Task
Prove that the only prime triple (i.e. three primes, each 2 from the next) is $3, 5, 7$.

Solution
First, observe that $(1, 3, 5)$ is non-prime since 1 is non-prime. Second, observe that, if a triple contains even number, then it is not prime. So let's consider only triples that contain only odd numbers, starting from triple $(3, 5, 7)$ and give them formal definition:

Definition 1. *A **triple**, denoted T_n, is three odd numbers, each 2 from the next, i.e. $T_n = (2n + 1, n + 3, 2n + 5)$, for any $n \in \mathbb{N}$.*

Definition 2. *A **non-prime triple** is a triple that has at least one non-prime element.*

Here are some examples of triples: $T_1 = (3, 5, 7), T_2 = (5, 7, 9), T_3 = (7, 9, 11), T_4 = (9, 11, 13), T_5 = (11, 13, 15)$ and so on.
Notice how $i - th$ element moves from position 3 to 2 to 1 in 3 consecutive triples:

$$\begin{aligned} T_n &= (2n + 1, 2n + 3, 2n + 5) \\ T_{n+1} &= (2n + 3, 2n + 5, 2n + 7) \\ T_{n+2} &= (2n + 5, 2n + 7, 2n + 9) \end{aligned}$$

If $2n+5$ is non-prime, then both T_n and T_{n+1} and T_{n+2} are non-prime, since $2n + 5$ is present in all these triples.

Definition 3. *Triples T_i and T_j **intersect** if they share at least one common element for all $i, j \in \mathbb{N}$.*

So whenever T_i and T_j intersect, they are equivalent with respect to being non-prime (if one is non-prime then another is also non-prime). Thus we can consider only *non-intersecting* triples, i.e. triples that have no common elements.
We can see that next nearest triple to T_n that does not interesect with it is T_{n+3}, since $T_{n+3} = (2n + 7, 2n + 9, 2n + 11)$, and it doesn't have common elements with T_n.
So the sequence of triples that do not intersect with each other is of this form:

$$[T_n, T_{n+3}, T_{n+6}, ...]$$

, i.e. all triples of the form T_{n+3k} for all $n, k \in \mathbb{Z}, n > 0, k \geq 0$.
We are only interested in such triples that start from (5, 7, 9), so let's give them a formal definition:

Definition 4. *A **non-intersecting triple** is a triple $\tau_k = T_{2+3k}$ for all $k \geq 0$.*

Examples of non-intersecting triples:

$$\tau_0 = T_{2+3\cdot 0} = T_2 = (5, 7, 9)$$
$$\tau_1 = T_{2+3\cdot 1} = T_5 = (11, 13, 15)$$
$$\tau_2 = T_{2+3\cdot 2} = T_8 = (17, 19, 21)$$

To prove the original statement we need to prove that all non-intersecting triples of the form $\tau_k = T_{2+3k}$ for all $k \geq 0$ are non-primes.
But first we need to prove several additional statements.

Definition 5. *τ_k^i is i-th element of non-intersecting triple τ_k*

Lemma 1. *τ_k^3 is divisible by 3 for all $k \geq 0$.*

Proof. By induction:
Initial step ($k = 0$): $\tau_0^3 = (T_{2+3\cdot 0})^3 = (T_2)^3 = (5, 7, 9)^3 = 9$ is divisible by 3.
Induction step: Assume τ_k^3 is divisible by 3, thus it can be expressed in a form $\tau_k^3 = 3n$, for some $n \in \mathbb{N}$.
$\tau_k^3 = (T_{3k+2})^3$ (by definition of τ_k)
$= (2(3k + 2) + 1, 2(3k + 2) + 3, 2(3k + 5) + 5)^3$ (by definition of T_n)
$= (6k + 5, 6k + 7, 6k + 9)^3$ (by algebra)
$= 6k + 9$ (by definition of τ_k^i)
$\tau_{k+1}^3 = (T_{3(k+1)+2})^3$ (by definition of τ_k)
$= (T_{3k+5})^3$ (by algebra)
$= (2(3k + 5) + 1, 2(3k + 5) + 3, 2(3k + 3) + 5)^3$ (By definitionof T_n)
$= (6k + 11, 6k + 13, 6k + 15)^3$ (By algebra)
$= 6k + 15$ (By definition of τ_k^i)
Hence, $\tau_{k+1}^3 = \tau_k^3 + 6$
$= 3n + 6$ (by induction hypothesis)
$= 3(n + 2)$ - this is divisible by 3.
Thus, by principle of mathematical induction, τ_k^3 is divisible by 3 for all $k \in \mathbb{Z}, k \geq 0$. ☐

Lemma 2. *τ_k is non-prime for all $k \geq 0$.*

Proof. By Lemma 1, τ_k^3 is divisible by 3 for all $k \geq 0$. This implies that τ_k^3 is non-prime for all $k \geq 0$ (by definition of prime number).
But then it implies that τ_k is non-prime for all $k \geq 0$ (since τ_k contains τ_k^3 for all k).
Thus, we proved that all non-intersecting triples are non-prime. □

Lemma 3. *For all $n \geq 2$ there exists $k \geq 0$ such that T_n and τ_k have common element. In other words, for each triple starting from $(5, 7, 9)$ there exists a non-intersecting triple that it shares element with.*

Proof. Take arbitrary $T_n = (2n+1, 2n+3, 2n+5)$ (by definition of T_n).
We need to find k such that $n = 2 + 3k$, i.e. $3k = n - 2$.
By division theorem, n can be expressed in one of these forms: $3m, 3m+1, 3m+2$ for some $m \geq 1$.
Case $n = 3m$: $T_n = T_{3m} = (6m+1, 6m+3, 6m+5)$
Case $n = 3m+1$: $T_n = T_{3m+1} = (6m+3, 6m+5, 6m+7)$
Case $n = 3m+2$: $T_n = T_{3m+2} = (6m+5, 6m+7, 6m+9)$
In each of three cases, take $k = m$.
Then $\tau(m) = T_{3m+2} = (6m+5, 6m+7, 6m+9)$
We can observe that $6m+5$ is shared by τ_m and T_n for each of three cases.
Since we have exhausted all cases, we proved that for all $n \geq 2$ there exists $k \geq 0$ such that T_n and τ_k have common element. □

Lemma 4. *All triples T_n where $n \in \mathbb{N}, n \geq 2$ are non-prime.*

Proof. By Lemma 2 all non-intersecting triples are non-prime, and by Lemma 3 any other triple T_n where $n \geq 2$ shares at least one common element with some non-intersecting triple.
Thus all triples T_n where $n \geq 2$ are non-prime.

□

Now we are in a position to prove the original proposition:

Original proposition. *For all $n \in \mathbb{N}$ among all triples of the form $(n, n+2, n+4)$ there exists only one prime triple, namely, $(3, 5, 7)$.*

Proof. Since even numbers are not prime, all triples of the form $(2n, 2n+2, 2n+4), n \in \mathbb{N}$, are non-prime. By Lemma 4, all triples of the form $(2n+1, 2n+3, 2n+5), n \in \mathbb{N}, n \geq 2$, are non-prime. Hence we are left with two triples: $(1, 3, 5)$ and $(3, 5, 7)$. Triple $(1, 3, 5)$ is non-prime since 1 is non-prime.
Hence, the only prime triple is $(3, 5, 7)$. This consitutes the proof of the original proposition. □