Task

Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Solution

Task is equavalent to proving/disproving the following theorem:

Theorem. $\forall n \in \mathbb{Z}, n > 0 : \left(5 \mid \sum_{i=0}^{4} (n+i)\right) \wedge \left(5 \mid \sum_{i=0}^{4} -(n+i)\right).$

Proof. By induction.

Initial step (n = 0): $\left(5 \mid \sum_{i=0}^{4} (0+i)\right) \wedge \left(5 \mid \sum_{i=0}^{4} -(0+i)\right)$

$= (5 \mid 10) \wedge (5 \mid -10)$ - this is true.

Induction step: Assume $\left(5 \mid \sum_{i=0}^{4} (n+i)\right) \wedge \left(5 \mid \sum_{i=0}^{4} -(n+i)\right).$

Then for $n+1$:

$\left(5 \mid \sum_{i=0}^{4} ((n+1)+i)\right) \wedge \left(5 \mid \sum_{i=0}^{4} -((n+1)+i)\right)$

$= \left(5 \mid \left(\sum_{i=0}^{4} (n+i) + \sum_{i=0}^{4} 1\right)\right) \wedge \left(5 \mid \left(\sum_{i=0}^{4} -(n+i) + \sum_{i=0}^{4} -1\right)\right)$

$= \left(5 \mid \left(\sum_{i=0}^{4} (n+i) + 5\right)\right) \wedge \left(5 \mid \left(\sum_{i=0}^{4} -(n+i) - 5\right)\right).$

Since, by induction hypothesis, $5 \mid \sum_{i=0}^{4} (n+i)$, then $\sum_{i=0}^{4} (n+i) = 5k$ for some $k \in \mathbb{Z}$. Similarly, $\sum_{i=0}^{4} -(n+i) = 5l$ for some $l \in \mathbb{Z}$.

Then we have $\left(5 \mid \left(\sum_{i=0}^{4} (n+i) + 5\right)\right) \wedge \left(5 \mid \left(\sum_{i=0}^{4} -(n+i) - 5\right)\right)$

$= (5 \mid (5k+5)) \wedge (5 \mid (5l-5))$

$= (5 \mid (5(k+1))) \wedge (5 \mid (5(l-1))$, which is obviously true.

Thus, by principle of induction,

$\forall n \in \mathbb{Z}, n > 0 : \left(5 \mid \sum_{i=0}^{4} (n+i)\right) \wedge \left(5 \mid \sum_{i=0}^{4} -(n+i)\right),$

which consitutes the proof of original claim. $\square$