Recommender Systems In Context: Singular Value Decomposition

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July 7, 2019

Introduction

- Matrix factorizations are ubiquitous in modern data science
- Our need for tools that can map a high dimensional space into a lower dimensional space of latent factors grows as the amount of data we collect increases
- Provide a brief overview of singular value decomposition (SVD)
- Walk through simple power method implementation for SVD on a dense matrix

Theory: Singular Value Decomposition (i.e., Spectral Decomposition)

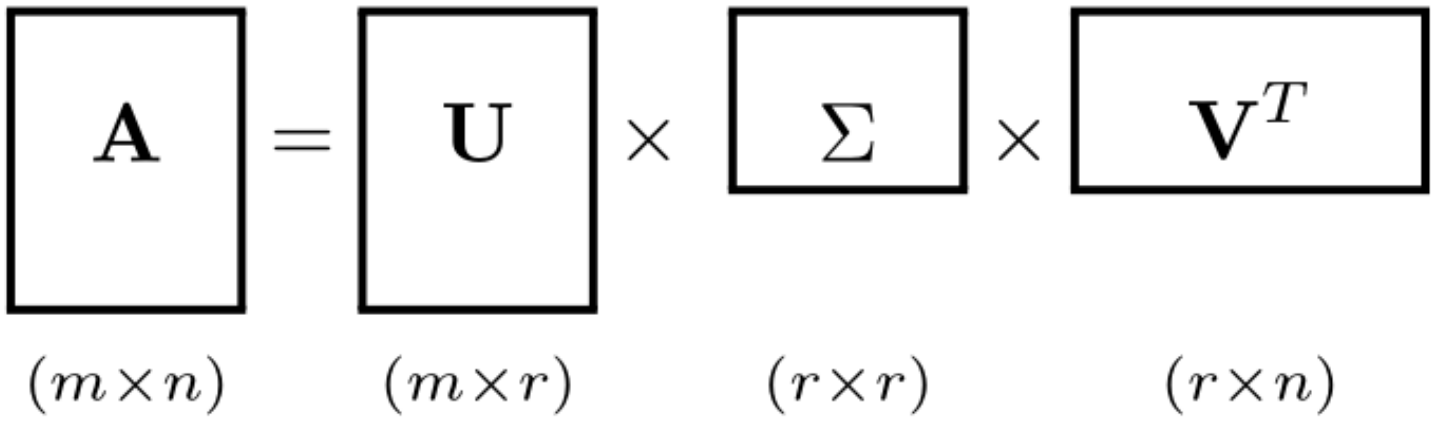
The singular value decomposition (SVD) of a matrix A with m rows and n columns is

$$A = U\Sigma V^T = \sum_i \sigma_i u_i \circ v_i^T$$

where

- $A = m \times n$ input data matrix ($A \in \mathbb{R}^{m \times n}$)
- $r =$ rank of matrix A (i.e., the number of linearly independent columns of A).
- $U = m \times r$ matrix of left singular vectors
- $\Sigma = r \times r$ diagonal matrix
- $V = n \times r$ matrix of right singular vectors. The superscript T indicates the transpose of matrix V.

Singular Value Decomposition (SVD) - Visualized



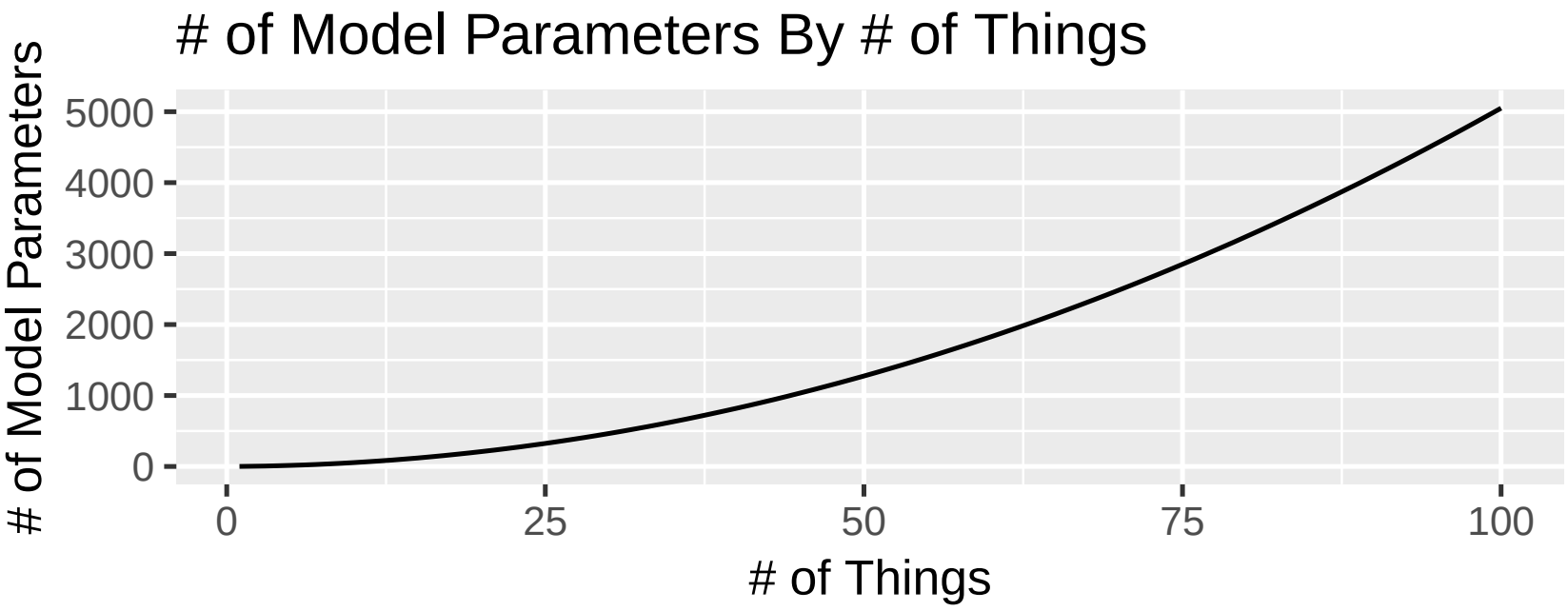
- If A has rank r, then U is $m \times r$, Σ is an $r \times r$ diagonal matrix with non-negative, non-increasing entries (i.e., sorted from largest to smallest), V is $n \times r$ and V^T is $r \times n$.
- U and V are *column orthonormal*
- By *column orthonormal*, we mean that $U^T U = I$ and $V^T V = I$, where I is an identity matrix
- Two vectors u and v whose dot product is $u \cdot v = 0$ (i.e., the vectors are perpendicular) are said to be orthogonal

$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$

σ_i ... scalar
$\mathbf{u}_i$ ... vector
$\mathbf{v}_i$ ... vector

- This is key to the algorithm we will implement later

Why Does Orthogonal Matter - Too Many Moving Parts



- Number of independent parameters to be estimated grows with square of number of things
- Number of data points available to estimate a covariance matrix grows only linearly with number of things
- Larger the number of things, the more data we typically need to estimate reliably

Importance of Normalization

– Magnitudes of attribute values comprising input data matrix should be scaled into roughly same range [Skillicorn 2007]

– Want to ensure that properties are compared in a way consistent with real world

- i.e., if measure some attributes in inches and others in miles, magnitude of numbers cannot be compared without rescaling
- Common to divide entries in each column of a dense matrix by standard deviation of that column

– May be more appropriate to normalize by keeping zero entries fixed when a matrix is sparse

- Mean of non-zero entries is typically subtracted from non-zero entries so that they become zero-centered
- Only non-zero entries are divided by standard deviation of the column mean

Importance of Normalization - Continued

– Must carefully consider form of normalization because selecting correct approach can significantly improve predictive performance

Implementation

- Modern libraries implementing SVD are based on extensive research using methods more complex than the method we are going to walk through here [GolubAndLoan 2013]
- Use simplest available approach for illustrative purposes
- There are more efficient and more numerically stable methods available [GolubAndLoan 2013]

Sample Data

```
# create sample matrix
A = np.array([
   [4, 1, 1],
   [2, 5, 3],
   [1, 2, 1],
   [4, 5, 5],
   [3, 5, 2],
   [2, 4, 2],
   [5, 3, 1],
   [2, 2, 5],
  ], dtype='float64')
# determine n rows and m columns
m,n = A.shape
print("A is a "+str(m)+", x "+str(n)+" matrix")
```

```
## A is a 8, x 3 matrix
```

Random Unit Vector

First, we define a function to create a random unit vector:

```
def randomUnitVector(n):
    unnormalized = [random.normalvariate(0,
      1) for _ in range(n)]
    xNorm = sqrt(sum(x * x for x in unnormalized))
    return [x / xNorm for x in unnormalized]
```

Create a random unit vector:

```
# build random unit vector
x = randomUnitVector(min(m,n))
print(np.round(x,4))
```

```
## [ 0.8037 -0.4859  0.3435]
```

One-Dimensional SVD

```
def svdPowerMethod_1d(A, epsilon=1e-10):
    m, n = A.shape
    x = randomUnitVector(min(m,n))
    lastV = None
    V = x
    if m > n:
        B = np.dot(A.T, A)
    else:
        B = np.dot(A, A.T)
    while True:
        lastV = V
        V = np.dot(B, lastV)
        V = V / norm(V)
        if abs(np.dot(V, lastV)) > 1 - epsilon:
            return V
```

```
def svdPowerMethodRow(A, k=None, epsilon=1e-10):
    A = np.array(A, dtype=float)
    m, n = A.shape
    svdEstimate = []
    if k is None:
        k = min(m, n)
    for i in range(k):
        matrixFor1D = A.copy()
        for singularValue, u, v in svdEstimate[:i]:
            matrixFor1D -= singularValue * np.outer(u, v)
        v = svdPowerMethod_1d(matrixFor1D,epsilon=epsilon)
        u_unnormalized = np.dot(A, v)
        sigma = norm(u_unnormalized)
        u = u_unnormalized / sigma
        svdEstimate.append((sigma, u, v))
    output = [np.array(x) for x in zip(*svdEstimate)]
    singularValues, us, vs = output
    return singularValues, us.T, vs
```

```
def svdPowerMethodColumn(A, k=None, epsilon=1e-10):
    A = np.array(A, dtype=float)
    m, n = A.shape
    svdEstimate = []
    if k is None:
        k = min(m, n)
    for i in range(k):
        matrixFor1D = A.copy()
        for singularValue, u, v in svdEstimate[:i]:
            matrixFor1D -= singularValue * np.outer(u, v)
        u = svdPowerMethod_1d(matrixFor1D,epsilon=epsilon)
        v_unnormalized = np.dot(A.T, u)
        sigma = norm(v_unnormalized)
        v = v_unnormalized / sigma
        svdEstimate.append((sigma, u, v))
    output = [np.array(x) for x in zip(*svdEstimate)]
    singularValues, us, vs = output
    return singularValues, us.T, vs
```

SVD of a Matrix, A, Using the Power Method

```
def svdPowerMethod(A, k=None, epsilon=1e-10):
    A = np.array(A, dtype=float)
    m, n = A.shape
    if m > n:
      s, uT, vs=svdPowerMethodRow(A,
      k, epsilon=1e-10)
    else:
      s, uT, vs=svdPowerMethodColumn(A,
      k, epsilon=1e-10)
    return s, uT, vs
```

Run svdPowerMethod()

```
# compute SVD
singularValues,U,V = svdPowerMethod(A)
```

The 8 by 3 matrix of left singular vectors U:

```
print(U)
```

```
## [[ 0.22155208 -0.52086662 -0.39333697]
##  [ 0.39458523  0.23924155  0.35445363]
##  [ 0.15830232  0.03055162  0.15299691]
##  [ 0.53347447  0.19168664 -0.19949776]
##  [ 0.39692635 -0.08648347  0.41053093]
##  [ 0.31660463  0.06110323  0.30599382]
##  [ 0.34630265 -0.64128879 -0.07381356]
##  [ 0.32840218  0.45969551 -0.62355828]]
```

The 3 by 3 diagonal singular value matrix Σ:

```
print(np.diag(singularValues))
```

```
## [[ 15.09626916   0.           0.        ]
##  [  0.           4.30056855   0.        ]
##  [  0.           0.           3.40701739]]
```

The 3 by 3 matrix of right singular vectors V:

```
print(V)
```

```
## [[ 0.54184843  0.67070986  0.50650623]
##  [-0.75152966  0.11682294  0.6492731 ]
##  [-0.37630233  0.73246207 -0.56735868]]
```

We reconstitute the matrix A:

```
# reconstitute matrix A
Sigma = np.diag(singularValues)
# reconstitute matrix A
AA=np.dot(U, np.dot(Sigma, V))
```

```
print(AA)
```

```
## [[ 4.  1.  1.]
##  [ 2.  5.  3.]
##  [ 1.  2.  1.]
##  [ 4.  5.  5.]
##  [ 3.  5.  2.]
##  [ 2.  4.  2.]
##  [ 5.  3.  1.]
##  [ 2.  2.  5.]]
```

We can see that the original and reconstituted matrices are the same to 10 decimal places:

```
print(np.round(A - AA, decimals=nDigits))
```

```
## [[ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]
##  [ 0.  0.  0.]]
```

- Have shown results from full-rank decomposition
- In many applications of SVD - including those involving recommender systems - our objective is to map a (sometimes sparse) high-dimensional space to a dense low-dimensional space
- For example, if we have a large user-by-item explicit ratings matrix, we may want to create a more compact representation of that ratings space using linearly independent latent factors
- We can think of the latent factors as an abstraction that allows us to represent each user and item by a linear combination of other users and items
- Classical SVD is one approach to solve for such latent factors, but unfortunately it can only be applied to a dense matrix
- Using independent latent factors often simplifies the interpretation of large datasets because it drastically reduces the number of interacting variables.

References

Gene H. Golub and Charles F. Van Loan. Matrix Computations. The John Hopkins University Press., 4th edition, 2013.

Jeremy Kun. Source Code for SVD, January 2018. URL https://github.com/ j2kun/svd.

David Skillicorn. Understanding Complex Datasets: Data Mining with Matrix Decompositions. Chapman and Hall/CRC, 2007.

Appendix A - Power Method

– Power method for SVD of a matrix $A \in \mathbb{R}^{m \times n}$ works as follows:

- Start by computing the first singular value σ_1 and the left and right singular vectors u_1 and v_1 of A, for which $\min_{i>j} \log(\sigma_i/\sigma_j) \geq \lambda$

1. Generate x_0 such that $x_0(i) \sim \mathrm{N}(0,1)$
2. for i in $[1, \ldots, k]$ where $k = \min(m, n)$:
3. $x_i \leftarrow A^T \leftarrow A x_{xi-1}$
4. $v_1 \leftarrow x_i / \|x_i\|$
5. $\sigma_1 \leftarrow \|A v_1\|$
6. $u_1 \leftarrow A v_1 / \sigma 1$
7. return (σ_1, u_1, v_1)

- Once we have computed (σ_1, u_1, v_1), we can repeat this process for $A - \sigma_1 u_1 v_1^T = \sum_{i=2}^{n} \sigma_i u_i v_i^T$

Appendix B - Matrix Approximation

We perform the matrix decomposition for the largest k singular values, then reconstitute the matrix using only the associated k latent user and item factors:

```
# compute SVD
kSingularValues,kU,kV = svdPowerMethod(A,k=2)
```

Using the 8 by 2 matrix of left singular vectors U, the 2 by 2 diagonal singular value matrix Σ, and the 2 by 3 matrix of right singular vectors V, we reconstitute the approximate matrix $\hat{A}$ using k factors (in this case, just two user factors and two item factors):

```
# reconstitute matrix A
kSigma = np.diag(kSingularValues)
# reconstitute matrix A
kAA=np.dot(kU, np.dot(kSigma, kV))
```

We can compare the original matrix A and the two-factor approximation to that matrix $\hat{A}$:

```
print(np.round(AA-kAA,2))
```

```
## [[ 0.5  -0.98  0.76]
##  [-0.45  0.88 -0.69]
##  [-0.2   0.38 -0.3 ]
##  [ 0.26 -0.5   0.39]
##  [-0.53  1.02 -0.79]
##  [-0.39  0.76 -0.59]
##  [ 0.09 -0.18  0.14]
##  [ 0.8  -1.56  1.21]]
```

Despite the use of an extremely small dataset, the two-factor approximation looks reasonable.