GS FOUNDATION (2023-24) Booklet 15
&
CSAT FOUNDATION 1.0 (2023-24) Booklet 14
COMBINATORICS

Contents

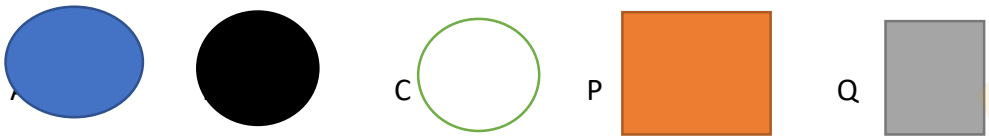
1) COUNTING: BASIS OF PERMUTATIONS AND COMBINATIONS

This topic is basically about counting. If you get the hang of how to count well, PnC is easy! Almost all questions from this topic would be in the form of 'in how many ways can we do it' – which is to count number of ways

For instance:

In how many ways can we pick one shape out of following 3 different circles and 2 different squares? – enlist it – 5 – so, that's your addition principle of counting!

Now, in how many ways can we pick 1 circle and 1 square – enlist – that's your multiplication principle!



Before answering it, we shall look at fundamental principles of counting:

1. Addition principle
2. Multiplication principle

2) FUNDAMENTAL PRINCIPLES OF COUNTING

These two principles form the basis of the entire chapter. UPSC asks very basic questions from this chapter which can be solved within seconds if you know these principles well.

Addition Principle:

The principle of addition states if a one task can be one done in 'm' ways and another task which is MUTUALLY EXCLUSIVE (Doing task 1 OR task 2 – we cannot do both at the same time) of the first task can be done in 'n' ways, then the number of possible ways in which either can be done is (m+n)

Ex: If I have 4 apples and 3 oranges in a basket, there are 3+4=7 ways I can pick one fruit from the basket

Here event is picking a fruit. I can pick either apple (out of 4) OR I can pick an orange (out of 3). Picking up apple or orange is totally unrelated to each other.

(**Basically, OR is addition**)

In above question on circles and squares, in how many ways can we pick a shape? 1 circle out of 3 OR 1 square out of 2 – We can pick a shape in 3+2=5 ways

Multiplication Principle:

The principle of multiplication states that if one task can be done in 'm' ways and another task which is INDEPENDENT of the first task can be done in 'n' ways, after the first task has been performed, then the number of possible ways in which both the tasks can be done is (m×n)

The ***independent*** **means** that no matter how one task is performed, the number of ways you can perform the second task is the same.

For example, the tasks "pick one apple" and "pick one orange" are independent in the previous example, since no matter what apple I pick, I still have the same 3 oranges to pick from.

Also, in circle and square example, no matter how we pick a circle from 3, we still have 2 squares to pick from.

In How many ways can we pick 1 circle AND 1 square? $3 \times 2 = 6$

In how many ways can we pick 1 apple and 1 orange from 4 apples AND 3 oranges in a basket? – are events independent? – YES

So, number of ways = $4 \times 3 = 12$

Basically, **AND is multiplication**

Thing to note here is that, in both above examples – events were independent.

Now, what if the events were dependent? We can simply modify our multiplication principle.

For example: Kritika wants to buy a car. The dealer says she can buy either Car A or car B. There are three models of each car viz. GXI, SXI, ZXI. There are 5 colour – red, black, green, yellow and white options for each car. How many options does she have.

What if the dealer says, car A does not come in black. How many options does she have now?

$$3 \times 5 + 3 \times 4 = 27$$

Notice that, Choice of colour is DEPENDENT on white kind of car Kritika choses to buy.

3) FORMING NUMBERS

Q. How many 3-digit numbers can be formed from 1, 2, 3, 4, 5

Notice that, there're 3 slots to be filled – hundreds place, tens place and unit's place.

- **If repetition is not allowed**

Hundred's place can be filled with 1 OR 2 OR 3 OR 4 OR 5 – 5 options.

Now that we've chosen any one (like we chose car in above example) we've 4 options left for tens digit & 3 options left for unit's digit

We've to chose Hundred's digit AND Ten's digit AND unit's digit

Total ways = $5 \times 4 \times 3 = 60$

(Will the answer change if we've to form 3-digit numbers from 0, 1, 2, 3, 4, 5) – just remove 2-digit numbers?

- **If repetition is allowed**

In this case, whatever restrictions we had are not there. Every choice of digit is again an INDEPENDENT event. We can directly apply multiplication principle.

For each digit, there are 5 choices – Total choices = $5 \times 5 \times 5 = 125$

(Will the answer change if we've to form 3-digit numbers from 0, 1, 2, 3, 4, 5)

Here, there's no condition on how digits are related.

Let's put some conditions and see how can we count the numbers.

In case of any condition: **WE ALWAYS START WITH THE CONDITION** – we first ensure that, while counting, the condition gets satisfied and then count.
For instance, if we're to find out 3-digit numbers out of 0, 1, 2, 3, 4 – we first have to ensure that, the number in fact is a 3-digit one. And thus, we began with hundred's place – put a non-zero digit there and proceed with the counting.

Other variants of the same question as above:

- How many 3-digit numbers divisible by 2 can be formed?
- How many 3-digit numbers not divisible by 2 can be formed?
- How many 3-digit numbers divisible by 3 can be formed?
- How many 3-digit numbers divisible by 5 can be formed?
- How many 3-digit numbers greater than 300 can be formed?

Let's do all the above questions for 2-digit; 4-digit and 5-digit numbers

What we did here for 5-digit case is simply an arrangement. Basically, symbols or things (digits in this case) remain same but we only re-arrange them in various ways.
As seen above, there are exactly n! ways of arranging 'n' symbols or objects

Q. How many factors of the number 28 * 36 * 54 * 105 are multiples of 120?
A. 540
B. 660
C. 594
D. 792

4) ANAGRAMS

Anagram word you spell by rearranging the letters of another word
For example, PCUS is an anagram of UPSC. UCSP is another example.
The question is, how many anagrams are possible from the given 'n' lettered word.
Because anagram is nothing but a re-arrangement, there are exactly n! anagrams possible.
Above question is same as having 'n' letters and finding out how many ways are there to arrange without repetition.

Other variants of anagram questions like

- Finding anagrams beginning or ending with vowel
- Having vowel or a consonant at particular places
- Having all vowels together etc.

Q. How many anagrams can be formed from DELHI?
Now comes the important part:

Q. How many anagrams can be formed from letters of MUMBAI?

- Notice that we don't have all letters as distinct.
- Had they been different, like if 2 Ms are M_1 and M_2, then we know that number of anagrams is simply 6! Right?
- But here, M_1UM_2BAI and M_2UM_1BAI are actually same words – and this is true for all the anagrams.
- 2 Ms being same, all the rearrangements of 2 Ms must be counted as 1.
- Q. How many re-arrangements are possible of 2 letters M_1 and M_2?
- Number of ways thus needs to divided by this as we don't want any double counting

Number of ways = $\frac{6!}{2!} = 360$

Q. How many anagrams can be formed from letters of KOLKATA?
Q. How many anagrams can be formed from letters of BENGALURU?
Q. How many anagrams can be formed from letters of QUEUE?
Q. How many anagrams can be formed from letters of ACCESS?

Q. How many anagrams can be formed from letters of EEIRIE?

- Only difference is that, E appears 3 times and I appear twice
- How many rearrangements possible of 3 E?
- How many of 2 I?
- Number of anagrams = $\frac{6!}{3!2!} = 60$

Q. How many anagrams can be formed from letters of TREE such that all vowels are together?

Q. How many anagrams can be formed from letters of KOLKATA such that all vowels are together?
Q. How many anagrams can be formed from letters of FOOTBALL such that all vowels are together?

- Arranging 6 when two are identical AND arranging 3 when 2 are identical

5) SEATING ARRANGEMENTS: LINEAR SEATING

Linear seating:
Q. How many ways are there to sit 6 people A, B, C, D, E, F on 6 chairs – same as re-arrangement: 6!
Now, let's put some restrictions:

- How many ways are there to sit 6 people A, B, C, D, E, F on 6 chairs if A and B always sit together?
- How many ways are there to sit 6 people A, B, C, D, E, F on 6 chairs if A and B don't sit together?
- How many ways are there to sit 6 people A, B, C, D, E, F on 4 chairs?

6) PERMUTATIONS AND COMBINATIONS

PERMUTATION:

a permutation of a set is, loosely speaking, an arrangement of its members into a sequence or linear order, or if the set is already ordered, a rearrangement of its elements.

In permutation, order is very important.

For instance, (1, 3, 5) is different than (3, 1, 5)

Let's take our linear seating example: How many ways are there to sit 6 people A, B, C, D, E, F on 4 chairs.

It's $6 \times 5 \times 4 \times 3 = \frac{6!}{2!} = \frac{6!}{(6-4)!}$

This is what permutation is. Notice that, order of persons sitting on the chair is important here: A – B – C – D is different from C – A – B – D

If we've to chose 'r' things out of 'n' without replacement/repetition and where order is important, there are: ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ Ways of doing it.

Let's revisit our forming numbers section:

How many 3-digit numbers can be formed from 1, 2, 3, 4, 5 without repetition?

Here n = 5 and r = 3 and without repetition and order is important as place values differ when we use different digits – means we've just use permutation formula

Number of ways = ${}^{5}P_{3} = \frac{5!}{(5-3)!} = 60\ ways$

Fundamentally all formulae like that of permutation, combination etc. follow from our counting principles albeit with some modifications.

Now, let's modify above question a bit.

How many ways are there to choose 3 numbers from 1, 2, 3, 4, 5?

- Note: we don't want to form 3-digit numbers, we just want to chose the numbers
- Is order important? Is choice (1, 2, 3) different from (2, 1, 3)?
- So, if all such permutations of any 3 chosen numbers are same, we need to ensure, they're not double counted
- What's the number of ways 3 chosen numbers can be rearranged?
- Thus, number of ways = ${}^{5}P_{3}/3! = \frac{5!}{2!3!}$ = 10

The combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter. In such case number of ways of choosing 'r' items out of 'n' = ${}^{n}C_{r} = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$

NOTE:

- ${}^{n}C_{r} = {}^{n}C_{n-r}$
- ${}^{n}P_{r} > {}^{n}C_{r}$

Permutations are for lists (order matters) and combinations are for groups (order doesn't matter). Combinations are easy going. Order doesn't matter. You can mix it up and it looks the same.

Permutation = Arrange – Order matters

Combination = select/choose – Order irrelevant

You know, a "combination lock" should really be called a "permutation lock". The order you put the numbers in matters.

If we've 8 different chocolates and we want to give 3 out of them away. How many ways are there to do this?

What if we want to chocolates to Raju, Shyam and Baburao?

Q. There are 7 boys and 9 girls. We want to choose 4 boys and 4 girls. How many ways are there to choose?

- If we select any 4, is the order important?
- AND is multiplication
- $WAYS = 7C4 \times 9C4$

Q. How many ways are there to choose 2 balls from 5 identical red balls?

Q. How many ways to choose 3 digits from numbers 0, 1, 2, 3, 4, 5?

Q. How many ways to choose ordered pairs of 3 digits from numbers 0, 1, 2, 3, 4, 5?

Q. How many 3-digit numbers can be formed from numbers 0, 1, 2, 3, 4, 5?

7) SEATING ARRANGEMENTS: CIRCULAR SEATING

How many ways are there to sit 5 people around a circular table with chairs equidistant?

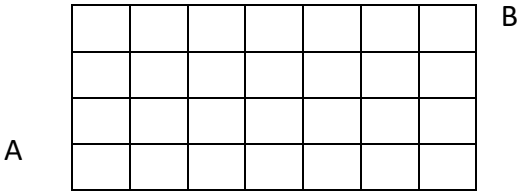
Recall how we counted linear arrangements: 5 seats: 5 options for 1st person; 4 for 2nd and so on.

In case of circular table, how many options does 1st person have?

- When the first person takes a seat, then we have order of chair with respect to that person.
- Now, when the 2nd person comes, he has 4 options and so on.

Number of ways N people can be seated around a circular table = (N – 1)!

8) NUMBER OF PATHS



- What is the number of ways can a person reach point B starting from point A along shortest paths?
- Along shortest path: person will go 7Right and 4Up without taking left or downwards turn
- He/she will always travel 11-unit distance

Basically, we've to choose in how many ways can a person who travels 11-unit distance take 7 rights (OR equivalently 4 Ups)

It's simply 11 choose 7 (Which is same as 11 Choose 4)

Alternatively:

The problem is same as finding different anagrams out of RRRRRRRUUUU

Which is again simply, $\frac{11!}{7!4!}$

9) NON-ZERO SOLUTIONS TO LINEAR EQUATION

Number of non-zero solutions to linear equation:

$x_1 + x_2 + \cdots + x_r = n;\ x_1, x_2, x_r \geq 0$

Are given by: ${}^{n+r-1}C_{r-1}$

Example: How many ways are there to distribute 4 chocolates among 5 boys?

Let, boy 1 gets x_1 chocolates, boy 2 gets x_2 chocolates and so on.

We want number of solutions of

$$x_1 + x_2 + \cdots + x_5 = 4$$

Which are = ${}^{4+5-1}C_{5-1} = {}^{8}C_4 = 70$

10) PYQS

CSE 2023: A flag has to be designed with 4 horizontal strips using some or all of the colours red, green and yellow. What is the number of different ways in which this can be done so that no two adjacent stripes have the same colour?

(a) 12 (b) 18 (c) 24 (d) 36

CSE 2023: In an examination, the maximum marks for each of the four papers namely P, Q, R and S are 100. Marks scored by the students are in integers. A student can score 99% in n different ways. What is the value of n?

(a) 16 (b) 17 (c) 23 (d) 35

CSE 2023: There are five persons, P, Q, R, S and T each one of whom has to be assigned one task. Neither P nor Q can be assigned Task-1. Task-2 must be assigned to either R or S. In how many ways can the assignment be done?

(a) 6 (b) 12 (c) 18 (d) 24

CSE 2023: How many natural numbers are there which given a remainder of 31 when 1186 is divided by these natural numbers?
(a) 6 (b) 7 (c) 8 (d) 9

CSE 2023: There are four letters and four envelopes and exactly one letter is to be put in exactly one envelope with the correct address. If the letters are randomly inserted into the envelopes, then
consider the following statements:
1. It is possible that exactly one letter goes into an incorrect envelope.
2. There are only six ways in which only two letters can go into the correct envelopes.
Which of the statements given above is/are correct?
(a) 1 only (b) 2 only (c) Both 1 and 2 (d) Neither 1 nor 2

CSE 2023: In how many ways can a batsman score exactly 25 runs by scoring single runs, fours and sixes only, irrespective of the sequence of scoring shots?
(a) 18 (b) 19 (c) 20 (d) 21

CSE 2023: What is the number of selections of 10 consecutive things out of 12 things in a circle taken in the clockwise direction?
(a) 3 (b) 11 (c) 12 (d) 66

CSE 2023: How many distinct 8-digit numbers can be formed by rearranging the digits of the number 11223344 such that odd digits occupy odd positions and even digits occupy even positions?
(a) 12 (b) 18 (c) 36 (d) 72

CSE 2023: Let x be a positive integer such that 7x + 96 is divisible by x. How many values of x are possible?
(a) 10 (b) 11 (c) 12 (d) Infinitely many

CSE 2022: The digits 1 to 9 are arranged in three rows in such a way that each row contains three digits, and the number formed in the second row is twice the number formed in the first row; and the number formed in the third row is thrice the number formed in the first row. **Repetition of digits is not allowed**. If only three of the four digits 2, 3, 7 and 9 are **allowed to use in the first row**, how many such combinations are possible to be arranged in the three rows? (a) 4
(b) 3
(c) 2
(d) 1

CSE 2022: How many 3-digit natural numbers (without repetition of digits) are there such that each digit is odd and the number is divisible by 5?
(a) 8
(b) 12
(c) 16
(d) 24

CSE 2022: The letters A, B, C, D and E are arranged in such a way that there are exactly two letters between A and E. How many such arrangements are possible?
(a) 12
(b) 18
(c) 24
(d) 36

CSE 2022: There are 9 cups placed on a table arranged in equal number of rows and columns out of which 6 cups contain coffee and 3 cups contain tea. In how many ways can they be arranged so that each row should contain at least one cup of coffee?
(a) 18
(b) 27
(c) 54
(d) 81

CSE 2022: Let A, B and C represent distinct non-zero digits. Suppose x is the sum of all possible 3-digit numbers formed by A, B and C without repetition.
Consider the following statements:
1. The 4-digit least value of x is 1332.
2. The 3-digit greatest value of x is 888
Which of the above statements is/are correct?
(a) 1 only
(b) 2 only
(c) Both 1 and 2
(d) Neither 1 nor 2

CSE 2022: There is a numeric lock which has a 3-digit PIN. The PIN contains digits 1 to 7. There is no repetition of digits. The digits in the PIN from left to right are in decreasing order. Any two digits in the PIN differ by at least 2. How many maximum attempts does one need to find out the PIN with certainty?
(a) 6
(b) 8
(c) 10
(d) 12

CSE 2022: One non-zero-digit, one vowel and one consonant from English alphabet (in capital) are to be used in forming passwords, such that each password has to start with a vowel and end with a consonant. How many such passwords can be generated?
(a) 105
(b) 525
(c) 945
(d) 1050

CSE 2022: What is the number of numbers of the form 0.XY, where X and Y are distinct non-zero digits?
(a) 72
(b) 81
(c) 90
(d) 100

CSE 2022: A, B and C are three places such that there are three different roads from A to B, four different roads from B to C and three different roads from A to C. In how many different ways can one travel from A to C using these roads.
(a) 10
(b) 13
(c) 15
(d) 36

CSE 2021: Consider all 3-digit numbers (without repetition of digits) obtained using three non-zero digits which are multiples of 3. Let S be their sum.
Which of the following is/are correct?
1. S is always divisible by 74.
2. S is always divisible by 9.
Select the correct answer using the code given below:
(a) 1 only
(b) 2 only
(c) Both 1 and 2
(d) Neither 1 nor 2

CSE 2021: On a chess board, in how many different ways can 6 consecutive squares be chosen on the diagonals along a straight path?
(a) 4
(b) 6
(c) 8
(d) 12

CSE 2021: Using 2, 2, 3, 3, 3 as digits, how many distinct numbers greater than 30000 can be formed?
(a) 3
(b) 6
(c) 9
(d) 12

CSE 2021: There are 6 persons arranged in a row. Another person has to shake hands with 3 of them so that he should not shake hands with two consecutive persons. In how many distinct possible combinations can the handshakes take place?
(a) 3
(b) 4
(c) 5
(d) 6

CSE 2021: How many different 5letter words (with or without meaning) can be c word 'DELHI' so that each word has to start with D and end with I?
(a) 24
(b) 18
(c) 12
(d) 6

CSE 2021: How many different sums can be formed with the denomination Rs. 50, Rs. 100, Rs. 200, Rs. 500, and Rs. 2,000 taking at least three denominations at a time?
(a) 16
(b) 15
(c) 14
(d) 10

CSE 2020: Four-digit numbers are to be formed using the digits 1, 2, 3 and 4; and none of these four digits are repeated in any manner. Further,
1. 2 and 3 are not to immediately follow each other
2. 1 is not to be immediately followed by 3
3. 4 is not to appear at the last place
4. 1 is not to appear at the first place
How many different numbers can be formed?
(a) 6
(b) 8
(c) 9
(d) None of the above

CSE 2020: Two statements S1 and S2 are given below followed by a Question:
S1: There are not more than two figures on any page of a 51-page book
S2: There is at least one figure on every page
Question: Are there more than 100 figures in that book?
Which one of the following is correct in respect of the above Statements and the Question?
(a) Both S1 and S2 are sufficient to answer the Question, but neither S1 alone nor S2 alone is sufficient to answer the Question
(b) S1 alone is sufficient to answer the Question. (c) S1 and S2 together are not sufficient to answer the Question.
(c) S1 and S2 together are not sufficient to answer the Question.
(d) S2 alone is sufficient to answer the Question.

CSE 2018: While writing all the numbers from 700 to 1000, how many numbers occur in which the digit at hundred's place is greater than the digit at ten's place, and the digit at ten's place is greater than the digit at unit's place?
(a) 61
(b) 64
(c) 85
(d) 91

CSE 2018: How many diagonals can be drawn by joining the vertices of an octagon?
a. 20
b. 24
c. 28
d. 64

CSE 2017: If 2 boys and 2 girls are to be arranged in a row so that the girls are not next to each other, how many possible arrangements are there?
(a) 3
(b) 6
(c) 12
(d) 24

CSE 2017: How many numbers arc there between 99 and 1000 such that the digit 8 occupies the units place?
(a) 64
(b) 80
(c) 90
(d) 104

CSE 2017: Certain 3-digit numbers following characteristics: 1. All the three digits are different. 2. The number is divisible by 7. 3. The number on reversing the digits is also divisible by 7. How many such 3-digit numbers are there?
(a) 2
(b) 4
(c) 6
(d) 8

CSE 2016: How many numbers are there between 100 and 300 which either begin with or end with 2?
(a) 110
(b) 111
(c) 112
(d) None of the above

CSE 2015: Twelve people form a club. By picking lots, one of them will host a dinner for all once in a month.
The number of dinners a particular member has to host in one year is
(a) One
(b) Zero
(c) Three
(d) Cannot be predicted

CSE 2015: There are 5 tasks and 5 persons. Task-I cannot be assigned to either person-I or person-2. Task-2 must be assigned to either person-3 or person-4. Every person is to be assigned one task. In how many ways can the assignment be done?
(a) 6
(b) 12
(c) 24
(d) 144

CSE 2015: In a society it is customary for friends of the same sex to hug and for friends of opposite sex to shake hands when they meet. A group of friends met in a party and there were 24 handshakes. Which one among the following numbers indicates the possible number of hugs?
(a) 39
(b) 30
(c) 21
(d) 20

CSE 2015: A selection is to be made for one post of Principal and two posts of Vice-Principal. Amongst the six candidates called for the interview, only two are eligible for the post of Principal while they all are eligible for the post of Vice-Principal. The number of possible combinations of selectees is

(a) 4

(b) 12

(c) 18

(d) None of the above

CSE 2015: A student has to opt for 2 subjects out of 5 subjects for a course, namely, Commerce, Economics, Statistics, Mathematics I and Mathematics II. Mathematics II can be offered only if Mathematics I is also opted. The number of different combinations of two subjects which can be opted is

(a) 5

(b) 6

(c) 7

(d) 8

11) COMPREHENSION

While awareness on use/misuse and abuse of antibiotics is common knowledge, as is the impact of dosing poultry with antibiotics, the environmental impact of antibiotics-manufacturing companies not treating their waste has scarcely been discussed at any length or seriousness thus far. Pollution from antibiotics factories is fuelling the rise of drug-resistant infections. The occurrence of drug-resistant bacteria surrounding the pharma manufacturing plants is well known.

Q. Which one of the following statements best reflects the most logical and practical message conveyed by the passage?

(a) It is necessary to put proper effluent treatment protocols in place.

(b) It is necessary to promote environmental awareness among people.

(c) Spread of drug-resistant bacteria cannot be done away with, as it is inherent in modern medical care.

(d) Pharma-manufacturing companies should be set up in remote rural areas, away from crowded towns and cities.

Sourcing food from non-agricultural lands (uncultivated systems such as forests, wetlands, pastures, etc) in addition to agricultural lands enables a systemic approach to food consumption. It allows rural and tribal communities to sustain themselves for the whole year and steer clear of natural disasters and season-inducted shortfalls of agricultural food. Since the productivity of trees is often more resilient to adverse weather conditions than annual crops, forest foods often provide a safety net during periods of food shortages caused by crop failure; forest foods also make important contributions during seasonal crop production gaps.

Q. Which one of the following statements best reflects the most logical and rational massage conveyed by the author of the passage?

(a) Food yielding trees should replace other trees in rural and tribal areas and community owned lands.

(b) Food security cannot be ensured in India with the present practice of conventional agriculture.

(c) Wastelands and degraded areas in India should be converted into agroforestry systems to help the poor.

(d) Agroecosystems should be developed in addition to or along with conventional agriculture.

India should ensure the growth of the digital economy while keeping personal data of citizens secure and protected. No one will innovate in a surveillance-oriented environment or in a place where an individual's personal information is compromised. The ultimate control of data must reside with the individuals who generate it; they should be enabled to use, restrict or monetize it as they wish. Therefore, data protection laws should enable the right kind of innovation–one that is user-centric and privacy protecting.

Q. Based on the above passage, the following assumptions have been made:

1. Protection of privacy is not just a right, but it has value to the economy.
2. There is a fundamental link between privacy and innovation.

Which of the above assumptions is/are valid?

(a) 1 only (b) 2 only (c) Both 1 and 2 (d) Neither 1 nor 2