Statistical Physics Homework 2

1. The occupation number of the ideal Fermi gas. Verify that for an ideal Fermi gas with Hamiltonian
$$H_0 = \sum_k e_k a_k^\dagger a_k, \tag{1}$$
the occupation number is
$$n_k^{(0)} = \left\langle a_k^\dagger a_k \right\rangle_0 = \frac{1}{e^{\beta(e_k-\mu)}+1}. \tag{2}$$

2. Matsubara representation of the noninteracting Green's function. Both for bosons and fermions the noninteracting Green's function is given by,
$$\mathcal{G}_{(0)}(x,\tau;x',\tau') = -\sum_k \varphi_k(x)\varphi_k^*(x')\, e^{-\frac{e_k-\mu}{\hbar}(\tau-\tau')} \times \begin{cases} 1 \pm n_k^{(0)}, & \text{if} \quad \tau > \tau', \\ \pm n_k^{(0)}, & \text{if} \quad \tau \leq \tau', \end{cases} \tag{3}$$
with $\varphi_k(x)$ is the wavefunction of the single-particle state k, e_k is its single-particle energy, and μ is the chemical potential.

   Show both for bosons and fermions, that the Green's function's Fourier coefficients read as
$$\mathcal{G}_{(0)}(x,x';i\omega_n) = \sum_k \frac{\varphi_k(x)\varphi_k^*(x')}{i\omega_n - \hbar^{-1}(e_k-\mu)}. \tag{4}$$

3. Calculate the noninteracting Green's function for the case of a homogeneous (translation invariant) system, both for bosons and fermions. Note, that the Hamiltonian in first quantized form is:
$$H_0^1 = -\sum_{i=1}^{N} \frac{\hbar^2 \Delta_i}{2M}. \tag{5}$$
Remember to use periodic boundary conditions and give $\mathcal{G}_{s,s'}(\mathbf{k}, i\omega_n)$ as the final result, which is defined as,
$$\mathcal{G}(\mathbf{r},s,\mathbf{r}',s',i\omega_n) = \frac{1}{V}\sum_k \mathcal{G}_{s,s'}(\mathbf{k},i\omega_n)e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}. \tag{6}$$
(That is, give the full Fourier transform of the noninteracting Green's function.)

Submission: at the beginning of the practice class on 12 Oct.