Exercises and solutions: *Matrix multiplication*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. Determine whether each of the following operations is valid, and, if so, the size of the resulting matrix (note that $\odot$ indicates Hadamard multiplication).
$\mathbf{A} \in \mathbb{R}^{2\times3}$, $\mathbf{B} \in \mathbb{R}^{3\times3}$, $\mathbf{C} \in \mathbb{R}^{3\times4}$

a) $\mathbf{CB}$
b) $\mathbf{C}^\mathsf{T}\mathbf{B}$
c) $(\mathbf{CB})^\mathsf{T}$
d) $\mathbf{C}^\mathsf{T}\mathbf{BC}$
e) $\mathbf{ABCB}$
f) $\mathbf{ABC}$
g) $\mathbf{C}^\mathsf{T}\mathbf{BA}^\mathsf{T}\mathbf{AC}$
h) $\mathbf{B}^\mathsf{T}\mathbf{BCC}^\mathsf{T}\mathbf{A}$
i) $\mathbf{AA}^\mathsf{T}$
j) $\mathbf{A}^\mathsf{T}\mathbf{A}$
k) $\mathbf{BBA}^\mathsf{T}\mathbf{ABBCC}$
l) $(\mathbf{CBB}^\mathsf{T}\mathbf{CC}^\mathsf{T})^\mathsf{T}$
m) $(\mathbf{A}+\mathbf{ACC}^\mathsf{T}\mathbf{B})^\mathsf{T}\mathbf{A}$
n) $\mathbf{C}+\mathbf{CA}^\mathsf{T}\mathbf{ABC}$
o) $\mathbf{C}+\mathbf{BA}^\mathsf{T}\mathbf{ABC}$
p) $\mathbf{B}+3\mathbf{B}+\mathbf{A}^\mathsf{T}\mathbf{A}-\mathbf{CC}^\mathsf{T}$
q) $\mathbf{A}\odot\mathbf{ABC}$
r) $\mathbf{A}\odot\mathbf{ABC}(\mathbf{BC})^\mathbf{T}$

2. Compute the following matrix multiplications. For each problem, apply two different methods of matrix multiplication and confirm that they give the same answer.

a) $\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}0&5\\2&2\end{bmatrix}$
b) $\begin{bmatrix}-3&2\\-2&3\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}$
c) $\begin{bmatrix}11&-5\\9&-13\end{bmatrix}\begin{bmatrix}3&1\\-8&.5\end{bmatrix}$
d) $\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$
e) $\begin{bmatrix}2&2\\1&3\end{bmatrix}\begin{bmatrix}10&1\\-5&4\end{bmatrix}$
f) $\begin{bmatrix}a&0\\b&0\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix}$
g) $\begin{bmatrix}a&b\\0&0\end{bmatrix}\begin{bmatrix}2&3\\4&1\end{bmatrix}$
h) $\begin{bmatrix}1&0&4\\0&1&1\\3&3&0\end{bmatrix}\begin{bmatrix}-2&-3&-1\\-1&-9&3\\0&1&5\end{bmatrix}$
i) $\begin{bmatrix}a&0&1\\0&b&0\\1&0&c\end{bmatrix}\begin{bmatrix}a&b&c\\1&2&3\\0&0&1\end{bmatrix}$

3. Compute the following matrix-vector products, if the operation is valid.

a) $\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$
b) $\begin{bmatrix}2\\3\end{bmatrix}^T\begin{bmatrix}2&0\\0&3\end{bmatrix}$
c) $\begin{bmatrix}2&1\\2&3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$
d) $\begin{bmatrix}2\\3\end{bmatrix}^T\begin{bmatrix}2&1\\2&3\end{bmatrix}$
e) $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$
f) $\begin{bmatrix}1&0&1\\0&-4&0\\1&0&1\end{bmatrix}\begin{bmatrix}5\\2\\1\end{bmatrix}$
g) $\begin{bmatrix}1\\2\\5\end{bmatrix}\begin{bmatrix}1&0&1\\0&-4&0\\1&0&1\end{bmatrix}$
h) $\begin{bmatrix}3\\3\\2\end{bmatrix}^T\begin{bmatrix}1&3&2\\6&1&5\\3&5&0\end{bmatrix}$

4. For the following pairs of matrices, vectorize and compute the vector dot product, then compute the Frobenius inner product as $tr(\mathbf{A}^\mathsf{T}\mathbf{B})$.

a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 5 \\ 7 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 13 & 14 \end{bmatrix}$

c) $\begin{bmatrix} 4 & -5 & 8 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}, \begin{bmatrix} 4 & -5 & 8 \\ 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

e) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

f) $\begin{bmatrix} 1 & 1 & 7 \\ 2 & 2 & 6 \\ 3 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

5. An $N \times N$ matrix $\mathbf{A}$ has N^2 elements. However, because the matrix product $\mathbf{A}^T\mathbf{A}$ is symmetric, not all elements are unique. Create two matrices (one 2×2 and one 3×3) that contain non-zero and non-repeating integers, and compute $\mathbf{A}^T\mathbf{A}$ for each matrix. Count the number of total elements and the number of unique elements. Then work out a formula for the number of unique elements in such a matrix.

Answers

1. -

a) no
b) yes: 4×3
c) no
d) yes: 4×4
e) no
f) yes: 2×4
g) yes: 4×4
h) no
i) yes: 2×2
j) yes: 3×3
k) no
l) no
m) yes: 3×3
n) no
o) yes: 3×4
p) yes: 3×3
q) no
r) yes: 2×3

2. -

a) $\begin{bmatrix} 0 & 5 \\ 2 & 17 \end{bmatrix}$
b) $\begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix}$
c) $\begin{bmatrix} 73 & 8.5 \\ 131 & 2.5 \end{bmatrix}$
d) $\begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$
e) $\begin{bmatrix} 10 & 10 \\ -5 & 13 \end{bmatrix}$
f) $\begin{bmatrix} 2a & 3a \\ 2b & 3b \end{bmatrix}$
g) $\begin{bmatrix} 2a+4b & 3a+b \\ 0 & 0 \end{bmatrix}$
h) $\begin{bmatrix} -2 & 1 & 19 \\ -1 & -8 & 8 \\ -9 & -36 & 6 \end{bmatrix}$
i) $\begin{bmatrix} a^2 & ab & ac+1 \\ b & 2b & 3b \\ a & b & 2c \end{bmatrix}$

3. -

a) $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$
b) $\begin{bmatrix} 4 & 9 \end{bmatrix}$
c) $\begin{bmatrix} 7 \\ 13 \end{bmatrix}$
d) $\begin{bmatrix} 10 & 11 \end{bmatrix}$
e) $\begin{bmatrix} b \\ c \\ a \end{bmatrix}$
f) $\begin{bmatrix} 6 \\ -8 \\ 6 \end{bmatrix}$
g) invalid
h) $\begin{bmatrix} 27 & 22 & 21 \end{bmatrix}$

4. -

a) $a + 2b + 3c + 4d$
b) 63
c) 135
d) $a^2 + b^2 + ca + bd$
e) $a^2 + b^2 + c^2 + d^2$
f) undefined

5. A symmetric matrix can have up to $N(N+1)/2$ unique elements.