# Statistical physics cheat sheet

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November 12, 2019

# 1 Fock-states

$$
\begin{aligned}
&|0\rangle = |0,0,0,\dots\rangle \\
&\langle 0|0\rangle = \langle \dots,0,0,0|0,0,0,\dots\rangle = 1 \\
&\langle \dots, n_i', \dots, n_2', n_1' | n_1, n_2, \dots, n_i, \dots\rangle = \dots \times \delta_{n_1 n_1'} \times \delta_{n_2 n_2'} \times \dots \times \delta_{n_i n_i'} \times \dots
\end{aligned}
$$

# 2 Creation and annihillation operators

## 2.1 Fermionic creation and annihillation operators

$$
\begin{aligned}
&\hat{a}_k^\dagger |n_1, \dots, n_k, \dots\rangle = \sqrt{1-n_k}(-1)^{\Sigma_k} |n_1, \dots, 1+n_k, \dots\rangle \\
&\hat{a}_k |n_1, \dots, n_k, \dots\rangle = \sqrt{n_k}(-1)^{\Sigma_k} |n_1, \dots, 1-n_k, \dots\rangle \\
&n_k \in \{0,1\},\ \Sigma_k = \sum_{j=1}^{k-1} n_j \\
&\hat{a}_k^\dagger |n_1, \dots, n_{k-1}, 1, n_{k+1}, \dots\rangle = 0 \\
&\hat{a}_k |n_1, \dots, n_{k-1}, 0, n_{k+1}, \dots\rangle = 0 \\
&\hat{a}_k = (\hat{a}_k^\dagger)^\dagger
\end{aligned}
$$

Anticommutation relations:

$$
\begin{aligned}
&\{\hat{a}_k, \hat{a}_l^\dagger\} = \hat{a}_k \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_k = \delta_{kl} \\
&\{\hat{a}_k, \hat{a}_l\} = \{\hat{a}_k^\dagger, \hat{a}_l^\dagger\} = 0
\end{aligned}
$$

## 2.2 Bosonic creation and annihillation operators

$$
\begin{aligned}
&\hat{a}_k^\dagger |n_1, \dots, n_k, \dots\rangle = \sqrt{1+n_k} |n_1, \dots, n_k+1, \dots\rangle \\
&\hat{a}_k |n_1, \dots, n_k, \dots\rangle = \sqrt{n_k} |n_1, \dots, n_k-1, \dots\rangle \\
&n_k \in \{0,1,\dots\} \\
&\hat{a}_k^\dagger |n_1, \dots, n_{k-1}, 1, n_{k+1}, \dots\rangle = 0 \\
&\hat{a}_k |n_1, \dots, n_{k-1}, 0, n_{k+1}, \dots\rangle = 0
\end{aligned}
$$

Commutation relations:

$$
\begin{aligned}
&[\hat{a}_k, \hat{a}_l^\dagger] = \hat{a}_k \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_k = \delta_{kl} \\
&[\hat{a}_k, \hat{a}_l] = [\hat{a}_k^\dagger, \hat{a}_l^\dagger] = 0
\end{aligned}
$$

For bosons only:

$$
|n_1, \dots, n_k, \dots\rangle = \frac{1}{\sqrt{\prod_{i=1}^{\infty} n_i!}} (\hat{a}_1^\dagger)^{n_1} \times (\hat{a}_2^\dagger)^{n_2} \times \dots \times (\hat{a}_k^\dagger)^{n_k} \times \dots |0\rangle .
$$

**For both bosons and fermions:**

- $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ gives the number of particles in the $k$-th eigenstate:

$$
\hat{n}_k |n_1, \dots, n_k, \dots\rangle = n_k |n_1, \dots, n_k, \dots\rangle .
$$

- $\hat{N} = \sum_k \hat{n}_k$ gives the total number of particles:

$$\hat{N} |n_1, \ldots, n_k, \ldots\rangle = \sum_{j=1}^{\infty} n_j |n_1, \ldots, n_k, \ldots\rangle = N |n_1, \ldots, n_k, \ldots\rangle .$$

- $\langle 0|0\rangle = 1$
- $\langle 0| \hat{a}_k^\dagger = 0$
- $\hat{a}_k |0\rangle = 0$

# 3 Field operators

$$\hat{\Psi}^\dagger(x) = \sum_{j=1}^{\infty} \varphi_j^*(x)\hat{a}_j^\dagger$$
$$\hat{\Psi}(x) = \sum_{j=1}^{\infty} \varphi_j(x)\hat{a}_j,$$

where $\varphi_j(x) = \langle x|j\rangle$ is the wavefunction of the $j$-th one-particle eigenstate.

The field operator satisfy the anticommutation relations for fermions:

$$\{\hat{\Psi}(x), \hat{\Psi}^\dagger(y)\} = \delta(x-y)$$
$$\{\hat{\Psi}^\dagger(x), \hat{\Psi}^\dagger(y)\} = \{\hat{\Psi}(x), \hat{\Psi}(y)\} = 0$$

And for bosons, the commutation relations:

$$[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] = \delta(x-y)$$
$$[\hat{\Psi}^\dagger(x), \hat{\Psi}^\dagger(y)] = [\hat{\Psi}(x), \hat{\Psi}(y)] = 0$$

The particle number operator is

$$\hat{N} = \int \mathrm{d}x \hat{\Psi}^\dagger(x)\hat{\Psi}(x) = \cdots = \sum_{k=1}^{\infty} \hat{a}_k^\dagger \hat{a}_k$$

Calculating the first-quantized wavefunction for an $N$-particle system: Let $|\Phi_N\rangle$ be a second quantized state for which

$$\hat{N} |\Phi_N\rangle = N |\Phi_N\rangle .$$

The first-quantized wavefunction for this state is

$$\Phi(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \left\langle 0 \left| \prod_{j=1}^{N} \hat{\Psi}(x_j) \right| \Phi_N \right\rangle .$$

# 4 Fock-space operators

If $\hat{o}$ is a single-particle operator in first-quantization, then its second quantized form is

$$\hat{O} = \int \mathrm{d}x \hat{\Psi}^\dagger(x)\hat{o}\hat{\Psi}(x)$$
$$= \sum_{j,k} \int \mathrm{d}x \varphi_j^*(x)\hat{o}\varphi_k(x)\hat{a}_j^\dagger \hat{a}_k.$$

If $\hat{o}$ is a two-particle operator in first-quantization, then its second-quantized form is

$$\begin{aligned}\hat{O} &= \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{o}\hat{\Psi}(x')\hat{\Psi}(x) \\ &= \frac{1}{2}\sum_{i,j,k,l}\int \mathrm{d}x\mathrm{d}x'\varphi_i^*(x)\varphi_j^*(x')\hat{o}\varphi_k(x')\varphi_l(x)\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l.\end{aligned}$$

# 5 Green function method

## 5.1 Grand canonical ensemble

$$\begin{aligned}&\hat{H} = \hat{H}_0 + \hat{H}_1 \\ &\hat{K} := \hat{H} - \mu\hat{N} \\ &\hat{K} = \hat{K}_0 + \hat{K}_1 \\ &\hat{K}_0 = \hat{H}_0 - \mu\hat{N},\ \ \hat{K}_1 = \hat{H}_1\end{aligned}$$

The trace of an operator:

$$\mathrm{Tr}\left[\hat{A}\right] = \mathrm{Tr}\left[\hat{A}\right] = \sum_{\{n_i\}}\langle\ldots,n_i,\ldots,n_1|\hat{A}|n_1,\ldots,n_i,\ldots\rangle\,,$$

where the sum is over the entire Fock-space, not just the $N$-particle subspace.

The grand canonical partition function is defined as

$$Z_G = e^{-\beta\Omega(T,V,\mu)} = \mathrm{Tr}\left[e^{-\beta\hat{K}}\right] = \sum_{\{n_i\}}\langle\ldots,n_i,\ldots,n_1|e^{-\beta\hat{K}}|n_1,\ldots,n_i,\ldots\rangle\,,$$

where $\beta = (1/k_BT)$.

The grand canonical density matrix is

$$\hat{\rho}_G = \frac{e^{-\beta\hat{K}}}{Z_G}.$$

The average of operator $\hat{O}$ *over the grand canonical ensemble* is

$$\langle\hat{O}\rangle = \mathrm{Tr}\left[\hat{\rho}_G\hat{O}\right] = \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}}\hat{O}\right].$$

## 5.2 Imaginary time and $K$-picture

The imaginary time is defined to be

$$\tau = -it.$$

The imaginary time dependendent form of an operator is defined as

$$\hat{A}_K(\tau) = e^{\frac{\hat{K}\tau}{\hbar}}\hat{A}e^{-\frac{\hat{K}\tau}{\hbar}}$$

This is also called the $K$-picture of the operator. It can be shown that $\hat{K}(\tau) = \hat{K}$:

$$\left[f(\hat{K}),\hat{K}\right] = 0 \implies \left[e^{\frac{\hat{K}\tau}{\hbar}},\hat{K}\right] = 0$$

Using the Baker-Campbell-Hausdorff formula:

$$\hat{K}(\tau) = Ke^{\frac{\hat{K}\tau}{\hbar}}\hat{K}e^{-\frac{\hat{K}\tau}{\hbar}} = \hat{K} + [e^{\frac{\hat{K}\tau}{\hbar}},\hat{K}] + \frac{1}{2!}[e^{\frac{\hat{K}\tau}{\hbar}},[e^{\frac{\hat{K}\tau}{\hbar}},\hat{a}_k]] + \frac{1}{3!}[e^{\frac{\hat{K}\tau}{\hbar}},[e^{\frac{\hat{K}\tau}{\hbar}},[e^{\frac{\hat{K}\tau}{\hbar}},\hat{K}]]] + \cdots = \hat{K}$$

Properties:

- $\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{A}_K(\tau) = \frac{1}{\hbar}[\hat{K}, \hat{A}_K(\tau)]$
- If $\hat{C} = \hat{A}\hat{B}$, then $\hat{C}(\tau) = \hat{A}(\tau)\hat{B}(\tau)$
- $\hat{\Psi}(x,\tau) = e^{\frac{\hat{K}\tau}{\hbar}}\hat{\Psi}(x)e^{-\frac{\hat{K}\tau}{\hbar}} = \sum_k \varphi_k(x)\hat{a}_k(\tau)$
- $\hat{\Psi}^\dagger(x,\tau) = e^{\frac{\hat{K}\tau}{\hbar}}\hat{\Psi}^\dagger(x)e^{-\frac{\hat{K}\tau}{\hbar}} = \sum_k \varphi_k(x)\hat{a}_k^\dagger(\tau)$
- $[\hat{A}(\tau), \hat{B}(\tau)] = e^{\frac{\hat{K}\tau}{\hbar}}[\hat{A}, \hat{B}]e^{-\frac{\hat{K}\tau}{\hbar}}$

# 6 Exercises

1. What is the first quantized wavefunction of $\sum_k c_k\hat{a}_k^\dagger |0\rangle$, where $\hat{a}_k^\dagger$ is a fermionic creation operator?
2. Calculate the first-quantized wavefunction for $\hat{a}_k^\dagger\hat{a}_l^\dagger |0\rangle$ for both fermionic and bosonic operators!
3. Consider 2 fermionic particles prepared in states

$$\Psi_1(x_1) = \sum_k b_k\varphi_k(x_1)$$
$$\Psi_2(x_2) = \sum_l c_l\varphi_l(x_2).$$

The state of the joint system in first quantization is

$$\Phi(x_1, x_2) = \frac{1}{\sqrt{2}}[\Psi_1(x_1)\Psi_2(x_2) - \Psi_1(x_2)\Psi_2(x_1)]$$

What is the Fock-space representation of $\Phi(x_1, x_2)$? Verify Your result by converting it back to first-quantization using the formula!

*Solution.*

$$\Psi_1(x_1) = \sum_k b_k\varphi_k(x_1) \to \sum_k b_k\hat{a}_k^\dagger |0\rangle$$
$$\Psi_2(x_2) = \sum_l c_l\varphi_l(x_2) \to \sum_l c_l\hat{a}_k^\dagger |0\rangle$$
$$\Phi(x_1, x_2) = \sum_{k,l} b_k c_l \frac{1}{\sqrt{2}}[\varphi_k(x_1)\varphi_l(x_2) - \varphi_k(x_2)\varphi_l(x_1)] \to \sum_{k,l} b_k c_l \hat{a}_k^\dagger a_l^\dagger |0\rangle$$

4. Find the eigenvalues and eigenvectors of $\hat{a}_k$ and $\hat{a}_k^\dagger$!
5. $\hat{\Psi}^\dagger(x)|0\rangle =?$
6. $[\hat{\Psi}(x), \hat{N}] =?$
7. Calculate the first-quantized wavefunction $\Phi(x_1, x_2, x_3)$ for bosonic three-particle system prepared in state $|2, 0, 1, 0, 0, \cdots\rangle$.
8. Prove that

$$[\hat{a}_j, f(\hat{a}_j^\dagger)] = \frac{\partial f(\hat{a}_j^\dagger)}{\partial \hat{a}_j^\dagger}$$
$$[\hat{a}_j^\dagger, f(\hat{a}_j)] = -\frac{\partial f(\hat{a}_j)}{\partial \hat{a}_j}$$

Hint: Use the Taylor-expansion formula:

$$f(\hat{A}) = \sum_{k=1}^{\infty} \frac{1}{k!} f^{(k)}(0)\hat{A}^k$$

And the derivative is

$$\frac{\partial f(\hat{A})}{\partial A} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} f^{(k)}(0)\hat{A}^{k-1}$$

9. Consider a fermionic system. What is the Fock-space representation of $\hat{S}_z$ and what is the meaning of this operator? What are the eigenstates of $\hat{S}_z$?

10. Calculate $\mathrm{Tr}\left[\hat{N}\right]$ for bosonic particles!

*Solution.*

$$\begin{aligned}
\mathrm{Tr}\left[\hat{N}\right] &= \sum_{\{n_i\}} \langle \ldots, n_i, \ldots, n_1 | \hat{N} | n_1, \ldots, n_i, \ldots \rangle \\
&= \sum_{\{n_i\}} \left\langle \ldots, n_i, \ldots, n_1 \left| \int \mathrm{d}x \hat{\Psi}^\dagger(x)\hat{\Psi}(x) \right| n_1, \ldots, n_i, \ldots \right\rangle \\
&= \sum_{\{n_i\}} \left\langle \ldots, n_i, \ldots, n_1 \left| \int \mathrm{d}x \sum_{k,l} \varphi_k^*(x)\varphi_l(x)\hat{a}_k^\dagger \hat{a}_l \right| n_1, \ldots, n_i, \ldots \right\rangle \\
&= \sum_{\{n_i\}} \sum_{k,l} \int \mathrm{d}x \varphi_k^*(x)\varphi_l(x) \langle \ldots, n_i, \ldots, n_1 | \hat{a}_k^\dagger \hat{a}_l | n_1, \ldots, n_i, \ldots \rangle \\
&= \sum_{\{n_i\}} \sum_{k,l} \int \mathrm{d}x \varphi_k^*(x)\varphi_l(x) \langle \ldots, n_i, \ldots, n_1 | \hat{a}_k^\dagger \sqrt{n_l} | n_1, \ldots, n_l - 1, \ldots \rangle \\
&= \sum_{\{n_i\}} \sum_{k,l} \int \mathrm{d}x \varphi_k^*(x)\varphi_l(x) \langle \ldots, n_i, \ldots, n_1 | \sqrt{n_k+1}\sqrt{n_l} | n_1, \ldots, n_k + 1, \ldots, n_l - 1, \ldots \rangle \\
&= \sum_{\{n_i\}} \sum_{k,l} \sqrt{n_k+1}\sqrt{n_l} \int \mathrm{d}x \varphi_k^*(x)\varphi_l(x) \langle \ldots, n_i, \ldots, n_1 | n_1, \ldots, n_k + 1, \ldots, n_l - 1, \ldots \rangle \\
&= \sum_{\{n_i\}} \sum_{k,l} \sqrt{n_k+1}\sqrt{n_l} \int \mathrm{d}x \varphi_k^*(x)\varphi_l(x)\delta_{kl} \\
&= \sum_{\{n_i\}} \sum_{k} n_k \underbrace{\int \mathrm{d}x \varphi_k^*(x)\varphi_k(x)}_{=1} \\
&= \sum_{\{n_i\}} \sum_{k} n_k
\end{aligned}$$

11. Calculate $\mathrm{Tr}\left[\hat{H}_0\right]$, where

$$\hat{H}_0 = \int \mathrm{d}x \hat{\Psi}^\dagger(x) \left[ -\frac{\hbar^2}{2M}\nabla^2 + U(x) \right] \hat{\Psi}(x)$$

12. Calculate $\langle \hat{n}_k \rangle$ for free non-interacting particles!

*Solution.* For non-interacting particles,

$$\hat{K} = \hat{K}_0 = \int \mathrm{d}x \hat{\Psi}^\dagger(x) \left[ -\frac{\hbar^2}{2M}\nabla^2 + U(x) - \mu \right] \hat{\Psi}(x)$$

$$\langle\hat{n}_k\rangle = \langle\hat{a}_k^\dagger\hat{a}_k\rangle = \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\hat{a}_k\right] = \frac{1}{Z_G}\mathrm{Tr}\left[\hat{a}_k e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\right] = \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}e^{\beta\hat{K}_0}\hat{a}_k e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\right].$$

We can use the imaginary time-dependence of operators ($K$-picture):

$$\hat{a}_k(\tau) = e^{\frac{\hat{K}\tau}{\hbar}}\hat{a}_k e^{-\frac{\hat{K}\tau}{\hbar}} \implies \hat{a}_k(\beta\hbar) = e^{\beta\hat{K}_0}\hat{a}_k e^{-\beta\hat{K}_0}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{a}_k(\tau) = \frac{1}{\hbar}[\hat{K}_0, \hat{a}_k(\tau)]$$

Using the identity $K_0 = K_0(\tau)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{a}_k(\tau) &= \frac{1}{\hbar}[\hat{K}_0(\tau), \hat{a}_k(\tau)] \\ &= \frac{1}{\hbar}\left[\sum_{l,m}\int \mathrm{d}x\varphi_l^*(x)\left(-\frac{\hbar^2}{2M}\nabla^2 + U(x) - \mu\right)\varphi_m(x)\hat{a}_l^\dagger(\tau)\hat{a}_m(\tau), \hat{a}_k(\tau)\right] \\ &= \frac{1}{\hbar}\left[\sum_l a_l^\dagger(\tau)\hat{a}_l(\tau)(e_l-\mu), \hat{a}_k(\tau)\right] = \frac{1}{\hbar}\sum_l(e_l-\mu)\left[a_l^\dagger(\tau)\hat{a}_l(\tau), \hat{a}_k(\tau)\right].\end{aligned}$$

$$\begin{aligned}[\hat{a}_l^\dagger(\tau)\hat{a}_l(\tau), \hat{a}_k(\tau)] &= \hat{a}_l^\dagger(\tau)\hat{a}_l(\tau)\hat{a}_k(\tau) - \hat{a}_k(\tau)\hat{a}_l^\dagger(\tau)\hat{a}_l(\tau) \\ &= e^{\frac{\hat{K}\tau}{\hbar}}\hat{a}_l^\dagger\hat{a}_l\hat{a}_k e^{-\frac{\hat{K}\tau}{\hbar}} - e^{\frac{\hat{K}\tau}{\hbar}}\hat{a}_k\hat{a}_l^\dagger\hat{a}_l e^{-\frac{\hat{K}\tau}{\hbar}} \\ &= e^{\frac{\hat{K}\tau}{\hbar}}\hat{a}_l^\dagger\hat{a}_l\hat{a}_k e^{-\frac{\hat{K}\tau}{\hbar}} - e^{\frac{\hat{K}\tau}{\hbar}}(\delta_{kl} - \hat{a}_l^\dagger\hat{a}_k)\hat{a}_l e^{-\frac{\hat{K}\tau}{\hbar}} \\ &= e^{\frac{\hat{K}\tau}{\hbar}}\hat{a}_l^\dagger\hat{a}_l\hat{a}_k e^{-\frac{\hat{K}\tau}{\hbar}} - e^{\frac{\hat{K}\tau}{\hbar}}(\delta_{kl}\hat{a}_l - \hat{a}_l^\dagger\hat{a}_l\hat{a}_k)e^{-\frac{\hat{K}\tau}{\hbar}} \\ &= -e^{\frac{\hat{K}\tau}{\hbar}}\delta_{kl}\hat{a}_l e^{-\frac{\hat{K}\tau}{\hbar}}.\end{aligned}$$

Using this result, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{a}_k(\tau) &= -\frac{1}{\hbar}\sum_l(e_l-\mu)\hat{a}_k(\tau) \\ \implies \hat{a}_k(\tau) &= \hat{a}_k e^{-\frac{(e_l-\mu)}{\hbar}\tau}\end{aligned}$$

$$\begin{aligned}\langle\hat{n}_k\rangle &= \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}e^{\beta\hat{K}_0}\hat{a}_k e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\right] = \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\hat{a}_k(\beta\hbar)\hat{a}_k^\dagger\right] \\ &= \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}(1 \pm \hat{a}_k^\dagger\hat{a}_k(\beta\hbar))\right] \\ &= \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\right] \pm \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\hat{a}_k(\beta\hbar)\right] \\ &= \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\right] \pm \frac{1}{Z_G}\mathrm{Tr}\left[e^{-\beta\hat{K}_0}\hat{a}_k^\dagger\hat{a}_k e^{-\beta(e_l-\mu)}\right] \\ &= 1 \pm \langle\hat{n}_k\rangle e^{-\beta(e_l-\mu)}\end{aligned}$$

$$\implies \langle\hat{n}_k\rangle = \frac{1}{1 \mp e^{-\beta(e_l-\mu)}}.$$

13. Calculate $\langle\hat{N}\rangle$ for bosonic particles!
14. Do something
15. asdfghjk

$$!, \nabla, \mathcal{G}(x,\tau,x',\tau')$$