Statistical physics cheat sheet

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1. Using the second quantized formalism, show that for noninteracting fermions

$$\Omega_0 = -k_B T \sum_i \ln\left(1 + e^{-\beta(\varepsilon_i - \mu)}\right) ,$$

where ε_i denotes the i-th one-particle level

Solution. The second quantized form of $\hat{K}_0$ is

$$\begin{aligned}
\hat{K}_0 &= \int \mathrm{d}x \hat{\Psi}^\dagger(x) \left[-\frac{\hbar^2}{2M}\triangle + U(x) - \mu\right] \hat{\Psi}(x) \\
&= \sum_{l,m} \int \mathrm{d}x \varphi_k^*(x) \underbrace{\left[-\frac{\hbar^2}{2M}\triangle + U(x) - \mu\right] \varphi_m(x)}_{(\varepsilon_m - \mu)\varphi_m(x)} \hat{a}_l^\dagger \hat{a}_m \\
&= \sum_{l,m} (\varepsilon_m - \mu)\delta_{lm} \hat{a}_l^\dagger \hat{a}_m \\
&= \sum_l (\varepsilon_l - \mu)\hat{a}_l^\dagger \hat{a}_l = \sum_l (\varepsilon_l - \mu)\hat{n}_l
\end{aligned}$$

Using this result,

$$\begin{aligned}
\Omega_0 &= -k_B T \ln Z_G \\
&= -k_B T \ln \mathrm{Tr}\left[e^{-\beta\hat{K}_0}\right] \\
&= -k_B T \ln \sum_{\{n_i\}} \left\langle n_1, \ldots, n_i, \ldots \middle| e^{-\beta\hat{K}_0} \middle| n_1, \ldots, n_i, \ldots \right\rangle \\
&= -k_B T \ln \sum_{\{n_i\}} \left\langle n_1, \ldots, n_i, \ldots \middle| e^{-\beta(\hat{H}_0 - \mu\hat{N})} \middle| n_1, \ldots, n_i, \ldots \right\rangle \\
&= -k_B T \ln \sum_{\{n_i\}} \left\langle n_1, \ldots, n_i, \ldots \middle| e^{-\beta\sum\limits_l (\varepsilon_l - \mu)\hat{a}_l^\dagger \hat{a}_l} \middle| n_1, \ldots, n_i, \ldots \right\rangle \\
&= -k_B T \ln \sum_{\{n_i\}} e^{-\beta\sum\limits_l (\varepsilon_l - \mu)n_l} \langle n_1, \ldots, n_i, \ldots | n_1, \ldots, n_i, \ldots \rangle
\end{aligned}$$

2. Using the result for Ω_0, calculate Ω_0 and N as a function of (T, V, μ) for a fermionic homogeneous system (noninteracting fermions in a box with periodic boundary conditions). Express your result with Fermi-Dirac integrals. Give the first three terms of the high temperature expansion for Ω_0 and for N.

3. For noninteracting fermions, one can define a characteristic temperature T_{deg} by which the chemical potential is zero:

$$\mu(T = T_{\mathrm{deg}}) = 0.$$

By dimensional analysis

$$k_B T_{\mathrm{deg}} = z \frac{\hbar^2}{2m} \left(\frac{N}{V}\right)^{2/3} ,$$

where z is a dimensionless number. Calculate z exactly and numerically.

4. Let us suppose that we have N noninteracting, spinless bosons confined in a 3 dimensional harmonic oscillator potential

$$V(\mathbf{r}) = \frac{m}{2}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2).$$

Calculate T_c, where Bose-Einstein condensation occurs.