

# Statistical physics cheat sheet

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1. Using the second quantized formalism, show that for noninteracting fermions

$$\Omega_0 = -k_B T \sum_i \ln \left[ 1 + e^{-\beta(\varepsilon_i - \mu)} \right],$$

where  $\varepsilon_i$  denotes the  $i$ -th one-particle level

*Solution.* The second quantized form of  $\hat{K}_0$  is

$$\begin{aligned} \hat{K}_0 &= \int dx \hat{\Psi}^\dagger(x) \left[ -\frac{\hbar^2}{2M} \Delta + U(x) - \mu \right] \hat{\Psi}(x) \\ &= \sum_{l,m} \int dx \varphi_k^*(x) \underbrace{\left[ -\frac{\hbar^2}{2M} \Delta + U(x) - \mu \right] \varphi_m(x)}_{(\varepsilon_m - \mu) \varphi_m(x)} \hat{a}_l^\dagger \hat{a}_m \\ &= \sum_{l,m} (\varepsilon_m - \mu) \delta_{lm} \hat{a}_l^\dagger \hat{a}_m \\ &= \sum_l (\varepsilon_l - \mu) \hat{a}_l^\dagger \hat{a}_l = \sum_l (\varepsilon_l - \mu) \hat{n}_l \end{aligned}$$

Using this result,

$$\begin{aligned} \Omega_0 &= -k_B T \ln Z_G \\ &= -k_B T \ln \text{Tr} \left[ e^{-\beta \hat{K}_0} \right] \\ &= -k_B T \ln \text{Tr} \left[ e^{-\beta(\hat{H}_0 - \mu \hat{N})} \right] \\ &= -k_B T \ln \text{Tr} \left[ e^{-\beta \sum_l (\varepsilon_l - \mu) \hat{a}_l^\dagger \hat{a}_l} \right] \\ &= -k_B T \ln \sum_{\{n_i\}} \left\langle n_1, \dots, n_i, \dots \left| e^{-\beta \sum_l (\varepsilon_l - \mu) \hat{a}_l^\dagger \hat{a}_l} \right| n_1, \dots, n_i, \dots \right\rangle \\ &= -k_B T \ln \sum_{\{n_i\}} e^{-\beta \sum_l (\varepsilon_l - \mu) n_l} \langle n_1, \dots, n_i, \dots | n_1, \dots, n_i, \dots \rangle \\ &= -k_B T \ln \sum_{\{n_i\}} \prod_{l=1}^{\infty} e^{-\beta(\varepsilon_l - \mu) n_l} = -k_B T \ln \sum_{n_1=0}^1 \sum_{n_2=0}^1 \dots \prod_{l=1}^{\infty} e^{-\beta(\varepsilon_l - \mu) n_l} \\ &= -k_B T \ln \prod_{l=1}^{\infty} \sum_{n_l=0}^1 e^{-\beta(\varepsilon_l - \mu) n_l} = -k_B T \ln \prod_{l=1}^{\infty} (1 + e^{-\beta(\varepsilon_l - \mu)}) \\ &= -k_B T \sum_{l=1}^{\infty} \ln \left[ 1 + e^{-\beta(\varepsilon_l - \mu)} \right] \end{aligned}$$

*Bonus:* for bosons, we have

$$\begin{aligned} \Omega_0 &= -k_B T \ln \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} e^{-\beta(\varepsilon_l - \mu) n_l} \\ &= -k_B T \ln \prod_{l=1}^{\infty} \frac{1}{1 - e^{-\beta(\varepsilon_l - \mu)}} \\ &= +k_B T \sum_{l=1}^{\infty} \ln \left[ 1 - e^{-\beta(\varepsilon_l - \mu)} \right] \end{aligned}$$

2. Using the result for  $\Omega_0$ , calculate  $\Omega_0$  and  $N$  as a function of  $(T, V, \mu)$  for a fermionic homogeneous system (noninteracting fermions in a box with periodic boundary conditions). Express your result with Fermi-Dirac integrals. Give the first three terms of the high temperature expansion for  $\Omega_0$  and for  $N$ .

*Solution.* For a homogeneous fermionic system, we have

$$\sum_l \rightarrow \sum_s \frac{V}{(2\pi)^3} \int d^3\mathbf{k} = (2S+1) \frac{V}{2\pi^2} \int_0^\infty dk k^2$$

$$\varepsilon_l \rightarrow \varepsilon(\mathbf{k}, s) = \frac{\hbar^2 k^2}{2m}$$

Using these, we have

$$\Omega_0 = (2S+1) \frac{V}{2\pi^2} \int_0^\infty dk k^2 \ln \left[ 1 + e^{-\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right)} \right]$$

We apply partial integration:

$$\Omega_0 = (2S+1) \frac{V}{2\pi^2} \left[ \left[ \ln \left( 1 + e^{-\beta \dots} \right) \frac{k^3}{3} \right]_0^\infty - \int_0^\infty dk \frac{k^3}{3} \frac{\partial}{\partial k} [\ln \dots] \right]$$

$$\frac{\partial}{\partial k} \ln \left[ 1 + e^{-\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right)} \right] = \frac{e^{-\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right)}}{1 + e^{-\beta \left( \frac{\hbar^2 k^2}{2m} - \mu \right)}} \left( \frac{-\beta \hbar^2}{m} \right) k$$

3. For noninteracting fermions, one can define a characteristic temperature  $T_{\text{deg}}$  by which the chemical potential is zero:

$$\mu(T = T_{\text{deg}}) = 0.$$

By dimensional analysis

$$k_B T_{\text{deg}} = z \frac{\hbar^2}{2m} \left( \frac{N}{V} \right)^{2/3},$$

where  $z$  is a dimensionless number. Calculate  $z$  exactly and numerically.

4. Let us suppose that we have  $N$  noninteracting, spinless bosons confined in a 3 dimensional harmonic oscillator potential

$$V(\mathbf{r}) = \frac{m}{2} (\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2).$$

Calculate  $T_c$ , where Bose-Einstein condensation occurs.