \sum w_n * \langle (UTC_n - UTC_k) \rangle + \langle (UTC_k * UTC_k) \rangle. \quad (B10)$$

Equation 11 gives the desired uncertainties, since every stochastic quantity in (B10) was estimated in the first step, and the weights used to create TAI are also known. It is the same equation for statistical uncertainties (Type A), however even uncalibrated labs should be included in that computation.

### An alternate Way to implement the same algorithm

With reference to equation A1, it is possible to write a computer program that considers every possible chain of calibrated links between the lab in question (k) and any two labs n and m, where n could equal m. In this case, the correlations or lack thereof for site-based and baseline-based links can be directly taken into account. Redundant links should be incorporated into  $\langle (U_n - U_k) * (U_m - U_k) \rangle$  by mimicking how they would be used in TAI generation. For example, parallel redundant links of the same type (site-based or link-based) will likely be handled by using the variance of the weighted sum of the links' Type A errors; these same weights should be applied to the Type B errors. The derived terms will not be the optimal ones for minimizing the systematic errors, however the goal is to correctly describe how accurately the laboratory time ticks are tied to their estimated  $UTC(k)$ .

Due to the complexity of the programming, this approach is not recommended. It is useful as a check for the more simple way described above.

### Appendix III. Two simple cases where the solutions can be found analytically (all GPS and all TWSTFT).

Case I. All N labs are equally weighted and do GPS only, with receiver calibration uncertainty G. The uncertainties are a property of the receiver (site-based).

$$\text{Then for } n \neq m, \langle (U_n - U_k) * (U_m - U_k) \rangle = G^2; \text{ if } n = m, \langle (U_n - U_k) * (U_m - U_k) \rangle = 2G^2. \quad (C1)$$

$$\text{Therefore } \langle (UTC - U_k)^2 \rangle = [(N-1)*(N-2)*G^2 + (N-1)*2G^2]/N^2 = [(N-1)/N]*G^2 \quad (C2)$$

$$\text{and } \sqrt{\langle (UTC - U_k)^2 \rangle} = GV[(N-1)/N], \text{ which approaches } G \text{ for large } N. \quad (C3)$$

Note that the uncertainty of a link is  $\sqrt{2} G$ .

Case II. All N labs are equally weighted and do TW only to a pivot lab P, with link calibration uncertainty T. No GNSS links are used. The difference between this and the first case is that here the uncertainties are link-based and not site-based.

$$\text{If } k = P, \langle (U_n - U_P) * (U_m - U_P) \rangle = T^2 \text{ if } n = m; \text{ zero otherwise} \quad (C4)$$

$$\text{Therefore } \langle (UTC - U_k)^2 \rangle = \sum (w_m * w_m * T^2) = (T^2(N-1)/N/N)^2 = T[V(N-1)]/N, \quad (C5)$$

which approaches  $T/VN$  for large  $N$  (C6)

If  $k \neq P$ ,  $\langle (U_n - U_k)^2 \rangle = T^2$  unless  $n=k$  or  $m=k$  (C7)

therefore  $\langle (UTC - U_k)^2 \rangle = [T^2(N-1)/N]^2$  (C8)

and  $V(\langle (UTC - U_k)^2 \rangle) = T V[(N-1)/N]$ , which approaches  $T$  for large  $N$ . (C9)