Digital Signal Processing

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EP20BTECH11011

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/dnshkmr7/EE3900
    −22/blob/main/Assignment%201/files/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

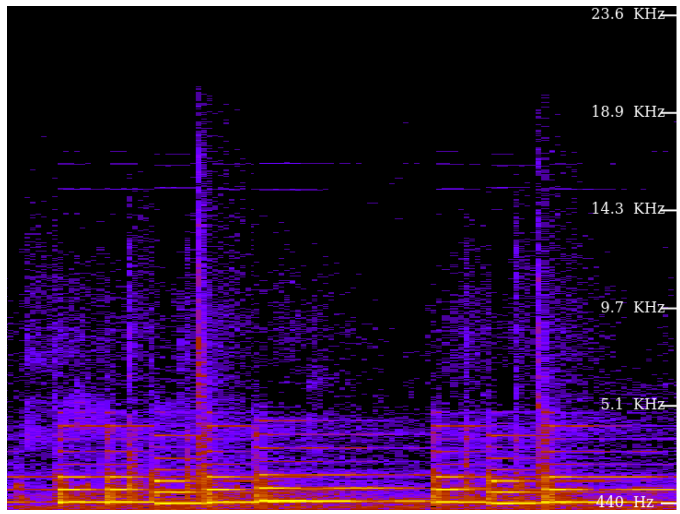

Fig. 2.2

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav
    ')
#sampling frequency of Input signal
sampl_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2∗cutoff_freq/sampl_freq
# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
```

```
#output signal = signal.lfilter(b, a,input signal
    )
#write the output signal into .wav file
sf.write('Sound With ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

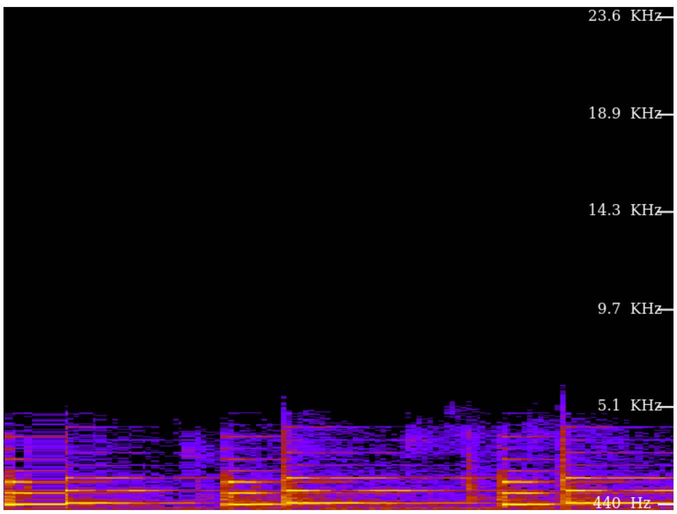

Fig. 2.4

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.3.

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
k = 20
y = np.zeros(20)

y[0] = x[0]
y[1] = -0.5*y[0]+x[1]

for n in range(2,k-1):
        if n < 6:
                y[n] = -0.5*y[n-1]+x[n]+x
                    [n-2]
        elif n > 5 and n < 8:
                y[n] = -0.5*y[n-1]+x[n-2]
        else:
                y[n] = -0.5*y[n-1]
print(y)

#subplots
plt.subplot(2, 1, 1)
plt.stem(range(0,6),x)
plt.title('Digital Filter Input-Output')
plt.ylabel('$x(n)$')
plt.grid()# minor

plt.subplot(2, 1, 2)
plt.stem(range(0,k),y)
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

#If using termux
plt.savefig('./figs/xnyn.pdf')
plt.savefig('./xnyn.eps')
subprocess.run(shlex.split("termux-open ./figs
    /xnyn.pdf"))
#else
#plt.show()
```

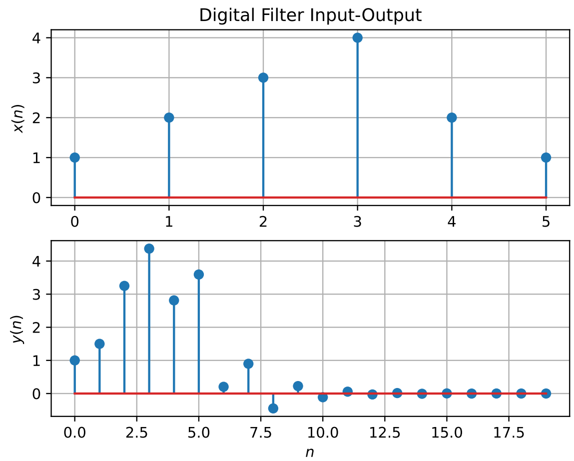

Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution: The following code yields Fig. 3.3.

```
wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section3_EP20B11011.c
```

```
wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section3_EP20B11011.py
```

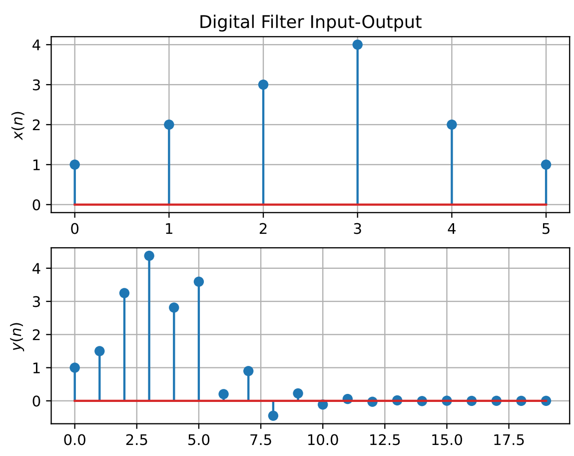

Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 4.1.
Solution:

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \tag{4.8}$$

$$x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.20}$$

$$= \sum_{n=-\infty}^{\infty} u(n)(az^{-1})^n \tag{4.21}$$

$$= \sum_{n=-\infty}^{\infty} (az^{-1})^n, \quad \left|az^{-1}\right| < 1 \tag{4.22}$$

(4.23)

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z| \tag{4.24}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.25}$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The graph is symmetric and periodic it is attending high of value 4 and minimum between (0 - 0.5). It is bounded between (0, 4) and periodic with period (2π) because in the below equation $\cos(\omega)$ is periodic function having period 2π

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.26}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.27}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.28}$$

$$= \frac{|1 + \cos 2\omega + j\sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.29}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \tag{4.30}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \tag{4.31}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.32}$$

The following code plots Fig. 4.6.

```
import numpy as np
import matplotlib.pyplot as plt
#if using termux
import subprocess
import shlex
#end if

#DTFT
def H(z):
        num = np.polyval([1,0,1],z**(−1))
        den = np.polyval([0.5,1],z**(−1))
        H = num/den
        return H

#Input and Output
omega = np.linspace(−3*np.pi,3*np.pi,100)

#subplots
plt.plot(omega, abs(H(np.exp(1j*omega))))
plt.title('Filter Frequency Response')
plt.xlabel('$\omega$')
plt.ylabel('$|H(e^{\jmath\omega})| $')
plt.grid()# minor

#if using termux
plt.savefig('./figs/dtft.pdf')
```

```
plt.savefig('./figs/dtft.eps')
subprocess.run(shlex.split("termux-open ./figs
    /dtft.pdf"))
#else
#plt.show()
```

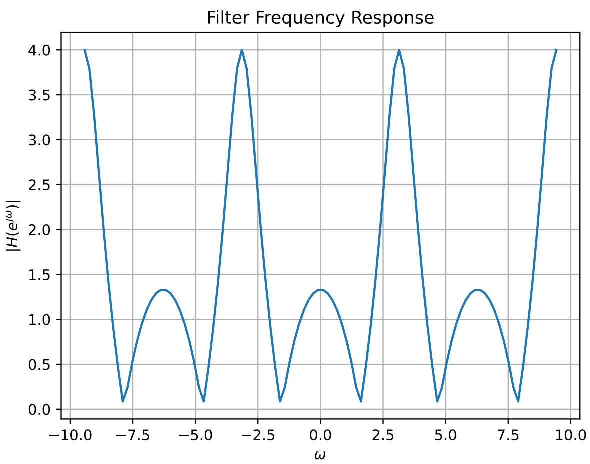

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $x(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.
Solution:

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.33}$$

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \tag{4.34}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega n} e^{j\omega k} d\omega \tag{4.35}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n)\, 2\pi \tag{4.36}$$

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = 2\pi h(n) \tag{4.37}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega k} d\omega = h(n) \tag{4.38}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).

Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Let $z^{-1} = x$,then, by polynomial long division we get

$$\begin{array}{r} 2x - 4 \\ \tfrac{1}{2}x + 1 \overline{\big) \; x^2 \qquad + 1} \\ -x^2 - 2x \\ \hline -2x + 1 \\ 2x + 4 \\ \hline 5 \end{array}$$

$$\implies (1 + z^{-2}) = (\frac{1}{2}z^{-1} + 1)(2z^{-1} - 4) + 5 \tag{5.3}$$

$$\implies \frac{(1 + z^{-2})}{\frac{1}{2}z^{-1} + 1} = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.4}$$

$$\implies H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.5}$$

Now, consider $\frac{5}{\frac{1}{2}z^{-1}+1}$
The denominator $\frac{1}{2}z^{-1} + 1$ can be expressed as sum of an infinite geometric progression, which as its first term equal to 1 and common ratio $\frac{-1}{2}z^{-1}$
Therefore, we can write $\frac{5}{\frac{1}{2}z^{-1}+1}$ as
$5\left(1 + \left(\frac{-1}{2}z^{-1}\right) + \left(\frac{-1}{2}z^{-1}\right)^2 + \left(\frac{-1}{2}z^{-1}\right)^3 + \left(\frac{-1}{2}z^{-1}\right)^4 + \dots\right)$
Therefore, H(z) can be given by,

$$H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.6}$$

$$\tag{5.7}$$

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.8}$$

$$\implies H(z) = 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.9}$$

Comparing the above expression to (4.1) we get h(n) for n<5 as,

$$h(0) = 1 \tag{5.10}$$
$$h(1) = \frac{-1}{2} \tag{5.11}$$
$$h(2) = \frac{5}{4} \tag{5.12}$$
$$h(3) = \frac{-5}{8} \tag{5.13}$$
$$h(4) = \frac{5}{16} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: Yes, it is bounded and convergent. We can clearly see the plot is remaining finite. The following code plots Fig. 5.3.

```
import numpy as np
import matplotlib.pyplot as plt
#if using termux
import subprocess
import shlex
#end if

n = np.arange(10)
fn=(−1/2)∗∗n
hn1=np.pad(fn, (0,2), 'constant',
     constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
     constant_values=(0))
plt.stem(np.arange(12), hn1+hn2)
plt.title('Filter Impulse Response')
plt.xlabel('$n$')
plt.ylabel('$h(n)$')
plt.grid()# minor

#if using termux
plt.savefig('./figs/hn.pdf')
plt.savefig('./figs/hn.eps')
subprocess.run(shlex.split("termux−open ./figs
     /hn.pdf"))
#else
#plt.show()
```

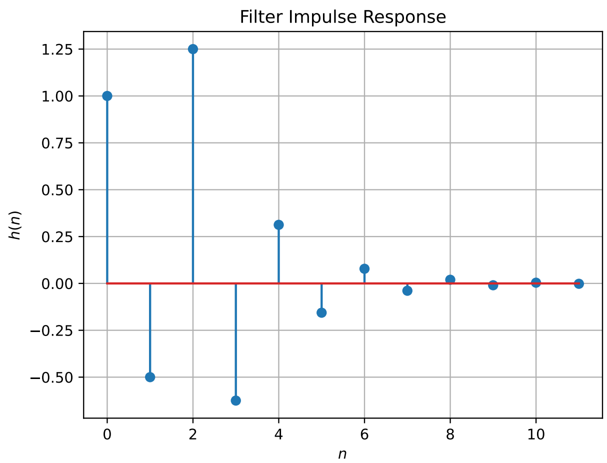

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

we know that

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.18}$$

Implies we can write that

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \ge 2 \end{cases} \tag{5.19}$$

A sequence is said to be bounded when

$$|x_n| \le M, \forall n \in \mathcal{N} \tag{5.20}$$

Now consider (5.19),
For n<0,

$$|h(n)| \le 0 \tag{5.21}$$

For $0 \le n < 2$,

$$|h(n)| = (\frac{1}{2})^n \tag{5.22}$$
$$\implies |h(n)| \le 1 \tag{5.23}$$

For $n \geq 2$,

$$|h(n)| = 5(\frac{1}{2})^n \tag{5.24}$$
$$\implies |h(n)| \leq 5 \tag{5.25}$$

From above we can say that,

$$M = max\{0, 1, 5\} \tag{5.26}$$
$$= 5 \tag{5.27}$$

Therefore since M exists and is a real value, we can say that h(n) is bounded.

5.4 Convergent? Justify using the ratio test.
Solution: We see that $h(n)$ is bounded. For large n, we see that

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \tag{5.28}$$
$$= \left(-\frac{1}{2}\right)^n (4+1) = 5\left(-\frac{1}{2}\right)^n \tag{5.29}$$
$$\implies \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} \tag{5.30}$$

and therefore, $\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1$. Hence, we see that $h(n)$ converges.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.31}$$

Is the system defined by (3.2) stable for the impulse response in (5.15)?
Solution: By using h(n) from 5.3

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.32}$$
$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.33}$$
$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.34}$$
$$= \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.35}$$
$$\tag{5.36}$$
$$= \frac{2}{3} + \frac{2}{3} < \infty \tag{5.37}$$

5.6 Verify the above result using a python code.
Solution:

```
wget https://github.com/dnshkmr7/EE3900
    −22/blob/main/Assignment%201/codes/
    Section5_EP20B11011.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.38}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

$$= h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \tag{5.39}$$
$$= H(z) + \frac{1}{2}z^{-1}H(z) = 1 + z^{-2} \tag{5.40}$$
$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.41}$$
$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.42}$$

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

k = 12
h = np.zeros(k)
h[0] = 1
h[1] = −0.5*h[0]
h[2] = −0.5*h[1] + 1

for n in range(3,k−1):
        h[n] = −0.5*h[n−1]

#subplots
plt.stem(range(0,k),h)
plt.title('Impulse Response Definition')
plt.xlabel('$n$')
plt.ylabel('$h(n)$')
plt.grid()# minor

#If using termux
plt.savefig('./figs/hndef.pdf')
subprocess.run(shlex.split("termux−open ./figs
    /hndef.pdf"))
```

```
#else
#plt.show()
```

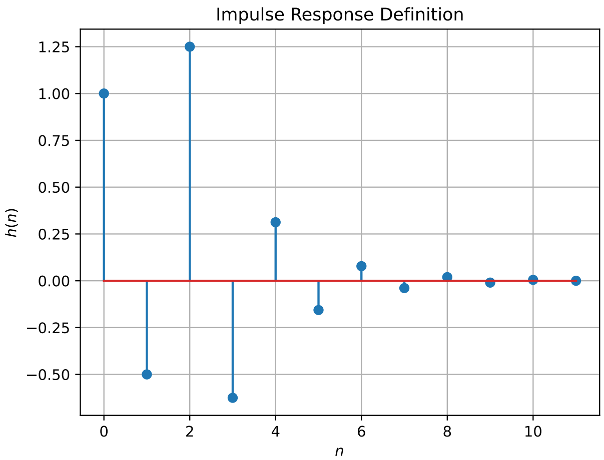

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.43)$$

Comment. The operation in (5.43) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.3.

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if

n = np.arange(14)
fn=(−1/2)**n
hn1=np.pad(fn, (0,2), 'constant',
     constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
     constant_values=(0))
h = hn1+hn2

nh=len(h)
x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
nx = len(x)

y = np.zeros(nx+nh−1)

for k in range(0,nx+nh−1):
        for n in range(0,nx):
                if k−n >= 0 and k−n < nh:
                        y[k]+=x[n]*h[k−n]


print(y)
#plots
plt.stem(range(0,nx+nh−1),y)
plt.title('Filter Output using Convolution')
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

#If using termux
plt.savefig('./figs/ynconv.pdf')
plt.savefig('./figs/ynconv.eps')
subprocess.run(shlex.split("termux−open ./figs
    /ynconv.pdf"))
#else
#plt.show()
```

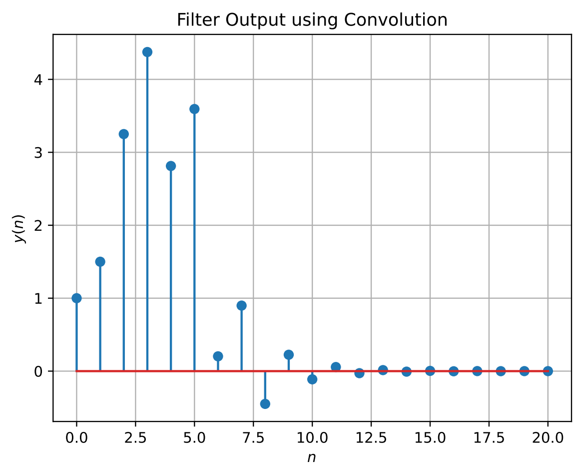

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution:
We know that from, (5.43),

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.44)$$

This can also be wrtten as a matrix-vector multiplication given by the expression,

$$y = T(h) * x \quad (5.45)$$

In the equation (5.45), $T(h)$ is a Teoplitz matrix.
The equation (5.45) can be expanded as,

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.46}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & 0 \\ h_n & h_{n-1} & h_{n-2} & . & . & h_1 \\ 0 & h_n & h_{n-1} & h_{n-2} & . & h_2 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & h_{n-1} \\ 0 & . & . & . & 0 & h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{5.47}$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.48}$$

Solution: From (5.43), we substitute $k := n-k$ to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.49}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.50}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.51}$$

6 DFT and FFT

6.1 Compute

$$X(k) \overset{\triangle}{=} \sum_{n=0}^{N-1} x(n) e^{-\mathrm{j}2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution:
We know that ,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{6.2}$$

Here, let, $\omega = e^{-j2\pi k}$. Then,

$$X(k) = 1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega \tag{6.3}$$

Similarly, we know from (5.19),

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \ge 2 \end{cases} \tag{6.4}$$

Now,again let, $\omega = e^{-j2\pi k}$. Then,

$$H(k) = 1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega \tag{6.5}$$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.6}$$

Solution:
Now, from (6.3) and (6.5), we know $X(k) and H(k)$. Now, given that,

$$Y(k) = X(k) * H(k) \tag{6.7}$$

$$Y(k) = (1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega) * \\ (1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega) \tag{6.8}$$

$$Y(k) = 1 + \frac{3}{2}\omega^{\frac{1}{5}} + \frac{13}{4}\omega^{\frac{2}{5}} + \frac{35}{8}\omega^{\frac{3}{5}} + \frac{45}{16}\omega^{\frac{4}{5}} \\ \frac{115}{32}\omega^{\frac{5}{5}} + \frac{1}{8}\omega^{\frac{6}{5}} + \frac{25}{32}\omega^{\frac{7}{5}} - \frac{5}{8}\omega^{\frac{8}{5}} \\ - \frac{5}{32}\omega^{5} \tag{6.9}$$

where, $\omega = e^{-j2k\pi}$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\mathrm{j}2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.10}$$

Solution: The following code plots Fig. 5.8 and computes $X(k)$ and $Y(k)$. Note that this is the same as $y(n)$ in Fig. 3.3.

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if
```

```
N = 14
n = np.arange(N)
fn=(-1/2)**n
hn1=np.pad(fn, (0,2), 'constant',
     constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
     constant_values=(0))
h = hn1+hn2

xtemp=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
x=np.pad(xtemp, (0,8), 'constant',
     constant_values=(0))

X = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
          for n in range(0,N):
                    X[k]+=x[n]*np.exp(-1j*2*
                         np.pi*n*k/N)

H = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
          for n in range(0,N):
                    H[k]+=h[n]*np.exp(-1j*2*
                         np.pi*n*k/N)

Y = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
          Y[k] = X[k]*H[k]

y = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
          for n in range(0,N):
                    y[k]+=Y[n]*np.exp(1j*2*np
                         .pi*n*k/N)

#print(X)
y = np.real(y)/N
#plots
plt.stem(range(0,N),y)
plt.title('Filter Output using DFT')
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor
#
#If using termux
plt.savefig('./figs/yndft.pdf')
plt.savefig('./figs/yndft.eps')
subprocess.run(shlex.split("termux-open ./figs
     /yndft.pdf"))
#else
#plt.show()
```

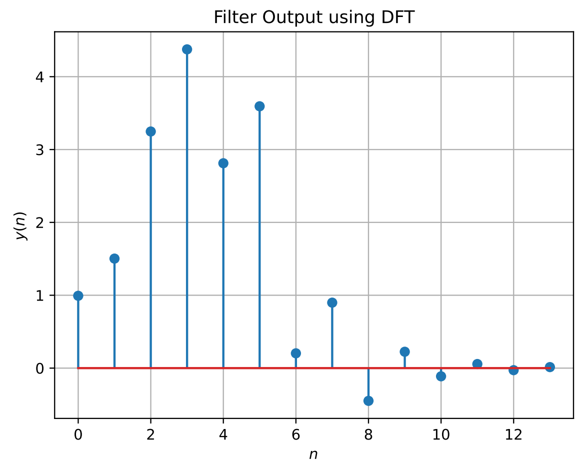

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT. **Solution:** Download the code from

```
wget https://github.com/dnshkmr7/EE3900
     -22/blob/main/Assignment%201/codes/
     Section6_EP20B11011.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.3).

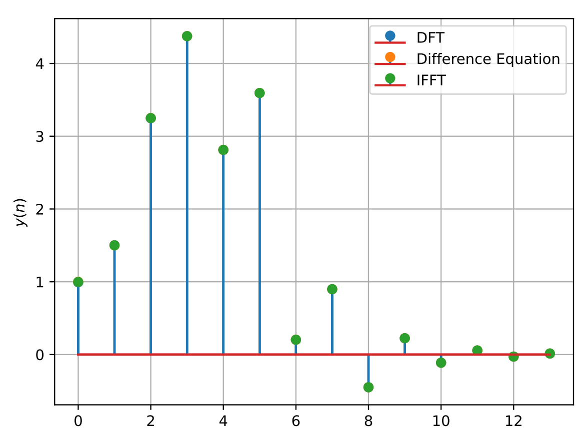

Fig. 6.4: $y(n)$ using FFT and IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_4^0 & W_N^1 & W_N^2 & W_N^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: We write

$$W_N^2 = \left(e^{-\frac{j2\pi}{N}}\right)^2 = e^{-\frac{j2\pi}{N/2}} = W_{N/2} \tag{7.8}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.9}$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (**??**),

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.10}$$

$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.11}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.12}$$

$$\implies -\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.13}$$

and

$$\mathbf{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.14}$$

$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.15}$$

Hence,

$$\mathbf{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.16}$$

$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2 F_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2 F_2 \end{bmatrix} \tag{7.17}$$

$$= \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.18}$$

Multiplying (7.18) by $\mathbf{P}_4$ on both sides, and noting that $\mathbf{W}_4\mathbf{P}_4 = \mathbf{F}_4$ gives us.

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.19}$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$, from (7.12) and (7.13),

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.20}$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.21}$$

Thus,

$$\begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.22}$$

and so,

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} = \mathbf{F}_N \tag{7.23}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.24}$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4\end{pmatrix}\begin{pmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{pmatrix} = \begin{pmatrix}x(0)\\x(2)\\x(1)\\x(3)\end{pmatrix} \tag{7.25}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.26}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.27}$$

$$X(1) = x(0) + x(1)e^{-\frac{j2\pi}{N}} + \ldots + + x(N-1)e^{-\frac{j2(N-1)\pi}{N}} \tag{7.28}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{j2(N-1)\pi}{N}} + \ldots + + x(N-1)e^{-\frac{j2(N-1)(N-1)\pi}{N}} \tag{7.29}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{j2mn\pi}{N}} \tag{7.30}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.31}$$

where $\mathbf{F}_N = \left[e^{-\frac{-j2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix}X(4)\\X(5)\\X(6)\\X(7)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.33}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0)\\X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix}X_1(2)\\X_1(3)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix}X_2(0)\\X_2(1)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix}X_2(2)\\X_2(3)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.37}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix} = \begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.38}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix} = \begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.39}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} = \begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.40}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} = F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} = F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} = F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} = F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.44}$$

Solution: We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n)e^{-\frac{j2kn\pi}{8}} \tag{7.45}$$

$$= \sum_{n=0}^{3}\left(x(2n)e^{-\frac{j2kn\pi}{4}} + e^{-\frac{j2k\pi}{8}}x(2n+1)e^{-\frac{j2kn\pi}{4}}\right) \tag{7.46}$$

$$= X_1(k) + e^{-\frac{j2k\pi}{4}}X_2(k) \tag{7.47}$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.48}$$
$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.49}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.50}$$
$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.51}$$
$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.52}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.53}$$
$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.54}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.55}$$
$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.56}$$

But observe that from (7.25),

$$\mathbf{P}_8 \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{7.57}$$
$$\mathbf{P}_4 \mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \tag{7.58}$$
$$\mathbf{P}_4 \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \end{pmatrix} \tag{7.59}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.60}$$

compute the DFT using (7.26)
Solution: Download the Python code from

```
$ wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section7_1_EP20B11011.py
```

12. Repeat the above exercise using the FFT after zero padding $\mathbf{x}$.
13. Write a C program to compute the 8-point FFT.
Solution: The C code for the above two problems can be downloaded from

```
$ wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section7_2_EP20B11011.c
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b,a,input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

```
wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section8_1_EP20B11011.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.

Solution: For the given values, the difference equation is

$$
\begin{aligned}
& y(n) - (4.44)\, y(n-1) + (8.78)\, y(n-2) \\
& - (9.93)\, y(n-3) + (6.90)\, y(n-4) \\
& - (2.93)\, y(n-5) + (0.70)\, y(n-6) \\
& - (0.07)\, y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\
& + \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\
& + \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\
& + \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\
& + \left(5.02 \times 10^{-5}\right) x(n-7) \qquad (8.2)
\end{aligned}
$$

From (8.1), we see that the transfer function can be written as follows

$$
H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \qquad (8.3)
$$

$$
= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \qquad (8.4)
$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.19),

$$
h(n) = \sum_i r(i) [p(i)]^n u(n) + \sum_j k(j) \delta(n - j) \qquad (8.5)
$$

Substituting the values,

$$
\begin{aligned}
h(n) = & \, [(2.76)\,(0.55)^n \\
& + (-1.05 - 1.84\jmath)\,(0.57 + 0.16\jmath)^n \\
& + (-1.05 + 1.84\jmath)\,(0.57 - 0.16\jmath)^n \\
& + (-0.53 + 0.08\jmath)\,(0.63 + 0.32\jmath)^n \\
& + (-0.53 - 0.08\jmath)\,(0.63 - 0.32\jmath)^n \\
& + (0.20 + 0.004\jmath)\,(0.75 + 0.47\jmath)^n \\
& + (0.20 - 0.004\jmath)\,(0.75 - 0.47\jmath)^n] u(n) \\
& + \left(-6.81 \times 10^{-4}\right) \delta(n) \qquad (8.6)
\end{aligned}
$$

Download the code from

```
wget https://github.com/dnshkmr7/EE3900
    -22/blob/main/Assignment%201/codes/
    Section8_2_EP20B11011.py
```

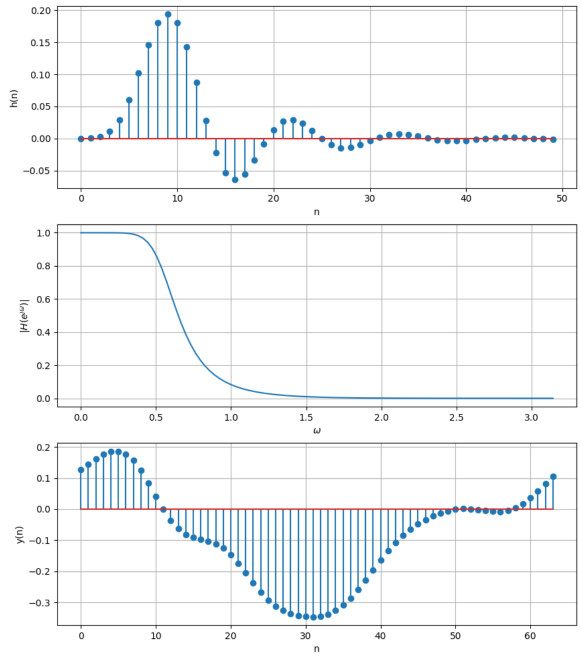

Fig. 8.2: Plot of $h(n)$, filter frequency response & $y(n)$

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: Setting the order of the filter to be 7 gives us the best possible output.