

# AN OPTIMAL FIR FILTERING ALGORITHM FOR A TIME ERROR MODEL OF A PRECISE CLOCK

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## Abstract

*This report\* is a comparative study of the GPS-based parameter estimates of the precise local clock linear time model. We estimate the time error  $x$  and fractional frequency offset  $y$  employing the finite impulse response (FIR) digital filters, namely: the simple moving average (MA), the second order low pass (LP), and the optimally unbiased (OU). The estimates are provided analytically, and we show that, in terms of a minimum root-mean-square error (RMSE), the simple MA is best to use for  $x$  estimates when  $0 \leq y < y_1$ , the second-order LP FIR is best if  $y_1 < y < y_2$ , and the OU is best for  $y_2 < y$ , where  $y_1$  and  $y_2$  are determined constants. Finally, we present an optimal FIR algorithm for the precise clock linear time error model and show that of all the estimators, including the Kalman, the presented algorithm yields minimum RMSE of the fractional frequency offset.*

## INTRODUCTION

GPS-based measurement of the local clock performance, the time error and the frequency offset, is now a key tool in timekeeping [1]. Herewith, it is shown that fast and accurate performance estimating is constrained by a large variance in the induced GPS noise. The standard deviation of the GPS noise using commercially available GPS time receivers is about 30 ns, can reach 10-20 ns [2], and may be improved by removal of systematic errors to 3-5 ns [2,3]. Even so, the time transfer random error has a much higher variance than that of a precise clock and the measured data cannot be used straightforwardly to steer the clock error. Certainly, this is a stochastic estimation problem, which is solved in practice with different filtering techniques requiring processing of the measurement for hours and even days. With this, traditionally, the time error model of a local clock is associated with the finite polynomial [4], which in discrete time is

$$x_n = x_0 + y_0 \Delta n + \frac{D}{2} \Delta^2 n^2 + \tilde{x}_n, \quad (1)$$

where  $x_0$  is an initial time error,  $y_0$  is an initial fractional frequency offset,  $D$  is a linear fractional frequency drift rate (basically representing oscillator aging),  $\tilde{x}(t)$  is a random time error component,  $\Delta = t_n - t_{n-1}$  is a sample time,  $t_n$  is discrete time, and  $n = 0, 1, \dots$

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Timekeeping employs several types of precise clocks, namely atomic (hydrogen, cesium, and rubidium) and crystal. The difference, respecting the model (1), is, first, in the magnitudes of  $y_0$  and  $D$ , which are strongly limited by physical nature and, second, in sensitivity to environment. In atomic clocks the aging rate  $D$  normally is small, the proper term in (1) is negligible for several days or even weeks, and the time error measurement behaves like a linear noisy process. Respecting this property, the clock performance estimators in modern timekeeping systems are usually designed based on either the two-state Kalman filter [5] or the FIR filter with a constant-weight (*simple*) MA or averaging [3].

Of all known filters, the simple MA is best in terms of minimum produced noise [6]. The estimate bias (inaccuracy), however, is great here. Aiming to improve accuracy, we have recently designed a new MA filter [7] with optimally compensated bias. Because the simple MA is optimal in terms of minimum noise [6] and the new filter [7] is optimal in terms of minimum bias, all the other possible FIR filters are intermediate. Among them, the second-order LP filter is the proximate simple rival.

In this report we focus attention on the aforementioned FIR filters, namely: simple MA, second-order LP, and OU, aiming to compare their performance (estimates of time error and fractional frequency offset) for the linear time error model,  $D = 0$ , perturbed by GPS noise. To determine the tradeoff, we first analytically provide the estimates and RMSE for each filter. Then we compare the filter performance for the typical magnitudes of the fractional frequency offset  $y_0$  and GPS noise related to the precise clock. Finally, we present the optimal FIR estimation algorithm for the linear time error model.

## TIME ERROR ESTIMATING WITH FIR FILTERS

In the time error measuring system, the GPS receiver induces into the signal (1) an extra noise  $v_n$ , which is Gaussian [8] with mean zero and of known constant variance  $\sigma_v^2$ . The GPS-based measurement is combined as an additive sum

$$z_n = x_n + v_n. \quad (2)$$

So long as the noise variance  $\sigma_v^2$  is much greater than that of the model (1), the term  $\tilde{x}_n$  in (1) may be neglected. Then for the linear case,  $D = 0$ , the time error function (1) simplifies to

$$x_n = x_0 + y_0 \Delta n, \quad (3)$$

in which aging may be accounted for by updating  $x_0$  and  $y_0$  for the start point of the estimation interval.

In view of (2), the FIR estimate of a time error is

$$\hat{x}_n = \sum_{i=0}^{N-1} W_i z_{n-i} = \mathbf{z}_N^T \mathbf{W}_N = \mathbf{x}_N^T \mathbf{W}_N + \mathbf{v}_N^T \mathbf{W}_N, \quad (4)$$

where  $W_i$  is a weighting function (kernel or impulse response);  $\mathbf{z}_N(n) = [z_n z_{n-1} \dots z_{n-N+1}]^T$  is the measurement data vector, of dimensions  $N \times 1$ ; and  $\mathbf{W}_N = [W_0(N), W_1(N), \dots, W_{N-1}(N)]^T$  is the filter weight matrix, of dimensions  $N \times 1$ . In accordance with (2), the measurement may be written as  $\mathbf{z}_N(n) = \mathbf{x}_N(n) + \mathbf{v}_N(n)$ , where  $\mathbf{x}_N(n) = [x_n x_{n-1} \dots x_{n-N+1}]^T$  is the time error data vector, of dimensions  $N \times 1$ , and

$\mathbf{v}_N(n) = [v_n \ v_{n-1} \dots v_{n-N+1}]^T$  is the GPS noise data vector, of dimensions  $N \times 1$ . It then may readily be shown that for the linear trend (3), the estimate (4) becomes

$$\hat{x}_n = x_0 + y_0 \Delta (n - \mathbf{k}_N^T \mathbf{W}_N) + w_{xn}, \quad (5)$$

where  $\mathbf{k}_N = [0 \ 1 \ \dots N-1]^T$  is a vector of integers of dimensions  $N \times 1$ . The noise  $w_{xn} = \mathbf{v}_N^T \mathbf{W}_N$  is formed by a weighted integration of the white Gaussian origin  $v_n$ , presenting samples of the weighted Wiener process. Let us note that, after integration, the low frequency components of the noise  $w_{xn}$  inherently dominate, promising some estimation trouble caused by temperature.

The first derivative of (5) produces an estimate of the fractional frequency offset; this is

$$\hat{y}_n = \frac{1}{\Delta} (\hat{x}_n - \hat{x}_{n-1}) = y_0 + v_{yn}, \quad (6)$$

in which increments of samples of the weighted Wiener process  $v_{yn} = \frac{1}{\Delta} [w_{xn} - w_{x(n-1)}]$   $= \frac{1}{\Delta} [\mathbf{v}_N^T(n) - \mathbf{v}_N^T(n-1)] \mathbf{W}_N$  represent the white Gaussian noise (Appendix A). To evaluate the noise values of  $v_n$ ,  $w_{xn}$  and  $v_{yn}$  just by simple MA, Figure 1 shows an example for  $\sigma_v = 30 \text{ ns}$ ,  $N = 30$ , and  $\Delta = 1 \text{ sec}$ . It then insures that the white Gaussian noise  $v_{yn}$  of the frequency estimate (6) is appreciably lower than the noise  $w_{xn}$  of the time error estimate (5).

Let us now evaluate the estimate error as follows:

$$\epsilon_{xn} = x_n - \hat{x}_n. \quad (7)$$

The RMSE is then calculated by  $E_{xn} = \sqrt{E[\epsilon_{xn}^2]} = \sqrt{\Delta \hat{x}_n^2 + \sigma_{\epsilon_{xn}}^2}$ , where  $\Delta \hat{x}_n = E[\epsilon_n]$  is a bias (inaccuracy), and  $\sigma_{\epsilon_{xn}}^2 = E[(\epsilon_{xn} - \Delta \hat{x}_n)^2]$  is a variance, representing precision of the filter. Use (3), (4), and (7), take into consideration that  $\mathbf{x}_N^T \mathbf{W}_N = \mathbf{W}_N^T \mathbf{x}_N$ , and write

$$\begin{aligned} E_{xn}^2 &= E[(y_0 \Delta \mathbf{k}_N^T \mathbf{W}_N - \mathbf{v}_N^T \mathbf{W}_N)^2] \\ &= y_0^2 \Delta^2 (\mathbf{k}_N^T \mathbf{W}_N)^2 + E[(\mathbf{v}_N^T \mathbf{W}_N)^2] - 2y_0 \Delta \mathbf{k}_N^T \mathbf{W}_N \mathbf{W}_N^T E[\mathbf{v}_N]. \end{aligned} \quad (8)$$

Because the noise  $v_n$  is mean zero,  $E[\mathbf{v}_N] = 0$ , and its variance is constant, then, first, the last term in (8) tends to zero, and, second, the transformation  $E[(\mathbf{v}_N^T \mathbf{W}_N)^2] = \sigma_v^2 \mathbf{W}_N^T \mathbf{W}_N$  brings (8) to

$$E_x^2 = y_0^2 \Delta^2 (\mathbf{k}_N^T \mathbf{W}_N)^2 + \sigma_v^2 \mathbf{W}_N^T \mathbf{W}_N. \quad (9)$$

Now note that the first term in (9) is a constant bias of the second order, which means that

$$\Delta \hat{x} = y_0 \Delta \mathbf{k}_N^T \mathbf{W}_N, \quad (10)$$

and the second term in (9) is in turn the noise constant variance of the filter:

$$\sigma_{ex}^2 = \sigma_v^2 \mathbf{W}_N^T \mathbf{W}_N. \quad (11)$$

Following the same scheme provides the error estimate for the fractional frequency offset (6); this is  $E_{yn}^2 = E[(y_0 - \hat{y}_n)^2]$ , and write

$$E_{yn}^2 = \frac{1}{\Delta^2} E[(\mathbf{w}_N^T \mathbf{W}_N)^2]. \quad (12)$$

It follows from (12) that first the estimate of the fractional frequency offset is unbiased,  $\Delta \hat{y}_n = 0$ , and, second, the standard deviation of the estimated noise is reduced by  $\Delta$  times, being dependent on the filter weight. It in turn means that the RMSE (12) is in fact a variance of the estimate error,  $E_y^2 = \sigma_{vy}^2$ . Now let us examine the selected FIR filters for the estimate errors.

## SIMPLE MA

A simple MA estimate is obtained via (5) with a constant weight  $\mathbf{W}_N = N^{-1} \mathbf{I}_N$ , where  $\mathbf{I}_N = [1 \ 1 \ \dots 1]^T$  is a unit matrix of dimensions  $N \times 1$ , and with a delay on the transient

$$\theta = \Delta(N-1). \quad (13)$$

Substituting the above-mentioned constant weight into (5) yields the time error estimate

$$\hat{x}_n = x_0 + y_0 \Delta \left( n - \frac{N-1}{2} \right) + w_{xn}, \quad (14)$$

where  $w_{xn}$  is a noise of the estimate [see comments for (5)] with a variance  $\sigma_{wx}^2 = \sigma_v^2 / N$ . It follows from (14) that the bias (10) is  $\Delta \hat{x} = 0.5 y_0 \theta$ , and the RMSE becomes

$$E_x^2 = 0.25 y_0^2 \theta^2 + \sigma_v^2 \frac{1}{N}. \quad (15)$$

The commonly known conclusion follows from (15): *the estimate variance is reduced by N times and the bias is 50% for the linear case*. The simple MA estimate of a fractional frequency offset is given by (6), where the estimate noise  $v_{yn}$  has a variance  $\sigma_{vy}^2$ , which in view of (12) is the second order of the RMSE of the estimate (6) (see Appendix A.A):

$$E_y^2 = \sigma_{vy}^2 = \frac{\sigma_v^2}{\Delta^2} \frac{2}{N^2}. \quad (16)$$

## SECOND-ORDER LP FILTER

The second-order LP filter is superior to the simple MA in terms of bias. Its truncated impulse response is calculated by

$$W_n = \begin{cases} k \frac{\Delta}{\tau} e^{-\frac{\Delta}{\tau}n}, & 0 \leq n \leq M-1, \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

where  $\tau$  is a time constant;  $M$  represents a reasonable weight length, which is normally taken to be  $3\tau$ ; and  $k = \left( \frac{\Delta}{\tau} \sum_{i=0}^{N-1} e^{-\frac{\Delta}{\tau}i} \right)^{-1}$  is an adjusting coefficient tending  $W_n$  to unit area. To compare errors, it is necessary to tune the filter for the transient (13) of a simple MA, which is  $\tau = \Delta(N-1)/3$ , and set  $M = N$ . Accordingly, the level  $\gamma$  at the truncated point of the impulse response is calculated by

$$\gamma = e^{-\frac{\Delta}{\tau}(N-1)} = e^{-3} \approx 0.05. \quad (18)$$

Substituting (17) into (5) yields the time error estimate. The bias, after routine manipulations using of the arithmetico-geometric progression and  $q = e^{-A\Delta} \neq 1$ , becomes

$$\Delta\hat{x} = y_0 \Delta \mathbf{k}_N^T \mathbf{W}_N = k \frac{\Delta^2 y_0 (N-1)}{\tau(1-e^{-\Delta/\tau})} \left\{ 1 - \frac{e^{-\Delta/\tau} [1 - e^{-\Delta(N-1)/\tau}]}{(N-1)(1-e^{-\Delta/\tau})} \right\}, \quad (19)$$

which taking into account (13) for an arbitrary  $\gamma$  reduces to

$$\Delta\hat{x} = y_0 \theta [1 - D(\theta, \gamma)], \quad (20)$$

where an auxiliary function  $D(\theta, \gamma)$  becomes constant for the great number of samples (observe that GPS-based measurement involves basically more than 100 points to achieve the accurate-enough estimate) and for  $\gamma = 0.05$  yields

$$D(\theta, \gamma) = \frac{\Delta \ln \gamma}{\theta (\gamma^{\Delta/\theta} - 1)^2} \left[ 1 - \gamma^{\frac{\Delta}{\theta}} \frac{\theta + \Delta(1-\gamma)}{\theta} \right]_{\substack{1 \ll N \\ \gamma=0.05}} \cong 0.684. \quad (21)$$

It then turns out that (21) simplifies (20) to a relation  $\Delta\hat{x} \cong 0.316 y_0 \theta$  with an error of 10%, 5%, and <1% for  $9 \leq N$ ,  $15 \leq N$ , and  $100 < N$ , respectively. That means that, for the sample time of  $\Delta = 100$  s and say 5 hours of averaging, one deals with  $N = 180$  samples, so, practically, the above-given approximate formula is adequately accurate.

A trivial statistical transformation of the random part in (5) for the weight (17) produces the estimate noise variance

$$\sigma_{wx}^2 = \frac{\Delta^2}{\tau^2} \sigma_v^2 \sum_{i=0}^{N-1} e^{-2\frac{\Delta}{\tau}(N-1-i)}. \quad (22)$$

To bring (22) to the final form, let us note that  $\sum_{i=0}^{N-1} e^{-2\frac{\Delta}{\tau}(N-1-i)} = \sum_{i=0}^{N-1} e^{-2\frac{\Delta_i}{\tau}}$  and, once the transient is over

with  $N < i$ , the unlimited sum  $\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1-a^x}$  may be considered instead. It brings (22) to the formula

$$\sigma_{wx}^2 = \sigma_v^2 \frac{\Delta^2}{\tau^2} \frac{1}{1-e^{-2\Delta/\tau}}, \text{ which, taking into account (13), produces}$$

$$\sigma_{wx}^2 = \sigma_v^2 \frac{\Delta}{\theta} G^2(\theta, \gamma), \quad (23)$$

in which  $G(\theta, \gamma)$ , like the case of (21), becomes constant for  $1 \ll N$ . This for  $\gamma = 0.05$  produces

$$G^2(\theta, \gamma) = \frac{\Delta \ln^2 \gamma}{\theta(1-\gamma^{2\Delta/\theta})} \underset{\gamma=0.05}{\underset{1 \ll N}{\cong}} 1.5. \quad \text{It then may be shown that the variance (23) is calculated by}$$

$$\sigma_{wx}^2 \cong 1.5 \sigma_v^2 \frac{\Delta}{\theta} \text{ with almost the same errors as in the case of (21).}$$

Based upon (5) and (20), we now may write the time error estimate provided with the second-order LP filter, which is  $\hat{x}_n = x_0 + y_0 \Delta \{n - k(N-1)[1 - D(\theta, \gamma)]\} + w_{xn}$ , that for  $1 \ll N$ ,  $\gamma = 0.05$ , and  $k \equiv 1$  simplifies to

$$\hat{x}_n \cong x_0 + y_0 \Delta \left( n - \frac{N-1}{3.16} \right) + w_{xn}, \quad (24)$$

where the variance of the estimate noise  $w_{xn}$  is given by (23).

Respectively, RMSE of the estimate (24) with account of (20) and (23) is obtained with  $E_x^2 = y_0^2 \theta^2 k^2 [1 - D(\theta, \gamma)]^2 + \sigma_v^2 \frac{\Delta}{\theta} G^2(\theta, \gamma)$  and simplifies to, as in the case of (24),

$$E_x^2 \cong 0.1 y_0^2 \theta^2 + \sigma_v^2 \frac{1.5}{N-1} \quad (25)$$

The estimate of the frequency offset has the common form (6), where the noise variance, in view of (12) and Appendix A.B, is equal to the second order of the RMSE of the estimate, that is:

$$E_y^2 = \sigma_{vy}^2 \cong \frac{\sigma_v^2}{\Delta^2} \frac{9}{(N-1)^2}. \quad (26)$$

## OPTIMALLY UNBIASED MA

The OU FIR filter was designed in [7] especially for the linear case (3) with the weighting function

$$W_i = \begin{cases} \frac{2(2N-1)-6i}{N(N+1)}, & 0 \leq i \leq N-1 \\ 0, & \text{otherwise} \end{cases}. \quad (27)$$

Substituting (27) into (5) shows that bias becomes identically zero, and then the time error estimate is determined by

$$\hat{x}_n = x_0 + y_0 \Delta n + w_{xn}, \quad (28)$$

where the noise variance is equal to the RMSE of the estimate, that is:

$$E_x^2 = \sigma_{wx}^2 = \sigma_v^2 \frac{2(2N-1)}{N(N+1)}. \quad (29)$$

It follows from (29) that, for a large  $N$ , the standard deviation of the estimate is 2 times greater than that of the simple MA. Finally, the estimate of the frequency offset (6) is accompanied here with the noise variance and RMSE obtained in Appendix A.C; these are equal to:

$$E_y^2 = \sigma_{vy}^2 \cong \frac{\sigma_v^2}{\Delta^2} \frac{4(5N^2 + 4N + 8)}{N^2(N+1)^2}. \quad (30)$$

## A COMPARATIVE ANALYSIS OF THE FILTERING ERRORS

Now examine all three FIR filters for the RMSE of the time error and fractional frequency offset estimates, relating results to the GPS-based measurement of the precise clock time error.

### ESTIMATING THE CLOCK TIME ERROR

Figure 2 qualitatively exhibits trends (15), (25), and (29) of the RMSE for the variable  $y_0$  and with the other parameters constant. The minimum RMSE of each filter separates values of  $y_0$  into three ranges. If  $0 \leq y_0 < y_1$ , the simple MA yields a minimum RMSE; the second-order LP filter is best in the range  $y_1 < y_0 < y_2$ ; and the optimally unbiased is most accurate if  $y_2 < y_0$  with its constant error. Solution of an equality of (15) and (24) produces the first cross-point coordinate

$$y_1 = \frac{\sigma_v}{\theta} \sqrt{\left| \frac{\frac{\Delta}{\theta} G^2(\theta, \gamma) - \frac{1}{N}}{0.25 - [1 - D(\theta, \gamma)]^2} \right|}_{\substack{1 \ll N \\ \gamma=0.05}} \cong 1.83 \sigma_v \sqrt{\frac{\Delta}{\theta^3}}, \quad (31)$$

Making the same calculations for (24) and (29), we obtain the second point

$$y_2 = \frac{\sigma_v}{\theta [1 - D(\theta, \gamma)]} \sqrt{\left| \frac{2(2N-1)}{N(N+1)} - \frac{\Delta}{\theta} G^2(\theta, \gamma) \right|}_{\substack{1 \ll N \\ \gamma=0.05}} \cong 5.00 \sigma_v \sqrt{\frac{\Delta}{\theta^3}}. \quad (32)$$

By definition, the minimum possible filtering error of the stationary process,  $y_0 = 0$ , is provided with simple MA, that is:

$$E_{x\min} = \sigma_v \frac{1}{\sqrt{N}} \cong \sigma_v \sqrt{\frac{\Delta}{\theta}}. \quad (33)$$

Substituting (31) into (15) or (25) either produces the first intermediate RMSE (Figure 1), namely

$$E_{x1} \cong 1.355\sigma_v \frac{1}{\sqrt{N-1}} = 1.355\sigma_v \sqrt{\frac{\Delta}{\theta}}. \quad (34)$$

Finally, calculating (25) or (29) either for (32) yields the second intermediate error:

$$E_{x2} \cong \sigma_v \frac{1}{\sqrt{N-1}} = 2\sigma_v \sqrt{\frac{\Delta}{\theta}}, \quad (35)$$

Consider, for example, the typical case of the GPS-based measurement of a clock time error; this is  $\sigma_v = 30$  ns,  $\Delta = 100$  sec,  $N = 865$ , and  $\theta = 24$  hours. The minimum RMSE (33) calculates for  $y_0 = 0$  to be  $E_{x\min} \cong 1$  ns, and then RMSE rises up to  $E_{x1} \cong 1.36$  ns (34) with  $y_0 = 2.16 \cdot 10^{-14}$ , proving that in the range of  $0 < y_0 < 2.16 \cdot 10^{-14}$  the simple MA is best. With  $y_0 = 5.91 \cdot 10^{-14}$  the RMSE reaches  $E_{x2} \cong 2$  ns (35), and then the second-order LP filter is best in the range of  $2.16 \cdot 10^{-14} < y_0 < 5.91 \cdot 10^{-14}$ . Assuming  $5.91 \cdot 10^{-14} < y_0$ , we conclude that the RMSE remains  $E_{x2} \cong 2$  ns using the optimally unbiased filter.

## ESTIMATING THE CLOCK FRACTIONAL FREQUENCY OFFSET

It follows from (16), (26), and (30) that the RMSE of the frequency offset estimate is  $y_0$  invariant, and its variance is appreciably reduced by  $\Delta^2$  and  $N$ . Furthermore, as the simple MA produces minimum noise among all the known filters [6], including the Kalman, it also provides the best estimate of the frequency offset in the linear case. For example, the above-considered case (Section IV.A) calculates the standard deviation  $E_y = 4.9 \cdot 10^{-13}$  with the simple MA,  $E_y = 1.04 \cdot 10^{-12}$  with the second-order LP FIR, and  $E_y = 1.55 \cdot 10^{-12}$  with the optimally unbiased FIR. Yet, in the reported GPS-based timekeeping system [3], the measurement noise is achieved with  $\sigma_v$  of about 5 ns, allowing the clock frequency steering of  $10^{-13}$  or better for an averaging time of  $\theta = 1$  day. The consistent estimate is calculated via (16) as  $E_y = 8.2 \cdot 10^{-14}$ .

## OPTIMAL FIR ALGORITHM

It now follows from the above-given analysis that the designed FIR filtering algorithm is optimal in the sense of minimum RMSE for the precise clock online performance estimate; this is

$$\hat{x}_n = \mathbf{z}_N^T \mathbf{W}_N, \quad (36)$$

$$\hat{y}_n = \frac{1}{\Delta N} [\mathbf{z}_N^T(n) - \mathbf{z}_N^T(n-1)] \mathbf{I}_N, \quad (37)$$

in which  $\mathbf{W}_N$  is given by  $\mathbf{W}_N = N^{-1} \mathbf{I}_N$ , (17), and (27) for  $y_0 \leq y_1$ ,  $y_0 \leq y_2$ , and  $y_2 \leq y_0$ , respectively. Herewith, the weight (17) may be considered for the other LP filter. The special feature of the algorithm is that it inherently produces the minimum possible RMSE of the linear model frequency offset estimate (37) among all the filters, including the Kalman.

## DISCUSSION

GPS-based measurement of the time error of a precise local clock looks like a linear noisy function, as the clock frequency drift rate is negligible for a few days or even decades. Respecting this property, we have set  $D = 0$  in (1) and examined the errors of the three FIR filters, which seem to be most accurate: the simple MA is optimal in a sense of minimum produced noise [6], the optimally unbiased FIR filter is optimal in a sense of a minimum bias [7], and the second-order LP FIR filter is the proximate simple rival with its intermediate performance. Depending on  $y_0$ , the following FIR filter produces a minimum RMSE of the time error estimate (Fig. 2), namely: the simple MA is best in the range  $0 < y_0 < y_1$ ; the second-order LP is best in the range  $y_1 < y_0 < y_2$ ; and the optimally unbiased yields a constant best estimate if  $y_2 < y_0$ . The estimation filtering algorithm (36) and (37) is optimal for the linear time error noisy function in a sense of minimum RMSE.

## APPENDIX A: ESTIMATE VARIANCE OF THE FRACTIONAL FREQUENCY OFFSET

### SIMPLE MA ESTIMATING

Consider the noisy part in (6) for the constant weight  $1/N$ ; this is

$$\begin{aligned} v_{yn} &= \frac{1}{\Delta} [\mathbf{v}_N^T(n) - \mathbf{v}_N^T(n-1)] \mathbf{W}_N = \frac{1}{\Delta N} \left[ \sum_{i=0}^{N-1} (v_{n-i} - v_{n-i-1}) \right] \\ &= \frac{1}{\Delta N} (v_n - v_{n-N}), \end{aligned} \quad (A1)$$

in which increments  $v_n - v_{n-N}$  of the white Gaussian noises form the white Gaussian noise (A1) of the variance

$$\sigma_{vy}^2 = \frac{1}{\Delta^2 N^2} E[(v_n - v_{n-N})^2] = \frac{1}{\Delta^2 N^2} [E(v_n^2) + E(v_{n-N}^2) - 2E(v_n v_{n-N})].$$

Observe that samples of the shifted stationary white Gaussian noise are uncorrelated,  $E(v_n v_{n-N}) = 0$ , and  $\sigma_v^2 = E(v_n^2) = E(v_{n-N}^2)$ . Then pass to the formula (16).

## SECOND-ORDER LP FIR ESTIMATING

Consider the noisy part in (6) for the weight (17) and  $\tau = \Delta(N-1)/3$ , this is

$$\begin{aligned} v_{yn} &= \frac{3k}{\Delta(N-1)} \sum_{i=0}^{N-1} (v_{n-i} - v_{n-i-1}) e^{-\frac{3}{N-1}i} \\ &= \frac{3k}{\Delta(N-1)} \left[ v_n - v_{n-1} \left( 1 - e^{-\frac{3}{N-1}} \right) - v_{n-2} e^{-\frac{3}{N-1}} \left( 1 - e^{-\frac{3}{N-1}} \right) - \dots + v_{n-N} e^{-3} \right]. \end{aligned}$$

Observe that for  $N \gg 1$  the decomposition holds true of  $e^{-\frac{3}{N-1}} \approx 1 - 3/(N-1)$ . Then neglect all the noise samples reduced by  $N-1$  in the brackets and write

$$v_{yn} \approx \frac{3k}{\Delta(N-1)} (v_n + 0.05v_{n-N}) \approx \frac{3k}{\Delta(N-1)} v_n. \quad (\text{A2})$$

The noise (A2) is white Gaussian, and (A2) readily produces the variance (26) for  $k = 1$ .

## OPTIMALLY UNBIASED FIR ESTIMATING

Consider the noisy part in (6) for the OU weight (27), this is

$$\begin{aligned} v_{yn} &= \frac{2}{\Delta N(N+1)} \left\{ \sum_{i=0}^{N-1} [(2N-1-3i)v_{n-i} - (2N-1-3i)v_{n-i-1}] \right\} \\ &= \frac{2}{\Delta N(N+1)} \sum_{i=0}^{N-1} [(2N-1)(v_{n-i} - v_{n-i-1}) - 3i(v_{n-i} - v_{n-i-1})] \\ &= \frac{2}{\Delta N(N+1)} \left\{ (2N-1)(v_n - v_{n-N}) + 3 \left[ v_n + (N-1)v_{n-N} - \sum_{i=0}^{N-1} v_{n-i} \right] \right\} \\ &= \frac{2}{\Delta N} \left[ 2v_n + \frac{N-2}{N+1} v_{n-N} - \frac{3N}{N+1} \left( \frac{1}{N} \sum_{i=0}^{N-1} v_{n-i} \right) \right]. \end{aligned} \quad (\text{A3})$$

Note that the expression in parenthesis of (A3) is zero-mean of the white Gaussian noise, then

$$v_{yn} = \frac{2}{\Delta N} \left( 2v_n + \frac{N-2}{N+1} v_{n-N} \right). \quad (\text{A4})$$

In a view of (A1) the noise (A4) is white Gaussian. The samples  $v_n$  and  $v_{n-N}$  are uncorrelated, so the variance of the noise (A4) is readily provided in a form of (30).

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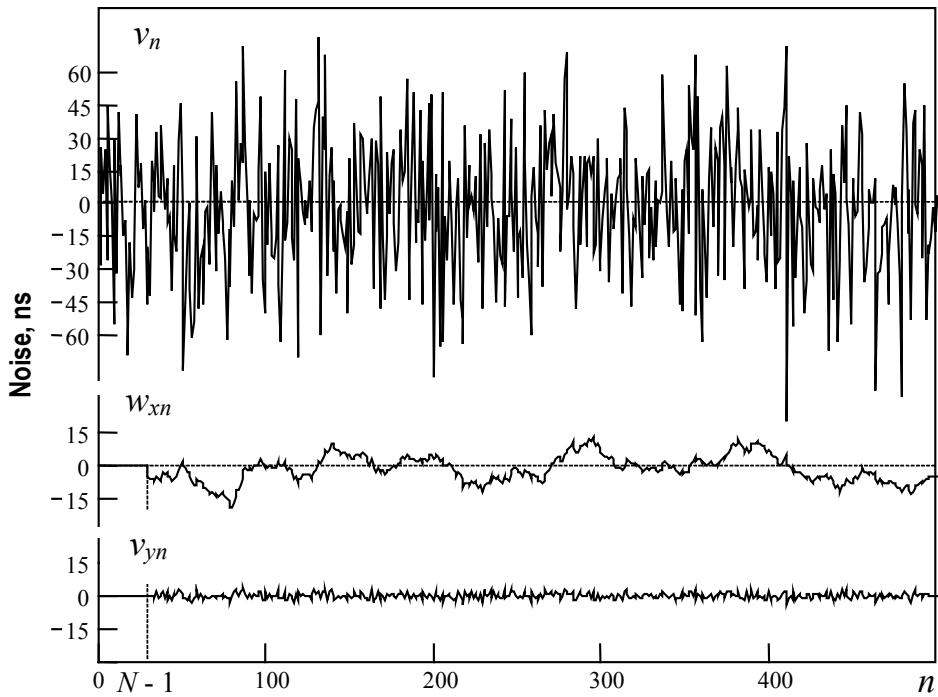


Figure 1. Noise example for the particular case of  $\sigma_v = 30$  ns,  $N = 30$ , and  $\Delta = 1$  sec.

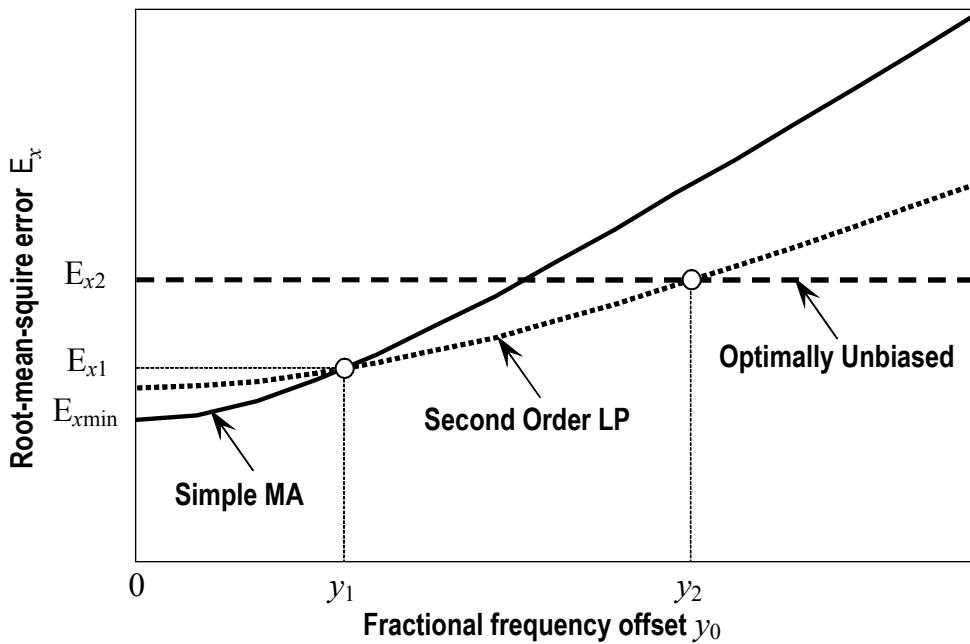


Figure 2. RMSE of the time error estimates provided with the FIR filters for different  $y_0$ .