From Kriging to large data methods

$$Z_k = Y(s_k) + \varepsilon_k \qquad k = 1, n$$

errors

$$\mathrm{Var}(Y(s)) = \sigma^2 \qquad \mathrm{Cov}(Y(s), Y(s')) = k(s, s')$$

e.g. $k(s, s') = \sigma^2 e^{-\|s-s'\|/\theta}$ ← Exp $\frac{1}{2}$

$\sigma^2 e^{-(\|s-s'\|/\theta)^2}$ ← Gaussian ∞

ε – i.i.d. variance 1 (τ^2)

Want to predict at s_0 $\quad Y(s_0)$

- Missing the linear model !!

Assembling the matrices $\underset{\sim}{z}$ - $n \times 1$

$$C_z = \text{cov}(\underset{\sim}{z}, \underset{\sim}{z}) \quad n \times n \text{ matrix.}$$

$$= (K + I) \qquad K_{ij} = k(s_i, s_j), \text{ e.g. } \underbrace{\sigma^2 e^{\frac{\|s_i - s_j\|}{\theta}}}_{\text{Exp}}$$

from iid errors

$$C_0 = \text{cov}(Y(s_0), \underset{\sim}{z}) \equiv \text{cov}\left(Y(s_0), \begin{pmatrix} Y(s_1) \\ \vdots \\ Y(s_n) \end{pmatrix}\right)$$

$$= k\left(s_0, \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}\right)$$

$$C_{00} = \text{Var}(Y(s_0)) = \sigma^2$$

$$\begin{pmatrix} Z \\ \hdashline Y(s_0) \end{pmatrix} = \mathrm{Gau}\left(0, \begin{pmatrix} \overset{n\times n}{C_Z} & \overset{n\times 1}{c_0} \\ \hdashline c_0' & \underset{1\times 1}{c_{00}} \end{pmatrix} \right)$$

$$[\, Y(s_0) \mid \underset{\sim}{Z} \,] \sim \mathrm{Gau}\Big(\underbrace{c_0' C_Z^{-1} \underset{\sim}{Z}}_{\text{kriging estimate}} ,\ \underbrace{c_{00} - c_0' C_Z^{-1} c_0}_{(\text{standard error})^2} \Big)$$

$$\hat{Y}(s_0) = c_0^T C_Z^{-1} \underset{\sim}{Z} = (\text{weights})^T \underset{\sim}{Z}$$

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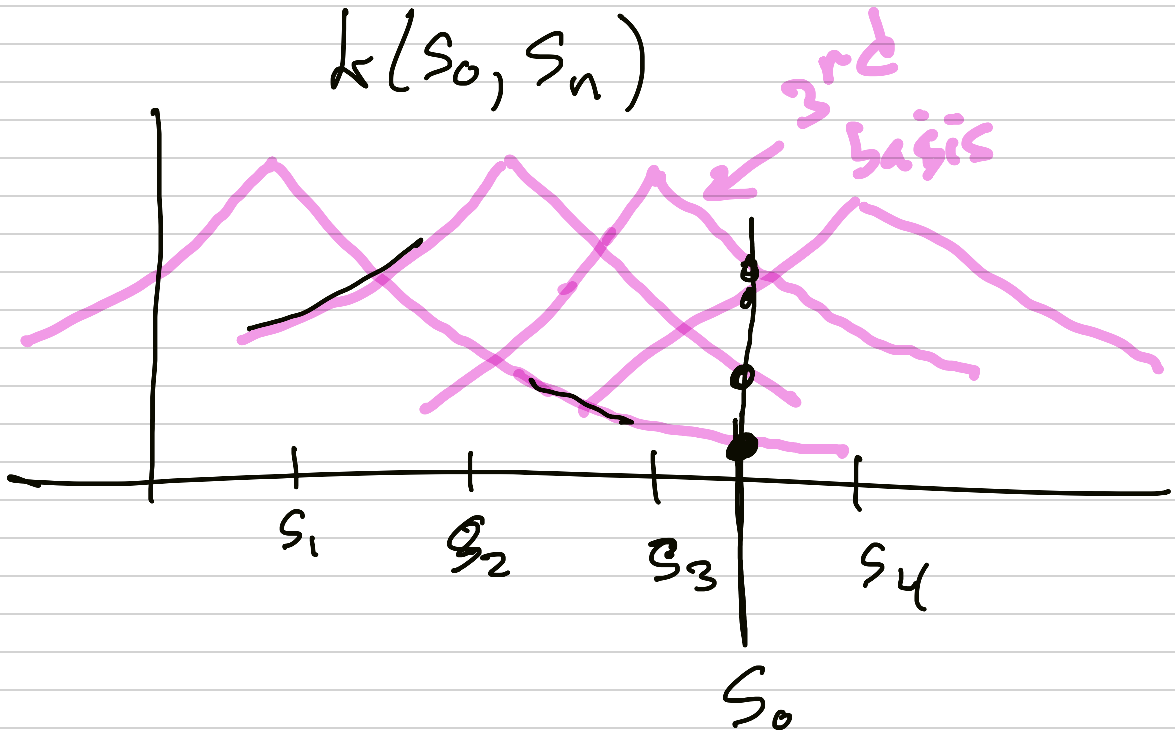
$$= c_0^T \left(C_Z^{-1} \underset{\sim}{Z} \right)$$

$$= \text{basis functions} \times \text{coefficients}$$

$$\hat{\alpha} = \left(C_{\tilde{z}}^{-1} \tilde{z} \right) \qquad C_0^T \hat{\alpha} = \sum_{k=1}^{n} k(s_0, s_k)\, \hat{\alpha}_k$$

$$C_0 = \begin{matrix} k(s_0, s_1) \\ \vdots \\ k(s_0, s_n) \end{matrix} \qquad = \sum_{k=1}^{n} \left[\sigma^2 e^{\frac{\|s_0 - s_k\|}{\theta}} \right] \hat{\alpha}_k$$



FRK - $Y(s)$ - Gaussian Process

$$Y(s) = \sum_{k=1}^{n} \phi_k(s)\alpha_k$$

$\underset{\sim}{\alpha} \sim Gau(0, C_\alpha)$ $\{\phi_k(s)\}$ fixed

chosen out of laziness

process at $s_1, \ldots, s_n$

$$\underset{\sim}{Z} = \underset{\sim}{Y} + \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{Z} = \Phi \underset{\sim}{\alpha} + \underset{\sim}{\varepsilon} \qquad \Phi_{ij} = [\phi_j(s_i)]$$

$$C_Z = \text{cov}(\underset{\sim}{Z}, \underset{\sim}{Z}) = (\Phi C_\alpha \Phi^T + I)$$

Fun fact

$\underset{\sim}{U} \sim Gau(0, \Sigma)$ A - matrix.

$$\text{Cov}(A\underset{\sim}{U}) = A \Sigma A^T$$

$$c_0 = C_\alpha \phi_0 \qquad \phi_0 = \begin{bmatrix} \phi_1(s_0) \\ \vdots \\ \phi_n(s_0) \end{bmatrix}$$

cov. driving spatial model

$$\hat{Y}(s_0) = \sum \phi_k(s_0)\, \hat{\alpha}_k$$

$$\underset{\sim}{\hat{\alpha}} = (C_\alpha \Phi^T)(C_z)\underset{\sim}{z} = \underline{C_\alpha \Phi^T} (\underline{\Phi C_\alpha \Phi^T + I})^{-1} \underset{\sim}{z}$$

basis function matrix

$$\phi_0^T \underset{\sim}{\hat{\alpha}} = \underbrace{\overset{c_0^T}{\sim\!\sim\!\sim\!\sim\!\sim\!\sim}}_{\text{weight ala Kriging}} \underset{\sim}{z}$$

Thinking about

$(C_z)^{-1}$ $\quad$ $C_z^{-1} = (\Phi C_\alpha \Phi^T + I)^{-1}$

ordinary kriging $\quad$ $C_z = (K + I)^{-1}$

n obs $\quad$ $K = n \times n$ $\quad$ $C_z - n \times n$

Sherman-Morrison-Woodbury formula.

$$\underbrace{(\Phi C_\alpha \Phi^T + I)^{-1}}_{\text{left } (C_z)^{-1}} = \underbrace{I - \Phi (\Phi^T \Phi + C_\alpha^{-1})^{-1} \Phi^T}_{\text{something else}}$$

right

$n - 10^5$ $\quad$ 100 basis functions $\quad$ right $\Phi^T \Phi - 100 \times 100$

left $\sim 10^5 \times 10^5$ matrix. $\quad$ $C_\alpha - 100 \times 100$

$\Rightarrow$ Φ - sparse matrix.