Controlling clocks with PID controllers

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Biography

Dr. Matsakis received his B.S. in physics from MIT, and Ph D in physics from U.C. Berkeley, where he studied under Charles Townes. Originally an astrophysicist, he switched to timekeeping about mid-way through his 40-year tenure at the U.S. Naval Observatory (USNO). During this period he worked on most aspects of timekeeping, including pulsar timekeeping, mercury trapped-ion clocks, atomic fountain construction, timescales, GPS, NTP, two-way satellite time-transfer (TWSTT, also known as TWSTFT), administration, and policy. He has published over 150 papers, and secured one patent. In July, 2019 he went from being Chief Scientist at the USNO to Chief Scientist for Masterclock, Inc.

Abstract

The PID approach to controlling clocks determines a frequency steered based upon the dot product of a 3-component gain vector with a vector whose components are optimal estimates of the phase (time, or P for proportional), frequency (derivative D), and integral of the phase (denoted I). Just as with a 2-component "PD" approach [1,2], which ignores the integral I, it is possible to compute the steady-state variances of any linear combination of the controlled clock's phase, frequency, and steers (hereafter PFS), as well as the time constants(s) for the response to a disturbance. The PD approach assumes unmodeled systematic errors, such as frequency drift, are small. If this is not the case, use of the integral helps allow for them. As with PD controllers, critical PID gains can be computed, which result in a decaying exponential response to a disturbance. The regions of PID stability can be mapped out, and it is found that increasingly higher choices of the integral's gain coefficient (I_gain) result in stable solutions only if increasingly higher minimum frequency-gain coefficients are selected. Since the minimum values of the three variances occurs with the I_gain set to 0, use of PID can be considered a way to ensure against non-stochastic errors, at the price of greater variance in the PFS.

I. Introduction

Control theory is often taught at the graduate level, appears in generalized form in textbooks and publications such as [3-6], and has been applied to disciplining oscillators [1,2,7]. The theory is not yet complete because it does not fully address the effects of suboptimal modelling (such as frequency drift), small variations in the loop gains, and other properties [8,9], although simulations can provide a useful supplement in this regard [10,11].

This paper extends the analysis published in [1,2, and 11], in which clocks are controlled by adjusting the frequency through steers computed as the frequency adjustments computed as the dot product of a gain vector and a system state vector. Two techniques were discussed: pole placement (PP) and Linear Quadratic Gaussian (LQG). LQG finds the gains that optimize performance as defined by a linear combination of the PFS, while the system response to a disturbance is uncontrolled. PP finds the gains that ensure a desired response to disturbance, such as a decaying exponential, while leaving the PFS to be uncontrolled consequences. The techniques were unified by showing how all their effects can be computed and graphically displayed on common plots that have the two components of the gain vector (phase gain and frequency gain) as the abscissa and ordinates.

An explicit requirement in the analysis is that the state of the disciplined clock (relative to its reference) be optimally estimated, and that the differences between the estimated and actual states be stochastic in nature. In this work we follow the Kalman formalism (Appendix 1) for state estimation, since it yields an optimal estimate of the current state under very general assumptions, such as independent noise effects and proper parameterization.

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Unfortunately, for many systems there are systematic effects that cannot be easily modelled. These can result in a controlled clock never reaching its desired set point. For example, assume steering has brought a clock to be exactly on time and on frequency, but the underlying oscillator has a frequency drift that is not removed by modelling. Over the next interval, no steer will be applied and the clock's time and frequency will reach higher values. Successive steering will bring it back towards 0, but every time the phase and frequency will be higher than anticipated, and there will be a consistent net offset in one direction. Another way to view this is to note that if a solution to the PD equation exists:

$$p_{n+1} = p_n + (p_n - p_{n-1}) - \tau g_P p_n - g_D(p_n - p_{n-1}) = (2 - \tau g_P - g_D)p_n - (1 - g_D)p_{n-1} \quad (1.1)$$

Then the addition of a drift term $\frac{d\tau^2}{2}$ on the right-hand side of equation (1.1), leads a solution of the original equation by substituting $p_{with\_drift,n} = p_n + \frac{d\tau}{2g_P}$ as follows:

$$p_{n+1} + \frac{d\tau}{2g_P} = (2 - \tau g_P - g_D)(p_n + \frac{d\tau}{2g_P}) + (g_D - 1)(p_{n-1} + \frac{d\tau}{2g_P}) + d\tau^2/2 \quad (1.2)$$

$$p_{n+1} + \frac{d\tau}{2g_P} = (2 - \tau g_P - g_D)p_n + (g_D - 1)p_{n-1} + (+2 - \tau g_P - g_D + g_D - 1)(\frac{d\tau}{2g_P}) + d\tau^2/2 \quad (1.3)$$

$$p_{n+1} = (2 - \tau g_P - g_D)p_n - (1 - g_D)p_{n-1} - \frac{\tau g_P d\tau}{2g_P} + \frac{d\tau^2}{2} = (2 - \tau g_P - g_D)p_n - (1 - g_D)p_{n-1} \quad (1.4)$$

A crude, but effective way to handle such situations is to steer so as to minimize the integral of the gain (I). In the example of equations 1.1-1.4, the integral of the phases will be greater than 0, which will result in larger steers, that will be much more effective in negating he effects of the unmodeled drift.

II. The expected variances of the phase, frequency, and frequency steers as functions of the gain vector, in steady-state.

In this section we first define the terms for the Kalman filter [1-3], with reference to Appendix A1. For the mathematics involving the filter, we use the phase integral I in the Gain function but ignore it in computing the uncertainties. This is because I is not a measurable quantity, and so in the Kalman filter it will just be a term whose uncertainty increases without bound. Following [5], we assume a system whose "true" and estimated states $x_{true,k}$ and x_k are stable and defined by the difference between their phase p_k and frequency f_k with respect to a reference clock. Vectors and matrices that include I will be designated as uppercase. After the measurement at time t_k, the estimated states are $x_k = \begin{pmatrix} p_k \\ f_k \end{pmatrix}$ and $X_k = \begin{pmatrix} I_k \\ p_k \\ f_k \end{pmatrix}$. and the covariances involving phase and frequency are P_k The steering goal is $p_k=f_k=I_k=0$. The measurements are at times t_k, spaced by intervals . The vector $x_{\_k}$ is the Kalman filter's estimate of the true state function $x_{true,k}$ at an instant just before the measurement at t_k, when its covariances are given in the matrix $P_{\_k}$. The measurement is assumed to be immediately used to create the state estimate x_k, and the frequency steering, based on that estimate, is immediately implemented. We utilize standard Kalman formulas such as

$$x_k = x_{-k} + K_g z_k = x_{-k} + K_g(\mathrm{H}\,\Delta x_k + \nu_k) \quad (2.1)$$

and $X_{-k} = (\Phi - BG)X_{k-1}$ (2.2)

Here H is the 2x1 measurement vector, which for our purposes is (1,0). K_g is the Kalman gain, ν_k and w_k are the measurement noise at time t_k and the process noise leading to time t_k respectively, $\Delta x_k = x_{true,k} - (x_-)_k$, and z_k is the innovation. The Kalman gain is computed using the process noise variance

Q and the measurement noise variance R, and the state evolution of x_k is given by $\varphi = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$. Each frequency steer $u_k = -GX_k$, which by t_{k+1} has altered the state by $-Bu_k$, where B=$\begin{pmatrix} \tau \\ \tau \\ 1 \end{pmatrix}$. The evolution of X_k is given by $\Phi = \begin{pmatrix} 1 & 1 & \tau \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix}$. The units are arbitrary in the sense that the time and integral coordinates can be turns of phase, nanoseconds, or radians. The frequency and control (steering) terms have units of the time coordinate divided by the sampling interval ?, which is taken to be 1 in the figures that follow.

Using the Kalman formalism, it is possible to compute the expected PFS for any set of gains that yield finite steady-state results in the long term. We first find the steady-state Kalman gain k_∞, and the steady state pre- and post-measurement covariances $p_{\_\infty}$ and p_∞, by requiring them all to be unchanged through a measurement cycle and applying Appendix 1's equations A1.8 and A1.19. This leads to a discrete-time algebraic Riccati equation for $p_{\_\infty}$, which can be solved using, for example the Matlab call $[p_{\_\infty}, L, F] = dare(\varphi^T, \mathrm{H}^T, Q, R)$. Note that this does not depend on the steering, because steering is a deterministic process that does not impact the uncertainties. After enough iterations to reach steady-state, the state evolution equation can be written with all the gain-dependent terms absorbed into A=$(\Phi - BG)$, so that

$$(X_{\_})_{k+1} = AX_k = A[(X_{\_})_k + Kz_k] = A(X_{\_})_k + AK\mathrm{H}\,\Delta x_k + AK\,v_k \tag{2.3}$$

Where K is a 3-component vector generated from the Kalman gain, with $(k_g)_{\_P} = (k_g)_{\_I}$:

$$K = \begin{pmatrix} (k_g)_{\_P} \\ (k_g)_{\_P} \\ (k_g)_{\_D} \end{pmatrix} \tag{2.4}$$

We write $\Sigma_{X_\_} = A\Sigma_X A^T$, where $\Sigma_{X_\_} = < (X_\_)_k (X_\_)_k^T >$ (2.5)

and $< (X_\_)_k (X)_{k-1}^T > = A\Sigma_x, where\ \Sigma_x = < (X)_k (X)_k^T >$ (2.6)

$$\Delta x_k = \varphi\left(x_{true,k-1} - (x_\_)_{k-1} - k_g H \Delta x_{k-1} - k_g v_{k-1}\right) + w_k + d$$

where $\Delta x_k = x_{true,k} - (x_\_)_k$ and the oscillator's systematic drift between measurements, $d = \begin{pmatrix} \frac{drift}{2} \\ drift \end{pmatrix}$, is distinguished from its stochastic component w_k.

$$\Delta x_k = \varphi\left(1 - k_g H\right)\Delta x_{k-1} - \varphi k_g v_{k-1} + w_k + d \tag{2.7}$$

Any drift component of oscillator's variation between measurements can be explicitly covered by considering w_k

$$\Sigma_{\Delta \mathrm{x}} = < \Delta x_k \Delta x_k^T > = \varphi(1 - k_g \mathrm{H})\Sigma_{\Delta \mathrm{x}}[\varphi(1 - k_g \mathrm{H})]^T + \varphi k_g \mathrm{R} k_g^T \varphi^T + \mathrm{Q} + \mathrm{D} + \varphi\left(1 - k_g \mathrm{H}\right) < \Delta x_{k-1} d > + (\varphi\left(1 - k_g \mathrm{H}\right) < \Delta x_{k-1} d >)^T \tag{2.8}$$

Where $D = < dd^T > = \begin{pmatrix} \frac{drift}{2} \\ drift \end{pmatrix}\begin{pmatrix} \frac{drift}{2} & drift \end{pmatrix} = \begin{pmatrix} drift^2/4 & drift^2/2 \\ drift^2/2 & drift^2 \end{pmatrix}$ (2.9)

But $< \Delta x_k d^T > = \varphi\left(1 - k_g H\right) < \Delta x_{k-1} d^T > + D$; $[1 - \varphi\left(1 - k_g H\right)] < \Delta x_{k-1} d^T > = D$ (2.10)

So $< \Delta x_{k-1} d^T > = [1 - \varphi\left(1 - k_g H\right)]^{-1} D$ (2.11)

Inserting 2.11 into 2.8,

$$\Sigma_{\Delta x} = \varphi(1 - k_g \mathrm{H})\Sigma_{\Delta x}[\varphi(1 - k_g \mathrm{H})]^T + \varphi k_g \mathrm{R} k_g^T \varphi^T + \mathrm{Q} + \mathrm{D} + \varphi(1 - k_g \mathrm{H})[1 - \varphi(1 - k_g H)]^{-1} D + (\varphi(1 - k_g \mathrm{H})[1 - \varphi(1 - k_g H)]^{-1} D)^T \tag{2.12}$$

Equation 2.12 is a Lyapunov equation, in Matlab: $\Sigma_{\Delta x} = dlyap(\varphi(1 - k_g H), \varphi k_g \mathrm{R} k_g^T \varphi^T + Q + \mathrm{D} + \varphi(1 - k_g \mathrm{H})[1 - \varphi(1 - k_g H)]^{-1} D + (\varphi(1 - k_g \mathrm{H})[1 - \varphi(1 - k_g H)]^{-1} D)^T)$. (2.13)

Note that the steering gains do not appear in the equations above, which is because the steering is deterministic. The following equation (2.14)computes that actual steady-state covariance matrix. Here the effect of the unmodeled drift vector shows itself only in the constant scalar $\mathrm{H}\,\Delta x_k$, which does not correlate with any quantity that changes sign, such as $(X\_)_k K^T$.

$$\Sigma_X = < X_k\, X_k^T > = < [(X\_)_k + K(\mathrm{H}\,\Delta x_k + \nu_k)][(X\_)_k + K(\mathrm{H}\,\Delta x_k + \nu_k)]^T \tag{2.14}$$

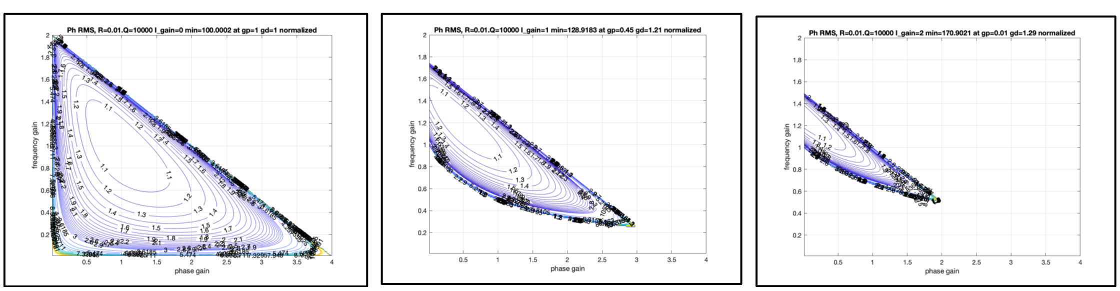
Since the noise parameter ν_k and the prediction error Δx_k are uncorrelated with each other and $(x\_)_k$ (though not x_k), and using equation 2.5, we have:

$$\Sigma_X = A \Sigma_X A^T + KH\Sigma_{\Delta x}(KH)^T + KRK^T \tag{2.15}$$

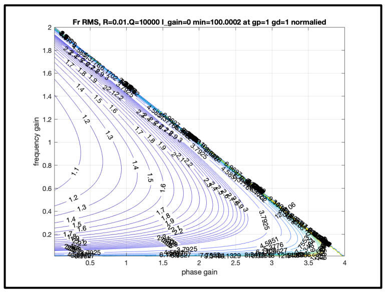
The variance of the control, u, is

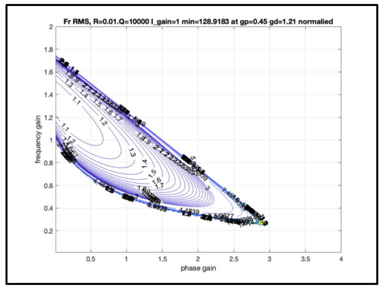
$$< u^2 > = < G\, X_k X_k^T G^T > = G \Sigma_x G^T \tag{2.16}$$

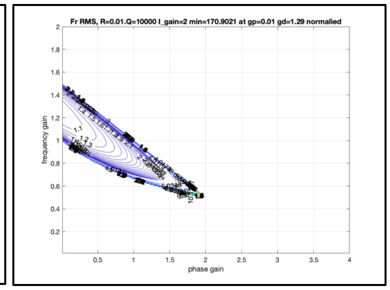
Using the equations developed above, the RMS of the PFS can be computed and plotted. The contour plots correspond to different values of the I_gain, and the contour spacings are 10% of the minimum value. Assuming 1-second measurements, they correspond to a measurement noise of .1 ns (R=.01), and an OCXO oscillator instability of roughly 10^{-12}, (Q=10,000, in units of (ns/day)^2. In the plot for g_I=0, the contours and values are of course equal to what would be computed for a PD controller. Except for this case of I_gain=0 (where plot quantization is a factor), the unfilled regions of low frequency gain indicate regimes of complete instability (unbounded behavior). The gains corresponding to the minima, for a fixed I_gain, do depend on its value. However, as shown below, the true minimum always occurs for I_gain=0, and their locations do not depend the noise modes. Figure 10 shows how the minimum depends on the value of I_gain. The value of the I_gain in handling unmodeled drift is not reflected in these figures.



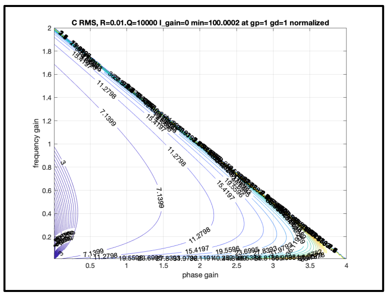
Figures 1-3 (above) plot the steady-state phase RMS as a function of P_gain and D_gain, for for zero drift and three different values of I_gain (0, 1, and 2). The lowest 10 contours are spaced 10% of the minimum.

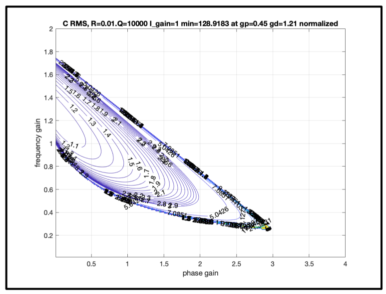


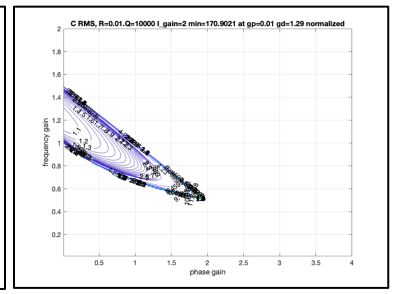




Figures 4-6 (above) plot the steady-state frequency RMS as a function of P_gain and D_gain, for for zero drift and three different values of I_gain (0, 1, and 2). The lowest 10 contours are spaced 10% of the minimum.







Figures 7-9 (above) plot the steady-state steer RMS as a function of P_gain and D_gain, for zero drift and three different values of I_gain (0, 1, and 2). The lowest 10 contours are spaced 10% of the minimum.

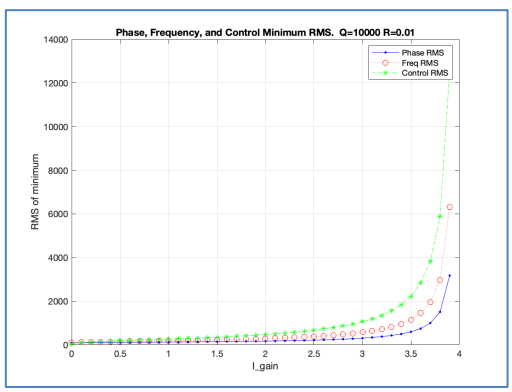


Figure 10. The RMS minimum value of all g_P and g_D combinations for fixed g_I.

The advantage of a PID controller is that it can remove frequency drift. As an example, Figure 11 assumes a large drift of 200 ns/day^2, and that the phase and derivatives gains are unity ($g_P=g_D=1$). In the absence of drift, the phase RMS of the controlled clock would be a minimum. However we see that a small amount of non-zero g_I makes a large improvement in the phase RMS, which itself vanishes when $g_I>1.5$.

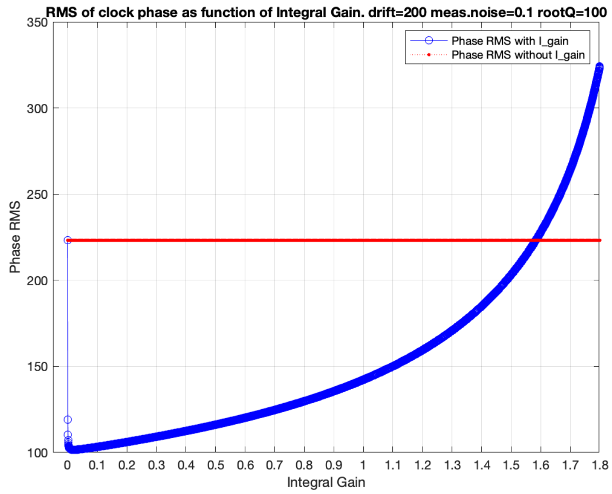


Figure 11. The blue curve shows how the phase RMS depends on the integral gain. The red curve is the phase RMS when the integral gain is zero. For critical gain examples, see Figures 14.

Perhaps one way to understand why the steady-state RMS phase of a PID controller will always be larger than a PD controller, in the absence of drift, is to consider the effect of a large phase jump (outlier) that is not identified so as to be ignored. This will increase the integral of the phase, so that when the phase is restored to its set point, the integral will be non-zero and cause the several future phase points to be negative, until finally both phase and integral become zero. Since noise can be considered a sum of many small "outliers", it should be expected that a non-zero integral gain will increase the steady-state RMS.

The stability regions and the role of the frequency gain are different in PID controls related to air conditioning and navigation, because those and other applications have what can be described as a friction term. Whereas the phase of an off-frequency oscillator will increase indefinitely, the temperature of a room will only cool to the temperature of the intake air and the motion of a ship is constantly slowed to that of the water.

III. Derivation of Time Constants and Critical Gains

Given a choice of gains, it is possible to compute how quickly and in what manner the system output will respond to a disturbance. In essence the pole placement technique chooses the gains on the basis of this response. The stochastic approach outlined in the previous section would give the variances associated with a stochastic set of disturbances. As in [1 and 2] we simplify the approach by solving for the time constants using only the phases for each time n. The basic equation then becomes:

$$p_{n+1} = p_n + (p_n - p_{n-1}) - \tau g_I \sum_{m=0}^{m=n} p_m - \tau g_P p_n - g_D(p_n - p_{n-1}) \tag{3.1}$$

$$p_n = p_{n-1} + (p_{n-1} - p_{n-2}) - \tau g_I \sum_{m=0}^{m=n-1} p_m - \tau g_P p_{n-1} - g_D(p_{n-1} - p_{n-2}) \tag{3.2}$$

Absorb the measurement interval τ into g_p *and* g_I for now. Show it explicitly after 2.14.

$$-g_I \sum_{m=0}^{m=n-1} p_m = p_n - p_{n-1} - (p_{n-1} - p_{n-2}) + g_P p_{n-1} + g_D(p_{n-1} - p_{n-2}) \tag{3.3}$$

$$p_{n+1} = p_n + (p_n - p_{n-1}) - g_P p_n - g_D(p_n - p_{n-1}) - g_I p_n + p_n - p_{n-1} - (p_{n-1} - p_{n-2}) + g_P p_{n-1} + g_D(p_{n-1} - p_{n-2}) \tag{3.4}$$

$$p_{n+1} = (3 - g_I - g_P - g_D)p_n + (-3 + g_P + 2g_D)p_{n-1} + (1 - g_D)p_{n-2} \tag{3.5}$$

$$0 = -p_n + (3 - g_I - g_P - g_D)p_{n-1} + (-3 + g_P + 2g_D)p_{n-2} + (1 - g_D)p_{n-3} \quad (3.6)$$

Now we desire each new p_n to be related to the previous point by r, which in the case of oscillatory behavior would be a complex number.

$$p_n = rp_{n-1}; r = e^{-\frac{1}{T}}e^{2\pi if};\ T = \frac{1}{1-|r|} = decay\ constant\ \text{; if T>>1}\ r = (1 - \tfrac{1}{T})e^{2\pi if} \quad (3.7)$$

$$Re(r) + iIm(r) = \cos(2\pi f) + isin(2\pi f);\ period = \frac{1}{f} = 2\pi/atan2(Im(r), Re(r)) \quad (3.8)$$

$$0 = -r^3 + (3 - g_I - g_P - g_D)r^2 + (-3 + g_P + 2g_D)r + (1 - g_D) \quad (3.9)$$

$$0 = r^3 + (-3 + g_I + g_P + g_D)r^2 + (3 - g_P - 2g_D)r + (-1 + g_D) \quad (3.10)$$

This cubic equation could have been derived by a z-transform of equation 3.1, substituting z for r. Either way, it describes how the controlled clock behaves for the applied gains. Complex roots will describe oscillatory behavior. If the magnitude of any root exceeds one, the clock will be unstable and its differences from the reference will be unbounded. Otherwise, the magnitudes provide time constants. Critical gains are those which generate only one time constant; for them the equation has just one root, which would be real as complex roots come in pairs. To have just one root, we require:

$$0 = (r - a)^3 = r^3 - 3ar^2 + 3a^2r - a^3\ \text{, where}\ a = e^{-\frac{\tau}{T}} \quad (3.11)$$

$$-a^3 = -1 + g_D\ ;\ \ g_D = 1 - a^3 \quad (3.12)$$

$$3a^2 = 3 - g_P - 2g_D\ \ ;\ \ g_P = 3 - 3a^2 - 2g_D = 1 - 3a^2 + 2a^3 \quad (3.13)$$

$$-3a = -3 + g_I + g_P + g_D\ ;\ g_I = 3 - 3a - g_P - g_D = 1 - 3a + 3a^2 - a^3 \quad (3.14)$$

These lead directly to the following conditions:

$$\tau g_P = 1 - 3e^{-2\tau/T} + 2e^{-3\tau/T} \quad (3.15)$$

$$\tau g_I = 1 - 3e^{-\tau/T} + 3e^{-2\tau/T} - e^{-3\tau/T} \quad (3.16)$$

$$g_D = 1 - e^{-3\tau/T} \quad (3.17)$$

Because equation 3.10 does not require a clean separation between r and a, other solutions are possible. For example $g_P = g_D = g_I = 1$ solves equation 3.10 trivially and is equivalent to a hard reset (T=0). If applied at time t_n and noise is ignored, the phase at t_{n+1} equals the sum of all previous phases, and the phase is zero thereafter. The PD critical gains also solve it and this is easily shown by inserting them into the equation as follows:

Let $g_I = 0\ ;\ g_P = (1 - r)^2 = 1 - 2r + r^2\ ; g_D = 1 - r^2\ ; g_P + g_D = 2 - 2r$ (3.18)

The right-hand side of equation 3.10 then becomes equal to 0 since

$$\begin{aligned} & r^3 + (-3 + g_I + g_P + g_D)r^2 + (3 - g_P - 2g_D)r + (-1 + g_D) \\ &= r^3 + (-3 + 2 - 2r)r^2 + (3 - 1 + 2r - r^2 - 2 + 2r^2)r + (-1 + 1 - r^2) \\ &= r^3 + (-1 - 2r)r^2 + (+2r + r^2)r - r^2 \\ &= r^3 - r^2 - 2r^3 + 2r^2 + r^3 - r^2 = 0 \end{aligned} \quad (3.19)$$

Figure 12a shows how the critical gains depend on time constant, with and without I_gain set to 0. Figure 12 c shows the PD critical gain with the projections PID gains projection on the g_I=0 plane. Figure 12c shows the associated RMS of the phase, frequency, and steers for R=1, Q=100,000. It is seen that the addition of an I_gain results in larger phase and frequency gain values where the time constant is large and the actual values for the gains correspondingly smaller. This results in lower steady-state RMS values for the phase and frequency, but larger ones for the steering (control).

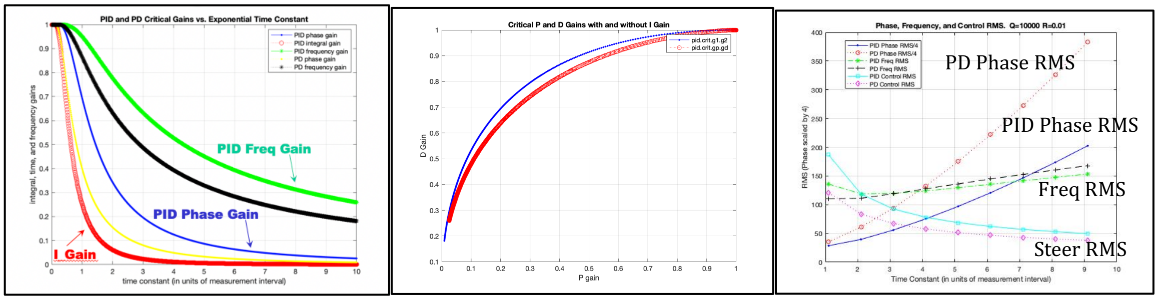


Figure 12a Critical PD and PID gains 12b) Critical gains in the g_I =0 plane 12c) Phase, Freq &Steer RMS

But the associated steady-state RMS's are of course not at their minimum value. In fact the minima of the PSF RMS's always occur when I_gain = 0, and they increase rapidly as a function of the I_gain. In other words, an LQG analysis for the PID case yields the same minimum as the PD case (where I_gain =0). Figure 13 shows how the minimum depends on the I_gain.

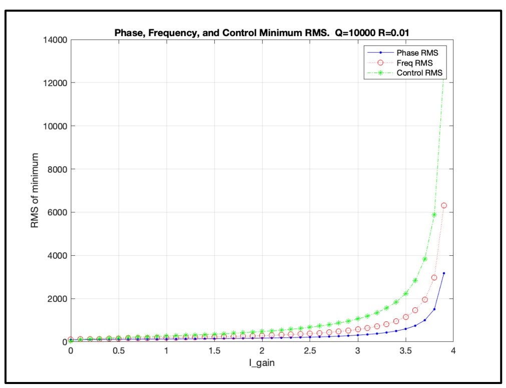
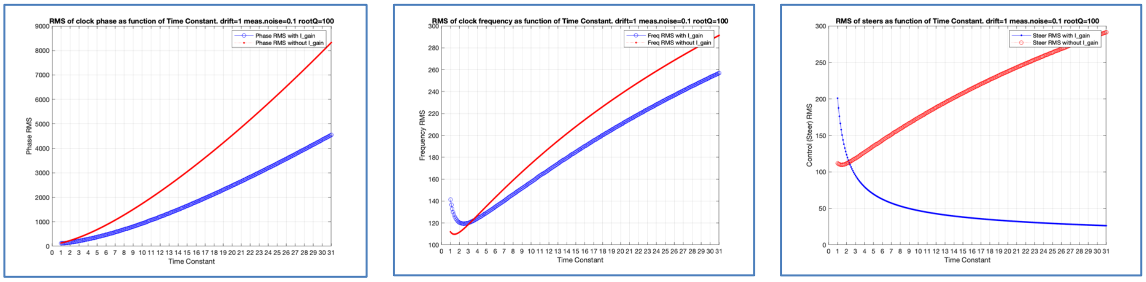


Figure 13. Steady-state RMS for critical gains, as a function of I_gain

Since the critical gains are not at the minima, the RMS of their PFC are always higher. In the case of non-zero drift, the situation is even more ambiguous because the states are not optimally estimated. Figures 14 show how the PFC RMS's behave for PD and PID controls, for a drift of 1 ns/day^2.



Figures 14. Steady-state phase, frequency, and control RMS as a function of time constant, for PD (red) and PID(blue) critical gainswith time constants of from 1 to 30 days, for a drift of 1 ns/ *day*^{2}, and measurement noise of 0.1 ns, and a frequency stability of 100 ns/day. The units are arbitrary.

IV. Transfer function

Following [11] the closed-loop transfer function H_{CL} can be computed using the Z-transform and the open-loop gain H_{OL} If the oscillator's phase is given by y_n

$$x_n = y_n - \sum_{m=0}^{\infty} m s_{n-m} \tag{4.1}$$

Where the steers are $s_n = (g_P + g_D)x_n - g_D x_{n-1} + g_I \sum_{k=0}^{\infty} x_{n-k}$ (4.2)

and they can be summed as follows:

$$y_n = \sum_{m=0}^{\infty} m(g_P + g_D)x_{n-m} - g_D \sum_{m=0}^{\infty} m x_{n-m-1} + g_I \sum_{m=0}^{\infty} m \sum_{k=0}^{\infty} x_{n-k-m} \tag{4.3}$$

The Z-transform of the sum of the steers up to each t_n is Y:

$$Y = (g_P + g_D)\sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{\infty} m x_{n-m} - g_D \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{\infty} m x_{n-m-1} + g_I \sum_{n=0}^{\infty} z^{-n} \sum_{m=0}^{\infty} m \sum_{k=0}^{\infty} x_{n-k-m} \tag{4.4}$$

$$Y = (g_P + g_D)\sum_{m=0}^{\infty} m z^{-m} \sum_{n=0}^{\infty} x_{n-m} z^{-(n-m)} - g_D \sum_{m=0}^{\infty} m z^{-m-1} \sum_{n=0}^{\infty} x_{n-m-1}\, z^{-(n-m-1)} + g_I \sum_{k=0}^{\infty} z^{-k} \sum_{m=0}^{\infty} m z^{-m} \sum_{n=0}^{\infty} x_{n-k-m} z^{-(n-k-m)} \tag{4.5}$$

$$Y = (g_P + g_D)X \sum_{m=0}^{\infty} m z^{-m} - g_D X \sum_{m=0}^{\infty} m z^{-m-1} + X g_I \sum_{k=0}^{\infty} z^{-k} \sum_{m=0}^{\infty} m z^{-m} \tag{4.6}$$

With $X = \sum_{n=0}^{\infty} z^{-n} X$ the open -loop gain is:

$$H_{OL} = Y/X = (g_P + g_D)\sum_{m=0}^{\infty} m z^{-m} - g_D z^{-1} \sum_{m=0}^{\infty} m z^{-m} + \frac{g_I z}{z-1} * z/(z-1)^2 \tag{4.7}$$

$$H_{OL} = (g_P + g_D) z/(z-1)^2 - g_D/(z-1)^2 + \frac{g_I z}{z-1} * z/(z-1)^2 \tag{4.8}$$

$$H_{OL} = [(g_P + g_D)z - g_D + \frac{g_I z^2}{z-1}]/(z-1)^2 \tag{4.9}$$

$$H_{OL} = [g_P z + (z-1) g_D + \frac{g_I z^2}{z-1}]/(z-1)^2 \tag{4.10}$$

$$H_{OL} = \frac{g_P z}{(z-1)^2} + \frac{g_D}{z-1} + \frac{g_I z^2}{(z-1)^3} = [g_P z(z-1) + g_D (z-1)^2 + g_I z^2]/(z-1)^3 \quad (4.11)$$

Note that if $g_I = 0$ we recover the open loop transfer function for $g_I = 0$ (the PD case) [8].

$$H_{OL,PD} = (g_P z/(z-1) + g_D)/(z-1) \quad (4.12)$$

The closed loop transfer function follows when feedback is taken into account

$$Y = H_{CL}X = H_{OL}(X - H_{CL}X) \;\; H_{CL} = H_{OL}(1 - H_{CL}) \;\; ; \;\; H_{CL} + H_{OL}H_{CL} = H_{OL} \quad (4.13)$$

$$H_{CL} = \frac{H_{OL}}{1+H_{OL}} \;\; ; \; H_{CL} + H_{OL}H_{CL} = H_{OL} \;\; ; \;\; H_{OL} = \frac{H_{CL}}{1-H_{CL}} \quad (4.14)$$

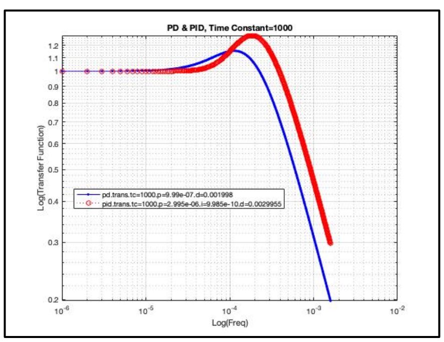
$$1 + H_{OL} = [(z-1)^3 + g_P z(z-1) + g_D(z-1)^2 + g_I z^2]/(z-1)^3 \quad (4.15)$$

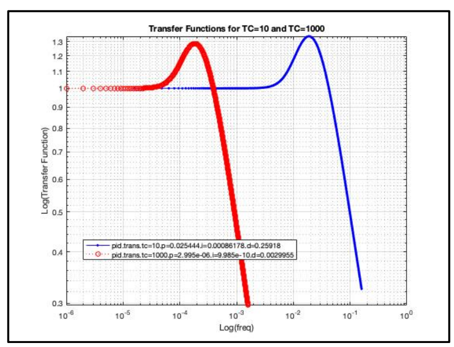
$$H_{CL} = [g_P z(z-1) + g_D(z-1)^2 + g_I z^2]/[(z-1)^3 + g_P z(z-1) + g_D(z-1)^2 + g_I z^2] \quad (4.16)$$

As in [11] the frequency dependence can be found after substituting $z = e^{2\pi i f}$ and the Allan variance can be computed if the power spectrum $S_{in}(f)$ of the oscillator is known and measurement noise is negligible:

$$\sigma_y^2(\tau) = \int_o^{f_h} df \, |H_{cl}(f)|^2 \, S_{in}(f) 2[\sin(\pi t f)]^4 \, /(\pi t f)^2 \quad (4.17)$$

Figure 15a shows the transfer functions for critical gains with a time constant of 1000 seconds, for both PID and PD steering, and Figure 15b compares the full PID transfer functions with time constant of 10 seconds and 1000 seconds.





Figures 15: Transfer functions for PID & PD critical gains and a time constant of 10000 (a), and PID critical gains with time constants of 10 and 1000 (b). Measurement noise is 0.1 and Q is 10,000.

V. Conclusions

Although the need for simulations always remains, particularly for non-Gaussian behavior, the analytic techniques for gross estimates of the behavior of PID-controlled clocks provide straightforward results. Critical gains can be calculated, and regions of instability are clearly demarked. In general, the use of PID steering provides protection against unmodeled behavior, with often-acceptable consequences such as larger steady-state RMS of the phase, frequency, and steers.

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VII. References

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Appendix 1. The Kalman and Steering formalism

This section is a review of the Kalman filter [1-3], in order to define the notation in the presence of proportional steering in frequency. Following [5], we assume a system whose "true" and estimated states $x_{true,k}$ and x_k are stable and defined by the difference between their phase and frequency with respect to a reference clock, $x_k = \binom{p_k}{f_k}$, and their covariances are P_k The steering goal is $p_k=f_k=I_k=0$, but steering is not invoked until after equation A.1.9. The discrete time series of state estimates, measured at times t_k, separated by intervals is denoted x_k. The vector $x_{\_,k}$ is the Kalman filter's estimate of the true state function $x_{true,k}$ at an instant just before the measurement at t_k, when its covariances are given in the matrix $P_{\_,k}$. The measurement is assumed to be immediately incorporated into the state estimate x_k, and the frequency steering is based upon that estimate, using the standard Kalman formula

$$x_k = (x_{\_})_k + k_g z_k = (x_{\_})_k + k_g(\mathrm{H}\,\Delta x_k + v_k) \quad \text{(A1.1)}$$

Where H is the 2x1 measurement vector, k_g is the Kalman gain, v_k is the measurement noise, and $\Delta x_k = x_{true,k} - (x_{\_})_k$.

The units are arbitrary in the sense that the "time coordinate" can be turns of phase, nanoseconds, or radians. The frequency and control (steering) terms have units of the time coordinate divided by the sampling interval, which is taken to be 1 in the figures that follow.

For measurement k,

$x_{true,k} =$ true value of x_k (not its estimate) (A1.2)

$\Delta x_k = x_{true,k} - (x_-)_k$ (a 2x1 column vector) (A1.3)

$z_k = H\Delta x_k + \nu_k$ (termed the innovation, a scalar) (A1.4)

$H = (1 \quad 0)$ = measurement row vector (A1.5)

ν_k is the measurement noise, a scalar with autocovariance R (A1.6)

Immediately after a measurement, the Kalman filter determines x_k

$x_k = (x_-)_k + k_g z_k$ (A1.7)

$k_g = \text{Kalman Gain} = (P_-)_k H^T [H(P_-)_k H^T + R]^{-1} = \frac{1}{[R+(P_-)_{1,1}]}\begin{pmatrix}(P_-)_{1,1}\\(P_-)_{2,1}\end{pmatrix}$ (A1.8)

P_k = covariance of state estimate uncertainties (a 2x2 matrix)

The state estimate and its covariance just before and just after incorporating the measurement at estimation time k are as follows:

$\varphi = \begin{pmatrix}1 & \tau\\0 & 1\end{pmatrix}$ = evolution matrix of x (A1.9)

$\Phi = \begin{pmatrix}1 & 1 & \tau\\0 & 1 & \tau\\0 & 0 & 1\end{pmatrix}$ = evolution matrix of X (A1.10)

$B = \begin{pmatrix}\tau\\\tau\\1\end{pmatrix} = \emptyset\begin{pmatrix}0\\1\\0\end{pmatrix}$ = steer implementation vector propagated forward (A1.11)

$b = \begin{pmatrix}\tau\\1\end{pmatrix} = \emptyset\begin{pmatrix}0\\1\end{pmatrix}$ (A1.12)

$G = (\,g_I \;\; g_P \quad g_D)$= gain row vector; g= $(g_P \quad g_D)$ (A1.13)

(g_P and g_I units are those of the phase divided by time, g_F is dimensionless)

$BG = \begin{pmatrix}\tau g_I & \tau g_P & \tau g_D\\\tau g_I & \tau g_P & \tau g_D\\g_I & g_P & g_D\end{pmatrix}$; $\qquad bg = \begin{pmatrix}\tau g_P & \tau g_D\\g_P & g_D\end{pmatrix}$ (A1.14)

$(\Phi - BG) = \begin{pmatrix}1-\tau g_I & 1-\tau g_P & \tau-\tau g_D\\-\tau g_I & 1-\tau g_P & \tau-\tau g_D\\-g_I & -g_P & 1-g_D\end{pmatrix} \quad \varphi - \text{bg} = \begin{pmatrix}1-\tau g_P & \tau-\tau g_D\\-g_P & 1-g_D\end{pmatrix}$ (A1.15)

$(X_-)_k = (\Phi - BG)X_{k-1} = \begin{pmatrix}1-\tau g_I & 1-\tau g_P & \tau-\tau g_D\\-\tau g_I & 1-\tau g_P & \tau-\tau g_D\\-g_I & -\tau g_P & \tau-g_D\end{pmatrix}\begin{pmatrix}p_k\\f_k\\I_k\end{pmatrix}$ (A1.16)

$x_k = (x_-)_k + k_g z_k = (x_-)_k + k_g(\text{H}\,\Delta x_k + \nu_k)$ (A1.17)

$x_k = \left(1 - k_g H\right)(x_-)_k + k_g H\, x_{true,k} + k_g \nu_k$ (A1.18)

$u_k = -GX_k = -G[(x_-)_k + Kz_k]$ =magnitude of steer (A1.19)