Exercises and solutions: *Matrix spaces*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. For each matrix-vector pair, determine whether the vector is in the column space of the matrix, and if so, the coefficients that map the vector into that column space.

a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$ **b)** $\begin{bmatrix}1&0\\2&0\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}$ **c)** $\begin{bmatrix}1&0\\2&0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}$ **d)** $\begin{bmatrix}1&1\\3&1\\0&1\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}$

e) $\begin{bmatrix}1&1\\3&1\\0&1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}$ **f)** $\begin{bmatrix}1&2\\2&1\end{bmatrix}, \begin{bmatrix}-3\\3\end{bmatrix}$ **g)** $\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}3\\6\\4\end{bmatrix}$ **h)** $\begin{bmatrix}-1&5&2\\-7&9&8\\-1&4&\pi\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$

2. Same as the previous exercise but for the row space.

a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}^T$ **b)** $\begin{bmatrix}1&0\\2&0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}^T$ **c)** $\begin{bmatrix}1&6\\2&12\end{bmatrix}, \begin{bmatrix}2\\9\end{bmatrix}^T$ **d)** $\begin{bmatrix}1&0&1\\2&2&0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}^T$

3. For each matrix-set pair, determine whether the vector set can form a basis for the column space of the matrix.

a) $\begin{bmatrix}1&0\\2&0\end{bmatrix}, \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$ **b)** $\begin{bmatrix}1&1\\2&1\end{bmatrix}, \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}4\\4\end{bmatrix}\right\}$ **c)** $\begin{bmatrix}3&6\\0&0\\1&2\end{bmatrix}, \left\{\begin{bmatrix}1\\0\\1/3\end{bmatrix}\right\}$

d) $\begin{bmatrix}0&0&3\\2&0&0\end{bmatrix}, \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ **e)** $\begin{bmatrix}e^{\pi}&3^{e}\\\sqrt[3]{3.7}&e^{e^2}\end{bmatrix}, \left\{\begin{bmatrix}-3\\-2\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$

4. Determine whether the following matrices have a null space. If so, provide basis vector(s) for that null space. Recall that a basis vector for the null space is a vector $\mathbf{v}$ such that $\mathbf{Av} = \mathbf{0}$ (excluding $\mathbf{v} = \mathbf{0}$).

a) $\begin{bmatrix}1&0\\2&0\end{bmatrix}$ **b)** $\begin{bmatrix}1&0\\2&2\end{bmatrix}$ **c)** $\begin{bmatrix}4&3\\1&1\\0&5\end{bmatrix}$ **d)** $\begin{bmatrix}3&1&5\\4&1&0\end{bmatrix}$

5. Fill in the blanks (dim=dimensionality) for matrix $\mathbf{A} \in \mathbb{R}^{2\times3}$

a) $dim(C(\mathbf{A})) = 0,\ dim(N(\mathbf{A}^T)) = \_\_\_$

b) $dim(C(\mathbf{A})) = 1,\ dim(N(\mathbf{A}^T)) = \_\_\_$

c) $dim(C(\mathbf{A})) = 2,\ dim(N(\mathbf{A}^T)) = \_\_\_$

d) $dim(C(\mathbf{A})) = 3,\ dim(N(\mathbf{A}^T)) = \_\_\_$

e) $dim(N(\mathbf{A})) = 0,\ dim(R(\mathbf{A})) = \_\_\_$

f) $dim(N(\mathbf{A})) = 1,\ dim(R(\mathbf{A})) = \_\_\_$

g) $dim(N(\mathbf{A})) = 2,\ dim(R(\mathbf{A})) = \_\_\_$

h) $dim(N(\mathbf{A})) = 3,\ dim(R(\mathbf{A})) = \_\_\_$

Answers

1. -

   a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ b) not in column space c) not in column space d) not in column space

   e) sizes don't match f) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ g) $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ h) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. -

   a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ c) Not in the row space d) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

3. -

   a) No. Any column space basis must be a single vector that is a multiple of $[1\ 2]^T$.
   b) Yes: $C(\mathbf{M}) = \mathbb{R}^2$, so any independent set of two vectors can be a basis.
   c) Yes
   d) Yes
   e) Yes for the same reason as (b).

4. -

   a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ b) No null space c) No null space d) $\begin{bmatrix} 1 \\ -4 \\ 1/5 \end{bmatrix}$

5. -

   a) 2
   b) 1
   c) 0
   d) Trick question; $dim(C(\mathbf{A}))$ cannot be greater than 2.
   e) Trick question; $dim(N(\mathbf{A}))$ must be >1.
   f) 2
   g) 1
   h) 0