Count outcomes, Poisson GLMs

Regression Models

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Key ideas

- Many data take the form of counts
  - Calls to a call center
  - Number of flu cases in an area
  - Number of cars that cross a bridge
- Data may also be in the form of rates
  - Percent of children passing a test
  - Percent of hits to a website from a country
- Linear regression with transformation is an option

Poisson distribution

- The Poisson distribution is a useful model for counts and rates
- Here a rate is count per some monitoring time
- Some examples uses of the Poisson distribution
  - Modeling web traffic hits
  - Incidence rates
  - Approximating binomial probabilities with small p and large n
  - Analyzing contigency table data

The Poisson mass function

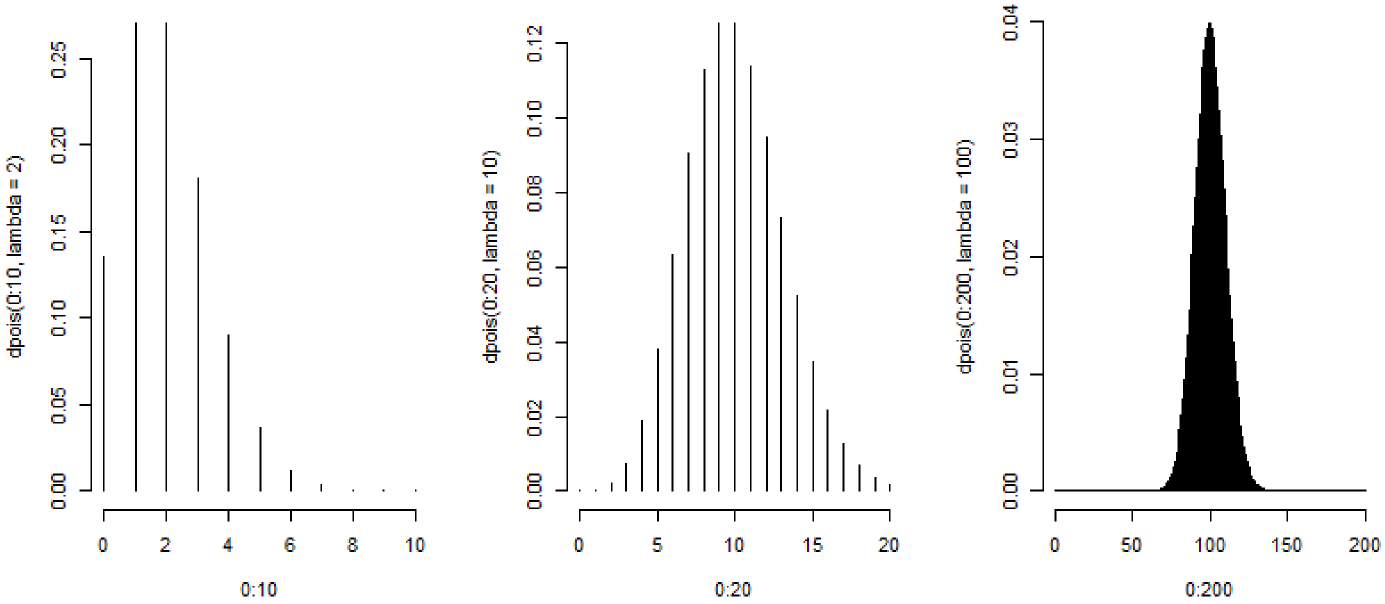
- $X \sim \text{Poisson}(t\lambda)$ if

$$P(X = x) = \frac{(t\lambda)^x e^{-t\lambda}}{x!}$$

  For $x = 0, 1, \ldots$.

- The mean of the Poisson is $E[X] = t\lambda$, thus $E[X/t] = \lambda$
- The variance of the Poisson is $Var(X) = t\lambda$.
- The Poisson tends to a normal as $t\lambda$ gets large.

```r
par(mfrow = c(1, 3))
plot(0 : 10, dpois(0 : 10, lambda = 2), type = "h", frame = FALSE)
plot(0 : 20, dpois(0 : 20, lambda = 10), type = "h", frame = FALSE)
plot(0 : 200, dpois(0 : 200, lambda = 100), type = "h", frame = FALSE)
```



Poisson distribution

Sort of, showing that the mean and variance are equal

```
x <- 0 : 10000; lambda = 3
mu <- sum(x * dpois(x, lambda = lambda))
sigmasq <- sum((x - mu)^2 * dpois(x, lambda = lambda))
c(mu, sigmasq)
```

```
[1] 3 3
```

Example: Leek Group Website Traffic

- Consider the daily counts to Jeff Leek's web site

http://biostat.jhsph.edu/~jleek/

- Since the unit of time is always one day, set $t = 1$ and then the Poisson mean is interpretted as web hits per day. (If we set $t = 24$, it would be web hits per hour).

Website data

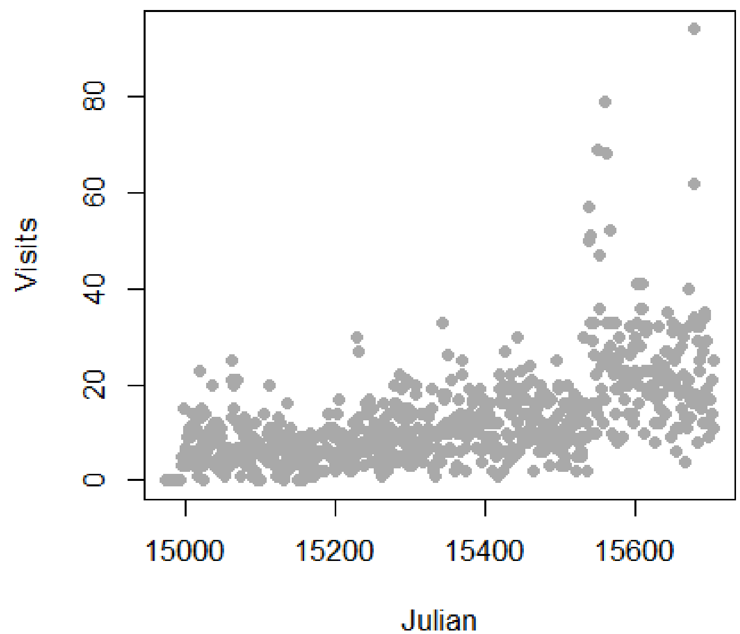
```
download.file("https://dl.dropboxusercontent.com/u/7710864/data/gaData.rda",destfile="./data/gaData.rda
load("./data/gaData.rda")
gaData$julian <- julian(gaData$date)
head(gaData)
```

```
        date visits simplystats julian
1 2011-01-01      0           0  14975
2 2011-01-02      0           0  14976
3 2011-01-03      0           0  14977
4 2011-01-04      0           0  14978
5 2011-01-05      0           0  14979
6 2011-01-06      0           0  14980
```

http://skardhamar.github.com/rga/

Plot data

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
```



Linear regression

$$NH_i = b_0 + b_1 JD_i + e_i$$

NH_i - number of hits to the website

JD_i - day of the year (Julian day)

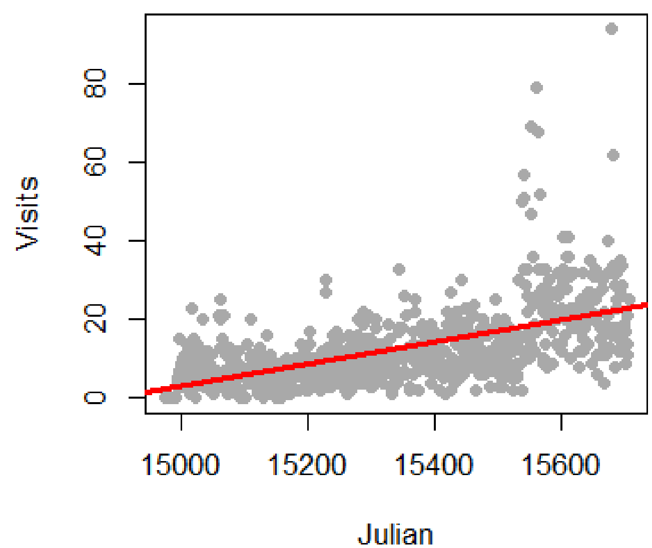
b_0 - number of hits on Julian day 0 (1970-01-01)

b_1 - increase in number of hits per unit day

e_i - variation due to everything we didn't measure

Linear regression line

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
lm1 <- lm(gaData$visits ~ gaData$julian)
abline(lm1,col="red",lwd=3)
```



Aside, taking the log of the outcome

- Taking the natural log of the outcome has a specific interpretation.
- Consider the model

$$\log(NH_i) = b_0 + b_1 JD_i + e_i$$

NH_i - number of hits to the website

JD_i - day of the year (Julian day)

b_0 - log number of hits on Julian day 0 (1970-01-01)

b_1 - increase in log number of hits per unit day

e_i - variation due to everything we didn't measure

Exponentiating coefficients

- $e^{E[\log(Y)]}$ geometric mean of Y.
  - With no covariates, this is estimated by $e^{\frac{1}{n}\sum_{i=1}^{n}\log(y_i)} = (\prod_{i=1}^{n} y_i)^{1/n}$
- When you take the natural log of outcomes and fit a regression model, your exponentiated coefficients estimate things about geometric means.
- e^{β_0} estimated geometric mean hits on day 0
- e^{β_1} estimated relative increase or decrease in geometric mean hits per day
- There's a problem with logs with you have zero counts, adding a constant works

```
round(exp(coef(lm(I(log(gaData$visits + 1)) ~ gaData$julian))), 5)
```

```
  (Intercept) gaData$julian
        0.000         1.002
```

Linear vs. Poisson regression

Linear

$$\mathrm{NH_i} = \mathrm{b_0} + \mathrm{b_1 JD_i} + \mathrm{e_i}$$

or

$$\mathrm{E[NH_i|JD_i, b_0, b_1]} = \mathrm{b_0} + \mathrm{b_1 JD_i}$$

Poisson/log-linear

$$\mathrm{log}\big(\mathrm{E[NH_i|JD_i, b_0, b_1]}\big) = \mathrm{b_0} + \mathrm{b_1 JD_i}$$

or

$$\mathrm{E[NH_i|JD_i, b_0, b_1]} = \mathrm{exp}\big(\mathrm{b_0} + \mathrm{b_1 JD_i}\big)$$

Multiplicative differences

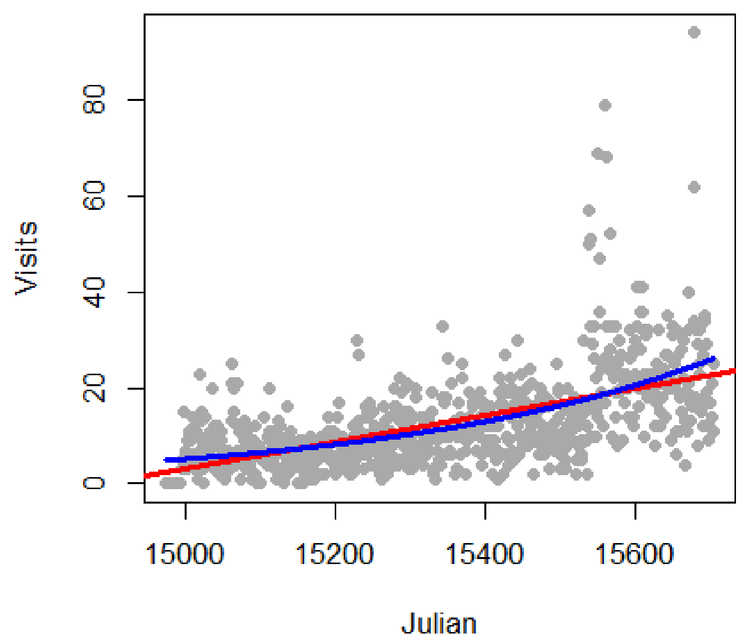
$$E[NH_i|JD_i, b_0, b_1] = \exp\left(b_0 + b_1 JD_i\right)$$

$$E[NH_i|JD_i, b_0, b_1] = \exp\left(b_0\right)\exp\left(b_1 JD_i\right)$$

If JD_i is increased by one unit, $E[NH_i|JD_i, b_0, b_1]$ is multiplied by $\exp\left(b_1\right)$

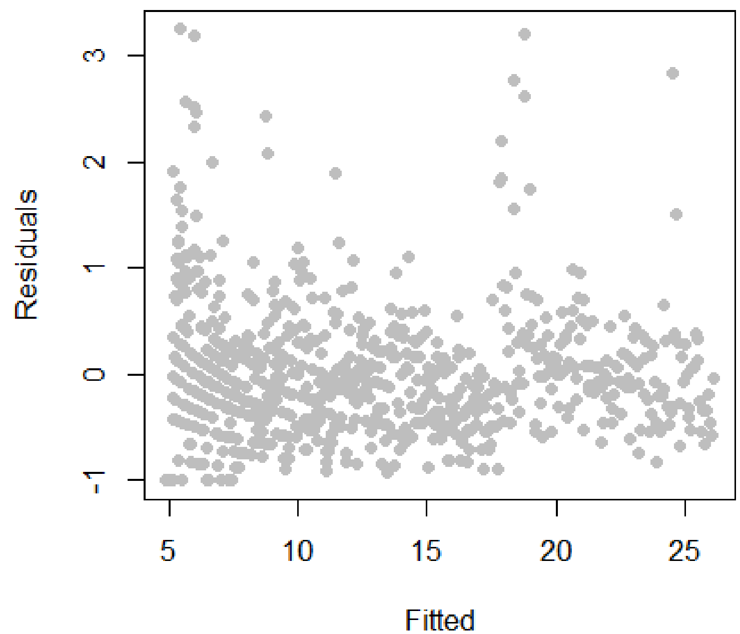
Poisson regression in R

```
plot(gaData$julian,gaData$visits,pch=19,col="darkgrey",xlab="Julian",ylab="Visits")
glm1 <- glm(gaData$visits ~ gaData$julian,family="poisson")
abline(lm1,col="red",lwd=3); lines(gaData$julian,glm1$fitted,col="blue",lwd=3)
```



Mean-variance relationship?

```
plot(glm1$fitted,glm1$residuals,pch=19,col="grey",ylab="Residuals",xlab="Fitted")
```



Model agnostic standard errors

```r
library(sandwich)
confint.agnostic <- function (object, parm, level = 0.95, ...)
{
    cf <- coef(object); pnames <- names(cf)
    if (missing(parm))
        parm <- pnames
    else if (is.numeric(parm))
        parm <- pnames[parm]
    a <- (1 - level)/2; a <- c(a, 1 - a)
    pct <- stats:::format.perc(a, 3)
    fac <- qnorm(a)
    ci <- array(NA, dim = c(length(parm), 2L), dimnames = list(parm,
                                                               pct))
    ses <- sqrt(diag(sandwich::vcovHC(object)))[parm]
    ci[] <- cf[parm] + ses %o% fac
    ci
}
```

http://stackoverflow.com/questions/3817182/vcovhc-and-confidence-interval

Estimating confidence intervals

```
confint(glm1)
```

```
                   2.5 %     97.5 %
(Intercept)   -34.34658 -31.159716
gaData$julian   0.00219   0.002396
```

```
confint.agnostic(glm1)
```

```
                    2.5 %     97.5 %
(Intercept)   -36.362675 -29.136997
gaData$julian   0.002058   0.002528
```

Rates

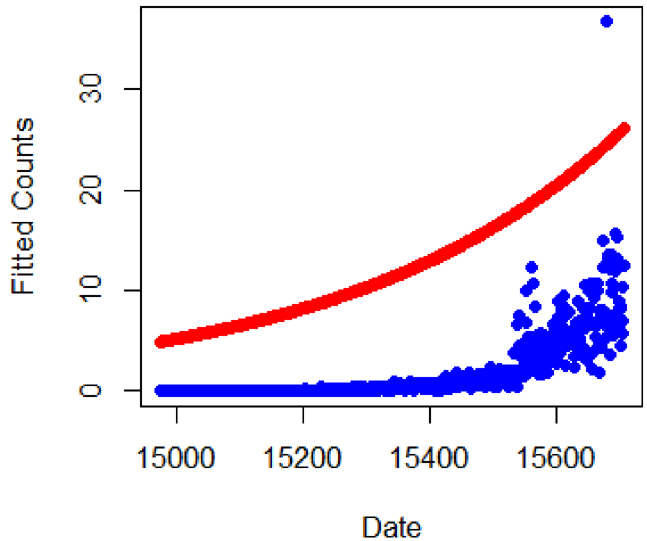
$$E[NHSS_i | JD_i, b_0, b_1]/NH_i = \exp\left(b_0 + b_1 JD_i\right)$$

$$\log\left(E[NHSS_i | JD_i, b_0, b_1]\right) - \log(NH_i) = b_0 + b_1 JD_i$$

$$\log\left(E[NHSS_i | JD_i, b_0, b_1]\right) = \log(NH_i) + b_0 + b_1 JD_i$$

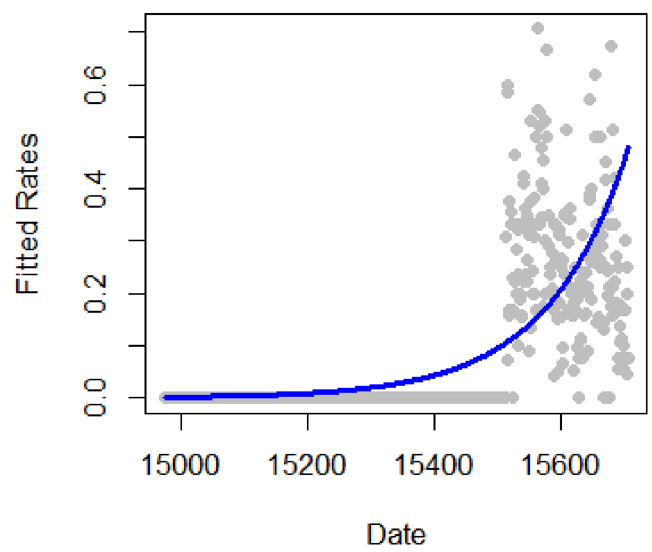
Fitting rates in R

```
glm2 <- glm(gaData$simplystats ~ julian(gaData$date),offset=log(visits+1),
            family="poisson",data=gaData)
plot(julian(gaData$date),glm2$fitted,col="blue",pch=19,xlab="Date",ylab="Fitted Counts")
points(julian(gaData$date),glm1$fitted,col="red",pch=19)
```



Fitting rates in R

```
glm2 <- glm(gaData$simplystats ~ julian(gaData$date),offset=log(visits+1),
            family="poisson",data=gaData)
plot(julian(gaData$date),gaData$simplystats/(gaData$visits+1),col="grey",xlab="Date",
     ylab="Fitted Rates",pch=19)
lines(julian(gaData$date),glm2$fitted/(gaData$visits+1),col="blue",lwd=3)
```



More information

- Log-linear models and multiway tables
- Wikipedia on Poisson regression, Wikipedia on overdispersion
- Regression models for count data in R
- pscl package - the function *zeroinfl* fits zero inflated models.