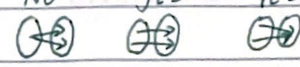
Unit 1: Functions

NO YES YES



Function - each x has one y

- VLT - Vertical Line Test - if crosses more than one point it is not a function (∵ pass/fail ∴ is/is not a f^n)

$f(x) = 2x+1$
$g(x) = x^2+3x+1$

Function Notation

ⓐ $f(g(y)) = f(y^2+3y+1)$
$= 2(y^2+3y+1)+1$
$= 2y^2+6y+3$

$y = f(x)$ } used in place of y in an equation

output ← y, x → input

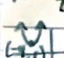
Domain & Range

Domain - set of all x-values (input)

Range - set of all y-values (output)

$f = \{(1,0),(2,0),(3,4)\}$
$D = \{1,2,3\}$
$R = \{0,4\}$

Finite - no duplicates, in order, a certain number of points



$D = \mathbb{R}$
$R = \{y \in \mathbb{R} \mid y \geq 1\}$

Infinite - line/graph, use set notation

Common Restrictions

① Cannot divide by 0

② Cannot squareroot negatives

eg. $\begin{pmatrix}1\\3\\5\end{pmatrix} \rightarrow \begin{pmatrix}2\\4\\7\end{pmatrix}$ $f(x)$ $\begin{pmatrix}2\\4\\7\end{pmatrix} \rightarrow \begin{pmatrix}1\\3\\5\end{pmatrix}$ $f^{-1}(x)$

Inverses - $f^{-1}(x)$ - the inverse function is the "reverse" of the original

- "inverse of f", "f inverse"
- $(x,y) \in f(x) \Leftrightarrow (y,x) \in f^{-1}(x)$

if $f(1)=2$ then $f^{-1}(2)=1$
if $f(3)=4$ then $2f^{-1}(4)$
$= 2(3)$
$= 6$

Note $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

[if $f(2)=3$ find x when $f^{-1}(3)=2x-1$]
$f^{-1}(3)=2$ $2=2x-1$
$x=3/2$

Steps to find the inverse:

① If given a list of points - flip the points

② If given an equation

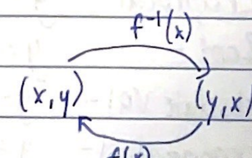
*① let $y=f(x)$

② Flip the x & y

③ solve for y

*④ use $f^{-1}(x)$

* - only do if given in f^n notation ($f(x)$)

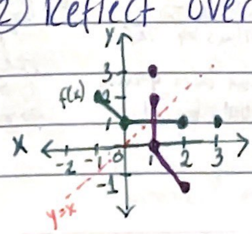


③ If given a graph

① Find key points - flip them & reconnect

② Reflect over $y=x$

eg.



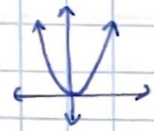
$(-1,2) \rightarrow (2,-1)$
$(0,1) \rightarrow (1,0)$
$(2,1) \rightarrow (1,2)$
$(3,1) \rightarrow (1,3)$

$D = \{x \in \mathbb{R} \mid -1 \leq x \leq 2, x=3\}$
$R = \{y \in \mathbb{R} \mid 1 \leq y \leq 2\}$
$D = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$
$R = \{y \in \mathbb{R} \mid -1 \leq y \leq 2, y=3\}$

∵ VLT fails ∴ inverse is not a f^n

Parent Functions

QUADRATIC



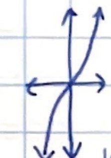
$y = x^2$

Domain $= \mathbb{R}$

Range $= \{y \in \mathbb{R} \mid y \geq 0\}$

Key Points $= \{$
(-2, 4)
(-1, 1)
(0, 0)
(1, 1)
(2, 4)
$\}$

CUBIC



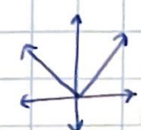
$y = x^3$

Domain $= \mathbb{R}$

Range $= \mathbb{R}$

Key points $= \{$
(-2, -8)
(-1, -1)
(0, 0)
(1, 1)
(2, 8)
$\}$

ABSOLUTE VALUE



$y = |x|$

Domain $= \mathbb{R}$

Range $= \{y \in \mathbb{R} \mid y \geq 0\}$

Key Points $= \{$
(-2, 2)
(-1, 1)
(0, 0)
(1, 1)
(2, 2)
$\}$

EXPONENTIAL

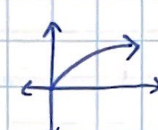
$y = 2^n$

Domain $= \mathbb{R}$

Range $= \{y \in \mathbb{R} \mid y > 0\}$

Key Points $= \{$
(-2, 1/4)
(-1, 1/2)
(0, 1)
(1, 2)
(2, 4)
$\}$

ROOT



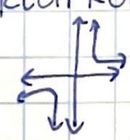
$y = \sqrt{x}$

$D = \{x \in \mathbb{R} \mid x \geq 0\}$

$R = \{y \in \mathbb{R} \mid y \geq 0\}$

Key Points $= \{$
(0, 0)
(1, 1)
(4, 2)
(9, 3)
$\}$

RECIPROCAL



$y = \frac{1}{x}$

$D = \{x \in \mathbb{R} \mid x \neq 0\}$

$R = \{y \in \mathbb{R} \mid y \neq 0\}$

Key Points $= \{$
(-2, -1/4)
(-1, -1)
(-1/2, -2)
(1/2, 2)
(1, 1)
(2, 1/2)
$\}$

Transformations

* a, c → affect y
* k, d → affect x

$$y = af(k(x-d)) + c$$

Quadratic – $y = a(k(x-d))^2 + c$

Cubic – $y = a(k(x-d))^3 + c$

Reciprocal – $y = \frac{a}{k(x-d)} + c$

Root – $y = a\sqrt{k(x-d)} + c$

Absolute Val – $y = a|k(x-d)| + c$

Exponential – $y = a2^{k(x-d)} + c$

Notes

↳ k must be factored out to get d

→ Order of transformations:
1. Stretch/compress
2. Reflect
3. Shift

Effects of Letters

a

$0 < |a| < 1$
↳ vertical compression by a factor of $|a|$
→ Multiply y values by $|a|$

$|a| > 1$
↳ Vertical stretch by a factor of $|a|$
→ multiply y values by $|a|$

$a < 0$
↳ Reflection over x-axis
→ Multiply y-values by -1

k

$0 < |k| < 1$
↳ horizontal stretch by a factor of $\frac{1}{|k|}$
→ mult. x values by $\frac{1}{|k|}$

$|k| > 1$
↳ horizontal compression by a factor of $\frac{1}{|k|}$
→ Mult. x values by $\frac{1}{|k|}$

$k < 0$
↳ Reflection over y-axis
→ Mult. x-vals by -1

c

$c > 0$:
↳ shift up c units
→ Add c to y values

$c < 0$:
↳ Shift down c units
→ Add c to y values

d

$d > 0$:
↳ Shift right d units
→ Add d to x-vals

$d < 0$:
↳ Shift left d-units
→ Add d to x-vals

Unit 2: Rational Expressions

Polynomials

Adding / Subtracting → Like terms

Multiplying → distributive property

$2+3x+4x=7x+2$

$-5(x-2)^2 = -5(x^2-2x+4)$
$= -5x^2+10x-20$

ⓐ $(2x+1)(3x-5)(4-x)$
$= (6x^2-7x-5)(4-x)$
$= 24x^2-28x-20-6x^3+7x^2+5x$
$= -6x^2+31x^2-23x-20$

Factoring Polynomials

↳ Greatest Common Factor (GCF)
- Divide out largest common coefficient & variable with max common exp.

ⓐ $5x^3y-3xy^2$
$= xy(5x^2-3y)$

↳ Grouping
- Group terms & factor GCF from each group

ⓑ a^3+3a^2+2a+6
$= a^2(a+3)+2(a+3)$
$= (a+3)(a^2+2)$

↳ Simple Trinomial
$x^2+(a+b)x+ab = (x+a)(x+b)$

ⓒ $m^2-mn-56n^2$
$= (m-8n)(m+7n)$

↳ Complex Trinomial
$ax^2+bx+c,\ a \neq 1$
- Find factors of a & place in each bracket
- Find factors of c & place in each bracket
- Check by expanding in your head

ⓓ $12x^2+17x+6$
$= (3x+2)(4x+3)$

↳ Difference of Squares
$a^2-b^2=(a-b)(a+b)$

ⓔ $(3a+2)^2-25b^2$
$= (3a+2+5b)(3a+2-5b)$

↳ Perfect Squares
$a^2+2ab+b^2 = (a+b)^2$
$a^2-2ab+b^2 = (a-b)^2$

ⓕ $9x^2+30x+25$
$= (3x+5)^2$

↳ Sum & Difference of Cubes
$a^3+b^3 = (a+b)(a^2-ab+b^2)$
$a^3-b^3 = (a-b)(a^2+ab+b^2)$

ⓖ $3x^5-3000x^2$
$= 3x^2(x^3-1000)$
$= 3x^2(x-10)(x^2+10x+100)$

Rational Expressions

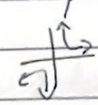
↳ An expression that is the ratio of two polynomials.

$f(x) = \frac{R(x)}{H(x)}$

* Rational expressions are equal IF they are the same for all possible values of the domain.

→ We have to state restrictions.

↳ Holes - missing value in graph
- occurs when restriction is removed through simplification



↳ Asymptotes - A line the graph approaches but never crosses
- remains after being simplified

eg. Simplify

ⓐ $f(x) = \frac{5x^3 - 5x^2}{6x^2 - 9x + 3}$
$= \frac{5x^2(x-1)}{3(2x^2-3x+1)}$
$= \frac{5x^2\cancel{(x-1)}}{3(2x-1)\cancel{(x-1)}}$
$= \frac{5x^2}{3(2x-1)}$

$x \neq 1$ (H)
$x \neq \frac{1}{2}$ (A)

Multiplying / Dividing
① Factor
② Reduce
③ State/Classify restrictions

ⓑ $\frac{-5x^3}{3y} \div \frac{y}{25x^2} = \frac{-5x^3}{3y} \cdot \frac{25x^2}{y}$
$= \frac{-125x^5}{3y^2}$

$x \neq 0$ (hole), $y \neq 0$ (asymptote)

ⓒ $\frac{2}{x} + \frac{3}{y}$
$= \frac{2y}{xy} + \frac{3x}{xy}$
$= \frac{2y+3x}{xy}$ $x \neq 0$ (A), $y \neq 0$ (A)

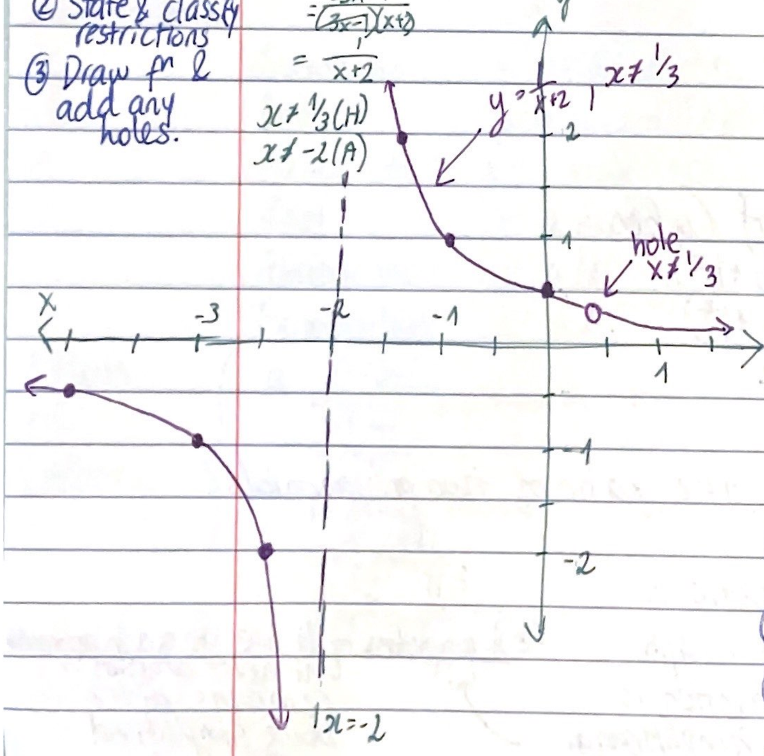
Adding / Subtracting
① Factor denominator
② Identify restrictions
③ Find lowest common denominator
④ Express with same LCD
⑤ Add/Sub numerators
⑥ Expand out & refactor top (if possible)

Steps:
① Simplify
② State & classify restrictions
③ Draw f^n & add any holes.

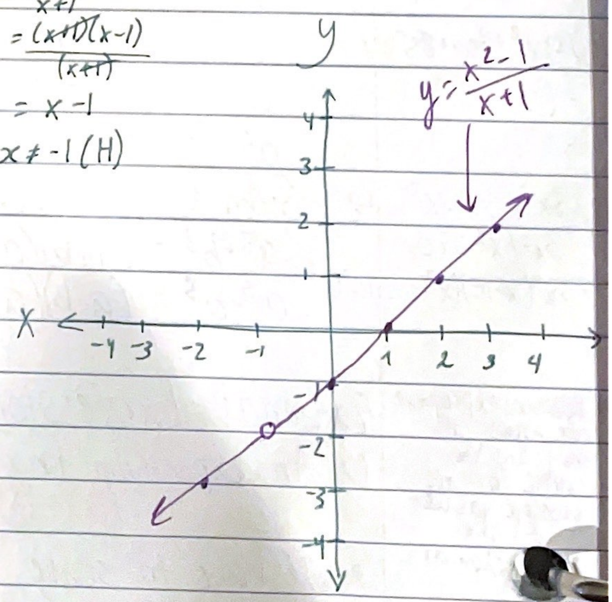
Graphs - usually come out to either linear (no as.) or reciprocal

ⓐ $y = \frac{3x-1}{3x^2+5x-2}$
$= \frac{\cancel{3x-1}}{\cancel{(3x-1)}(x+2)}$
$= \frac{1}{x+2}$

$x \neq \frac{1}{3}$ (H)
$x \neq -2$ (A)



ⓑ $y = \frac{x^2-1}{x+1}$
$= \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$
$= x - 1$
$x \neq -1$ (H)



Unit 3: Quadratic Functions

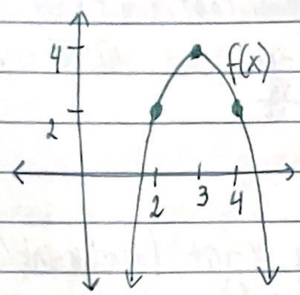
Quadratic Functions

Forms:
1. Standard $y=ax^2+bx+c$ (y-int)
2. Vertex $y=a(x-h)^2+k$ (vertex)
3. Factored $y=a(x-s)(x-t)$ (x-ints)

Graphing them:
1. Vertex form
   - gr10 (a) use step pattern
   - gr 11 (b) plot parent & transform

First & Second Differences
- 1^{st} diff the same → linear
- 2^{nd} diff the same → quadratic

eg.



1. Vertex - (3, 4)
2. Dir of opening - down
3. Axis of Sym. - $x=3$
4. Domain - $D=\mathbb{R}$
5. Range - $R=\{y \in \mathbb{R} \mid y \leq 4\}$
6. Max/min - Vertex is a max.

eg. Determine eq^n & classify f^n

x	y	1st diff	2nd diff
0	-32	18	-4
1	-14	14	-4
2	0	10	-4
3	10	6	-4
4	16		

∴ 2nd diffs the same ∴ f^n is quadratic

$y=ax^2+bx+c$ y-int

$y=ax^2+bx-32$

sub in (2, 0) ← any point

$0=4a+2b-32$

$32=4a+2b$

$\boxed{(1)\ 16=2a+b}$

↳ Sub in (1, -14)

$-14=a+b-32$

$\boxed{(2)\ 18=a+b}$

(1) $18=a+b$
(2) $16=2a+b$

$\boxed{-2=a}$ (2)-(1)

→ sub a into (2)

$18=-2+b$

$\boxed{b=20}$

∴ $y=-2x^2+20x-32$

Completing the Square
(standard ax^2+bx+c → vertex $y=a(x-h)^2+k$)

1. Factor a out of the first two terms
2. Take the coefficient of the second term, divide by 2 & square it
3. Take the ans from (2) & add/sub inside the brackets
4. Factor the trinomial, move the negative outside & simplify

$y=2x^2+4x+1$

$y=2(x^2+2x)+1$

$y=2(x^2+2x+1-1)+1$

$y=2(x+1)^2-1$

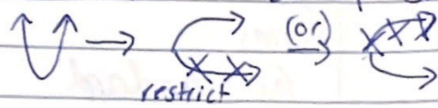
Changing Form
- factored → standard : expand
- standard → factored : factor it
- vertex → standard : expand
- standard → vertex : complete the square

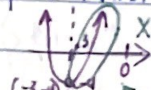
(a) $f(x)=2(x+3)^2-4$
let $y=2(x+3)^2-4$
$\vdots$
$\frac{x+4}{2}=(y+3)^2$
$\pm\sqrt{\frac{x+4}{2}}=y+3$
$y=-3\pm\sqrt{\frac{x+4}{2}}$
$f^{-1}(x)=-3+\sqrt{\frac{x+4}{2}}$ $x\geq -3$ for $f(x)$
$f^{-1}(x)=-3-\sqrt{\frac{x+4}{2}}$ $x<-3$ for $f(x)$

Inverse of a Quadratic (given an eq^n)

↳ let $y=\ldots$ (remove f^n notation)
→ switch x & y
→ Solve for y
→ go back to f^n notation: $f^{-1}(x)=\ldots$

NOTE - has to be a f^n so VLT pass





Radicals - a square, cube, or higher root ($\sqrt{\ }$ - radical symbol)

Properties:
① $\sqrt{a}\sqrt{b}=\sqrt{ab}$
② $a\sqrt{b}\cdot c\sqrt{d}=ac\sqrt{bd}$
③ $a\sqrt{b}+c\sqrt{b}=(a+c)\sqrt{b}$
④ $\sqrt{a}\sqrt{a}=a$
⑤ $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

(a) $\sqrt{50}+2\sqrt{2}-\sqrt{3}=5\sqrt{2}+2\sqrt{2}-\sqrt{3}$
$=7\sqrt{2}-\sqrt{3}$

(a) $f(x)=-0.5x^2+8x-24$
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$x=\frac{-8\pm\sqrt{64(-48)}}{-1}$
$x=\frac{-8\pm 4}{-1}$
$x_1=4,\ x_2=12$

(b) $3x^2+5x-9=0$
$D=b^2-4ac$
$=5^2-4(3)(-9)$
$\therefore D>0\ \therefore$ 2 real roots

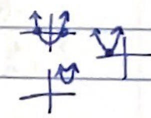
Solving Quadratics (* finding x-int) (find x-int/zeros/roots)

① Solve by factoring (move to one side, get a zero, factor)
② Graph it & look (only works if they're integers)
③ Quadratic Formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The Discriminant

$D=b^2-4ac$

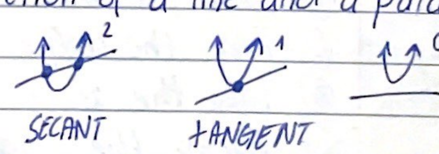
If D is positive → 2 REAL ROOTS
zero → 1 REAL ROOT
negative → NO REAL ROOTS
perf. square → FACTORABLE



Families of Quadratics - Quadratics that share a common property (e.g. same vertex)

Linear Quadratic Systems - Intersection of a line and a parabola

① Sub into quadratic
② Move everything to one side
③ Factor/quadratic formula to solve



Study Notes: Unit 4

Exponent Laws

① $x^0 = 1$
② $x^1 = x$
③ $x^a x^b = x^{a+b}$
④ $x^a / x^b = x^{a-b}$
⑤ $(x^a)^b = x^{ab}$
⑥ $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
⑦ $x^{-a} = \frac{1}{x^a}$
⑧ $x^{-1} = \frac{1}{x}$
⑨ $\sqrt[a]{x} = x^{1/a}$
⑩ $\sqrt{x} = x^{1/2}$
⑪ $\sqrt[a]{x^b} = x^{b/a}$
⑫ $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$

e.g. $x^y (x^{y+1})^{y+2} \left(\frac{1}{x}\right)^{6y} (x^{y+1})^2$
$= x^y x^{y^2+3y+2} x^{-6y} x^{2y+2}$
$= x^{y^2+3y-6y+2y+y+2+2}$
$= x^{y^2+4}$

Solving Exponential Equations

↳ Goal: single variable on both sides
* get the bases to match, drop base, set exp. equal
* use log if bases don't match

Exponential F^n:
$y = a b^{k(x-d)} + c$
base (2, 3, ..) can change your key points

Types:

simple	divide 1st	common fact	hidden quad.
$27^{-2} = 3^{x+1}$	$20 = 5 \cdot 2^{3x}$	$3^{x+2} - 3^x = 216$	$2^{2x} - 2^x = -12$
$3^{-6} = 3^{x+1}$	$4 = 2^{3x}$	$3^2 3^x - 3^x = 216$	$(2^x)^2 - 2^x + 12 = 0$
$\boxed{x = -7}$	$\boxed{x = 2/3}$	$9 \cdot 3^x - 3^x = 216$	$(2^x - 4)(2^x + 3) = 0$
		$8 \cdot 3^x = 216$	$2^x = 4$ $2^x = -3$
		$3^x = 27$	$\boxed{x = 2}$ no sol
		$\boxed{x = 3}$	

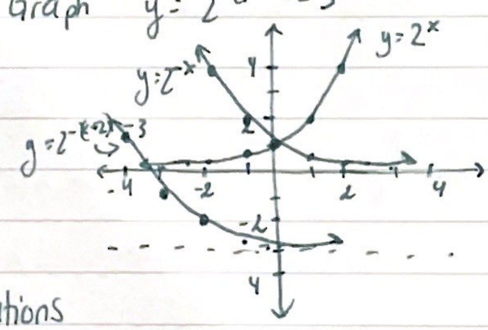
Rational Exponential Equations

→ isolate the variable first
→ flip the exponent
in general: $a = b + cx^{e/f}$

$\frac{a-b}{c} = x^{e/f}$
$\left(\frac{a-b}{c}\right)^{f/e} = \left(x^{e/f}\right)^{f/e}$
$\left(\frac{a-b}{c}\right)^{f/e} = x$

ⓐ $2 = x^{2/3}$
$(2)^{3/2} = (x^{2/3})^{3/2}$
$\sqrt{2}^3 = x$
$2\sqrt{2} = x$

e.g. Graph $y = 2^{-(x+2)} - 3$



Properties

- the greater the b - the steeper
- always going through (0,1)

e.g. Simplify & state a, k, d, c & write transformations

ⓐ $y = 3\left(\left(\frac{1}{3}\right)^{1/2 x}\right) - 1$
$y = 3 \cdot 3^{-1/2 x} - 1$
$y = 3^{-1/2 x + 1} - 1$
$y = 3^{-1/2 (x-2)} - 1$

$a = 1$
$k = -1/2$
$d = 2$
$c = -1$

- shift right 2
- shift 1 up
- horizontal stretch by a factor of 2
- reflect over y-axis

Study Notes - Ch 5 Part 1 (Trig Ratios)

Primary Trig Ratios

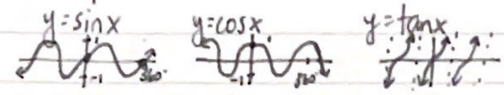
(SOHCAHTOA) $\sin\theta = {opp}/{hyp}$ $\cos\theta = {adj}/{hyp}$ $\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta}$

Reciprocal Trig Ratios

"cosecant" $\csc\theta = {hyp}/{opp}$
"secant" $\sec\theta = {hyp}/{adj}$
"cotangent" $\cot\theta = \frac{adj}{opp}$

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

Graphs



Notation

$\cancel{sx} \to \sin x, \sin\theta, \sin 40°$
$\cancel{si\theta^2} \to \sin^2 40°, (\sin 40°)^2$
* $\sin^{-1}\theta \neq \frac{1}{\sin\theta} = \csc\theta$
* sin40 → radians
sin40° → angles

5, 4, 3
13, 12, 5

Input / Output

$y = \sin x$ x → angles
y → numbers
$y = \sin^{-1} x$ x → numbers
y → angles

Cosine Law

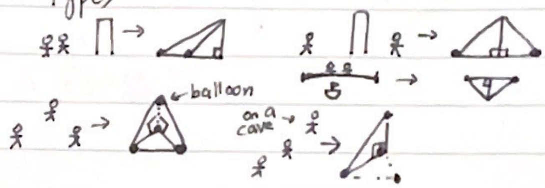
SSS SAS

$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Word Problems

<u>angle of elevation</u> - from sight line UP to an object
<u>angle of depression</u> - from sight line DOWN to an object

Types



Sine Law

* side & corresponding angle + → one other side/angle
* SSA, * SAA

finding sides → $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
finding angles → $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Sine Law - Ambiguous Case

With SSA there are 3 cases:
→ no possible triangle → 2 possible triangles
→ 1 possible triangle

$\sin\theta = \sin(180° - \theta)$

2 angles (θ and $180° - \theta$) have the same value when inverted.
How do we know which one???:
① Does it look acute/obtuse
② SATT (angles must add to 180)
③ larger angles correspond to larger sides

$\sin 70 = 0.94$
$\sin 110 = 0.94$

Grade 10 Theorems

Z pattern (Z) w=z, x=y
C pattern (C) w+x=180°, y+z=180°
F pattern (F) x=y, w=z

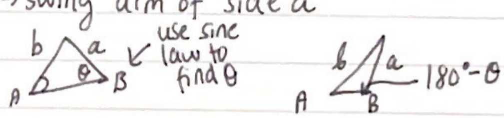
SATT - Sum of All angles in a Triangle theorem: x+y+z=180°
SAT - supplementary angle theorem: x+y=180°, x+y+z=180°
CAT - complementary angle theorem: x+y=90°, x+y+z=90°

OAT - opposite angle theorem: w=y, x=z
ITT - Isosceles triangle theorem: x=y
ETT - Equilateral triangle t.: x=y=z, x=60°, y=60°, z=60°
EAT - Exterior Angle t.: x+y+z=360°

Given an acute angle ($A < 90°$):

Case 1: $a < b\sin A$, no solution
Case 2: $a = b\sin A$, one sol - right △
Case 3: $a > b\sin A$ & $a < b$, 2 sols (1 acute, 1 obtuse)
Case 4: $a \geq b$, one sol (check this case first)

→ How to draw case 3: $a > b\sin A$, $a < b$
↳ swing arm of side a



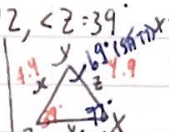
Given an obtuse angle:

Case 1: $a \leq b$, no sol.
Case 2: $a > b$, one sol. (obtuse)

eg. In △XYZ, ∠Z = 39°, x = 7.4cm, z = 4.9cm

∵ Z < 90°
∴ first 4 cases

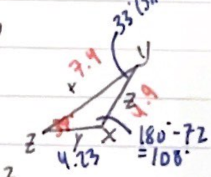
x sin Z = 4.66
∵ z > x sin Z
∴ case 3.
<u>2 sol</u>



① $\frac{\sin X}{x} = \frac{\sin Z}{z}$
$\frac{\sin X}{7.4} = \frac{\sin 39°}{4.9}$
<u>$X = 72°$</u>

69° (SATT)

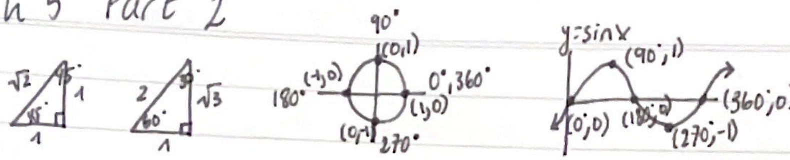
② $y^2 = x^2 + z^2 - 2xz\cos Y$
<u>$y \approx 7.27$cm</u>



① $\frac{y}{\sin Y} = \frac{z}{\sin Z}$
$\frac{y}{\sin 33} = \frac{4.9}{\sin 39°}$
<u>$y \approx 4.23$cm.</u>

33° (SATT)
180° - 72 = 108°

Ch 5 - Part 2

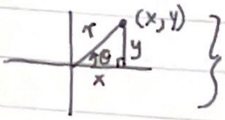


e.g. $\dfrac{\sqrt{2}\sin 45° - \sec 180°}{\csc 45°}$

$= \dfrac{\sqrt{2}(1/\sqrt{2}) + 1}{\sqrt{2}}$

$= \dfrac{2}{\sqrt{2}} = \dfrac{2\sqrt{2}}{2}$

$= \sqrt{2}$



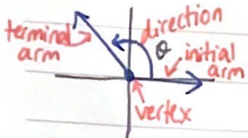
$\sin\theta = y/r$ $\quad \csc\theta = r/y$
$\cos\theta = x/r$ $\quad \sec\theta = r/x$
$\tan\theta = y/x$ $\quad \cot = x/y$

on unit circle $(r=1)$ →
$\sin\theta = y$ $\quad \csc\theta = 1/y$
$\cos\theta = x$ $\quad \sec\theta = 1/x$
$\tan\theta = y/x$ $\quad \cot = x/y$

$x<0$, $y>0$ | $x,y>0$
Q2 | Q1
Q3 | Q4
$x,y<0$ | $x>0$, $y<0$

Definitions

standard position - an angle is in standard position if:
① vertex is at origin
② initial arm is on positive x-axis

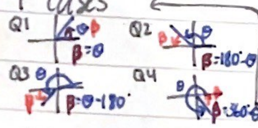


positive angle - formed by a counter-clockwise rotation of terminal arm

negative angle - formed by a clockwise rotation of terminal arm

coterminal angles - coterminal angles are:
→ in standard position
→ share same terminal arm

$\theta = 100°$
$\gamma = 360° + 100° = 460°$
$\alpha = 360° - 100° = 260°$

4 cases


principal angle - the smallest positive angle θ in a set of coterminal angles $0 \le \theta \le 360°$

related acute angle - β - the angle between the terminal arm & x-axis $0 < \theta < 90°$

CAST Rule - a trick to determine which quadrant trig ratios are pos. in.

S	A
T	C

A - all
S - sine T - tangent
C - cosine

① Given a point in the xy plane:
- draw triangle & determine quadrant
- find x, y, r to find ratios
β - choose a ratio & invert
θ - use quadrant diagrams to find θ

NOTE
$\sin\beta = |\sin\theta|$
$\cos\beta = |\cos\theta|$
$\tan\beta = |\tan\theta|$

e.g. Given P(1, -1) find $\sin\theta$, θ, β
$\sin\theta = y/r$ $\quad \beta = 45°$
$= -1$ $\quad \theta = 315°$

② Evaluate trig ratios
① Find the principal angle ($\pm 360°$ until $0° \le \theta \le 360°$)
② Determine which quadrant (sketch)
③ Apply CAST rule & convert to β (if on unit-circle, skip)

② $\sin 270° = -1$

ⓐ $\cos 120°$
$= -\cos 60°$
$= -1/2$

ⓑ $\cos 480° = \cos 120°$
$= -\cos 60°$
$= -1/2$

③ Given a trig ratio find another
e.g. If $\cos\theta = \frac{3}{2\sqrt{10}}$ determine $\tan\theta$, if $\theta \in Q4$
$\because \cos\theta = x/r$ $\quad x^2 + y^2 = r^2$ $\quad \because \theta \in Q4$
$\therefore x = 3, r = 2\sqrt{10}$ $\quad 9 + y^2 = 40$ $\quad \therefore y = -\sqrt{31}$
$y = \pm\sqrt{31}$
$\tan\theta = y/x = -\sqrt{31}/3$

① use formulas to find distances/values
② use $x^2 + y^2 = r^2$ to find others
③ make ratio

e.g. $\sin\theta = -1/2$, find $\tan\theta$
$y = -1$ $\quad x^2 = r^2 - y^2$ $\quad$ in Q3: $\quad$ in Q4:
$r = 2$ $\quad x = \pm\sqrt{3}$ $\quad \tan\theta = -\sqrt{3}/2$ $\quad \tan\theta = \sqrt{3}/2$

④ Work backwards to find angle
e.g. If $\sin\theta = -\frac{7}{\sqrt{74}}$ determine $\theta \ni 0 \le \theta \le 360°$
$\because \sin\theta < 0$ $\quad \sin\beta = \left|-\frac{7}{\sqrt{74}}\right|$
$\therefore \theta \in Q3$ or Q4 $\quad \beta = \sin^{-1}\left(\frac{7}{\sqrt{74}}\right)$
$\beta = 54°$

Q3 Q4
$\boxed{\theta_1 = 234°}$ $\quad \boxed{\theta_2 = 306°}$

e.g. $\sin\theta = 1/2$, find θ
$\because \sin\theta > 0$ $\quad \sin\beta = 1/2$ $\quad$ Q1 $\theta_1 = 30°$ $\quad$ Q2 $\theta_2 = 150°$
$\therefore$ Q1 or Q2 $\quad \beta = 30°$

STEPS
① Find β
② Find quadrants using CAST
③ use θ rules

CHANGING RANGE
$0° \le \theta \le 720°$ (add 360)
$\theta_1 = 234°$ $\quad \theta_3 = 594°$
$\theta_2 = 306°$ $\quad \theta_4 = 660°$
$-360° \le \theta \le 360°$ (sub. 360)

⑤ Trig Identities
① split into LS/RS
② Make them match
③ $\therefore$ LS = RS. $\therefore$ ...

tips
- work on more complicated side
- turn everything into $\sin\theta$ & $\cos\theta$
- use algebra
- $\because$ & $\therefore$

formulas
① $\csc x = \frac{1}{\sin x}$
② $\sec x = \frac{1}{\cos x}$ } reciprocal identities
③ $\cot x = \frac{1}{\tan x}$
④ $\tan x = \sin x / \cos x$ } quotient identity
⑤ $\sin^2 x + \cos^2 x = 1$
⑥ $\sin^2 x = 1 - \cos^2 x$ } pythagorean identity
⑦ $\cos^2 x = 1 - \sin^2 x$

eg. $\dfrac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

LS $= \dfrac{\sin^2\theta}{1-\cos\theta}$ $\qquad$ RS $= 1 + \cos\theta$

$= \dfrac{1-\cos^2\theta}{1-\cos\theta}$ $\qquad = \dfrac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}$

$= \dfrac{1-\cos^2\theta}{1-\cos\theta}$ $\qquad = \dfrac{1-\cos^2\theta}{1-\cos\theta}$

$\therefore$ LS = RS

$\therefore \dfrac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

$y = a\sin(k(x-d)) + c$

$y = a\cos(k(x-d)) + c$

Ch 6 (Sinusodial Functions)

Periodic Function - self-repeating graph

Cycle - the smallest complete repeating pattern

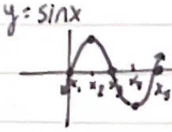
Period - the length of 1 cycle

Axis: the horizontal line that is halfway between min & max $y = \frac{max + min}{2}$

Amplitude: the vertical distance from axis to max or min $y = \frac{max - min}{2}$

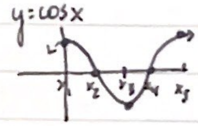
Point of inflection - a point where a graph changes concavity concave up ∪ conc. down ∩

$y = \sin x$



key pts: (0°,0), (90°,1), (180°,0), (270°,-1), (360°,0)
axis, max, axis, min, axis

$y = \cos x$



key pts: (0°,1), (90°,0), (180°,-1), (270°,0), (360°,1)
max, axis, min, axis, max

Graphing

① Find the amplitude: amp $= |a|$

② Find the axis: $y = c$

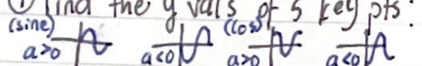
③ Find max & min: min = axis − amp, max = axis + amp

④ Find the phase shift: d

⑤ Find the period: $360°/k$

⑥ Find the x vals of 5 key points:
- ① $x_1 = d$
- ④ $x_2 = (x_1 + x_3) \div 2$
- ③ $x_3 = (x_1 + x_5) \div 2$
- ⑤ $x_4 = (x_3 + x_5) \div 2$
- ② $x_5 = d + 360°/k$ or d + period

⑦ Find the y vals of 5 key pts:



(sine) a>0: y_1 - axis, y_2 - max, y_3 - axis, y_4 - min, y_5 - axis

a<0: y_1 - axis, y_2 - min, y_3 - axis, y_4 - max, y_5 - axis

(cos) a>0: y_1 - max, y_2 - axis, y_3 - min, y_4 - axis, y_5 - max

a<0: y_1 - min, y_2 - axis, y_3 - max, y_4 - axis, y_5 - min

Plug in and solve Q

- sub in x or y to solve for other value
- need CAST knowledge

e.g. A sinusodial f^n has an amp of 4, a period of 120°, and a max at (0,9), determine the eq^n.

+cos
- $y = a\cos(k(x-d)) + c$
- amp $= 4 \to a = 4$
- period $= 120° \to k = 3$
- axis: $y = 5$
- shift: $d = 0$

* the only things that can change are sign of a & d

$y = 4\cos 3x + 5$

e.g. Determine m if (90°, m) is a point on the graph $y = -2\sin(2x + 45°) - 1$

sub in $x = 90°$, $y = m$

$m = -2\sin(2(90°) + 45°) - 1$

$m = -2\sin 225° - 1$

$m = -2(-\sin 45°) - 1$

$m = -2(-\sqrt{2}/2) - 1$

$m = \sqrt{2} - 1$

Finding the Equation

↳ Determine whether cos or sin

→ Determine if a is pos or neg

→ Find letter values:

a = amp (but ±)

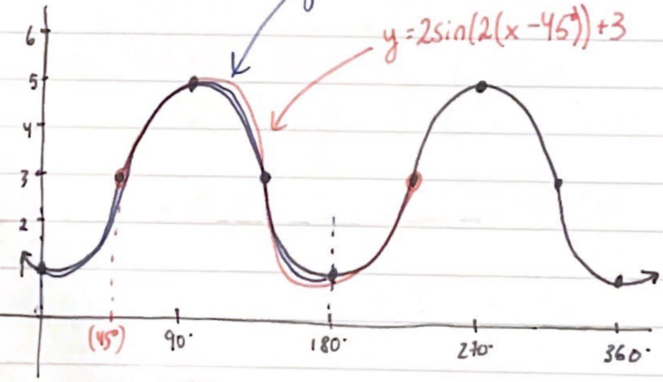
$k = 360°/\text{period}$

d = shift

$c = \frac{max + min}{2}$

→ with table of vals, just graph it

e.g. →



① axis: $y = 3 \to \boxed{c = 3}$

② amp: $|a| = 2 \to \boxed{a = \pm 2}$

③ $d = 0°$ *

④ $k = 360°/\text{period} = 360°/180° = 2 \to \boxed{k = 2}$

* for multiple equations

Unit 7: Sequences & Series

Sequences

$\{2, 4, 6, 8\}$ Sequence - An ordered list of numbers

$\{2, 4, 6, 8\}$ Finite seq. - a sequence with a definite (countable) num. of terms.

$\{2, 4, 6, 8 \ldots\}$ Infinite seq. - a sequence that continues indefinitely (uncountable)

$\{2, 4, 6, \}$ $\underset{+2}{\curvearrowright}$ $\underset{+2}{\curvearrowright}$ $t_n = t_{n-1} + 2$ Recursive Seq. - terms determined from previous term

$t_n = 2n$ (non-recursive seq. - from pos.)

$\{t_1, t_2, t_3, \ldots\}$ Terms - the numbers in a sequence

↑ subscripts for pos.

t_1 = first term = a $\quad$ t_n = n^{th} term = general term (used for formulas)

Pascal's Triangle

coefficients,
a exp. count ↓
b exp. count ↑

1 ← row 0
1 1 ← row 1
1 2 1 ← row 2
1 3 3 1 ← row 3
1 4 6 4 1 ← row 4
1 5 10 10 5 1 ← row 5
1 6 15 20 15 6 1
⋮

$(3x+4y)^3$
$= (3x)^3 + 3(3x)^2(4y) + 3(3x)(4y)^2 + (4y)^3$
$= 27x^3 + 108x^2y + 144xy^2 + 64y^3$

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$

Arithmetic & Geometric Sequences

(non-recursive)
$a = t_1$ $\quad$ n = position
t_n = general term
d = common difference
r = common ratio

*r is the thing that is multiplied.
so, if:
16, 8, 4, 2, ...
$r = \frac{1}{2}$

Arithmetic Sequence - sequence with common difference

$$\boxed{t_n = a + (n-1)d}$$

eg. Determine general term.

ⓐ 3, 9, 15, 21 (+6, +6, +6)

$t_n = 3 + (n-1)6$

$t_n = 6n - 3$

eg. For an ARITHMETIC sequence, $t_7 = 9$ and $t_{22} = 54$ determine t_1

$t_n = a + (n-1)d$
$9 = a + (7-1)d \Rightarrow$ ① $9 = a + 6d$
$54 = a + (22-1)d \Rightarrow$ ② $54 = a + 21d$
$45 = 15d$ ② − ①
$\boxed{d = 3}$

sub d into ①
$9 = a + 6(3)$
$a = -9$

$\therefore t_1 = -9$

Geometric Sequence - sequence with a common ratio

$$\boxed{t_n = ar^{n-1}}$$

eg. Determine the number of terms in the sequence.

16, 7, −2, −11, ..., −245 (−9)

$t_n = a + (n-1)d$
$-245 = 16 + (n-1)(-9)$
$\boxed{n = 30}$

eg. If $t_3 = 18$ & $t_8 = 4374$ of a GEOMETRIC sequence, determine t_1 and t_2.

$t_n = ar^{n-1}$
① $18 = ar^2$
② $4374 = ar^7$
$243 = r^5$ ② ÷ ①
$\sqrt[5]{243} = r$
$r = 3$

sub r into ①
$18 = a(3)^2$
$18 = 9a$
$a = 2$
$\therefore t_1 = 2$

$t_2 = (2)(3)^1$
$t_2 = 6$
$\therefore t_2 = 6$

↓

Series - sum of all values in a sequence

Arithmetic & Geometric Series

Arithmetic Series

$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$

Geometric Series

$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$

eg. Determine the sum of
$3+8+13+\ldots+248$:
$+5 \quad +5$

$S_n = \frac{n}{2}[2a+(n-1)d]$
$S_{50} = \frac{50}{2}[2(3)+49(5)]$
$\boxed{S_{50} = 6275}$

$t_n = a+(n-1)d$
$248 = 3+(n-1)5$
$248 = -2+5n$
$250 = 5n$
$\boxed{n=50}$

eg. How many terms of the series $2+5+8+\ldots$ add to the sum of 495?
$+3 \quad +3$

$S_n = \frac{n}{2}[2a+(n-1)d]$
$495 = \frac{n}{2}[2(2)+(n-1)3]$
$990 = n(4+3n-3)$
$0 = 3n^2+n-990$
$0 = (3n+55)(n-18)$
$n_1 = -\frac{55}{3}$ or $\boxed{n_2 = 18}$

$\therefore n_1$ is invalid
$\therefore$ There are 18 terms.

eg. Determine the sum of the first 12 terms of the series:
$1-2+4-8+\ldots$
$\times -2 \quad \times -2 \quad \times -2$

$S_n = \frac{a(r^n-1)}{r-1}$
$S_{12} = \frac{(-2)^{12}-1}{-3}$
$S_{12} = \frac{4095}{-3}$
$\boxed{S_{12} = -1365}$

Summation Notation (Bonus)

Also called sigma notation

"add up" $\displaystyle\sum_{i=1}^{34}(4i+31) = 35+39+\ldots+167$

34 ← ending term
i=1 ← starting term
rule (t_n) (gen. term)
167 is 4(34)+31

eg. Write in sigma notation
1,3,5,7,9,11,13
$\displaystyle\sum_{i=1}^{7}(2i-1)$

Never ending - use ∞

$\displaystyle\sum_{i=2}^{\infty}\frac{1}{i} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$

Counting DOWN:

$\displaystyle\sum_{n=3}^{0}(n+1) = 4+3+2+1$

multiplying:

"multiply" $\displaystyle\prod_{j=0}^{4} j^2 = 0\times1\times4\times9\times16 = 0$ ☺

Study Notes - Unit 8

Common Time Periods:

annually - 1
semi-annually - 2
quarterly - 4
monthly - 12
semi-monthly - 24
bi-weekly - 26
weekly - 52
daily - 365

Letters:

A - FV (new amount)
P - principal
R - rate (per annum)
T - time (years*)
I - interest earned
i - rate per compound period (divide)
n - # of compound periods (multiply) (years* comp.p.)

examples
- look in notes & homework

Simple Interest "simple"

$\boxed{I = PRT}$

$A = P + I$

Compound Interest "compounded"

$\boxed{A = P(1+i)^n}$

$I = A - P$

$FV = PV(1+i)^n$

FV of an Annuity
- compound, multiple payments, large amount at end "payments" "compounded"

$\boxed{FV = \frac{R[(1+i)^n - 1]}{i}}$

$\boxed{I = FV - Rn}$

PV of an Annuity
- large amount at start

$\boxed{PV = \frac{R[1-(1+i)^{-n}]}{i}}$

$\boxed{I = Rn - PV}$