$$\int Norm_x[a,A]Norm_x[b,B]dx$$

$$= \int \frac{1}{\sqrt{2\pi}|A|^{1/2}} \exp\left(-\frac{1}{2}(x-a)A^{-1}(x-a)^T\right) \cdot \frac{1}{\sqrt{2\pi}|B|^{1/2}} \exp\left(-\frac{1}{2}(x-b)B^{-1}(x-b)^T\right) dx$$

$$= \frac{1}{\sqrt{2\pi}|A|^{1/2}|B|^{1/2}} \int \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left((x-a)A^{-1}(x-a)^T + (x-b)B^{-1}(x-b)^T\right)\right) dx$$

We analyze the formula in exp() now

$$(x-a)A^{-1}(x-a)^T + (x-b)B^{-1}(x-b)^T$$

$$= xA^{-1}x^T - xA^{-1}a^T - aA^{-1}x^T + aA^{-1}a^T + xB^{-1}x^T - xB^{-1}b^T - bB^{-1}x^T + bB^{-1}b^T$$

$$= x(A^{-1}+B^{-1})x^T - x(A^{-1}a^T + B^{-1}b^T) - (aA^{-1}+bB^{-1})x^T + aA^{-1}a^T + bB^{-1}b^T$$

$$= x(A^{-1}+B^{-1})x^T - x(A^{-1}+B^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T)$$

$$-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})x^T$$

$$+aA^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})a^T + bB^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})b^T$$

$$= x(A^{-1}+B^{-1})x^T - x(A^{-1}+B^{-1})\left((aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}\right)^T$$

$$-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})x^T$$

$$+aA^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T - B^{-1}b^T + B^{-1}a^T)$$

$$+bB^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}b^T - A^{-1}a^T + A^{-1}a^T + B^{-1}b^T)$$

$$= x(A^{-1}+B^{-1})x^T - x(A^{-1}+B^{-1})\left((aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}\right)^T$$

$$-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})x^T$$

$$+aA^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T) + aA^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}(a^T - b^T)$$

$$+bB^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}(b^T - a^T) + bB^{-1}(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T)$$

$$= x(A^{-1}+B^{-1})x^T - x(A^{-1}+B^{-1})\left((aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}\right)^T$$

$$-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}+B^{-1})x^T$$

$$+(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}(A^{-1}a^T + B^{-1}b^T)$$

$$+aA^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}(a^T - b^T) - bB^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}(a^T - b^T)$$

$$= (x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})(A^{-1}+B^{-1})(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})^T$$
$$+a(B(A^{-1}+B^{-1})A)^{-1}(a^T-b^T)-b(A(A^{-1}+B^{-1})B)^{-1}(a^T-b^T)$$
$$= (x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})(A^{-1}+B^{-1})(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})^T$$
$$+a(A+B)^{-1}(a^T-b^T)-b(A+B)^{-1}(a^T-b^T)$$
$$= (x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})(A^{-1}+B^{-1})(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})^T$$
$$+(a-b)(A+B)^{-1}(a-b)^T$$

Substitute above formula:

$$\int Norm_x[a,A]Norm_x[b,B]dx$$

$$= \frac{1}{\sqrt{2\pi}|A|^{1/2}|B|^{1/2}}\int \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\big((x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})(A^{-1}+B^{-1})(x-(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1})^T+(a-b)(A+B)^{-1}(a-b)^T\big)\right)dx$$

Let $\mu=(aA^{-1}+bB^{-1})(A^{-1}+B^{-1})^{-1}$, $\sum_*=(A^{-1}+B^{-1})^{-1}$

$$= \frac{|(A^{-1}+B^{-1})^{-1}|^{1/2}}{\sqrt{2\pi}|A|^{1/2}|B|^{1/2}}\int \frac{1}{\sqrt{2\pi}|(A^{-1}+B^{-1})^{-1}|^{1/2}}\exp\left(-\frac{1}{2}\big((x-\mu)\Sigma_*^{-1}(x-\mu)^T+(a-b)(A+B)^{-1}(a-b)^T\big)\right)dx$$

$$= \frac{1}{\sqrt{2\pi}(|A||A^{-1}+B^{-1}||B|)^{1/2}}\exp\left(-\frac{1}{2}(a-b)(A+B)^{-1}(a-b)^T\right)\int Norm_x[\mu,\Sigma_*]dx$$

$$= \frac{1}{\sqrt{2\pi}(A+B)^{1/2}}\exp\left(-\frac{1}{2}(a-b)(A+B)^{-1}(a-b)^T\right)\int Norm_x[\mu,\Sigma_*]dx$$

$$= Norm_a[b,A+B]\int Norm_x[\mu,\Sigma_*]dx$$

Because A, B are diagonal, so $\mu=\sum_*(A^{-1}a+B^{-1}b)$

So

$$\int Norm_x[a,A]Norm_x[b,B]dx = Norm_a[b,A+B]\int Norm_x[\Sigma_*(A^{-1}a+B^{-1}b),\Sigma_*]dx$$