THE FUNDAMENTAL STRUCTURE FUNCTION OF OSCILLATOR NOISE MODELS*

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ABSTRACT

Continuous-time models of oscillator phase noise $x(t)$ usually have stationary nth differences, for some n. The covariance structure of such a model can be characterized in the time domain by the structure function:

$$D_n(t;\tau_1,\tau_2) = E \, \Delta_{\tau_1}^n x(s+t) \, \Delta_{\tau_2}^n x(s) \tag{0}$$

Although formulas for the special case $D_2(0;\tau,\tau)$ (the Allan variance times $2\tau^2$) exist for power-law spectral models, certain estimation problems require a more complete knowledge of (0).

We exhibit a much simpler function of one time variable, $D(t)$, from which (0) can easily be obtained from the spectral density by uncomplicated integrations. Believing that $D(t)$ is the simplest function of time that holds the same information as (0), we call $D(t)$ the fundamental structure function.

We compute $D(t)$ for several power-law spectral models. Two examples are $D(t) = K|t|^3$ for random walk FM, $D(t) = Kt^2 \ln|t|$ for flicker FM. Then, to demonstrate its use, we exhibit a BASIC program that computes means and variances of two Allan variance estimators, one of which incorporates a method of frequency drift estimation and removal. Except for a one-line function definition of $D(t)$, the program is independent of the phase noise spectrum. The outputs were used for assigning confidence intervals to the results of recent hydrogen maser performance tests at JPL.

THE STRUCTURE FUNCTION

The purpose of this paper is to demonstrate an easy way of computing a class of time-domain oscillator stability measures known collectively as the structure function. Let the phase of an oscillator with nominal frequency ν_0 be modeled by

$$2\pi\nu_0(t + x(t)) \, ,$$

where $x(t)$ is a random process representing the "phase time" of the

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oscillator. The most restricted version of the structure function, discussed by Lindsey and Chie [1], is

$$D_n(\tau) = E[\Delta_\tau^n x(s)]^2 \; , \tag{1}$$

where Δ_τ is the backwards difference operator defined by $\Delta_\tau f(t) = f(t) - f(t-\tau)$, and E denotes the mathematical expectation. It is to be understood that the nth difference process $\Delta_\tau^n x(t)$ is wide-sense stationary, so that (1) does not depend on s. For n = 2 we obtain the Allan variance as

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} D_2(\tau) \; .$$

A more general version of the structure function is

$$D_n(t; \tau_1,\tau_2) = E \, \Delta_{\tau_1}^n x(s+t) \, \Delta_{\tau_2}^n x(s) \; , \tag{2}$$

which was used by Yaglom [2] as a basis for a theory of processes with stationary nth differences. For the application to be given at the end of this paper, we shall need the yet more general version

$$D(t; \underline{a},\underline{b}) = E \, \Delta_{\underline{a}} x(s+t) \, \Delta_{\underline{b}} x(s) \; , \tag{3}$$

where

$$\underline{a} = (a_1,\ldots,a_n), \; \Delta_{\underline{a}} = \Delta_{a_1} \ldots \Delta_{a_n} \; ,$$

and likewise for $\underline{b}$, $\Delta_{\underline{b}}$.

By using the transfer function $1 - e^{-i\omega\tau}$ of the operator Δ_τ, one can express all of these quantities as integrals involving the two-sided spectral density $S_x(\omega)$ of the process $x(t)$, which is assumed to have stationary nth differences for some n. For example, if the long-term frequency drift rate is zero, then

$$D_2(\tau) = \int_{-\infty}^{\infty} \left|1 - e^{-i\omega\tau}\right|^4 S_x(\omega) \, \frac{d\omega}{2\pi}$$

$$= 16 \int_{-\infty}^{\infty} \sin^4(\tfrac{1}{2}\omega\tau) \, S_x(\omega) \, \frac{d\omega}{2\pi} \; , \tag{4}$$

$$D(t; a,b,c,d) = E \, \Delta_a\Delta_b \, x(s+t) \, \Delta_c\Delta_d \, x(s)$$

$$= \int_{-\infty}^{\infty} (1 - e^{-i\omega a})(1 - e^{-i\omega b})(1 - e^{i\omega c})(1 - e^{i\omega d}) \, e^{i\omega t} \, S_x(\omega) \, \frac{d\omega}{2\pi} \; . \tag{5}$$

(This is (3) with n = 2.)

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Because of the importance of the Allan variance as a stability measure, the integral (4) has been evaluated in closed form for power-law spectral models

$$S_x(\omega) = K_\alpha |\omega|^{\alpha-2}, \quad \alpha = -2, -1, 0, 1, 2,$$

with a high-frequency cutoff when $\alpha \geq 1$ ("PM" noises) [3]. The expressions (5) and (3) are more complex, yet we shall show how to compute any of these in two easy steps: 1) By means of simple integrations involving only the spectral density, evaluate a single function $D(t)$ of one time variable; 2) apply a certain difference operator of order $2n$ to $D(t)$. This function $D(t)$, depending only on the spectral density, will be called the <u>fundamental</u> structure function because, merely by taking linear combinations of values of $D(t)$, one can generate the entire covariance structure of the differences of $x(t)$.

STATIONARY PHASE

If $x(t)$ is stationary, then $D(t)$ turns out to be just the autocovariance function:

$$D(t) = Cov(x(s+t), x(s)) = \int_{-\infty}^{\infty} e^{i\omega t} S_x(\omega) \frac{d\omega}{2\pi}. \tag{6}$$

Indeed, by expanding the differences in the middle expression of (5), one can show directly that

$$D(t; a,b,c,d) = \Delta_a \Delta_b \Delta_{-c} \Delta_{-d} D(t). \tag{7}$$

It is just this form of expression that will be extended to the nonstationary situation, where $S_x(\omega)$ is not integrable near zero frequency.

THE FUNDAMENTAL STRUCTURE FUNCTION IN GENERAL

Let us be given a process $x(t)$ with stationary nth differences. It is known [2] that x has a spectral density $S_x(\omega)$ with the properties

$$\int_1^{\infty} S_x(\omega)\, d\omega < \infty,$$

$$\int_0^1 \omega^k S_x(\omega)\, d\omega < \infty, \tag{8}$$

for some integer k, $0 \leq k \leq 2n$. (We ignore the more general case of a spectral distribution function.) Here is one way to compute the fundamental structure function $D(t)$: Define the function

$$B(z) = \int_0^{\infty} e^{i\omega z} (i\omega)^k S_x(\omega) \frac{d\omega}{2\pi} \tag{9}$$

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on the upper half-plane Im z > 0. Let C(z) be any kth integral of B(z), i.e., any function such that $C^{(k)}(z) = B(z)$ on Im z > 0. It will always be found that C(z) can be extended to the real line by continuity. Then let

$$D(t) = 2 \text{ Re } C(t), \qquad t \text{ real} \tag{10}$$

If k > 0, then D(t) is not unique; any polynomial of degree < k may be added to it. If x is stationary then k = 0, and D(t) reduces to the covariance function (6).

RANDOM WALK FM

To see how easy it is to carry out the above procedure for power-law noise types, consider the spectral density $S_x(\omega) = \omega^{-4}$, and take k = 4. Then

$$B(z) = \int_0^\infty e^{i\omega z} \frac{d\omega}{2\pi} = - \frac{1}{2\pi i z} \qquad (\text{Im } z > 0) \ ,$$

a fourth integral of which is

$$C(z) = - \frac{1}{2\pi i} \frac{z^3 \ln z}{6} \qquad (\text{Im } z \geq 0) \ .$$

(One can throw out a polynomial of degree less than 4.) Taking ln z = $\ln|z| + i \text{ Arg } z$ for Im z ≥ 0 gives

$$D(t) = 2 \text{ Re } C(t) = 0 \qquad\qquad (t > 0)$$

$$= -t^3/6 \qquad\qquad (t \leq 0).$$

Adding $t^3/12$ to this gives the alternate form

$$D(t) = |t|^3/12 \ .$$

OTHER POWER-LAW NOISE TYPES

We now give D(t) for the power-law spectrum

$$S_x(\omega) = K_\alpha |\omega|^{\alpha-2} \ , \quad K_\alpha = \frac{h_\alpha}{2(2\pi)^\alpha} \ .$$

The forms for the constant K_α and the power of ω link the results to an accepted notation for the noise spectrum, namely

$$S_y(f) = h_\alpha f^\alpha$$

[3], where y = dx/dt, f = $\omega/(2\pi)$, and $S_y(f)$ is the one-sided spectral density of y. Fractional values of α are allowed. We can take k to be the integer part of 2 − α (refer to (8)).

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Case 1. $\alpha < 1$, not an odd integer.

$$D(t) = K_\alpha \frac{-|t|^{1-\alpha}}{2\Gamma(2-\alpha)\,\cos(\pi\alpha/2)} . \tag{11}$$

This case includes white FM ($\alpha = 0$) and random walk FM ($\alpha = -2$).

Case 2. $\alpha = -1, -3, -5, \ldots$

$$D(t) = \frac{K_\alpha}{\pi}\,(-1)^{(3-\alpha)/2}\,\frac{t^{1-\alpha}\,\ln|t|}{(1-\alpha)!} . \tag{12}$$

This case, which includes flicker FM ($\alpha = -1$), is actually the easiest of all.

Case 3. $\alpha = 1$ (Flicker PM).

Here we use the spectrum

$$S_x(\omega) = K_1\,|\omega|^{-1}\,\exp\,(-|\omega/\omega_h|) .$$

The exponential high-frequency cutoff yields the elementary result

$$D(t) = -\frac{K_1}{2\pi}\,\ln(t^2 + 1/\omega_h^2) , \tag{13}$$

whereas the usual rectangular cutoff yields a cosine integral.

Case 4. $\alpha > 1$.

This phase noise process (provided with a high-frequency cutoff) is stationary; thus $D(t)$ is just the autocovariance function of x.

DERIVING THE STRUCTURE FUNCTION FROM $D(t)$

Having the fundamental structure function $D(t)$ in hand, we now show how to use it. Let $x(t)$ be a noise with spectral density $S_x(\omega)$, k the smallest integer satisfying (8), and n_0 the smallest integer such that $2n_0 \geq k$. The differences of x of order n_0 and higher are stationary, and to simplify matters let us assume that they are ergodic and have mean zero. For $n_0 = 2$, this means that an oscillator with phase time $x(t)$ has no long-term average frequency drift. Our main result is the following formula for the general structure function (3) of order $n \geq n_0$:

$$D(t; \underline{a},\underline{b}) = \Delta_{\underline{a}}\Delta_{-\underline{b}}D(t). \tag{14}$$

The proof, which has appeared elsewhere [4], starts from the spectral integral

$$D(t; \underline{a}, \underline{b}) = \int_{-\infty}^{\infty} \Delta_{\underline{a}} \Delta_{-\underline{b}} \; e^{i\omega t} \; S_x(\omega) \; \frac{d\omega}{2\pi} \; , \qquad (15)$$

in which the differences operate on $e^{i\omega t}$ as a function of t. (Refer to (5) for the case $n = 2$.) One cannot simply pull the difference operators outside the integral, for the Fourier transform integral of $S_x(\omega)$ diverges (unless $k = 0$) if the singularity at $\omega = 0$ is not neutralized. One way to do this is through the function $B(z)$ given in (9). Another embodiment of $D(t)$ is given directly by the formula

$$D(t) = \int_{-\infty}^{\infty} [\cos \omega t - \frac{1}{1 + \omega^{2n}} \sum_{j=0}^{n-1} \frac{(i\omega t)^{2j}}{(2j)!}] \; S_x(\omega) \; \frac{d\omega}{2\pi} \qquad (16)$$

where $n \geq n_0$.

MOMENTS OF STABILITY ESTIMATORS

The remainder of this paper presents a nontrivial application of the preceding theory. Let the phase time $x(t)$ have a long-term frequency drift component:

$$x(t) = x_0(t) + \frac{1}{2}ct^2 \; ,$$

where c is the constant rate of frequency drift, and $x_0(t)$ is a Gaussian process whose 2nd differences are stationary, ergodic, and have mean zero. Observing $x(t)$ for $0 \leq t \leq T$, we wish to estimate c and the quantities

$$\sigma_y^2(\tau) = \frac{1}{2\tau^2} E [\Delta_\tau^2 x(t)]^2 \qquad \text{(Gross Allan variance)} \; ,$$

$$\sigma_{y0}^2(\tau) = \frac{1}{2\tau^2} E [\Delta_\tau^2 x_0(t)]^2 \qquad \text{(Net Allan variance)} \; ,$$

which are the theoretical Allan variances before and after removal of drift.

To define and manipulate the estimators, it is necessary to set up some notation as follows: Write

$$C(a,b,t) = \frac{1}{ab} \; \Delta_a \Delta_b x(t) \; .$$

(This has nothing to do with (10).) Then $EC = c$ for any a,b,t. A particular one of these is used for estimating c, namely

$$\hat{c} = C(\tau_c, T-\tau_c, T) \qquad (17)$$

where $\tau_c = T/6.29$, a value chosen to minimize the variance of $\hat{c}$ in the presence of flicker FM noise [5]. The interpretation of (17) is that the estimated drift rate equals the average frequency near the end of the record, minus the average frequency near the beginning, divided by the intervening time $T - \tau_c$.

Let us introduce the additional quantities

$$c_j = C(\tau,\tau,j\tau), \quad c_\tau = C(\tau,T-\tau,T) , \tag{18}$$

where j is an integer. To estimate $(2/\tau^2) \, \sigma_y^2(\tau)$, we use the unbiased statistic

$$V = \frac{1}{m-1} \sum_{j=2}^{m} c_j^2 , \tag{19}$$

where it is now assumed that T/τ is an integer m. This is just the usual Allan variance estimator involving the sum of squares of adjacent second differences. (The unusual scale factor $2/\tau^2$ is convenient for these calculations.) To estimate $(2/\tau^2) \, \sigma_{y0}^2(\tau)$, we use

$$V_0 = \frac{1}{m-1} \sum_{j=2}^{m} (c_j - \hat{c})^2 . \tag{20}$$

Since

$$\frac{1}{m-1} \sum_{j=2}^{m} c_j = c_\tau ,$$

we have

$$V_0 = V - 2\hat{c}c_\tau + \hat{c}^2 . \tag{21}$$

Our goal is to compute the mean and variance of V_0, a biased estimator. First, we see that V_0 does not depend on the true drift rate c, and so for this purpose we can take $c = 0$. Then

$$EV = (2/\tau^2) \, \sigma_{y0}^2(\tau) , \tag{22}$$

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$$\text{Var } V = \frac{1}{(m-1)^2} \sum_j \sum_k \text{Cov}(c_j^2, c_k^2) \ , \tag{23}$$

$$EV_0 = EV - 2E\hat{c}c_\tau + E\hat{c}^2 \ , \tag{24}$$

$$\text{Var } V_0 = \text{Var } V + 4 \text{ Var}(\hat{c}c_\tau) + \text{Var } \hat{c}^2 - 4 \text{ Cov}(V, \hat{c}c_\tau) + 2 \text{ Cov}(V, \hat{c}^2)$$

$$- 4 \text{ Cov}(\hat{c}c_\tau, \hat{c}^2) \ . \tag{25}$$

Because $x(t)$ is Gaussian, the fourth moments in (25) are determined from the second moments of $\hat{c}$, c_j, c_τ, and the general formula

$$\text{Cov}(uv, xy) = Eux \ Evy + Euy \ Evx \ , \tag{26}$$

where u, v, x, y are zero-mean random variables with a joint Gaussian distribution. In turn, the required second moments are just special cases of the structure function formula (14), which here takes the form

$$E \ C(a, b, s+t) \ C(c, d, s) = \frac{1}{abcd} \ \Delta_a \Delta_b \Delta_{-c} \Delta_{-d} D(t) \ , \tag{27}$$

where $D(t)$ is the fundamental structure function of the phase-time noise $x(t)$. This expression, when written out, is a sum of 16 terms involving D. (See line 1040 of the BASIC program in Fig. 1.)

A BASIC PROGRAM

The tedious but elementary task of computing (23) – (25) by means of (26) and (27) is carried out by the BASIC program shown in Fig. 1. The main point to observe about this program is that the whole algorithm is independent of the phase noise spectrum, which enters only through the one-line function definition of $D(t)$ in line 1030 (and the print statement 130). The only requirement on $S_x(\omega)$ is that the integer k in (8) be at most 4. This guarantees that $x(t)$ has stationary second differences. Because of the way the outputs are scaled, a multiplicative constant in $D(t)$ can be neglected; thus we can let $D(t) = |t|^3$ for random walk FM. For flicker FM, line 1030 becomes

```
DEF DD(T) = T*T*LOG(ABS(T) + (T=0)) .
```

Line 1040 evaluates (27), and the subsequent definitions evaluate the required special cases. Note that the length of the test run is used as a unit of time. It must be admitted that computational efficiency has been sacrificed to gain ease of coding.

Figure 2 shows the output of the program for random walk FM. The column MEAN(NET) gives the expected value of $(\tau^2/2)V_0$ relative to the true net Allan

variance $c_{y0}^2(\tau)$. The variances of V and V_0, the estimators of gross and net Allan variances, are presented in terms of "degrees of freedom" (DF), defined for a positive random variable X by

$$DF = \frac{2(EX)^2}{Var\ X},$$

as if X had a χ^2 distribution. The DF(GROSS) result is valid only if the true drift has been subtracted from the phase data. When $T/\tau = 1$, the gross and net DF are both 1, since both V and V_0 are squares of mean-zero Gaussians.

CONFIDENCE INTERVALS FOR ALLAN VARIANCES

Following Howe et al. [6], we use the results shown in Fig. 2 to assign confidence intervals based on the χ^2 distribution, usually with a fractional DF. The author has not estimated the error caused by pretending that V and V_0 are proportional to χ^2 variables. Random walk FM was used as a noise model for $\tau > 10^4$ s. Fig. 3 shows the estimates of gross Allan variance (before drift removal) and net Allan variance (after drift removal) for a 72-day test run of a pair of hydrogen masers at JPL.

Different drift estimators are used in Figs. 3a and 3b. In Fig. 3a, we use a value of drift measured by retuning the masers over a period much longer than the 72-day test run. Regarding this value as close to the "true" value c, and using it in (21) in place of $\hat{c}$, we can use the gross DF numbers from Fig. 2 to assign confidence intervals to net Allan variance. In Fig. 3b, the estimated value $\hat{c}$ was used, so that one must use the net mean and DF numbers to compute the confidence intervals. The negative bias of V_0 pushes the confidence intervals up; in fact, for $T/\tau = 2$ (one sample of second phase difference), the bias is so great that the 90% confidence interval does not contain the estimate itself. This is pushing things too far, perhaps.

The basic problem is that one cannot remove estimated drift (via (21), for example) without also taking a bite out of the long-term random fluctuations. The outputs of the above computer program for $\omega^{\alpha-2}$ phase noise ($-2.5 \leq \alpha \leq 0$) support the conjecture that the bias of the net Allan variance estimator V_0 is always negative.

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100 !MOMENTS OF ALLAN VARIANCE ESTIMATORS
110 GOSUB 1000
120 MC=6.29
130 PRINT \'NOISE TYPE = RANDOM WALK FM'
140 PRINT 'MEAN & D.F. FOR GROSS & NET A.V.'\\
    'T/TAU','MEAN(NET)','D.F.(GROSS)','D.F.(NET)'
150 FOR M=2 TO 9,10 TO 18 BY 2,20 TO 50 BY 5
160   GOSUB 1080
170   DF=2/VARV,DF0=2*EV0*EV0/VARV0
180   PRINT M,EV0,DF,DF0
190   NEXT M
200 STOP

1000 !DEFINITIONS
1030 DEF DD(T)=ABS(T*T*T)    !FUNDAMENTAL STRUCTURE FUNCTION
                             !FOR RANDOM WALK FM
1040 DEF MM(A,B,C,D,T)=(DD(T)-DD(T-A)-DD(T-B)-DD(T+C)-DD(T+D)
    +DD(T-A-B)+DD(T-A+C)+DD(T-A+D)+DD(T-B+C)+DD(T-B+D)+DD(T+C+D)
    -DD(T-A-B+C)-DD(T-A-B+D)-DD(T-A+C+D)-DD(T-B+C+D)
    +DD(T-A-B+C+D))/(A*B*C*D)
1050 DEF FF(A,B)=MM(A,1-A,B,1-B,0)
1060 DEF GG(A,B,J)=MM(A,1-A,B,B,1-J*B)
1070 DEF HH(A,J)=MM(A,A,A,A,J*A)
1073 RETURN

1080 !COMPUTATION SUBROUTINE
     !INPUTS: M=T/TAU, MC=T/TAUC (=6.29)
     !OUTPUTS: EV0/A.V., VAR(V)/A.V.**2, VAR(V0)/A.V.**2
                EV0          VARV              VARV0
1090 U=1/M,UC=1/MC  !U STANDS FOR TAU
1100 FUU=FF(U,U),FCU=FF(UC,U),FCC=FF(UC,UC)
1110 HU0=HH(U,0),HU02=HU0*HU0  !HU0 IS SCALED ALLAN VARIANCE
1120 !
1130 EV0=HU0-2*FCU+FCC
1140 M1=M-1,X=0\ H=HH(U,J),X=X+(M1-J)*H*H FOR J=1 TO M-2\
     X=2*X+M1*HU02,VARV=X*2/(M1*M1)
1150 VARCU=FCC*FUU+FCU*FCU, VARCC=2*FCC*FCC
1160 X=0\ X=X+GG(UC,U,J)*GG(U,U,J) FOR J=2 TO M\ COVVCU=2*X/M1
1170 X=0\ G=GG(UC,U,J),X=X+G*G FOR J=2 TO M\ COVVCC=2*X/M1
1180 CVCUCC=2*FCC*FCU
1190 !
1200 VARV0=VARV+4*VARCU+VARCC-4*COVVCU+2*COVVCC-4*CVCUCC
1210 RETURN WHERE EV0=EV0/HU0,VARV=VARV/HU02,VARV0=VARV0/HU02
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Figure 1. A BASIC program for computing mean and variance
of an estimator of net Allan variance.

NOISE TYPE = RANDOM WALK FM
MEAN & D.F. FOR GROSS & NET A.V.

T/TAU	MEAN(NET)	D.F.(GROSS)	D.F.(NET)
2	.11213718	1	1.0000011
3	.4131003	1.882353	1.2011257
4	.56608639	2.7692308	1.9797428
5	.65837896	3.6571431	2.8213698
6	.72007427	4.5454549	3.6927653
7	.76417726	5.4339623	4.5779951
8	.7970189	6.3225806	5.4662905
9	.82222714	7.2112679	6.3534235
10	.84209356	8.1000005	7.2390502
12	.87125838	9.8775517	9.0083684
14	.89153524	11.655173	10.777728
16	.90639572	13.432836	12.546251
18	.91772997	15.210527	14.314574
20	.92664775	16.988236	16.084209
25	.9423454	21.432559	20.511747
30	.95254386	25.876923	24.943548
35	.9596919	30.321313	29.378236
40	.96497606	34.765708	33.814985
45	.96903914	39.210128	38.253179
50	.97225997	43.654528	42.692561

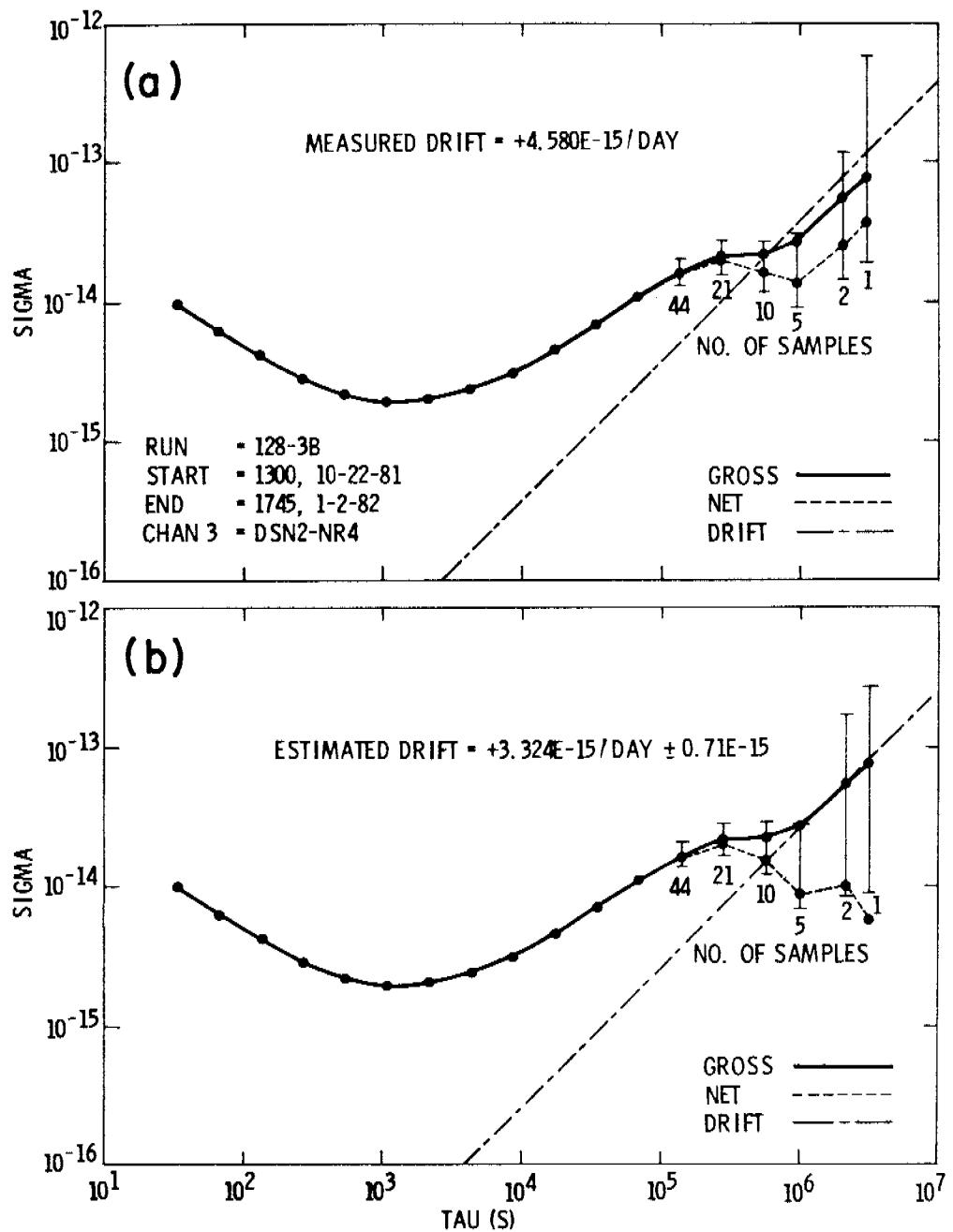Figure 2.   Output of the program of Fig. 1 for
random walk FM noise.

Figure 3. Allan variance of a pair of hydrogen masers (DSN2 and NR4) before and after removal of drift, with 90% confidence intervals for net Allan variance.

QUESTIONS AND ANSWERS

None for Paper #12