Task

Prove that for any natural number n,

$$2 + 2^2 + 2^3 + 2^4 + \ldots + 2^n = 2^{n+1} - 2$$

Solution

Task is equavalent to proving the following theorem:

Theorem. $\forall n \in \mathbb{N} : \sum_{i=1}^{n} 2^n = 2^{n+1} - 2$

Proof. By induction.

Initial step (n = 1): $\sum_{i=1}^{1} 2^1 = 2^{1+1} - 2$
$\Rightarrow 2 = 4 - 2$ (by algebra)
$\Rightarrow 2 = 2$, so equality holds.
Induction step:
Assume $\sum_{i=1}^{n} 2^n = 2^{n+1} - 2$.
Then for $n + 1$:
$\sum_{i=1}^{n+1} 2^{n+1} = \sum_{i=1}^{n} 2^n + 2^{n+1}$ (pulling out last member)
$= 2^{n+1} - 2 + 2^{n+1}$ (by induction hypothesis)
$= 2 \cdot 2^{n+1} - 2$
$= 2^{n+2} - 2$.
Thus $\sum_{i=1}^{n+1} 2^{n+1} = 2^{n+2} - 2$, which is induction hypothesis for $n + 1$.

Hence, by principle of induction, $\forall n \in \mathbb{N} : \sum_{i=1}^{n} 2^n = 2^{n+1} - 2$ $\square$