Statistical Physics Homework 1

1. Prove the following commutation relations of the field operators:

$$[\Psi(\mathbf{r},s),\Psi^\dagger(\mathbf{r}',s')] = \delta_{s,s'}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{1a}$$
$$[\Psi(\mathbf{r},s),\Psi(\mathbf{r}',s')] = [\Psi^\dagger(\mathbf{r},s),\Psi^\dagger(\mathbf{r}',s')] = 0, \tag{1b}$$

   for bosons, and

$$\{\Psi(\mathbf{r},s),\Psi^\dagger(\mathbf{r}',s')\} = \delta_{s,s'}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{2a}$$
$$\{\Psi(\mathbf{r},s),\Psi(\mathbf{r}',s')\} = \{\Psi^\dagger(\mathbf{r},s),\Psi^\dagger(\mathbf{r}',s')\} = 0, \tag{2b}$$

   for fermions. Here $[A,B]=AB-BA$ is the commutator, while $\{A,B\}=AB+BA$ is the anticommutator.

2. We know that the particle number operator is given by

$$\hat{N} = \sum_k a_k^\dagger a_k, \tag{3}$$

   both for bosons and fermions. Prove that with the help of field operators it reads as,

$$\hat{N} = \sum_s \int d^3r\, \Psi^\dagger(\mathbf{r},s)\Psi(\mathbf{r},s). \tag{4}$$

3. Prove that for bosons the following relation holds,

$$[a_k,(a_l^\dagger)^n] = n\,\delta_{k,l}\,(a_l^\dagger)^{n-1}, \tag{5}$$

   where n is positive integer. What is the difference for fermions?

4. Calculate the wavefunction of the following Fock states (both for bosons and fermions)

   (a) the single particle state with quantum number i:

$$\left|\Phi_1\right\rangle = \left|0,\dots,0,\overset{i}{1},0,\dots\right\rangle, \tag{6}$$

   (b) the two particle state with quantum numbers i and j ($i \neq j$):

$$\left|\Phi_2\right\rangle = \left|0,\dots,0,\overset{i}{1},0,\dots,0,\overset{j}{1},0,\dots\right\rangle, \tag{7}$$

   (c) the two particle state with the quantum number i occupied twice:

$$\left|\Phi_2'\right\rangle = \left|0,\dots,0,\overset{i}{2},0,\dots\right\rangle, \tag{8}$$

   (d) the three particle state with the first three states singly occupied:

$$\left|\Phi_3\right\rangle = \left|1,1,1,0,0,\dots\right\rangle. \tag{9}$$

Submission: at the beginning of the practice class on 28 Sep.