EP3227 - Project Report
Analysis of Nonlinear Dynamical Systems: Logistic Map

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Abstract

Chaos theory is a branch of mathematics that deals with nonlinear dynamical systems. These systems may contain very few controlling parameters and may follow very simple rules, but they all have a very sensitive dependence on their initial conditions. Despite their deterministic simplicity, over time these systems can produce totally unpredictable and chaotic behaviour. This report will be discussing the chaos that arises from the logistic map and how this seemingly simple equation can be found everywhere from population to biology to fluid convection and even a dripping faucet.

Keywords
Nonlinear Dynamical Systems, Chaos Theory, Logistic Map

1 Logistic Map

The logistic equation is a differential equation that treats time as continuous. The logistic map instead uses a nonlinear difference equation to look at discrete time-steps or generations. It's called the logistic map because it maps the population value at any time-step to its value at the next time-step. [1]

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

Here x represents the percentage of the maximum population at a certain epoch n, while , r represents the rate of growth. The term $(1 - x)$ represents the limitations of the system. If the growth rate is low, the population will eventually diminish. If the growth rate is high, the population may stabilize or experience fluctuations/chaotic behavior for specific growth rates.

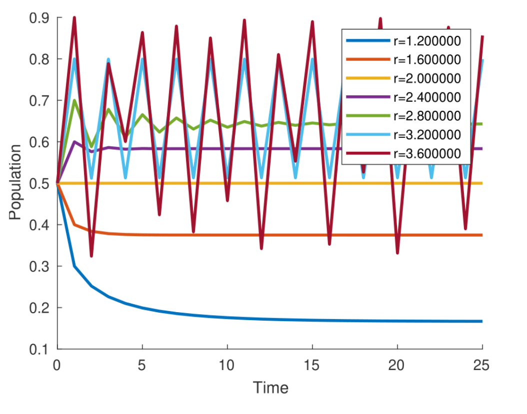

Figure 1: MATLAB simulation was used to iterate the map for 25 generations with r varying from 1.2 to 3.6 (step size: 0.4) and an initial population of 0.5. The results were visualized as a line chart.[2, 3]

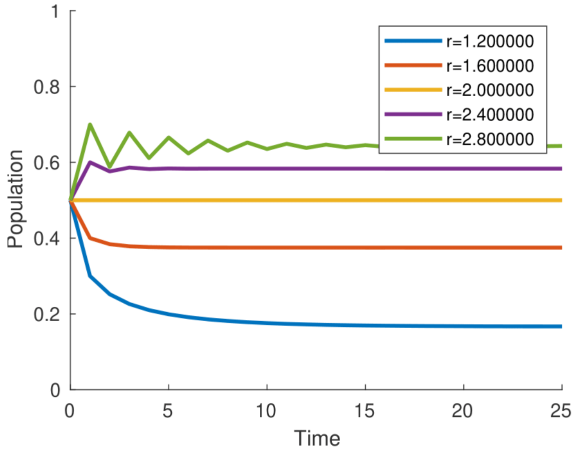

Figure 2: We observe as growth rate changes from 1.2 to 2.8 the population always ends up stabilising at a fixed value (depends on r) after some generations. In fact, for any growth rate less than 3 population always reach equilibrium and this is also proven true as this is what we observe in the population of the wild.

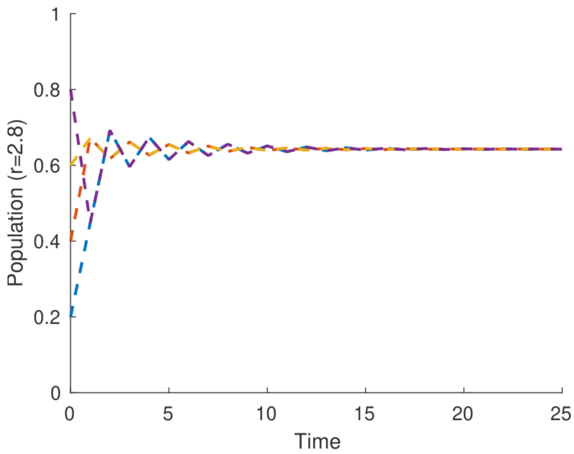

Figure 3: It has been established that growth rate affects the equilibrium population, but the effect of initial population on the equilibrium/stable population hasn't been confirmed. Again, MATLAB was used to this time to run the logistic map for 25 generations with population varying from 0.2 to 0.8 in steps of 0.2 with growth rate being 2.8. The values from the iterations are visualised as line chart.

Observing the results, we can confidently say that the initial population does not have an effect on the stable population which has a fixed growth rate. Coming back to varying the growth rate. If the growth rate is varied 3 and above we start to observe that the population never reaches a single stable population instead it goes generation cycles between multiple populations until growth rate reaches 3.57.

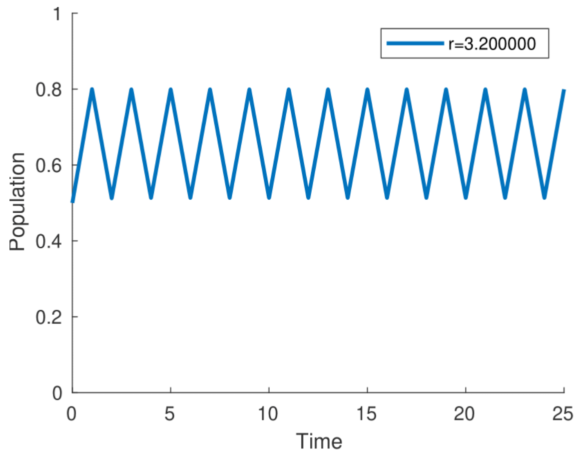

Figure 4: For $r = 3.2$, it is observed that the population value oscillates back and forth between 2 values.

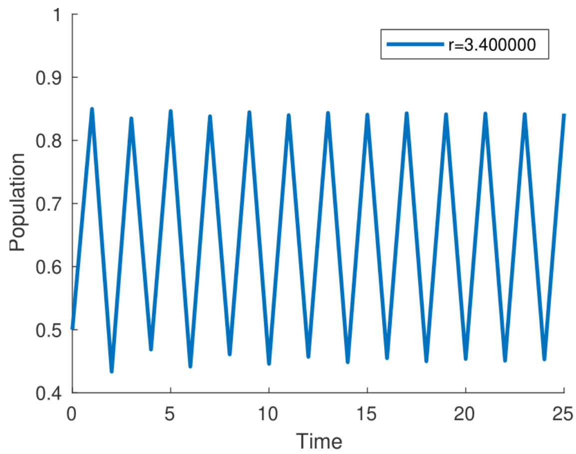

Figure 5: For $r = 3.4$, instead of the population value oscillating back and forth between 2 values population goes through a 4 generation cycle before repeating.

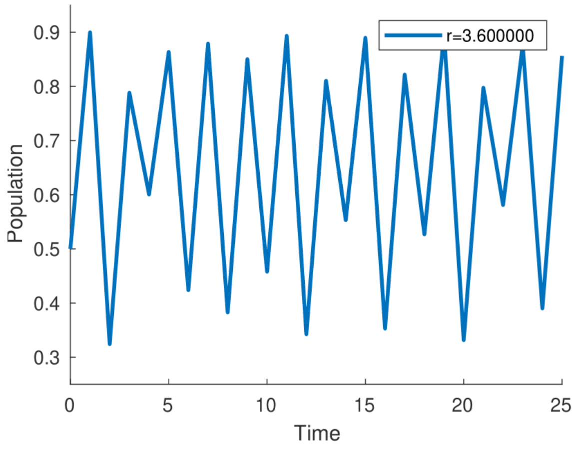

Figure 6: As r is increased population goes through an 8 generation cycle then 16 then 32, 64 and when r finally is 3.57 chaos ensues. There is no sign of repeating cycles, and it never even hits the same point twice except when order arrives amidst chaos.

2 Bifurcation & the path to Chaos

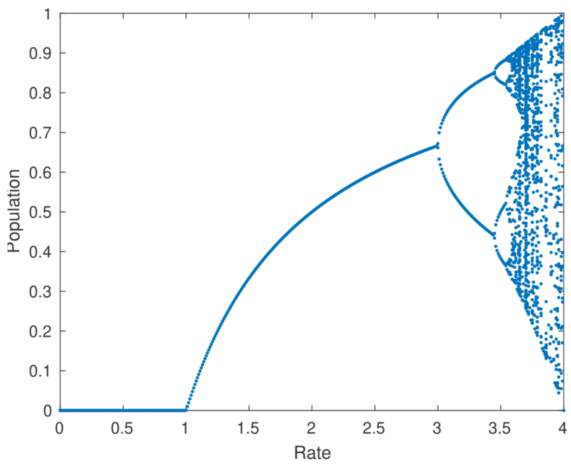

Figure 7: A graph is plotted with r on the x-axis and population on the y-axis for 2000 generations across 400 growth rates between 0 and 4 with initial population being 0.5, and we will be visualising this data using a bifurcation diagram.

The nature of stability of logistic map from $r = 0$ to $r = 3.57$ has been discussed previously. The study of logistic map as chaos ensues is continued. It was previously mentioned that order arrives amidst chaos, and it can be observed.

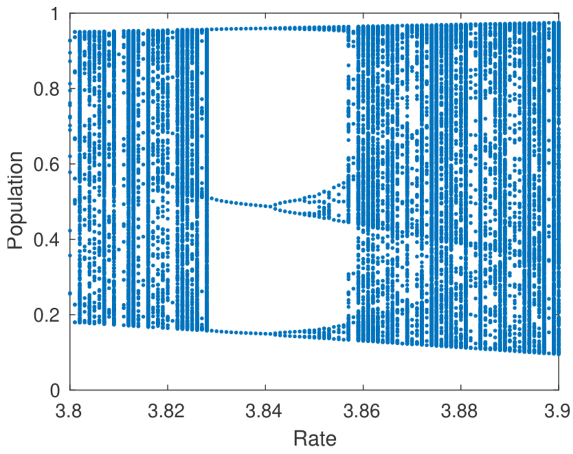

Figure 8: For example, around $r = 3.83$ there is a stable cycle with a period of 3 generations and there are these windows of stable periodic behaviour observed everywhere on the path to chaos in bifurcation diagram. Again, similar to what happened earlier this period 3 cycle splits into 6,12,24 and so on, and then returns to chaos. In fact, this bifurcation diagram contains period cycles of every length. By the time we cross growth rate 3.86, it has bifurcated so many times that the system now appears to go through random population values, and this is stated because if one did know the exact initial conditions they can be calculated and determined. So, this model is pseudo-random, and this behaviour is classified as chaos: deterministic and aperiodic.[4]

3 Fractals in Bifurcation Diagram

We notice large scale features being repeated on the smaller scale as we zoom in to observe r=3.83. This bifurcation indeed is a fractal. Fractals are self-similar, meaning that they have the same structure at every scale. As you zoom in on them, you find smaller copies of the larger

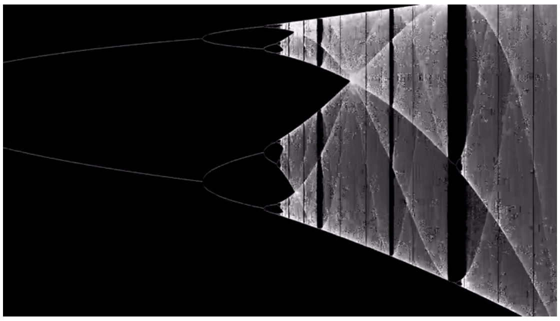

Figure 9: An animated GIF of zooming into bifurcation.[5]

macro-structure. Here, at this fine scale, you can see a tiny reiteration of the same bifurcations, chaos, and limit cycles we saw in the first bifurcation diagram of the full range of growth rates.

Physicist Mitchell Feigenbaum's study on the fractals in bifurcation diagram and chaos theory as a whole gave rise to a new constant. The Feigenbaum constant which is defined as[3]:

$$\delta \approx 4.6692$$

He found this constant by dividing the width of each bifurcation section with the width of next bifurcation section.

$$\lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \tag{2}$$

4 Logistic map in real-world

A major experimental confirmation of nonlinear behaviour in natural phenomena came from Lichchaber where they induced convection inside mercury in a controlled setup and observed period doubling behaviour of temperature spikes in mercury.

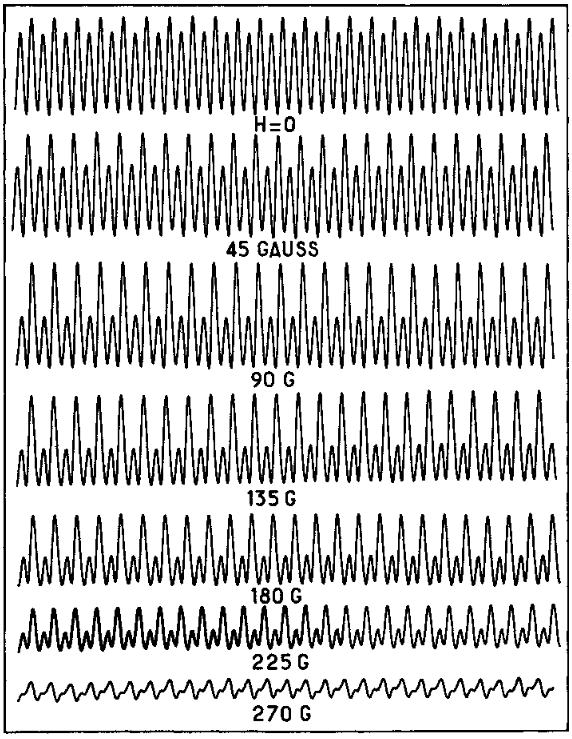

Figure 10: Period doubling in temperature spikes measure in mercury.[6]

Scientists studied the response of human eyes to flickering of lights. Period doubling was found there as well as human eyes went from responding to every flicker to every other flicker to once in 4 flickers and so on.

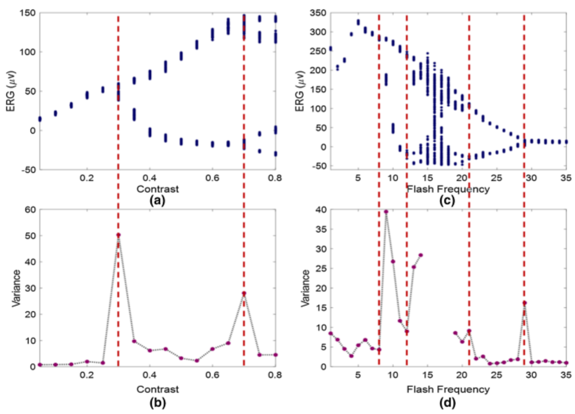

Figure 11: Bifurcation seen in response time measured against flash frequency & contrast.[7, 8]

Atrial fibrillation/irregular heartbeat which was thought to be irregular was actually found to be a nonlinear dynamic phenomenon. On the path to fibrillation scientists observed period doubling of heartbeat before it became irregular. They applied chaos theory to give controlled electrical shocks to the heart at the right moment so that they bring back stability in heartbeat and make it normal. This study helped scientists to reduce danger and improve the success rate of this procedure to a great extent.[9]

Other real world nonlinear systems include leaky faucets, firing up of neurons, random number generators, social sciences, and urban planning.[10]

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