Task

Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap\limits_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\} = \emptyset$. Prove that your example has the stated property.

Solution

Let's take $A_n = (0, \frac{1}{n})$

Theorem. *Given a family of intervals* $A_n = \left(0, \frac{1}{n}\right), n \in \mathbb{N}$, *for all* $n \in \mathbb{N}$ $A_{n+1} \subset A_n$ *and* $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$.

Proof. First let's prove that for all $n \in \mathbb{N}$ $A_{n+1} \subset A_n$:
Clearly, $\forall n \in \mathbb{N} : \frac{1}{n+1} < \frac{1}{n}$ and hence $\forall n \in \mathbb{N} : \left(0, \frac{1}{n+1}\right) \subset \left(0, \frac{1}{n}\right)$.

Now let's prove that $\bigcap\limits_{n=1}^{\infty} A_n = \{x | (\forall n)(x \in A_n)\} = \emptyset$.

Clearly, $0 \notin \bigcap\limits_{n=1}^{\infty} \left(0, \dfrac{1}{n}\right)$ for all $n \in \mathbb{N}$. We will prove that $\forall p \in \mathbb{R} \left[p \notin \bigcap\limits_{n=1}^{\infty} A_n\right]$, where p is meant to be arbitrarily small number.

Assume, on the contrary, $\exists p \in \mathbb{R} \left[p \in \bigcap\limits_{n=1}^{\infty} A_n\right]$.

We will show that there is an interval $A_m = \left(0, \frac{1}{m}\right)$, for some $m \in \mathbb{N}$, such that $p \notin A_m$.

Let's take m such that $\frac{1}{m} < p$, i.e. $m > \frac{1}{p}$. We can always find such m by Archimedean property of real numbers. Then p is upper bound of A_m. Since $lub(A_m) = \frac{1}{m}$ and $\frac{1}{m} < p$ then $p \notin A_m$ (by definition of least upper bound).

Hence, $\exists m \in \mathbb{N} : p \notin A_m$. But then it means that $p \notin \left(A_m \cap \bigcap\limits_{n=1, n\neq m}^{\infty} A_n\right)$ (by definition of intersection), which is the same as $p \notin \bigcap\limits_{n=1}^{\infty} A_n$. But it contradicts our assumption that $\exists p \in \mathbb{R} \left[p \in \bigcap\limits_{n=1}^{\infty} A_n\right]$.

Hence, $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$. $\square$