Exercises and solutions: *Vectors*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*. Checking your work on a computer is a recommended second step.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. Based on the notation, determine whether each of the following symbols refers to a matrix, vector, or scalar.

a) $\mathbf{M}$ **b)** $\mathbf{w}$ **c)** $\mathbf{W}$ **d)** σ **e)** λ **f)** $\mathbf{q}$

2. State the type and dimensionality of the following vectors (e.g., "four-dimensional column vector"). For 2D vectors, additionally draw the vector starting from the origin.

a) $\begin{bmatrix}1\\2\\3\\1\end{bmatrix}$ **b)** $\begin{bmatrix}1 & 2 & 3 & 1\end{bmatrix}$ **c)** $\begin{bmatrix}-1\\3\end{bmatrix}$ **d)** $\begin{bmatrix}7 & 1/3\end{bmatrix}$

3. Solve the following operations. For 2D vectors, draw both vectors in standard position, and the vector sum (also in standard position).

a) $\begin{bmatrix}4 & 5 & 1 & 0\end{bmatrix} + \begin{bmatrix}-4 & -3 & 3 & 10\end{bmatrix}$

b) $\begin{bmatrix}4\\2\\0\end{bmatrix} - \begin{bmatrix}6\\-4\\60\end{bmatrix} + \begin{bmatrix}2\\-5\\40\end{bmatrix}$

c) $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$

d) $\begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}3\\4\end{bmatrix}$

e) $\begin{bmatrix}-3\\1\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix}$

f) $\begin{bmatrix}1\\4\end{bmatrix} + \begin{bmatrix}2\\8\end{bmatrix}$

4. Compute scalar-vector multiplication for the following pairs:

a) $-2\begin{bmatrix}4 & 3 & 0\end{bmatrix}$

b) $(-9 + 2{\cdot}5)\begin{bmatrix}0 & 4 & 3\end{bmatrix}$

c) $0\begin{bmatrix}3\\3.14{\cdot}\pi^{3.14}\\9\\-234987234\end{bmatrix}$

d) $\lambda\begin{bmatrix}0\\3\\1\\11\end{bmatrix}$

5. Compute the dot product between the following pairs of vectors.

a) $\begin{bmatrix}-4\\-2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}1\\3\end{bmatrix}$ **b)** $\begin{bmatrix}1 & 2i\\3 & \text{-}i\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}1 & 2i\\3 & i\end{bmatrix}$ **c)** $\begin{bmatrix}7\\-2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}-7\\-24\end{bmatrix}$ **d)** $\begin{bmatrix}3/2\\4/5\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}2/3\\-5/4\end{bmatrix}$

e) $\begin{bmatrix}0\\1\\2\end{bmatrix}^{\mathsf{T}} \begin{bmatrix}-2\\-1\\0\end{bmatrix}$ **f)** $\begin{bmatrix}4\\1\\3\end{bmatrix}^{\mathsf{T}} \begin{bmatrix}4\\1\\3\end{bmatrix}$ **g)** $\begin{bmatrix}7/2\\-3\\6\end{bmatrix}^{\mathsf{T}} \begin{bmatrix}10\\3.5\\-4\end{bmatrix}$ **h)** $\begin{bmatrix}81\\3\\9\end{bmatrix}^{\mathsf{T}} \frac{1}{3} \begin{bmatrix}1\\1\\1\end{bmatrix}$

6. Compute the outer product between the following pairs of vectors.

a) $\begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}^{\mathsf{T}}$ **b)** $\begin{bmatrix}-1\\3\\0\end{bmatrix} \begin{bmatrix}0\\1\\3\end{bmatrix}^{\mathsf{T}}$ **c)** $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix}^{\mathsf{T}}$ **d)** $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \begin{bmatrix}5\\6\\7\\8\end{bmatrix}^{\mathsf{T}}$

e) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \begin{bmatrix}10\\20\\30\\40\end{bmatrix}^{\mathsf{T}}$ **f)** $\begin{bmatrix}1\\1\\2\\2\end{bmatrix} \begin{bmatrix}a\\b\\c\\d\end{bmatrix}^{\mathsf{T}}$ **g)** $\begin{bmatrix}2\\6\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix}^{\mathsf{T}}$ **h)** $\begin{bmatrix}-4\\1\\3\end{bmatrix} \begin{bmatrix}2\\1\\3\end{bmatrix}^{\mathsf{T}}$

7. Compute a unit vector in the same direction as the following vectors. Compute the magnitude of the resulting vector to make sure it equals 1.

a) $\begin{bmatrix}1\\3\end{bmatrix}$ **b)** $\begin{bmatrix}3\\4\end{bmatrix}$ **c)** $\begin{bmatrix}6\\-8\end{bmatrix}$ **d)** $\begin{bmatrix}.1\\.2\\.4\\.2\end{bmatrix}$

8. Identify whether the following subsets are also subspaces. If they are not subspaces, find a vector in the set and a scalar such that $\lambda\mathbf{v}$ is outside the set.

a) All points in $\mathbb{R}^2$

b) All points on the line $y = 2x + .1$

c) Points satisfying $x^2 + y^2 + z^2 = 1$

d) All points on the line $y = 2x + 0$

9. Label the following sets as independent or dependent. For dependent sets, determine whether it is possible to modify only one element of one vector to change it to an independent set.

a) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ **b)** $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}\right\}$ **c)** $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\-2\end{bmatrix}\right\}$

d) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-1\\-2\end{bmatrix}\right\}$ **e)** $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$ **f)** $\left\{\begin{bmatrix}5\\12\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix}, \begin{bmatrix}10\\23\end{bmatrix}\right\}$

g) $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}\right\}$ **h)** $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}8\\16\\24\end{bmatrix}\right\}$ **i)** $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$

10. Prove that the algebraic and geometric formulas for the dot product are equivalent:
$\mathbf{v}^T\mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos(\theta_{vw})$

11. What is the magnitude (length) of vector $\mu\mathbf{v}$ for the following μ?

a) $\mu = 0$ **b)** $\mu = \|\mathbf{v}\|$ **c)** $\mu = 1/\|\mathbf{v}\|$ **d)** $\mu = 1/\|\mathbf{v}\|^2$

12. What is the dimensionality of the subspace spanned by $V = \lambda\mathbf{v} + \gamma\mathbf{w}$ for the following $\mathbf{v}, \mathbf{w}$?

a) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\0\\0\\0\end{bmatrix} \right\}$ **b)** $\left\{ \begin{bmatrix}1\\2\\6\\0\end{bmatrix}, \begin{bmatrix}2\\4\\12\\6\end{bmatrix} \right\}$ **c)** $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$ **d)** $\left\{ \begin{bmatrix}6\\9\\3\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}4\\6\\2\end{bmatrix} \right\}$

13. Remove one vector in the following sets to create a basis set for a 2D subspace.

a) $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix} \right\}$ **b)** $\left\{ \begin{bmatrix}1\\4\\3\end{bmatrix}, \begin{bmatrix}0\\-4\\5\end{bmatrix}, \begin{bmatrix}3\\12\\9\end{bmatrix} \right\}$ **c)** $\left\{ \begin{bmatrix}-3\\2\\13\end{bmatrix}, \begin{bmatrix}4.5\\-3\\-19.5\end{bmatrix}, \begin{bmatrix}-1.5\\1\\6\end{bmatrix} \right\}$

14. Determine whether the following vector is in the set spanned by the bracketed vectors, in other words, whether $\mathbf{u} \in S = \{\mathbf{v_1}, ..., \mathbf{v_n}\}$

a) $\begin{bmatrix}5\\1\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$ **b)** $\begin{bmatrix}4\\1\\12\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix} \right\}$ **c)** $\begin{bmatrix}0\\1\\3\end{bmatrix}, \left\{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix} \right\}$

Answers

1. -

a) matrix **b)** vector **c)** matrix **d)** scalar **e)** scalar **f)** vector

2. -

a) 4D column **b)** 4D row **c)** 2D column **d)** 2D row

3. ("Standard position" means the vector starts at the origin.)

a) $\begin{bmatrix} 0 & 2 & 4 & 10 \end{bmatrix}$

b) $\begin{bmatrix} 0 \\ 1 \\ -20 \end{bmatrix}$

c) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

f) $\begin{bmatrix} 3 \\ 12 \end{bmatrix}$

4. -

a) $\begin{bmatrix} -8 & -6 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 4 & 3 \end{bmatrix}$

c) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 \\ \lambda 3 \\ \lambda \\ \lambda 11 \end{bmatrix}$

5. -

a) -10 **b)** 7 $4i$ **c)** -1 **d)** 0

e) -1 **f)** 26 **g)** 1/2 **h)** $31 = (81{+}3{+}9)/3$

6. -

a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 0 & -1 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \\ 15 & 18 & 21 & 24 \\ 20 & 24 & 28 & 32 \end{bmatrix}$

e) $\begin{bmatrix} 10 & 20 & 30 & 40 \\ 20 & 40 & 60 & 80 \\ 30 & 60 & 90 & 120 \\ 40 & 80 & 120 & 160 \end{bmatrix}$

f) $\begin{bmatrix} a & b & c & d \\ a & b & c & d \\ 2a & 2b & 2c & 2d \\ 2a & 2b & 2c & 2d \end{bmatrix}$

g) $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

h) $\begin{bmatrix} -8 & -4 & -12 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

7. -

a) $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

c) $\begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$

d) $\frac{1}{\sqrt{.25}} \begin{bmatrix} .1 \\ .2 \\ .4 \\ .2 \end{bmatrix} = \begin{bmatrix} .2 \\ .4 \\ .8 \\ .4 \end{bmatrix}$

8. This question seems tricky because the question discusses *vector* $\mathbf{v}$ while the problems list *variables* x, y, and z. The insight is that $x = \mathbf{v}_1$, $y = \mathbf{v}_2$, and $z = \mathbf{v}_3$.

a) subspace and subset
b) subset (consider $\mathbf{v} = \mathbf{0}$)
c) subset
d) subspace and subset

9. -

a) independent
b) independent
c) dependent. Any element could be changed for independence.
d) dependent. Not possible to create an independent set.
e) dependent. Change a 0 to non-zero.
f) dependent. Not possible to create an independent set.
g) independent
h) dependent. Change any element to create an independent set.
i) dependent. Could change, e.g., 3 to 4 for independence.

10. This is explained in the video "Dot product from a geometric perspective," but you should try to reconstruct it on your own. *Hint: Here's the formula for the Law of Cosines:*
$\|\mathbf{q} - \mathbf{r}\|^2 = \|\mathbf{q}\|^2 + \|\mathbf{r}\|^2 - 2\|\mathbf{q}\|\|\mathbf{r}\|\cos(\theta_{qr})$

11. -

a) 0
b) $\|\mathbf{v}\|^2$
c) 1
d) $1/\|\mathbf{v}\|$

12. -

a) 1D
b) 2D
c) 3D
d) 1D

13. -

a) First or second
b) First or third
c) First or second

14. -

a) no
b) yes
c) no