Unit 1- FUNCTIONS

Function - each x has one y

- Vertical Line Test (VLT) - if a vertical line crosses more than one point it is not a function ($\because$ pass/fail $\therefore$ is/is not f^n)

$f(x)=2x+1$
$g(x)=x^2+3x+1$
ⓐ $f(g(y))=f(y^2+3y+1)$
$=2(y^2+3y+1)+1$
$=2y^2+6y+3$

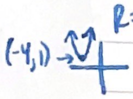
Function Notation

$y = f(x)$ — y: output, x: input — } Used in place of y in an equation

Domain & Range

Domain - set of all x values (input)
Range - set of all y values (output)
Finite - has to be in order, no duplicates, a certain number of points
Infinite - line/graph - use set notation

$f=\{(1,0),(2,0),(3,4)\}$
$D=\{1,2,3\}$
$R=\{0,4\}$



$D=\mathbb{R}$
$R=\{y\in\mathbb{R} \mid y\geq 1\}$

Common Restrictions
① Cannot divide by 0
② Cannot square root negatives.

Inverses - $f^{-1}(x)$ - the inverse function is the "reverse" of the original
- "inverse of f", "f inverse"
- $(x,y)\in f(x) \Longleftrightarrow (y,x)\in f^{-1}(x)$
- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

e.g. $\begin{pmatrix}1\\3\\5\end{pmatrix}\rightarrow\begin{pmatrix}2\\4\\7\end{pmatrix}$ $f(x)$ $\begin{pmatrix}2\\4\\7\end{pmatrix}\rightarrow\begin{pmatrix}1\\3\\5\end{pmatrix}$ $f^{-1}(x)$

ⓐ if $f(3)=4$ then $2f^{-1}(4)=2(3)=6$
ⓑ if $f(2)=3$, find x when $f^{-1}(3)=2x-1$
$f^{-1}(3)=2$ $\quad 2=2x-1$
$x=3/2$

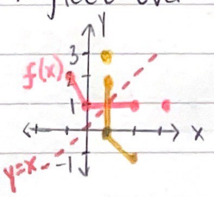
Steps to find the inverse:
① If given a list of points - flip the points
② If given an equation:
 *① Let $y=f(x)$
 ② Flip the x & y
 ③ Solve for y
 *④ Use $f^{-1}(x)$

*only do if question is given in function notation

$(x,y) \xrightarrow{f^{-1}(x)} (y,x)$, $(y,x) \xrightarrow{f(x)} (x,y)$

e.g. Find inverse
ⓐ $f(x)=\frac{3}{x+1}$
let $y=\frac{3}{x+1}$
$x=\frac{3}{y+1}$
$(y+1)x=3$
$y=\frac{3}{x}-1$
$f^{-1}(x)=\frac{3}{x}-1$

③ If given a graph
 ① Find key points - flip them & reconnect
 ② Reflect over $y=x$



$(-1,2)\rightarrow(2,-1)$
$(0,1)\rightarrow(1,0)$
$(2,1)\rightarrow(1,2)$
$(3,1)\rightarrow(1,3)$

$D=\{x\in\mathbb{R} \mid -1\leq x\leq 2, x=3\}$
$R=\{y\in\mathbb{R} \mid 1\leq y\leq 2\}$

$D=\{x\in\mathbb{R} \mid 1\leq x\leq 2\}$
$R=\{y\in\mathbb{R} \mid -1\leq y\leq 2, y=3\}$
$\because$ VLT fails $\therefore$ inverse is not a function.

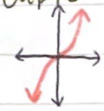
Parent Functions

Quadratic - $y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

$D = \mathbb{R}$
$R = \{y \in \mathbb{R} \mid y \geq 0\}$

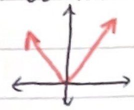
Cubic - $y = x^3$



x	y
-2	-8
-1	-1
0	0
1	1
2	8

$D = \mathbb{R}$
$R = \mathbb{R}$

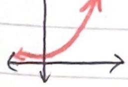
Absolute Value - $y = |x|$



x	y
-2	2
-1	1
0	0
1	1
2	2

$D = \mathbb{R}$
$R = \{y \in \mathbb{R} \mid y \geq 0\}$

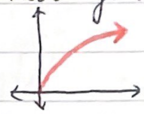
Exponential - $y = 2^x$



x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

$D = \mathbb{R}$
$R = \{y \in \mathbb{R} \mid y > 0\}$

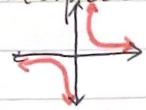
Root - $y = \sqrt{x}$



x	y
0	0
1	1
4	2
9	3

$D = \{x \in \mathbb{R} \mid x \geq 0\}$
$R = \{y \in \mathbb{R} \mid y \geq 0\}$

Reciprocal - $y = 1/x$



x	y
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2

$D = \{x \in \mathbb{R} \mid x \neq 0\}$
$R = \{y \in \mathbb{R} \mid y \neq 0\}$

Transformations $y = af(k(x-d)) + c$

Quadratic - $y = a(k(x-d))^2 + c$
Cubic - $y = a(k(x-d))^3 + c$
Reciprocal - $y = \frac{a}{k(x-d)} + c$
Root - $y = a\sqrt{k(x-d)} + c$
Absolute Val - $y = a|k(x-d)| + c$
Exponential - $y = a2^{k(x-d)} + c$

Notes

↳ k must be factored out to get d
→ order of transformations:
1. Stretch/compress
2. Reflect
3. Shift

Effects of Letters

a - multiply y values by a
↳ $0 < |a| < 1$
Vertical compression by a factor of $|a|$
→ $|a| > 1$
Vertical stretch by a factor of $|a|$
→ $a < 0$
Reflection over x-axis

k - multiply x values by $1/k$
↳ $0 < |k| < 1$
Horizontal stretch by a factor of $\frac{1}{|k|}$
→ $|k| > 1$
Horizontal compression by a factor of $\frac{1}{|k|}$
→ $k < 0$
Reflection over y-axis

c - add c to y values
↳ $c > 0$
Shift up c units
→ $c < 0$
Shift down c units

d - add d to x values
↳ $d > 0$
Shift right d units
→ $d < 0$
Shift left d units

Unit 2: RATIONAL EXPRESSIONS

Polynomials

Adding/subtracting → Like terms

Multiplying → Distributive property

$2+3x+4x=7x+2$

$-5(x-2)^2=-5(x^2-2x+4)$

$=-5x^2+10x-20$

Factoring Polynomials

↳ Greatest Common Factor (GCF)

- Divide out largest common coefficient & variable with max. common exponent

→ Grouping

- Group terms & Factor GCF from each group

→ Simple Trinomial

- $x^2+(a+b)x+ab=(x+a)(x+b)$

→ Complex Trinomial $ax^2+bx+c,\ a\neq 1$

- Find factors of a & place in each bracket
- Find factors of c & place in each bracket
- Check by expanding in your head.

→ Difference of Squares

- $a^2-b^2=(a-b)(a+b)$

→ Perfect Squares

- $a^2+2ab+b^2=(a+b)^2$, $a^2-2ab+b^2=(a-b)^2$

→ Sum & Difference of Cubes

- $a^3+b^3=(a+b)(a^2-ab+b^2)$, $a^3-b^3=(a-b)(a^2+ab+b^2)$

ⓐ $5x^3y-3xy^2$
$=xy(5x^2-3y)$

ⓑ a^3+3a^2+2a+6
$=a^2(a+3)+2(a+3)$
$=(a+3)(a^2+2)$

ⓒ $m^2-mn-56n^2$
$=(m-8n)(m+7n)$

ⓓ $12x^2+17x+6$
$=(3x+2)(4x+3)$

ⓔ $(3a+2)^2-25b^2$
$=(3a+2+5b)(3a+2-5b)$

ⓕ $9x^2+30x+25$
$=(3x+5)^2$

ⓖ $3x^5-3000x^2$
$=3x^2(x^3-1000)$
$=3x^2(x-10)(x^2+10x+100)$

Rational Expressions

Rational expressions are equal if they are the same for all values of the domain.

↳ An expression that is the ratio of two polynomials. $f(x)=\frac{R(x)}{H(x)}$

→ Restrictions have to be stated:

↳ Holes - missing value in graph
- occurs when a restriction is removed through simplification

↳ Asymptotes - a line the graph approaches but never crosses
- remains after being simplified

e.g. Simplify:

$$f(x)=\frac{5x^3-5x^2}{6x^2-9x+3}=\frac{5x^2(x-1)}{3(2x^2-3x+1)}=\frac{5x^2\cancel{(x-1)}}{3(2x-1)\cancel{(x-1)}}=\frac{5x^2}{3(2x-1)}$$

$x\neq 1$ (hole)

$x\neq \frac{1}{2}$ (asymptote)

Multiplying / Dividing

1. Factor
2. Reduce
3. State/classify restrictions

ⓐ $\frac{-5x^3}{3y} \div \frac{y}{25x^2} = \frac{-5x^3}{3y} \cdot \frac{25x^2}{y}$
$= \frac{-125x^5}{3y^2}$

$x \neq 0$ (h.)
$y \neq 0$ (a.)

Adding / Subtracting

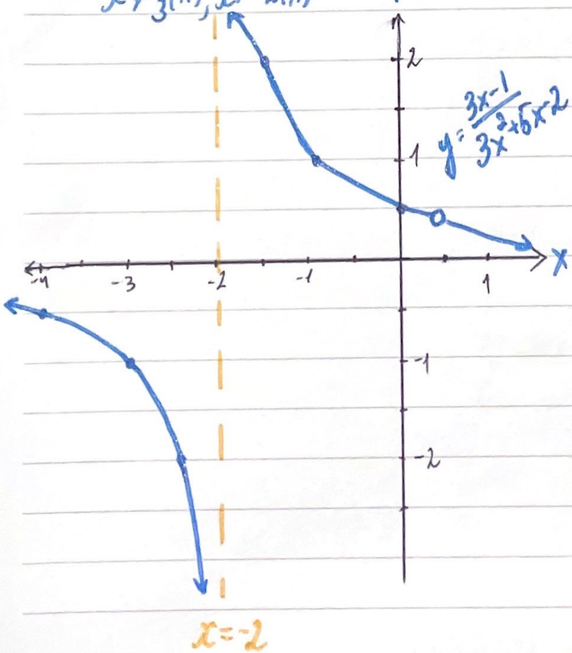
1. Factor denominator
2. Identify restrictions
3. Find LCD. lowest common denominator
4. Express with same LCD.
5. Add/sub numerators.
6. Expand & refactor top (if possible)

ⓑ $\frac{2}{x} + \frac{3}{y} = \frac{2y}{xy} + \frac{3x}{xy}$
$= \frac{2y+3x}{xy}$, $x \neq 0$ (A.) $y \neq 0$ (A.)

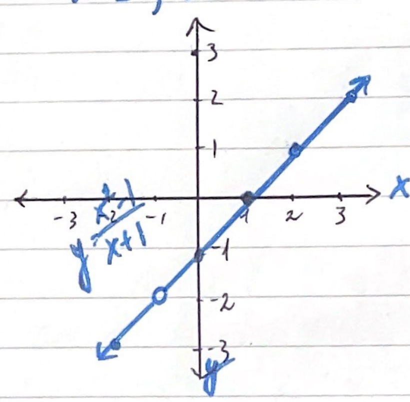
Graphs - usually come out as either linear (no asymptote) or reciprocal.

Steps:
1. Simplify
2. State & classify restrictions
3. Draw f^n & add any holes

ⓐ $y = \frac{3x-1}{3x^2+5x-2}$
$= \frac{\cancel{3x-1}}{(\cancel{3x-1})(x+2)}$
$= \frac{1}{x+2}$,
$x \neq \frac{1}{3}$(H), $x \neq -2$(A)



ⓑ $y = \frac{x^2-1}{x+1}$
$= \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$
$= x+1$, $x \neq -1$ (hole)



Unit 3: QUADRATIC FUNCTIONS

Forms:

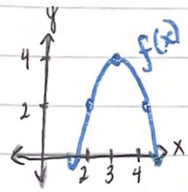
① Standard - $y = ax^2 + bx + c$ - y-int (c)
② Vertex - $y = a(x-h)^2 + k$ - vertex (h, k)
③ Factored - $y = a(x-s)(x-t)$ - x-ints (s, t)

Graphing:

① Vertex form:
gr 10 ⓐ use step pattern (1, 3, 5, ..)
gr 11 ⓑ plot parent & transform

First & Second Differences - 1^{st} diff the same → linear
- 2^{nd} diff the same → quadratic

eg.



ⓘ Vertex - (3, 4)
ⓘⓘ Direction of opening - Down
ⓘⓘⓘ Axis of symmetry - $x = 3$
ⓘⓥ Domain - $D = \{x \in \mathbb{R}\}$
ⓥ Range - $R = \{y \in \mathbb{R} \mid y \leq 4\}$
ⓥⓘ Max/min? - Vertex is a max.

eg. Determine eq^n & classify f^n

x	y	1st diff	2nd diff
0	-32		
		18	
1	-14		-4
		14	
2	0		-4
		10	
3	10		-4
		6	
4	16		

∵ 2nd diffs the same ∴ f^n is quadratic.

$y = ax^2 + bx + c$ ← y-int
$y = ax^2 + bx - 32$
Sub in (2, 0) ← any point
$0 = 4a + 2b - 32$
① $16 = 2a + b$

sub in (1, -14)
$-14 = a + b - 32$
② $18 = a + b$

① $18 = a + b$
② $16 = 2a + b$
$\boxed{-2 = a}$ ② - ①

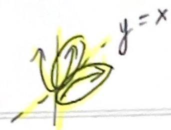
sub a into ②
$18 = -2 + b$
$\boxed{b = 20}$
$\therefore y = -2x^2 + 20x - 32$

Completing the Square (standard ax^2+bx+c → vertex $y=a(x-h)^2+k$)

$y = 2x^2 + 4x + 1$
① Factor a out of the first two terms — $y = 2(x^2 + 2x) + 1$
② Take the coefficient of the second term, divide by 2 & square it — $* \left(\frac{2}{2}\right)^2 = 1$
③ Take the answer from ② & add/subtract inside brackets. — $y = 2(x^2 + 2x + 1 - 1) + 1$
④ Factor the trinomial, move the negative outside & simplify — $y = 2(x+1)^2 - 1$

Changing Form

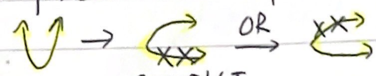
- factored → standard : expand
- standard → factored : factor it
- vertex → standard : expand
- standard → vertex : complete the square



Inverse of a Quadratic (given an eq^n)

① let $y = \ldots$ (remove f^n notation)
② Switch x & y
③ solve for y
④ go back to f^n notation: $f^{-1}(x) = \ldots$

NOTE - Inverse has to be a function so VLT passes

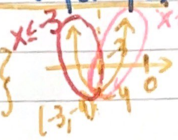


eg. $f(x) = 2(x+3)^2 - 4$
let $y = 2(x+3)^2 - 4$
$\vdots$
$\frac{x+4}{2} = (y+3)^2$
$\pm\sqrt{\frac{x+4}{2}} = y+3$
$y = -3 \pm \sqrt{\frac{x+4}{2}}$

$\rightarrow f^{-1}(x) = -3 + \sqrt{\frac{x+4}{2}}$, $x \geq -3$ for $f(x)$ }

OR (choose one, both ans are ok)

$f^{-1}(x) = -3 - \sqrt{\frac{x+4}{2}}$, $x \leq -3$ for $f(x)$ } $x \geq -3$ (pos. side)



Properties:

Radicals - a square, cube, or higher root ($\sqrt{\ }$ - radical symbol)

① $\sqrt{a}\sqrt{b} = \sqrt{ab}$
② $a\sqrt{b} * c\sqrt{d} = ac\sqrt{bd}$
③ $a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$
④ $\sqrt{a}\sqrt{a} = a$
⑤ $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

eg. $\sqrt{50} + 2\sqrt{2} - \sqrt{3}$
$= 5\sqrt{2} + 2\sqrt{2} - \sqrt{3}$
$= 7\sqrt{2} - \sqrt{3}$

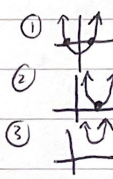
Solving Quadratics (/ finding x-int(s)), (/finding zeroes), (/finding roots)

eg. $f(x) = -0.5x^2 + 8x - 24$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-8 \pm \sqrt{64-48}}{-1}$
$x = (-8 \pm 4)/-1$
$x_1 = 4$, $x_2 = 12$

① Solve by factoring (move to one side, get a "= 0", factor)
② Graph it & look (only works if x-ints $\in \mathbb{Z}$)
③ Quadratic Formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

The Discriminant

$D = b^2 - 4ac$ —— If D is positive $\rightarrow$ 2 REAL ROOTS ①
zero $\rightarrow$ 1 REAL ROOT ②
negative $\rightarrow$ NO REAL ROOTS ③
perfect square $\rightarrow$ FACTORABLE



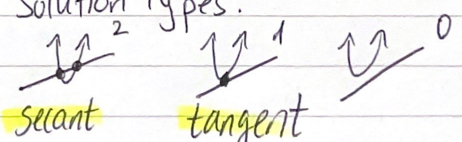
eg. $3x^2 + 5x - 9 = 0$
$D = b^2 - 4ac$
$= 5^2 - 4(3)(-9)$
$\because D > 0 \therefore 2$ real roots.

Families of Quadratics - Quadratics that share a common property (eg same vertex)

Linear Quadratic Systems - Intersection of a line & a parabola

① Sub line into quadratic
② Move everything to one side
③ Factor/quadratic formula to solve

Solution Types:



Study Notes: Unit 4

Exponent Laws

① $x^0 = 1$
② $x^1 = x$
③ $x^a x^b = x^{a+b}$
④ $x^a / x^b = x^{a-b}$
⑤ $(x^a)^b = x^{ab}$
⑥ $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
⑦ $x^{-a} = \frac{1}{x^a}$
⑧ $x^{-1} = \frac{1}{x}$
⑨ $\sqrt[a]{x} = x^{1/a}$
⑩ $\sqrt{x} = x^{1/2}$
⑪ $\sqrt[a]{x^b} = x^{b/a}$
⑫ $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$

e.g. $x^y (x^{y+1})^{y+2} \left(\frac{1}{x}\right)^{6y} (x^{y+1})^2$
$= x^y x^{y^2+3y+2} x^{-6y} x^{2y+2}$
$= x^{y^2+3y-6y+2y+y+2+2}$
$= x^{y^2+4}$

Solving Exponential Equations

↳ Goal: single variable on both sides
* get the bases to match, drop base, set exp. equal
* use log if bases don't match

Exponential F^n:
$y = ab^{k(x-d)} + c$
base (2, 3, ..)
can change your key points

Types:

simple	divide 1^{st}	common fact	hidden quad.
$27^{-2} = 3^{x+1}$	$20 = 5 \cdot 2^{3x}$	$3^{x+2} - 3^x = 216$	$2^{2x} - 2^x = -12$
$3^{-6} = 3^{x+1}$	$4 = 2^{3x}$	$3^2 3^x - 3^x = 216$	$(2^x)^2 - 2^x + 12 = 0$
$\boxed{x = -7}$	$\boxed{x = 2/3}$	$9 \cdot 3^x - 3^x = 216$	$(2^x - 4)(2^x + 3) = 0$
		$8 \cdot 3^x = 216$	$2^x = 4$; $2^x = -3$
		$3^x = 27$	$\boxed{x = 2}$; no sol
		$\boxed{x = 3}$	

Rational Exponential Equations

→ isolate the variable first
→ flip the exponent

in general: $a = b + cx^{e/f}$

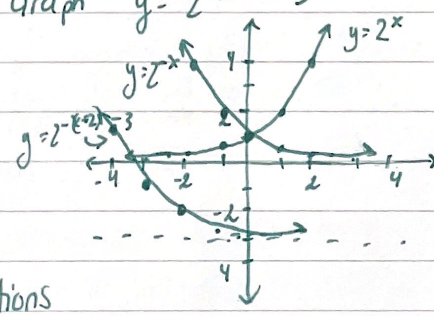
$\frac{a-b}{c} = x^{e/f}$

$\left(\frac{a-b}{c}\right)^{f/e} = \left(x^{e/f}\right)^{f/e}$

$\left(\frac{a-b}{c}\right)^{f/e} = x$

ⓐ $2 = x^{2/3}$
$(2)^{3/2} = (x^{2/3})^{3/2}$
$\sqrt{2}^3 = x$
$2\sqrt{2} = x$

e.g. Graph $y = 2^{-(x+2)} - 3$



Properties

- the greater the b - the steeper
- always going through (0,1)

eg. Simplify & state a, k, d, c & write transformations

ⓐ $y = 3\left(\left(\frac{1}{3}\right)^{\frac{1}{2}x}\right) - 1$
$y = 3 \cdot 3^{-\frac{1}{2}x} - 1$
$y = 3^{-\frac{1}{2}x + 1} - 1$
$y = 3^{-\frac{1}{2}(x-2)} - 1$

$a = 1$
$k = -\frac{1}{2}$
$d = 2$
$c = -1$

- shift right 2
- shift 1 up
- horizontal stretch by a factor of 2
- reflect over y-axis

Study Notes - Ch 5 Part 1 (Trig Ratios)

Primary Trig Ratios

(SOHCAHTOA) $\sin\theta = opp/hyp$ $\cos\theta = adj/hyp$ $\tan\theta = opp/adj = \frac{\sin\theta}{\cos\theta}$

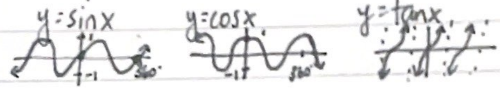
Reciprocal Trig Ratios

"cosecant" "secant" "cotangent"

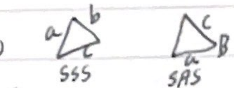
$\csc\theta = hyp/opp$ $\sec\theta = hyp/adj$ $\cot\theta = adj/opp$

$\csc\theta = 1/\sin\theta$ $\sec\theta = 1/\cos\theta$ $\cot\theta = 1/\tan\theta$

Graphs



Cosine Law



$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

Notation

5, 4, 3
13, 12, 5

$\cancel{\sin} \rightarrow \sin x, \sin\theta, \sin 40°$

$\sin 40°^2 \rightarrow \sin^2 40°, (\sin 40°)^2$

+ $\sin^{-1}\theta \neq \frac{1}{\sin\theta} = \csc\theta$
+ $\sin 40 \rightarrow$ radians
  $\sin 40° \rightarrow$ angles

Input / Output

$y = \sin x$ x → angles
y → numbers

$y = \sin^{-1} x$ x → numbers
y → angles

Word Problems

angle of elevation - from sight line UP to an object

angle of depression - from sight line DOWN to an object

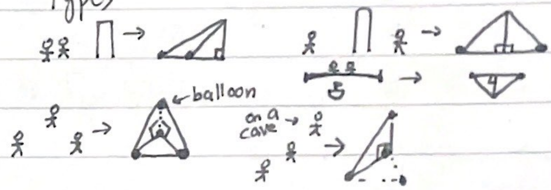
Sine Law

* side & corresponding angle + → one other side/angle
* SSA, * SAA

finding sides → $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

finding angles → $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Types



Sine Law - Ambiguous Case

With SSA there are 3 cases:
→ no possible triangle → 2 possible triangles
→ 1 possible triangle

$\sin\theta = \sin(180° - \theta)$

2 angles (θ and $180° - \theta$) have the same value when inverted.

How do we know which one???:
1. Does it look acute/obtuse
2. SATT (angles must add to 180°)
3. larger angles correspond to larger sides

$\sin 70° = 0.94$
$\sin 110° = 0.94$

Grade 10 Theorems

Z pattern (Z) w=z, x=y

C pattern (C) w+x=180°, y+z=180°

F pattern (F) x=y, w=z

SATT - Sum of All angles in a Triangle Theorem: x+y+z=180°

SAT - Supplementary angle theorem: x+y=180°, x+y+z=180°

CAT - complementary angle theorem: x+y=90°, x+y+z=90°

OAT - opposite angle theorem: w=y, x=z

ITT - Isosceles triangle theorem: x=y

ETT - Equilateral triangle T.: x=y=z x=60°, y=60°, z=60°

EAT - Exterior Angle T.: x+y+z=360°

Given an acute angle ($A < 90°$):

Case 1: $a < b\sin A$, no solution

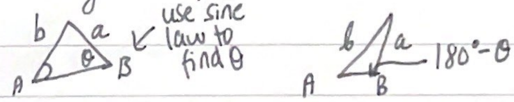
Case 2: $a = b\sin A$, one sol - right Δ

Case 3: $a > b\sin A$ & $a < b$, 2 sols (1 acute, 1 obtuse)

Case 4: $a \geq b$, one sol (check this case first)

→ How to draw case 3: $a > b\sin A$, $a < b$

↳ swing arm of side a



Given an obtuse angle:

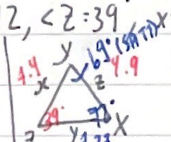
Case 1: $a \leq b$, no sol.

Case 2: $a > b$, one sol. (obtuse)

eg. In ΔXYZ, ∠Z=39°, x=7.4cm, z=4.9cm

∵ z<90° ∴ first 4 cases

x sin Z = 4.66
∵ z > x sin Z
∴ case 3: 2 sols



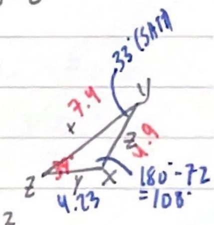
① $\frac{\sin X}{x} = \frac{\sin Z}{z}$

$\frac{\sin X}{7.4} = \frac{\sin 39°}{4.9}$

$X = 72°$

② $y^2 = x^2 + z^2 - 2xz\cos Y$

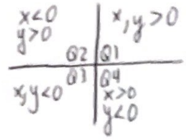
$y \approx 7.27cm$



① $\frac{y}{\sin Y} = \frac{z}{\sin Z}$

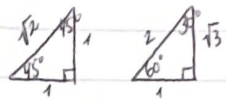
$\frac{y}{\sin 33°} = \frac{4.9}{\sin 39°}$

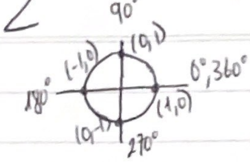
$y \approx 4.23cm$

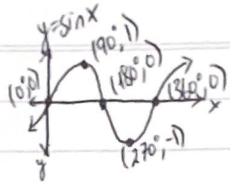


NOTE
$\sin\beta = |\sin\theta|$
$\cos\beta = |\cos\theta|$
$\tan\beta = |\tan\theta|$

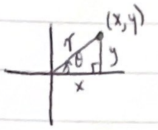
UNIT 5 - PART 2







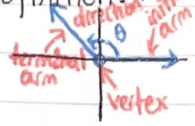
eg. $\dfrac{\sqrt{2}\sin 45° - \sec 180°}{\csc 45°}$
$= \dfrac{\sqrt{2}(1/\sqrt{2})+1}{\sqrt{2}}$
$= \dfrac{2}{\sqrt{2}}$
$= \sqrt{2}$



$\sin\theta = y/r$ $\csc\theta = r/y$
$\cos\theta = x/r$ $\sec\theta = r/x$
$\tan\theta = y/x$ $\cot\theta = x/y$

on unit circle ($r=1$) → $\sin\theta = y$ $\csc\theta = 1/y$
$\cos\theta = x$ $\sec\theta = 1/x$
$\tan\theta = y/x$ $\cot\theta = x/y$

Definitions:



standard position - an angle is in standard position if:
① vertex is at origin (0,0)
② initial arm is on positive x-axis

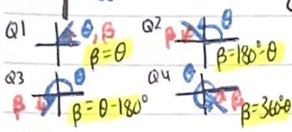
positive angle - formed by a counter-clockwise rotation of terminal arm

negative angle - formed by a clockwise rotation of terminal arm

$\theta = 100°$
$\gamma = 360° + 100° = 460°$
$\alpha = 360° - 100° = 260°$

coterminal angles - coterminal angles are:
↳ in standard position
→ share same terminal arm

4 cases:



θ - **principal angle** - the smallest positive angle θ in a set of coterminal angles $0° \le \theta \le 360°$

β - **related acute angle** - the angle between the terminal arm & x-axis $0° < \beta < 90°$

CAST Rule - A trick to determine which quadrant trig ratios are pos. in

S	A
T	C

A - all
S - sine
C - cosine
T - tangent

Question Types:

① **Given a point in the xy plane**:
- draw triangle & determine quadrant
- find x, y, r to find ratios
- β - choose a ratio & invert to find
- θ - use quadrant diagrams to find

eg. Given P(1,-1) find $\sin\theta$, θ, β

(1,-1)

$\sin\theta = y/r$
$= -1/\sqrt{2}$
$= -\sqrt{2}/2$

$\beta = 45°$
$\theta = 360° - 45°$
$= 315°$

② **Evaluate trig ratios**: ① Find the principal angle ($\pm 360°$ until $0° \le \theta \le 360°$)
② Determine which quadrant (sketch)
③ Apply CAST rule & convert to β (if on unit circle, skip)

ⓐ $\sin 270° = -1$
ⓑ $\cos 120° = -\cos 60°$ β
$= -1/2$
cos is − in Q2
ⓒ $\cos 480° = \cos 120°$ ← ①
$= -\cos 60°$ ← ③
$= -1/2$

③ **Given a trig ratio, find another**
① use formulas to find distances/values
② use $x^2+y^2=r^2$ to find others
③ make ratio

eg. If $\cos\theta = \frac{3}{2\sqrt{10}}$ determine $\tan\theta$, if $\theta \in Q4$
① ∵ $\cos\theta = x/r$
∴ $x = 3$, $r = 2\sqrt{10}$
② $x^2+y^2=r^2$
$9+y^2 = 40$
$y = \pm\sqrt{31}$
∵ $\theta \in Q4$
∴ $y = -\sqrt{31}$
$\tan\theta = y/x$
③ $= -\sqrt{31}/3$

eg. $\sin\theta = -\frac{1}{2}$, find $\tan\theta$
① $y=-1$, $r=2$
② $x^2 = r^2 - y^2$
$x = \pm\sqrt{3}$
③ IN Q3: $\tan\theta = -\sqrt{3}/2$
IN Q4: $\tan\theta = \sqrt{3}/2$

In Q3, $x<0$ & in Q4, $x>0$
2 possible answers.

④ **Work backwards to find angle**
① Find β
② Find quadrants using CAST
③ Use θ rules

eg. If $\sin\theta = -7/\sqrt{74}$ determine $\theta \ni 0° \le \theta \le 360°$
② ∵ $\sin\theta$ is negative
∴ $\theta \in Q3$ or Q4
$\sin\beta = |-7/\sqrt{74}|$
① $\beta = \sin^{-1}(7/\sqrt{74})$
$\beta = 54°$
③ Q3: 54° $\boxed{\theta_1 = 234°}$
Q4: 54° $\boxed{\theta_2 = 306°}$

eg. $\sin\theta = 1/2$, find θ such that $0° \le \theta \le 360°$
② ∵ $\sin\theta > 0$
∴ Q1 or Q2
$\sin\beta = \frac{1}{2}$
① $\beta = 30°$
③ Q1: $\boxed{\theta_1 = 30°}$
Q2: $\boxed{\theta_2 = 150°}$

↳ CHANGING RANGE: $0° \le \theta \le 720°$ (add 360°)
$\theta_1 = 234°$ $\theta_3 = 594°$
$\theta_2 = 306°$ $\theta_4 = 660°$

If $-360° \le \theta \le 360°$
↳ (sub 360°)

⑤ Trig Identities

Steps: ① Split into LS/RS
② Make them match
* ③ $\because LS = RS \therefore$ ~~~

Tips: - work on more complicated side
- turn everything into $\sin\theta$ & $\cos\theta$
- use algebra

Formulas:

① $\csc x = \frac{1}{\sin x}$
② $\sec x = \frac{1}{\cos x}$
③ $\cot x = \frac{1}{\tan x}$
] reciprocal identities

④ $\tan x = \frac{\sin x}{\cos x}$] quotient identity

⑤ $\sin^2 x + \cos^2 x = 1$
⑥ $\sin^2 x = 1 - \cos^2 x$
⑦ $\cos^2 x = 1 - \sin^2 x$
] pythagorean identity

eg. $\frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

$LS = \frac{\sin^2\theta}{1-\cos\theta}$

$= \frac{1-\cos^2\theta}{1-\cos\theta}$

$RS = 1 + \cos\theta$

$= \frac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}$

$= \frac{1-\cos^2\theta}{1-\cos\theta}$

$\because LS = RS \therefore \frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$

$y = a\sin(k(x-d)) + c$

$y = a\cos(k(x-d)) + c$

Ch 6 (Sinusodial Functions)

Periodic Function - self-repeating graph

Cycle - the smallest complete repeating pattern

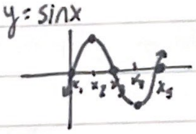
Period - the length of 1 cycle

Axis: the horizontal line that is halfway between min & max $y = \frac{max + min}{2}$

Amplitude: the vertical distance from axis to max or min $y = \frac{max - min}{2}$

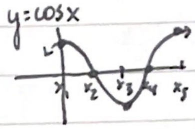
Point of inflection - a point where a graph changes concavity (concave up ∪, conc. down ∩)

$y = \sin x$



key pts: $(0,0)$, $(90,1)$, $(180,0)$, $(270,-1)$, $(360,0)$
axis, max, axis, min, axis

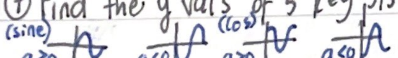
$y = \cos x$



key pts: $(0,1)$, $(90,0)$, $(180,-1)$, $(270,0)$, $(360,1)$
max, axis, min, axis, max

Graphing

1. Find the amplitude: amp $= |a|$
2. Find the axis: $y = c$
3. Find max & min: min = axis − amp, max = axis + amp
4. Find the phase shift: d
5. Find the period: $360°/k$
6. Find the x vals of 5 key points:
   1. $x_1 = d$
   4. $x_2 = (x_1 + x_3) \div 2$
   3. $x_3 = (x_1 + x_5) \div 2$
   5. $x_4 = (x_3 + x_5) \div 2$
   2. $x_5 = d + 360°/k$ or d + period
7. Find the y vals of 5 key pts:



(sine) $a > 0$: y_1 - axis, y_2 - max, y_3 - axis, y_4 - min, y_5 - axis

$a < 0$: y_1 - axis, y_2 - min, y_3 - axis, y_4 - max, y_5 - axis

(cos) $a > 0$: y_1 - max, y_2 - axis, y_3 - min, y_4 - axis, y_5 - max

$a < 0$: y_1 - min, y_2 - axis, y_3 - max, y_4 - axis, y_5 - min

Plug in and solve Q

- sub in x or y to solve for other value
- need CAST knowledge

eg. A sinusodial f^n has an amp of 4, a period of 120°, and a max at $(0,9)$, determine the eq^n.

+cos

$y = a\cos(k(x-d)) + c$
amp $= 4 \rightarrow a = 4$
period: $120° \rightarrow k = 3$
axis: $y = 5$
shift: $d = 0$

* the only things that can change are sign of a & d

$y = 4\cos 3x + 5$

e.g. Determine m if $(90°, m)$ is a point on the graph $y = -2\sin(2x + 45°) - 1$

sub in $x = 90°$, $y = m$

$m = -2\sin(2(90°) + 45°) - 1$

$m = -2\sin 225° - 1$

$m = -2(-\sin 45°) - 1$

$m = -2(-\sqrt{2}/2) - 1$

$m = \sqrt{2} - 1$

Finding the Equation

↳ Determine whether cos or sin

→ Determine if a is pos or neg

→ Find letter values:

a = amp (but ±)

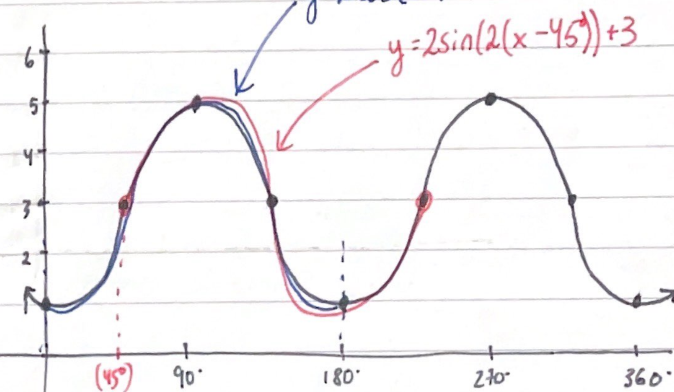
$k = 360°/\text{period}$

d = shift

$c = \frac{max + min}{2}$

→ with table of vals, just graph it

e.g. →



1. axis: $y = 3 \rightarrow \boxed{c = 3}$
2. amp: $|a| = 2 \rightarrow \boxed{a = \pm 2}$
3. $d = 0°$ *
4. $k = 360°/\text{period} = 360°/180° = 2 \rightarrow \boxed{k = 2}$

* for multiple equations

Unit 7: Sequences & Series

Sequences

$\{2,4,6,8\}$ Sequence - An ordered list of numbers

$\{2,4,6,8\}$ Finite seq. - a sequence with a definite (countable) num. of terms.

$\{2,4,6,8\ldots\}$ Infinite seq. - a sequence that continues indefinitely (uncountable)

$\{2,4,6,\}$ $\underbrace{}_{+2\ +2}$ $t_n = t_{n-1}+2$ Recursive Seq. - terms determined from previous term

$t_n = 2n$ (non-recursive seq. - from pos.)

$\{t_1, t_2, t_3, \ldots\}$ Terms - the numbers in a sequence

↑ subscripts for pos.

t_1 = first term = a t_n = n^{th} term = general term (used for formulas)

Pascal's Triangle

coefficients,
a exp. count ↓,
b exp. count ↑

$$
\begin{array}{ccccccccccccc|l}
 & & & & & & 1 & & & & & & & \leftarrow \text{row } 0 \\
 & & & & & 1 & & 1 & & & & & & \leftarrow \text{row } 1 \\
 & & & & 1 & & 2 & & 1 & & & & & \leftarrow \text{row } 2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow \text{row } 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow \text{row } 4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow \text{row } 5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \vdots \\
 & & & & & & \vdots & & & & & & &
\end{array}
$$

$(3x+4y)^3$
$=(3x)^3+3(3x)^2(4y)+3(3x)(4y)^2+(4y)^3$
$=27x^3+108x^2y+144xy^2+64y^3$

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = 1a^3+3a^2b^1+3a^1b^2+1b^3$

Arithmetic & Geometric Sequences

Arithmetic sequence - sequence with common difference

Geometric sequence - sequence with a common ratio

(non-recursive)

$$\boxed{t_n = a+(n-1)d}$$

$$\boxed{t_n = ar^{n-1}}$$

$a = t_1$ n = position
t_n = general term
d = common difference
r = common ratio

eg. Determine general term.

ⓐ $3, 9, 15, 21$ (+6, +6, +6)

$t_n = 3+(n-1)6$

$t_n = 6n-3$

eg. Determine the number of terms in the sequence.

$16, 7, -2, -11, \ldots, -245$ (−9)

$t_n = a+(n-1)d$

$-245 = 16+(n-1)(-9)$

$\boxed{n=30}$

*r is the thing that is multiplied.
so, if:
$16, 8, 4, 2, \ldots$
$r = \frac{1}{2}$

eg. For an ARITHMETIC sequence, $t_7 = 9$ and $t_{22} = 54$ determine t_1

$t_n = a+(n-1)d$

$9 = a+(7-1)d \Rightarrow$ ① $9 = a+6d$

$54 = a+(22-1)d \Rightarrow$ ② $54 = a+21d$

$45 = 15d$ ② − ①

$\boxed{d=3}$

sub d into ①

$9 = a+6(3)$

$a = -9$

$\therefore t_1 = -9$

eg. If $t_3 = 18$ & $t_8 = 4374$ of a GEOMETRIC sequence, determine t_1 and t_2.

$t_n = ar^{n-1}$

① $18 = ar^2$

② $4374 = ar^7$

$243 = r^5$ ② ÷ ①

$\sqrt[5]{243} = r$

$r = 3$

sub r into ①

$18 = a(3)^2$

$18 = 9a$

$a = 2$

$\therefore t_1 = 2$

$t_2 = (2)(3)^1$

$t_2 = 6$

$\therefore t_2 = 6$

↓

Series - sum of all values in a sequence

Arithmetic & Geometric Series

Arithmetic Series

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

Geometric Series

$$\boxed{S_n = \frac{a(r^n-1)}{r-1}}$$

eg. Determine the sum of $3+8+13+\ldots+248$:

$\underbrace{3+8}_{+5}\underbrace{+13}_{+5}$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$S_{50} = \frac{50}{2}[2(3)+49(5)]$

$\boxed{S_{50} = 6275}$

$t_n = a+(n-1)d$

$248 = 3+(n-1)5$

$248 = -2+5n$

$250 = 5n$

$\boxed{n=50}$

eg. How many terms of the series $2+5+8+\ldots$ add to the sum of 495?

$\underbrace{2+5}_{+3}\underbrace{+8}_{+3}$

$S_n = \frac{n}{2}[2a+(n-1)d]$

$495 = \frac{n}{2}[2(2)+(n-1)3]$

$990 = n(4+3n-3)$

$0 = 3n^2+n-990$

$0 = (3n+55)(n-18)$

$n_1 = -\frac{55}{3}$ or $\boxed{n_2 = 18}$

$\therefore n_1$ is invalid

$\therefore$ There are 18 terms.

eg. Determine the sum of the first 12 terms of the series:

$1-2+4-8+\ldots$

$\times -2$ $\times -2$ $\times -2$

$S_n = \frac{a(r^n-1)}{r-1}$

$S_{12} = \frac{(-2)^{12}-1}{-3}$

$S_{12} = \frac{4095}{-3}$

$\boxed{S_{12} = -1365}$

Summation Notation (Bonus)

Also called sigma notation

"add up" $\left\{ \sum_{i=1}^{34} (4i+31) \right.$ $= 35+39+\ldots+167$

34 ← ending term

i=1 ← starting term

← Rule (t_n) (gen. term)

167 is 4(34)+31

eg. Write in sigma notation

1, 3, 5, 7, 9, 11, 13

$\sum_{i=1}^{7}(2i-1)$

Never ending - use ∞

$$\sum_{i=2}^{\infty} \frac{1}{i} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

multiplying:

"multiply" $\left\{ \prod_{j=0}^{4} j^2 \right.$ $= 0\times1\times4\times9\times16 = 0$:(

Counting Down:

$$\sum_{n=3}^{0}(n+1) = 4+3+2+1$$

Study Notes - Unit 8

Common Time Periods:

annually - 1	semi-monthly - 24
semi-annually - 2	bi-weekly - 26
quarterly - 4	weekly - 52
monthly - 12	daily - 365

letters:

- A - FV (new amount)
- P - principal
- R - rate (per annum)
- T - time (years*)
- I - interest earned
- i - rate per compound period (divide)
- n - # of compound periods (multiply) (years * comp.p.)

examples
- look in notes & homework

Simple Interest "simple"

$\boxed{I = PRT}$

$A = P + I$

Compound Interest "compounded"

$\boxed{A = P(1+i)^n}$

$I = A - P$

$FV = PV(1+i)^n$

FV of an Annuity
- compound, multiple payments, large amount at end

"payments" "compounded"

$\boxed{FV = \frac{R[(1+i)^n - 1]}{i}}$

$\boxed{I = FV - Rn}$

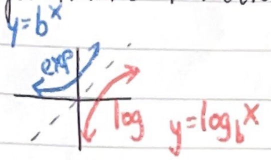
PV of an Annuity
- large amount at start

$\boxed{PV = \frac{R[1-(1+i)^{-n}]}{i}}$

$\boxed{I = Rn - PV}$

Extra?

Logarithmic Functions



$y=\log_b x$

- logs are the inverses of exponentials
- "logarithm of x with base b"
- "log base b of x"

$y=2^x \xrightarrow{\text{inverse}} y=\log_2 x$

- think: "what exponent gives an ans of x?"

$a^x=y$

$\log_a y = x$

- If you don't write a base, it is assumed to be 10

Properties of Logarithms:

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$
5. $\log_a(mn) = \log_a m + \log_a n$
6. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
7. $\log_a m^n = n\log_a m$
8. $\log_a b = \frac{\log_x b}{\log_x a}$ ← change of base formula. You choose x