Task

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

Solution

Task is equavalent to proving the following theorem:

Theorem. $\forall M \in \mathbb{R}, M > 0 : \lim\limits_{n\to\infty} \{a_n\}_{n=1}^{\infty} = L \Rightarrow \lim\limits_{n\to\infty} \{Ma_n\}_{n=1}^{\infty} = ML.$

Proof. Let's take arbitrary $M \in \mathbb{R}, M > 0$. By definition of limit,
$(\forall \epsilon \in \mathbb{R}, \epsilon > 0)\,(\exists n \in \mathbb{N})\,(\forall m > n)\,[|a_n - L| < \epsilon]$.
Let's take arbitrary $\delta \in \mathbb{R}, \delta > 0$ and take $\epsilon = \frac{\delta}{M}$. Then, for this ϵ,
$(\exists n \in \mathbb{N})\,(\forall m > n)\,[|a_n - L| < \epsilon]$
$\Rightarrow (\exists n \in \mathbb{N})\,(\forall m > n)\,[M\,|a_n - L| < M\epsilon]$ (multiplying by $M > 0$)
$\Rightarrow (\exists n \in \mathbb{N})\,(\forall m > n)\,[|Ma_n - ML| < M\epsilon]$
$\Rightarrow (\exists n \in \mathbb{N})\,(\forall m > n)\,\left[|Ma_n - ML| < M\left(\frac{\delta}{M}\right)\right]$ (by definition of ϵ)
$\Rightarrow (\exists n \in \mathbb{N})\,(\forall m > n)\,[|Ma_n - ML| < \delta]$.
Since δ was taken arbitrarily, $(\forall \delta \in \mathbb{R}, \delta > 0)\,(\exists n \in \mathbb{N})\,(\forall m > n)\,[|Ma_n - ML| < \delta]$.
But this is precisely definition of $\lim\limits_{n\to\infty} \{Ma_n\}_{n=1}^{\infty} = ML$.
Hence, $\forall M \in \mathbb{R}, M > 0 : \lim\limits_{n\to\infty} \{a_n\}_{n=1}^{\infty} = L \Rightarrow \lim\limits_{n\to\infty} \{Ma_n\}_{n=1}^{\infty} = ML$.

$\square$