

# THE RANGE COVERED BY A CLOCK ERROR IN THE CASE OF WHITE FM

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## Abstract

*The range covered by the time error of a clock affected by a white frequency noise is studied by means of the theory of the Wiener process and its probability distribution is inferred. The application to atomic clocks and the MTIE characterization used in the telecommunication standards is also examined.*

## 1. INTRODUCTION

In these last years, mostly due to the input of the telecommunication community, it became of interest to know of the possible range spanned by the time error of a clock, since it helps in correctly designing the memory buffers. The problem may be illustrated as follows: suppose we have an atomic clock used as a synchronization unit in a telecommunication network and we know that the clock signal is mostly affected by a certain random noise. Which is the "time error," i.e. the phase deviation, that such a clock may accumulate in a certain time interval? Apart from deterministic trends, the answer regarding the random component may only be a probabilistic one in view of the stochastic the nature of the process. So the problem can be better expressed as: knowing the spectral density of phase fluctuations, which is the probability law of the range spanned by such phase fluctuations?

The case of white phase noise was recently examined and it was possible to infer the probability law of the spanned range [1, 2, 3]. Also the relationship between the amount of white PM noise stated by the Allan deviation or the spectral density and the Maximum Time Interval Error MTIE, largely used in the telecom community and recently defined as a percentile quantity [4], was evinced. In this paper, the case of Gaussian white frequency modulation, which results in a phase random walk is considered and the range probability law is inferred, by the study of Wiener

processes and their characteristics. This study yields to the estimation of the "maximum" range that can be spanned by the clock error by the identification of a certain percentile in the range distribution, i.e. a range value which is not exceeded more than a certain small percentage of times. The relationship of the percentile range and the MTIE is again investigated and some examples of atomic clock white frequency noise and possible percentile MTIE are given.

## 2 MATHEMATICAL BACKGROUND

In the following the theoretical definition and characterization of the Wiener process are introduced [5]. Symbols used hereafter are typical in the description of stochastic processes and are different from the common symbols used in PTTI descriptions. The application to the time error of a clock and to the telecommunication standards will be addressed in the next section.

Let  $\{X_k\}$  be a sequence of mutually independent random variables with a common distribution, zero mean and variance  $\sigma^2$ . The discrete variable  $S_n$

$$S_n = X_1 + \dots + X_n, \quad (S_0 = 0),$$

denoting the position at time  $n$  (integer) of a moving particle, describes a random walk. The range  $R_n$  spanned by the discrete process  $S_n$  is defined as the difference between the maximum and the minimum value, therefore:

$$R_n = \max [0, S_1, \dots, S_n] - \min [0, S_1, \dots, S_n].$$

The discrete sums  $S_n$  are asymptotically normally distributed and can be considered as the value at time  $t=n$  of a continuous Wiener process. For the evaluation of the spanned range, the continuous approximation is more convenient, therefore, the sum  $S_n$  is replaced by the Wiener process  $S(t)$  where  $S(0) = 0$ . Moreover, the introduction of the Wiener process is not only useful for the following analytical development, but also because the Wiener process itself can be a convenient description of reality, for example when describing the error of an atomic clock. Therefore, in the following only the continuous Wiener process will be examined. At any instant  $t$ ,  $S(t)$  is a normal variable with zero mean and variance  $\sigma^2 t$ , therefore probability that the process  $S(t)$  is in the position  $s$  can be described by the probability law:

$$p(s; t) = \text{Prob}\{S(t)=s\} = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{s^2}{2\sigma^2 t}\right\} \quad (1)$$

A Wiener process with drift  $\mu$ , can also be introduced as  $Y(t) = S(t) + \mu t$ , but for sake of simplicity the case with  $\mu=0$  is here examined. Nevertheless, the range distribution can be found also in case of  $\mu \neq 0$ . In the new notation, the range of  $S(t)$  is defined as:

$$R(0,t) = \max_{0 \leq T \leq t} S(T) - \min_{0 \leq T \leq t} S(T).$$

assuming that  $S(0) = 0$ . Studies on the peculiarities of the Wiener processes are reported in many papers and reference texts [5,6,7]. Particular attention is devoted to the study of the *survival probability* of a restricted process, i.e. the probability that the Wiener process evolves till the instant  $t$  without having touched upper and lower boundaries  $a$  and  $b$  ( $a > 0$ ,  $b < 0$ ). The survival probability benefits from many analytical results and it can be seen that there is an intimate relationship between the probability distribution of the range and the survival probability [8, 9, 10, 11]. We will investigate such a relationship. The study of the range probability distribution requires evaluating the joint probability distribution  $F_{m,M}(t)$  of the maximum and the minimum value of the process, indicated respectively with  $M$  and  $m$ .  $F_{m,M}(t,b,a)$  represents the probability that the minimum value  $m$  doesn't exceed the value  $b$  and the maximum  $M$  doesn't exceed the value  $a$ :

$$F_{m,M}(t;b,a) = P\{M(t) \leq a, m(t) \leq b\}.$$

This probability can be expressed as:

$$F_{m,M}(t;b,a) = P\{M(t) \leq a, m(t) \leq b\} = P\{M(t) \leq a\} - P\{M(t) \leq a, m(t) > b\}.$$

where the survival probability  $P\{M(t) \leq a, m(t) > b\}$  of the process restricted by the barriers  $a$  and  $b$  is introduced. Let's consider the joint density function  $f_{m,M}(b,a)$  of the maximum and the minimum, i.e. the probability that the larger value falls between  $a$  and  $a+da$  and that the smaller value falls between  $b$  and  $b+db$ , obtained, by definition, as:

$$f_{m,M}(b,a) = \frac{\partial^2 F_{m,M}(t;b,a)}{\partial b \partial a}$$

Since we consider the Wiener process  $S(t)$  with  $S(0)=0$ , the existence field of the joint density function  $f_{m,M}(b,a)$  is given by the region in which  $a$  is positive and  $b$  negative. Moreover, in such a region, a certain sub-region  $D_r$  can be identified where the following relationship concerning the range holds:

$$D_r = \{ 0 \leq R(0,t) = M(t) - m(t) \leq r \}$$

The region  $D_r$  can be identified by the following relationships:

$$\begin{cases} b \leq 0, a \geq 0, \\ a - b \leq r. \end{cases} \quad \text{that is} \quad \begin{cases} -r \leq b \leq 0, \\ 0 < a \leq r + b. \end{cases}$$

The range probability distribution, i.e. the probability that the range  $R(0,t)$  doesn't exceed a certain value  $r$ , is given by the integral extended to the region  $D_r$  of the joint density function of the maximum and the minimum, i.e.:

$$F_R(r) = P\{R(0,t) \leq r\} = \iint_{D_r} f_{m,M}(b,a) db da = \int_{-r}^0 \left[ \frac{\partial F_{m,M}(t;b,a)}{\partial b} \right]_{a=0}^{a=b+r} db$$

From this writing it is intuitive that the range probability distribution depends on the survival probability, which is known in case of a Wiener process. By several laborious calculations, the range probability distribution can be obtained as:

$$F_R(r) = P\{R(0,t) \leq r\} = \quad (2)$$

$$= \sum_{k=1}^{+\infty} \left\{ -6kErf\left[\frac{2kr}{\sigma\sqrt{2t}}\right] + 4kErf\left[\frac{(1+2k)r}{\sigma\sqrt{2t}}\right] + 4kErf\left[\frac{(2k-1)r}{\sigma\sqrt{2t}}\right] + k \left( Erf\left[\frac{2(1-k)r}{\sigma\sqrt{2t}}\right] - Erf\left[\frac{2(1+k)r}{\sigma\sqrt{2t}}\right] \right) \right\},$$

where  $Erf[y] = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx$  stands for the error function.

An analogous but more complicated expression holds for the range distribution in case of a Wiener process with drift  $\mu$  [10]. In Fig. 1 the range probability distribution (2) is represented for a Wiener process with variance  $\sigma^2=1$ . We can note that, for fixed  $t$ , the probability distribution increases when the value  $r$  is rising. It means that, for a fixed  $t$ , it becomes more and more probable to observe a range below the threshold level  $r$ , if  $r$  is high. For fixed  $r$ , instead, it can be seen that the probability that the covered range is below the threshold value  $r$ , is initially high, but then it decreases with time. If we consider an horizontal section of the Fig.1, we can identify the curves relating  $t$  and  $r$  that guarantee a certain percentile in the range distribution, i.e., for each  $t$ , a range threshold  $r$  which is not exceeded

more than a certain percentage of times. By fixing the probability level at the values 95%, 90%, and 80%, the Fig. 2 is obtained, where the range thresholds  $r$  are on the vertical axis and the time instants  $t$  on the horizontal one. It can be observed that the curve referred to the largest percentile increases more rapidly, that makes sense because, for fixed  $t$ , the threshold range  $r$  that guarantees to be larger than the observable ranges in 95 out of 100 cases, should be larger than the threshold ranges corresponding to smaller percentages.

### 3 RELATIONSHIP WITH THE MAXIMUM TIME INTERVAL ERROR (MTIE)

The study of the probability of the range spanned by a Wiener process can find immediate application in the characterization of clocks. This is of particular interest in case of digital telecommunication networks. Digital switching equipment in fact require synchronization in order to avoid slips in the input elastic stores [12,13,14]. To specify the clock stability requirements in telecommunication standards, the International Telecommunication Union (ITU-T) defined the quantity MTIE (Maximum Time Interval Error) [4]. It measures the range covered by the error of a clock with respect to a known reference. Let  $x(t)$  be the time error of a clock and  $\tau$  the observation time, the range of the clock error is defined as (Fig. 3):

$$\text{MTIE}_{t_0}(\tau) = \max_{t_0 \leq t \leq t_0 + \tau} (x(t)) - \min_{t_0 \leq t \leq t_0 + \tau} (x(t)).$$

Recently ITU-T defined  $\text{MTIE}(\tau, \beta)$  as a specified  $\beta$ -percentile of the random variable  $\text{MTIE}(\tau)$ , that is to say as the range value which is not exceeded more than a certain small percentage  $(1-\beta)$  of times, for any  $t_0$ .

The clock phase error  $x(t)$  is usually due to deterministic variations and to stochastic noises of different nature. In most of the commercially available clocks and reference oscillators one of the dominant noises, over certain observation intervals, is due to a white frequency modulation, which results in a phase random walk. Therefore, the previous study of the range covered by a Wiener process is of immediate utilization to study the range covered by the phase error of a clock affected by white FM. Let's consider a white FM with zero average that corresponds, in the stochastic process language, to a phase error described by a Wiener process  $S(t)$  without drift ( $\mu=0$ ) and with variance  $\sigma^2$ . The range spanned by the phase error  $x(t)$  is thus the range spanned by the process  $S(t)$  as studied in the previous section, where we replace the elapsed time  $t$  with the observation interval  $\tau$ . For sake of convenience, let's consider the new variable  $R_N = R/\sigma$ , i.e. the range normalized over the square root of the variance for unit of time of the Wiener process,  $R_N$  is dimensionless. According to the results obtained in the previous section, the probability distribution (2) of  $R_N$  can be written as :

$$\begin{aligned}
F_{R_N}(r_N) &= P\{R_N(\tau) \leq r_N\} = \\
&= \sum_{k=1}^{+\infty} \left\{ -6kErf\left[\frac{2kr_N}{\sqrt{2\tau}}\right] + 4kErf\left[\frac{(1+2k)r_N}{\sqrt{2\tau}}\right] + 4kErf\left[\frac{(2k-1)r_N}{\sqrt{2\tau}}\right] + \right. \\
&\quad \left. + k\left(Erf\left[\frac{2(1-k)r_N}{\sqrt{2\tau}}\right] - Erf\left[\frac{2(1+k)r_N}{\sqrt{2\tau}}\right]\right) \right\}.
\end{aligned} \tag{3}$$

Such distribution probability allows the interpretation of the percentile range as contained in the percentile definition of MTIE( $\tau, \beta$ ), in fact, by fixing the percentile level  $F_{R_N}(r_N) = \beta = 0.80, 0.90$ , and  $0.95$  respectively, as done before, the percentile curves of Fig. 2 are obtained and they describe the range threshold values that are not exceed in the  $\beta$  percentage of observations. The same percentile curves are also represented in Fig. 4 in logarithmic co-ordinates and with the normalized range values. From Fig. 4 some numerical estimations of the MTIE( $\tau, \beta$ ) are possible, when the level of random walk noise is known; for example, for  $\tau=10^5$  units of time, the normalized range threshold level corresponding to the 90<sup>th</sup> percentile is about equal to the value 750. From the expressions (3) of the probability distribution  $F_{R_N}(r_N)$ , it is difficult to analytically solve for the expression relating  $\tau$  and  $r$  for a fixed probability. The percentile curve represented in Fig. 2 and 4 are thus found by numerical evaluations, but it can be seen that such percentile curves are nicely approximated by the curves

$$r_N = k_\beta \sqrt{2\tau},$$

where  $k_\beta$  is a real number depending on the probability levels and that in the represented cases amounts to:  $k_{80} = 1.39$ ,  $k_{90} = 1.59$ , and  $k_{95} = 1.77$

This approximated relationship allows to find a direct, through approximated, connection between the percentile range, thus the MTIE( $\tau, \beta$ ) and the noise variances. Let's evaluate that.

In the language of stochastic processes, the Wiener process is described by a drift  $\mu$  and a variance  $\sigma^2$ . In clock stability characterization, we are more familiar to Allan variances or spectral densities. The relationship between the Allan variance AVAR and the  $\sigma^2$  of the Wiener process is given by [15]  $\sigma^2 = AVAR(\tau) \cdot \tau$ , where the dimension of  $\sigma$  are [ps/ $\sqrt{s}$ ], when the phase error  $x(t)$  is measures in ps and  $\tau$  in s. The square root of the variance for unit of time of the Wiener process, used for normalizing the range, is therefore  $\sigma \cdot \sqrt{1s}$ . By using the approximated relationship above, the known [16] time domain/frequency domain relationships, by choosing the

range and  $\tau$  units in ps and s respectively, the percentile range MTIE( $\tau, \beta$ ) as a function of  $\tau$  can be written as:

$$\text{MTIE}(\tau, \beta) \approx k_\beta \sqrt{2\tau} \cdot \sigma = k_\beta \sqrt{2\tau} \cdot (\text{ADEV}(\tau) \cdot \sqrt{\tau}) = k_\beta \sqrt{2\tau} \cdot \frac{\sqrt{h_0}}{\sqrt{2}}$$

where  $h_0$  is the constant that determines the amount of white frequency noise in the polynomial model for the frequency spectral density  $S_y(f)$ . It remains to evaluate what can be the numerical values of MTIE( $\tau, \beta$ ) for some typical clocks. This is discussed in the next section.

#### 4 NUMERICAL EXAMPLE

Let's evaluate the percentile range, thus MTIE( $\tau, \beta$ ), for example of a typical commercial high stability cesium clock. Let's assume that the noise of the clock is due to a white FM with zero average, that means that the cesium clock is considered on observation intervals of about  $1 \leq \tau \leq 10^6$  s, and that its frequency deviation is equal to zero (corresponding to  $\mu=0$ ). Let's suppose that the WFM noise of such a Cs clock amounts to a typical value given by

$$\text{ADEV}(\tau) = 1 \cdot 10^{-11} \cdot \tau^{-1/2} \quad 1 \leq \tau \leq 10^6 \text{ s}$$

By using the relationship between the Allan variance AVAR and the  $\sigma^2$  of the Wiener process, it is found:

$$\sigma = \text{ADEV}(1\text{s}) \cdot \sqrt{1\text{s}} = 1 \cdot 10^{-11} \cdot \sqrt{1\text{s}} = 10 \frac{\text{ps}}{\text{s}} \cdot \sqrt{1\text{s}} = 10 \frac{\text{ps}}{\sqrt{\text{s}}}$$

The square root of the variance for unit of time of the Wiener process, used for normalizing the range, is therefore  $\sigma \cdot \sqrt{1\text{s}} = 10 \text{ ps}$ . Now the values of the threshold percentile range reported in Fig. 4 can be interpreted as a percentile range, thus MTIE( $\tau, \beta$ ), provided that  $\tau$  is measured in seconds and the normalized range values (dimensionless) reported in Fig. 4 are multiplied by the normalization factor that is, in this case,  $\sigma \cdot \sqrt{1\text{s}} = 10 \text{ ps}$ . Therefore, the MTIE( $\tau, \beta$ ) that could be observed on the phase error  $x(t)$  of the considered Cs clock are reported in Fig. 5 and, for example, for  $\tau=10^5 \text{ s}$ , it amounts to about 7.5 ns, which makes sense considering the stability of the considered clock. By using the approximate relationship discussed in the previous section it can also be written that, for the particular clock:  $\text{MTIE}(\tau, \beta) \approx k_\beta \sqrt{2\tau} \cdot 10 \text{ ps}$ .

It can be worth comparing there results with the MTIE( $\tau, \beta$ ) prescribed by ITU standards. For example, the case of a Cs clock is considered in the standard [17] concerning primary reference clocks(PRC), which reports the limits for MTIE( $\tau, \beta$ ) values,giving:

$$\begin{aligned} \text{MTIE} &= 0.275 \cdot 10^{-3} \cdot \tau + 0.025 \text{ } \mu\text{s} & \text{for} & \quad 0.1 \leq \tau \leq 1000 \text{ s} \\ &= 10^{-5} \cdot \tau + 0.29 \text{ } \mu\text{s} & \text{for} & \quad \tau > 1000 \text{ s} \end{aligned}$$

Some numerical values are for example:

$\tau = 1 \text{ s}$	$\text{MTIE} \approx 25 \text{ ns}$
$\tau = 10 \text{ s}$	$\text{MTIE} \approx 27.8 \text{ ns}$
$\tau = 100 \text{ s}$	$\text{MTIE} \approx 52.5 \text{ ns}$
$\tau = 1000 \text{ s}$	$\text{MTIE} \approx 300 \text{ ns}$
$\tau = 10000 \text{ s}$	$\text{MTIE} \approx 390 \text{ ns}$

which are represented by a dotted line in Fig. 5 and that are largely achieved by the stability of the considered Cs standard.

## CONCLUSION

By the analytical study of the properties of random walks and Wiener processes, it was possible to infer the probability distribution of the range covered by the process. This helps in understanding the percentile MTIE( $\tau, \beta$ ) used in telecommunication for describing the range covered by a clock time error, when the noise of the clock is due to a white FM and thus random walk of phase. An example concerning a high stability commercial Cs clock gives estimates of the expected MTIE( $\tau, \beta$ ) which largely comply with the requests of telecommunication standards.

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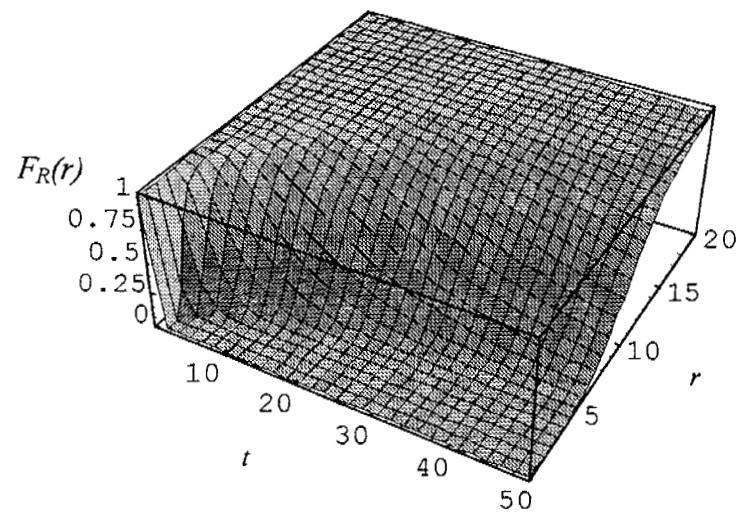


Fig. 1: range probability distribution  $F_R(r)$  for a Wiener process with zero average and  $\sigma^2=1$ .

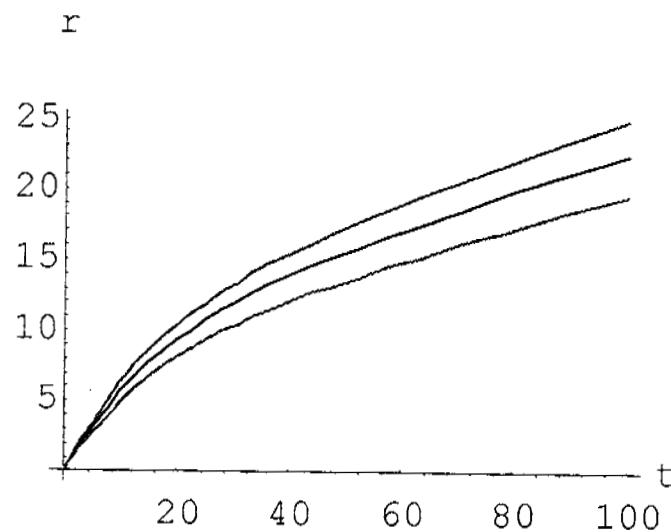


Fig. 2: Horizontal section of the surface in Fig. 1 representing, for each  $t$ , the threshold percentile value of the range  $r$  in the case of the 80<sup>th</sup>, 90<sup>th</sup>, and 95<sup>th</sup> percentile.

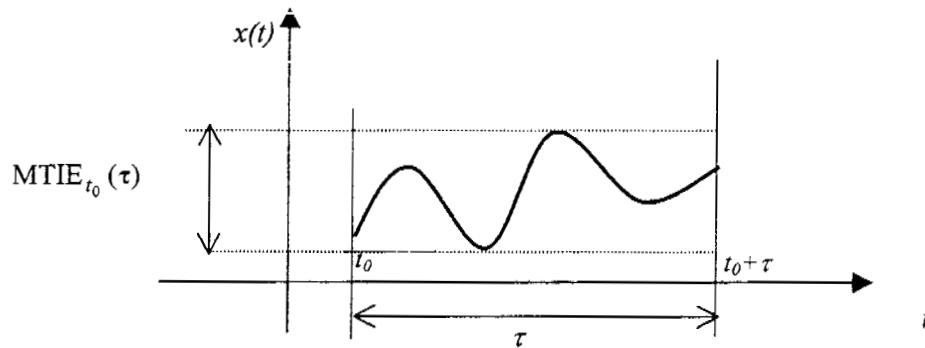


Fig. 3: graphical representation of the quantity MTIE [4]

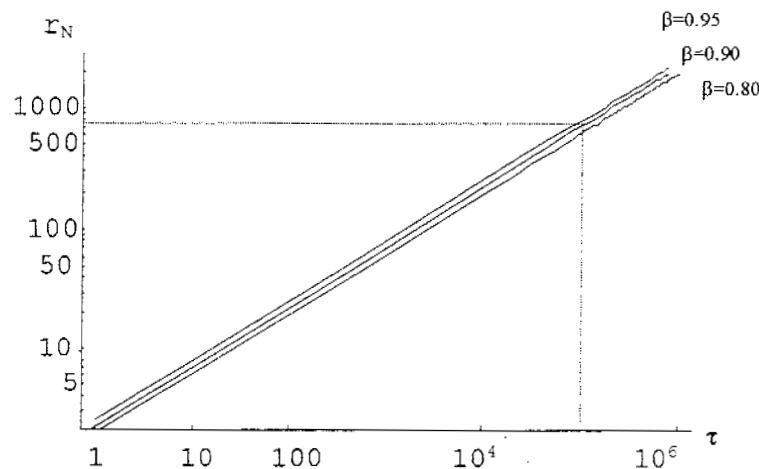


Fig. 4: normalized range threshold values, versus observation time for different percentile levels.

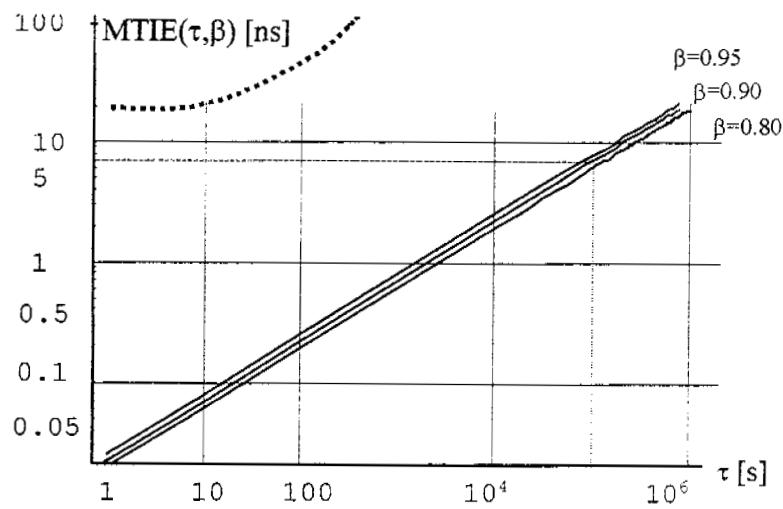


Fig. 5: MTIE( $\tau, \beta$ ) as expected from an high stability Cs standard and ITU requests (dotted line, top left)

### **Questions and Answers**

JUDAH LEVINE (NIST): The anti-specification usually has a requirement on the frequency accuracy as well, but you do not have that in your method.

PATRIZIA TAVELLA (IEN): No, the frequency accuracy for such a finite standard is that the frequency deviation should not exceed  $10^{-11}$  over the long term. In the case of cesium, I think it is very well done.

JUDAH LEVINE : But, it might not be true for rubidium.

PATRIZIA TAVELLA: Yes, you are right.