Task
Say whether the following is true or false and support your answer by a proof: For any integer n, the number n^2+n+1 os odd.

Solution
Task is equavalent to proving/disproving the following theorem:

Theorem. $\forall n \in \mathbb{Z}, n \geq 0 : \big(2 \nmid \big(n^2+n+1\big)\big) \wedge \big(2 \nmid \big((-n)^2+(-n)+1\big)\big)$.

Proof. By induction.
Initial step (n = 0): $\big(2 \nmid \big(0^2+0+1\big)\big) \wedge \big(2 \nmid \big((-0)^2+(-0)+1\big)\big)$
$= (2 \nmid 1) \wedge (2 \nmid (1))$ - this is true.
Induction step: Assume $\big(2 \nmid \big(n^2+n+1\big)\big) \wedge \big(2 \nmid \big((-n)^2+(-n)+1\big)\big)$.
Then for $n+1$:
$\big(2 \nmid \big((n+1)^2+(n+1)+1\big)\big) \wedge \big(2 \nmid \big((-(n+1))^2+(-(n+1))+1\big)\big)$
$= \big(2 \nmid \big(\big(n^2+n+1\big)+2\,(n+1)\big)\big) \wedge \big(2 \nmid \big(n^2+n+1\big)\big)$
$= 2 \nmid \big(\big(n^2+n+1\big)+2\,(n+1)\big)$ (by induction hypothesis $2 \nmid \big(n^2+n+1\big)$).
Since, by induction hypothesis, $2 \nmid \big(n^2+n+1\big)$, then $n^2+n+1 = 2k+1$ for some $k \in \mathbb{Z}$.
Then we have $2 \nmid \big(\big(n^2+n+1\big)+2\,(n+1)\big)$
$= 2 \nmid (2k+1+2\,(n+1))$
$= 2 \nmid (2\,(k+n+1)+1)$
(define $l = k+n+1 \in Z$)
$= 2 \nmid (2l+1)$, where $l = k+n+1 \in Z$ - this is clearly true.
Hence, by principle of induction,
$\forall n \in \mathbb{Z}, n \geq 0 : \big(2 \nmid \big(n^2+n+1\big)\big) \wedge \big(2 \nmid \big((-n)^2+(-n)+1\big)\big)$,
which constitutes the proof of original claim. $\square$