



## A multi-resolution census algorithm for calculating vortex statistics in turbulent flows

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**Summary.** The fundamental equations that model turbulent flow do not provide much insight into the size and shape of observed turbulent structures. We investigate the efficient and accurate representation of structures in two-dimensional turbulence by applying statistical models directly to the simulated vorticity field. Rather than extract the coherent portion of the image from the background variation, as in the classical signal-plus-noise model, we present a model for individual vortices using the non-decimated discrete wavelet transform. A template image, which is supplied by the user, provides the features to be extracted from the vorticity field. By transforming the vortex template into the wavelet domain, specific characteristics that are present in the template, such as size and symmetry, are broken down into components that are associated with spatial frequencies. Multivariate multiple linear regression is used to fit the vortex template to the vorticity field in the wavelet domain. Since all levels of the template decomposition may be used to model each level in the field decomposition, the resulting model need not be identical to the template. Application to a vortex census algorithm that records quantities of interest (such as size, peak amplitude and circulation) as the vorticity field evolves is given. The multiresolution census algorithm extracts coherent structures of all shapes and sizes in simulated vorticity fields and can reproduce known physical scaling laws when processing a set of vorticity fields that evolve over time.

**Keywords:** Multivariate multiple linear regression; Non-decimated discrete wavelet transform; Penalized likelihood; Turbulence; Vorticity

### 1. Introduction

The large-scale fluid motion of planetary atmospheres and oceans is extremely turbulent and is strongly influenced by the planetary rotation and the planet's gravitational field. The planetary rotation and gravitation render the resulting fluid motion primarily horizontal. For example, in the Earth's atmosphere, vertical motion is typically at speeds of several centimetres per

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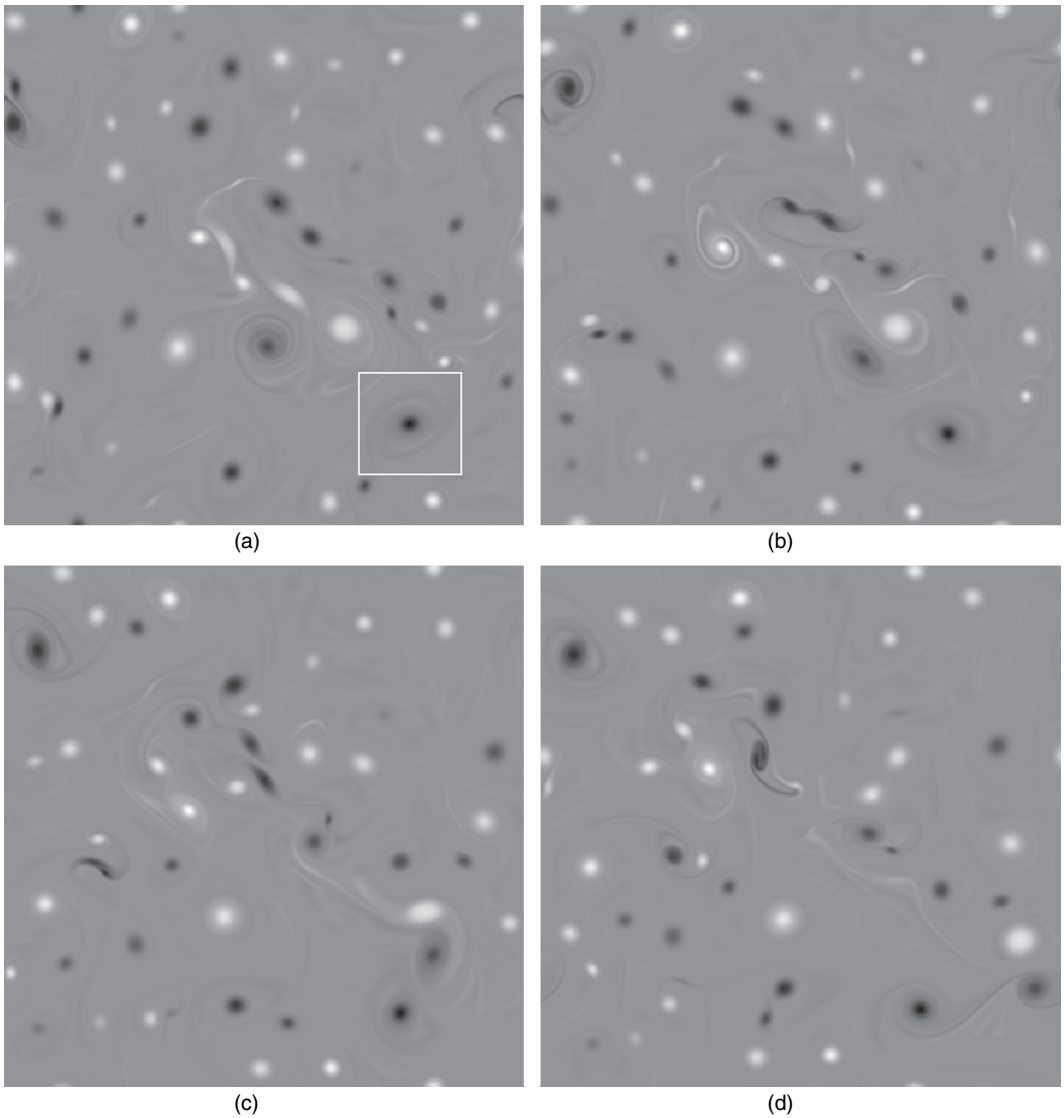
second, whereas strong horizontal motions such as the jetstream can exceed speeds of  $100 \text{ m s}^{-1}$ . Turbulent fluids are characterized by having a wide range of spatial scales with complex non-linear interactions between the scales. One noteworthy feature of turbulent fluids is that they self-organize into coherent features. The two main categories of large-scale coherent structures are vortices, such as hurricanes, tornados, oceanic vortices and Jupiter's Great Red Spot, and jets, such as the Earth's atmospheric jetstream and the Gulf Stream in the north Atlantic Ocean. With respect to the scope of this paper we are interested only in identifying vortices, so the term coherent structure will be synonymous with vortex.

The reason for the formation of coherent structures is poorly understood. Owing to the quasi-horizontal nature of atmospheres and oceans, the energy cascades from small to large scales, and the accumulation of energy at large scales is associated with large-scale coherent structures (McWilliams and Weiss, 1994). Structures may also be formed from the growth of instabilities in the flow, with the scale of the structure determined by the scale of the instability. Regardless of their mechanism of formation, accepting that such structures exist in turbulent flows and analysing their behaviour and effect has led to significant advances in understanding turbulence.

In many instances we wish to know the statistics of vortex properties. Whereas traditional theories of turbulence are framed in terms of energy spectra, more recent theories are based around the statistics of the vortex population (Carnevale *et al.*, 1991; McWilliams and Weiss, 1994). Coherent vortices in the ocean with spatial scales of tens of kilometers can live for more than a year and travel across the ocean, affecting the energetics, salinity and biology of the ocean. The number and strength of such vortices are often determined by manually identifying vortices. One method of validating atmospheric models is to determine whether they capture the statistics of atmospheric vortices such as storms and hurricanes. In all these areas, a robust efficient method to calculate the vortex statistics would represent a major advance.

Fig. 1, which displays observations from a numerical simulation of turbulent flow, provides examples of such vortices. A vortex is a spinning turbulent flow that has anomalously high (in absolute value) vorticity. Following the definition that is common for the northern hemisphere, positive vorticity (lighter shades in the images) corresponds to spinning in the counter-clockwise direction and negative vorticity (darker shades in the images) corresponds to spinning in a clockwise direction. Regions of high vorticity exhibit a peak near its centre and decays back to zero vorticity in all directions from that centre. The self-organization of a fluid into vortices is an emergent phenomenon that can only be partially understood by analysis of the governing partial differential equations. Determining the details of a vortex population requires analysing a time-evolved field, either from numerical simulations, laboratory experiments or observations of natural systems, and requires a pattern recognition (or census) algorithm.

To determine the statistics of the vortices, we must first identify individual vortices and measure their properties. In two-dimensional turbulence the structure of the vortices is relatively simple and a broad variety of census algorithms have been successful (McWilliams, 1990; Farge and Philipovitch, 1993; Siegel and Weiss, 1997). However, we would like to develop methods of structure identification that work in more realistic fluid situations ranging from three-dimensional idealized planetary turbulence to the most realistic general circulation model. In these situations, the structures include jets as well as vortices, and these structures exist in a more complex fluctuating environment. The goal of the current work is to develop a census algorithm for identifying vortices in two-dimensional turbulence that is sufficiently general to handle, with modifications, these more realistic situations.



**Fig. 1.** Observed vorticity fields ( $512 \times 512$  pixels) from a numerical simulation at times  $t$  (a) 15 s, (b) 16 s, (c) 17 s and (d) 18 s: negative vorticity is seen as darker shades of grey approaching black whereas positive vorticity is seen as lighter shades of grey approaching white; the background, roughly zero vorticity, is a medium grey; the white square in the lower right-hand corner of the vorticity field in (a) is centred at the largest coherent structure, in peak absolute vorticity

The ‘data mining’ of turbulent fluid flow, both simulated and observed, can be thought of as a statistical problem of feature extraction. In this work we focus on the detection of coherent structures from a simulated scalar field of rotational motion. The reason that we focus on simulated fields is that real data at the required resolution are difficult to obtain in practice. Our approach to this problem consists of two parts. The first step is to develop a flexible model for a single coherent ‘template’ structure (vortex) by using multi-resolution analysis (MRA) and the second identifies individual vortices through a stepwise model selection procedure. Although the model for the template function embodies prior information from the scientist, it is sufficiently

flexible to capture a broad range of features that are associated with coherent structures that we might observe in a fluid. Once a suitable template has been chosen, the second step provides an objective approach to identifying features of interest through classic statistical methodology, allowing the information in the vorticity field to dictate what is and what is not a vortex. With individual coherent structures efficiently summarized through a set of parameters, such as diameters and centroid locations, the next step of modelling movement and vortex interaction may take place, e.g. Storlie *et al.* (2004). This has the potential to advance our understanding of many complex natural systems, such as hurricane formation and storm evolution.

In the past, coherent structures have been quantified by simply decomposing the flow into signal and background noise by using wavelet-based techniques; see, for example, Farge *et al.* (1992, 1996) and Wickerhauser *et al.* (1994, 1997). The focus of the methodology that was used in those references has an engineering orientation, i.e. the wavelet transform is used to discriminate between signal and noise by using the compression rate as a classifier, the idea being that the coherent structures (vortices) that are observed in the turbulent flow will be well represented by only a few wavelet coefficients and therefore most coefficients may be discarded. From one vorticity field, two fields are produced from such a procedure: one based on the largest wavelet coefficients that is meant to capture the largest coherent structures in the original field; and another, based on the remaining wavelet coefficients, is a mixture of background structures (i.e. filaments) and noise.

Siegel and Weiss (1997) developed a wavelet-based census algorithm for two-dimensional turbulence that relied on specific attributes of coherent structures. First, the vorticity field was separated into coherent and background fields by using an iterative wavelet thresholding technique; then the surviving wavelet coefficients in the coherent part are grouped according to the spatial support of the Haar basis. Luo and Jameson (2001) investigated a wavelet-based technique for identifying, labelling and tracking ocean vortices and a database was built of wavelet signatures based on the amount of energy that is contained at each scale. They found that Gaussian densities were adequate to model observed vortices.

Our methodology differs significantly from the signal-plus-noise model and other wavelet-based techniques. Our main scientific contribution is the development and efficient implementation of a statistical method to extract a wide variety of coherent structures from two-dimensional turbulent fluid flow. We achieve this through the formulation of a flexible statistical model for individual coherent structures (vortices) and obtain a fixed number of isolated vortices from the original field via regression methodology. From these models a completely different set of summary statistics may be calculated; for example, instead of global summary statistics like the enstrophy spectrum over the entire image we can look at local statistics for each structure such as the average size, amplitude, circulation and enstrophy of the individual vortices.

This work has led to new models for two-dimensional multiscale features. Although this work concentrates on vortices, the methodology proposed is quite generic and can be transferred to model other types of coherent structures. The adaptation of model selection methods for segmenting images is made possible by formulating our model in terms of multiple linear regression. Finally, we note that this work also depends on the development of computing algorithms whereby we fit many local regressions simultaneously by using equivalences with image convolution filtering. The statistical computing aspects of this problem are important to analyse large problems that are of scientific interest. As a test of these ideas we can carry out a census of vortices from a high resolution simulation of two-dimensional turbulence.

The next section provides a brief introduction to two-dimensional turbulence and the numerical simulation that was used. Section 3 outlines how we model individual coherent structures and the entire vorticity field. The two-dimensional MRA approach to images is introduced and

the user-defined vortex template function is also discussed. Section 4 provides the methodology to estimate a single coherent structure and also multiple structures in the vorticity field. Section 5 looks at how well the technique performs at estimating multiple coherent structures and at the temporal scaling of vortex statistics.

## 2. Two-dimensional fluid turbulence

Turbulence in atmospheres and oceans occurs in many forms and at many scales. Two-dimensional turbulent flow is an idealization that captures many of the features of planetary turbulence and, in particular, has vortex behaviour that is common to many atmospheric and oceanic phenomena. Thus, two-dimensional turbulence often serves as a laboratory to study aspects of planetary turbulence and as a test bed for developing theories and algorithms. Here we follow this strategy and use two-dimensional turbulence to develop and test a new multi-resolution census algorithm for vortex statistics. Although there are experimental fluid systems that approximate two-dimensional fluid flow, numerical simulations are the most effective way to develop and test such algorithms. Treating the output of numerical simulations as data which are analysed just as we would with observations or laboratory experiments is routine in the field of turbulence and we follow this route.

Two-dimensional fluid dynamics has a long history and many aspects are well understood (Kraichnan and Montgomery, 1980; Frisch, 1995; Lesieur, 1997). The equations of motion for two-dimensional turbulence are written in terms of the fluid velocity  $\mathbf{u} = (u_x, u_y)$  and its scalar vorticity

$$\zeta(\mathbf{u}) = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}},$$

where  $\nabla$  is the gradient operator and  $\hat{\mathbf{z}}$  is a unit vector in the direction perpendicular to the plane containing the two-dimensional velocity  $\mathbf{u}$ . In general the curl of a vector field is given by  $\nabla \times \mathbf{u}$  and, hence, vorticity is the curl of the velocity field. Following the ‘right-hand rule’ vorticity is positive when the flow is rotating anticlockwise and negative otherwise. A related concept is circulation which is related to vorticity by Stoke’s theorem

$$\Gamma(\mathbf{u}) = \int_S \zeta(\mathbf{u}) dS,$$

where  $S$  is a surface in two dimensions. The units of circulation are length squared over time and vorticity is the circulation per unit area. Enstrophy is given by

$$\mathcal{E}(\mathbf{u}) = \frac{1}{A} \int_S |\zeta(\mathbf{u})|^2 dS,$$

where  $A$  is the area of the fluid, and is a measure of the mean-square vorticity. Enstrophy is a quantity that is similar to energy (which is the mean-square velocity) and plays an important role in turbulence theory despite being somewhat non-intuitive.

It is important to understand fluid flow at a macroscopic scale and, in particular, to create more accurate models for the flow of the atmosphere and ocean. Recent advances in computing have allowed scientists to produce more realistic simulations of turbulent flows (Ferziger, 1996; Moin and Mahesh, 1998). Numerical simulations of two-dimensional turbulence show that random initial conditions will self-organize into a collection of coherent vortices which subsequently dominate the dynamics (McWilliams, 1984, 1990). Owing to this self-organization, traditional scaling theories of turbulence fail to describe the dynamics correctly, whereas scaling theories that are based on the statistics of the vortices are much more successful (Carnevale *et al.*, 1991; Weiss and McWilliams, 1993; Bracco *et al.*, 2000).

### 2.1. Description of the data

The equations of motion for two-dimensional fluid flow are

$$\begin{aligned}\frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla) \zeta &= \mathcal{D}, \\ \zeta &= (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}},\end{aligned}$$

where  $\mathcal{D}$  is a general dissipation operator. Despite being small, dissipation is an important component of turbulence and is necessary for numerical simulations. Owing to the nature of two-dimensional turbulence, energy cascades to large scales and is not dissipated in the limit of small dissipation, whereas enstrophy cascades to small scales and has finite dissipation in the limit of small dissipation. The dissipation of enstrophy is governed by the evolution of the vortex population. These equations are simulated with doubly periodic boundary conditions by using a pseudospectral algorithm, using a hyperviscous diffusion operator  $\mathcal{D} = -\nu \nabla^4 \zeta$  on a  $512 \times 512$  grid. The simulations start from small-scale random initial conditions.

As time proceeds, the random initial conditions self-organize into coherent vortices. The initial scale of the vortices is governed by the scale of the initial conditions. Subsequently, vortices grow through vortex mergers until a final end state is reached with two vortices, one of each sign. During the time period after self-organization but before the number of vortices becomes too small, the turbulence can be modelled as a population of interacting coherent vortices. We emphasize that the vortices are not introduced into the flow by any external forcing, but rather they arise through the natural self-organizing nature of two-dimensional turbulence.

Here we apply our algorithm solely to data from numerical experiments. The equations of motion and the methods for solving them are firmly established and previous work in fluid dynamics has shown that numerical simulations, laboratory experiments and observations of natural systems all produce vortices with similar properties. Numerical simulations provide the most complete and accurate representation of vortex dynamics and thus provide the most stringent test for the algorithm proposed.

## 3. Models for vorticity fields

The goal of our multi-resolution census algorithm is to identify all coherent structures (vortices) from a given vorticity field and then to compute summary statistics from each structure identified. As discussed in Section 1, this approach differs from previous approaches in that we are not interested in merely separating the coherent portion of the field from the background, as in the traditional signal-plus-noise model, but instead isolate individual coherent structures in the image for further analysis. For the simple vortex fields that are considered here, our proposed method performs similarly to previous methods. This is quantitatively demonstrated by the scaling relationships that are shown in Fig. 8 later. However, previous methods are not easily extended to flows with more complex structure. The general statistical framework of the current approach provides a flexible modelling framework in which to implement generalizations in a relatively straightforward manner.

For clarity we first outline a continuous version of the problem and then follow with a more practical discrete approximation. Let  $\zeta_t(\mathbf{x})$  denote an observed vorticity field at time  $t$  and location  $\mathbf{x} \in \mathbb{R}^2$ . Given  $S$  vortices,  $\zeta_t(\mathbf{x})$  can be decomposed as

$$\zeta_t(\mathbf{x}) = \sum_{s=1}^S v_s(\mathbf{x}) + e_t(\mathbf{x}) \quad (1)$$

where  $v_s$  is the localized vorticity that is associated with each coherent structure and  $e_t$  is the background variation from other types of structure. Given the dominance of vortices in our

application, it is useful to assume a stochastic character for the residual component  $e_t$ . The statistical problem is, given the observed field  $\zeta_t$ , estimate  $S$  and  $\{v_s\}$ .

The main modelling component in this work is to expand the individual vortex field in a finite basis

$$v_s = \sum_{i=1}^M \alpha_{s,i} z_i(\mathbf{x} - \boldsymbol{\mu}_s) \quad (2)$$

where  $\boldsymbol{\mu}_s$  approximates the centre of the vortex and the (linear) coefficients  $\{\alpha_{s,i}\}$  determine the shape. Estimation procedures for  $\boldsymbol{\mu}_s$  and  $\{\alpha_{s,i}\}$  are discussed in Section 4. The basis functions  $\{z_i\}$  are designed to provide a multiscale representation of coherent structures and build in prior knowledge of the shape of the vortex. Their specification is motivated in the next two sections.

### 3.1. Multi-resolution analysis

The use of wavelets for two-dimensional image analysis and compression has a large literature and at its heart is the decomposition of an image into different scales or levels of resolution (Vetterli and Kovačević, 1995; Mallat, 1998). In our work we assume the scale to be in powers of 2 and within a scale consider a further decomposition that divides features into three orientations: vertical, horizontal and diagonal. The net result is that, for a given initial vorticity field  $\zeta_t$  and  $J$  levels of resolution ( $J=6$  is used when applying the model in practice), we decompose the field into the sum of  $3J+1$  distinct components: three different orientations for each level of resolution and a smoothed field. Heuristically, as the levels increase, the main features at each level will increase in size by a factor of 2. Strong horizontal features at a given scale are represented by the horizontal component whereas similar correspondences hold for the vertical and diagonal components. Later, it will be exploited that some of these level-specific features may correspond to features at (possibly different) levels of the decomposition of the vorticity field.

We assume that the vorticity field  $\zeta(\mathbf{x})$  has finite energy, i.e.  $\int_{\mathbf{x}} \zeta^2(\mathbf{x}) < \infty$ , and let  $\phi$  be a scaling function and  $\psi$  be the corresponding wavelet generating an orthonormal basis on  $L^2(\mathbb{R})$ . Define the three separable two-dimensional wavelets

$$\begin{aligned} \psi^h(x_1, x_2) &= \phi(x_1) \psi(x_2), \\ \psi^v(x_1, x_2) &= \psi(x_1) \phi(x_2), \\ \psi^d(x_1, x_2) &= \psi(x_1) \psi(x_2), \end{aligned}$$

corresponding to the horizontal, vertical and diagonal directions respectively. This follows from the fact that the two-dimensional wavelets are the outer product of two one-dimensional wavelet and scaling functions, where the scaling function averages (smooths) across its spatial direction whereas the wavelet function differences across its spatial direction. The two-dimensional wavelet  $\psi^h(x_1, x_2) = \phi(x_1) \psi(x_2)$  will therefore smooth across the first dimension ( $x_1$ -axis) and difference across the second dimension ( $x_2$ -axis), thus favouring horizontal features. The two-dimensional wavelet basis function  $\psi^v(x_1, x_2) = \psi(x_1) \phi(x_2)$  differences across the  $x_1$ -axis and smooths across the  $x_2$ -axis, thus favouring vertical features, and  $\psi^d(x_1, x_2) = \psi(x_1) \psi(x_2)$  differences across both directions and favours non-vertical–non-horizontal (i.e. diagonal) features. The separable scaling function  $\phi(x_1, x_2) = \phi(x_1) \phi(x_2)$  is associated with the approximation space.

Although the two-dimensional discrete wavelet transform (DWT), as a decimated orthonormal transform, would be a more efficient representation of the vorticity field we find it advantageous to utilize the two-dimensional maximal overlap DWT (MODWT). Unlike the

orthonormal transform, the two-dimensional MODWT produces a redundant non-orthogonal transform. The reason for this discrepancy is that the two-dimensional MODWT does not subsample in either dimension; it only filters the original image. The advantages are that the transform is translation invariant to integer shifts in space and it reduces potential artefacts that are caused by how the wavelet filter represents abrupt changes in the image.

Assume that both  $\psi$  and  $\phi$  have been rescaled so that the squared norm of the wavelet coefficients equals the squared norm of the original observations. The separable wavelet functions that are associated with specific scale and spatial directions  $D \in \{h, v, d\}$  are given by

$$\psi_{j,k,l}^D(x_1, x_2) = \frac{1}{2^{2j}} \psi^D\left(\frac{x_1 - k}{2^j}, \frac{x_2 - l}{2^j}\right), \quad j = 1, \dots, J, \quad k = 1, \dots, M, \quad l = 1, \dots, N.$$

The separable scaling function  $\phi_{j,k,l}$  is defined similarly. Hence, each level in the transform will have the same spatial dimension as the original field ( $M \times N$ ) and represent a redundant set of wavelet coefficients. The two-dimensional MODWT begins with the original vorticity field  $\zeta$  (the spatial location will be omitted when implied), and at all scales we denote  $\omega_j^\phi = \langle \zeta, \phi_{j,k,l} \rangle$  and  $\omega_j^D = \langle \zeta, \psi_{j,k,l}^D \rangle$  for  $D \in \{h, v, d\}$ , where  $\langle x, y \rangle$  is the two-dimensional inner product. The vorticity field  $\zeta(\mathbf{x})$  may now be decomposed into  $3J + 1$  subfields: three fields of wavelet coefficients at each level of resolution corresponding to distinct spatial directions and one field containing the scaling coefficients at the final level. The scaling (approximation) field for level  $j$  may be obtained from the four fields at level  $j + 1$  via

$$\omega_j^\phi(\mathbf{x}) = \omega_{j+1}^\phi * \phi_{j,k,l} + \sum_D \omega_{j+1}^D * \psi_{j,k,l}^D, \quad (3)$$

where the asterisk denotes the convolution operator.

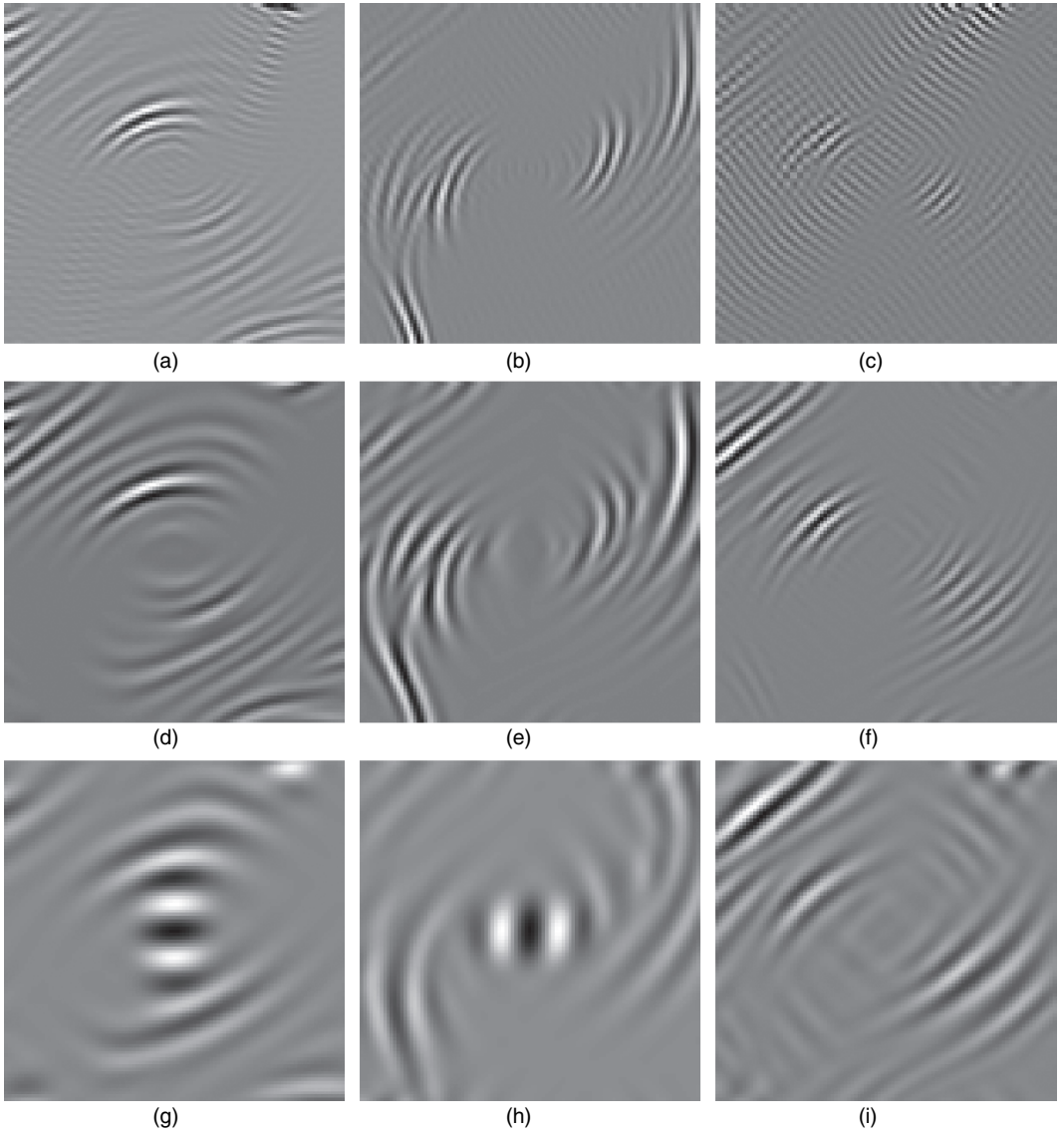
The two-dimensional MRA of the vorticity field is an additive decomposition that is given by recursively applying equation (3) over all  $j$ , i.e.

$$\zeta(\mathbf{x}) = \omega_J^\phi * \phi_{J,k,l} + \sum_{j=1}^J \sum_D \omega_j^D * \psi_{j,k,l}^D = \alpha_J^\phi(\mathbf{x}) + \sum_{j=1}^J \sum_D \alpha_j^D(\mathbf{x}),$$

where  $\alpha_J^\phi$  is the wavelet approximation field and  $\alpha_j^D$  is the wavelet detail field that is associated with the spatial direction  $D \in \{h, v, d\}$ . The MRA of  $\zeta(\mathbf{x})$  provides a convenient way of isolating features at different scales and directions with coefficients in the spatial domain *versus* the wavelet domain. This is advantageous since reconstruction is now reduced from the full inverse two-dimensional MODWT to simple addition and potential phase adjustments are eliminated.

Figs 2 and 3 display the six scales from a two-dimensional MRA of the sample vorticity field in Fig. 1 (at  $t = 15$ ), defined by a  $100 \times 100$  pixel section centred at  $(x, y) = (399, 101)$ . Each row displays the wavelet detail fields that are associated with the three spatial directions: horizontal, vertical and diagonal. It is clear that each of the two-dimensional wavelet filters captures distinct spatial directions at a fixed spatial scale. Given that the filaments from this particular vortex are elliptical in shape, it is not surprising to see the detail coefficients of the filament structures strongest in the north-east–south-west directions. It is interesting to note that the coherent structure (i.e. the dark region in the centre of the image) is not seen until the third scale (Figs 2(g)–2(i)) and then only in the horizontal and vertical directions. At higher scales, corresponding to larger spatial areas and lower spatial frequencies, the coherent structure is apparent in all three directions. This is most probably due to the spatial extent (size) of the structure at time  $t = 15$  in the simulation.

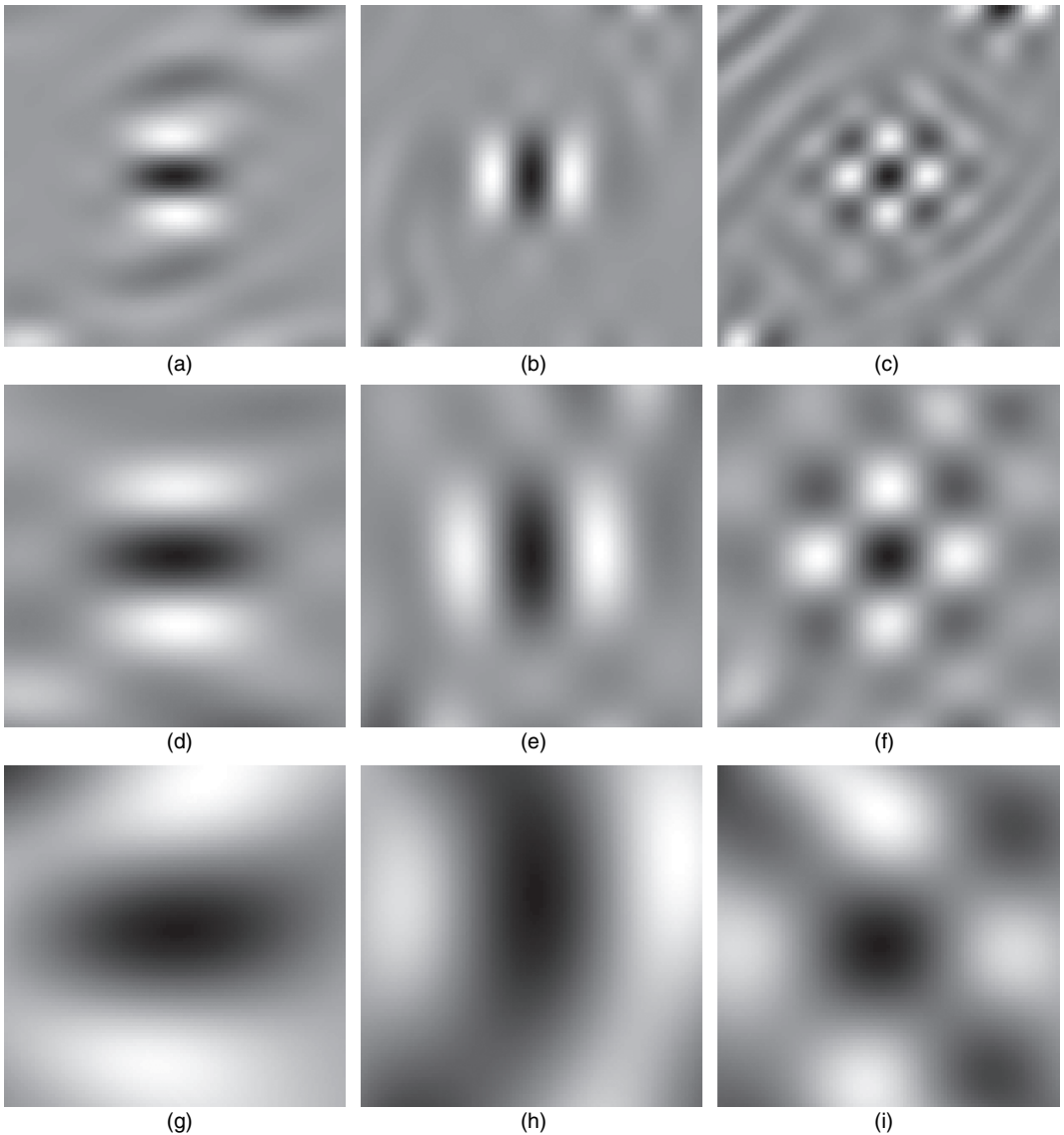




**Fig. 2.** Two-dimensional MRA ( $j = 1, 2, 3$ ) of a  $100 \times 100$  pixel section of the vorticity field centred at  $(x, y) = (399, 101)$  by using the Daubechies least asymmetric wavelet filter ( $L = 8$ ) (LH corresponds to horizontal, HL to vertical and HH to diagonal wavelet coefficients; this pixel section is the area that is highlighted by the white square in Fig. 1(a)): (a) LH1; (b) HL1; (c) HH1; (d) LH2; (e) HL2; (f) HH2; (g) LH3; (h) HL3; (i) HH3

### 3.2. Single-vortex model

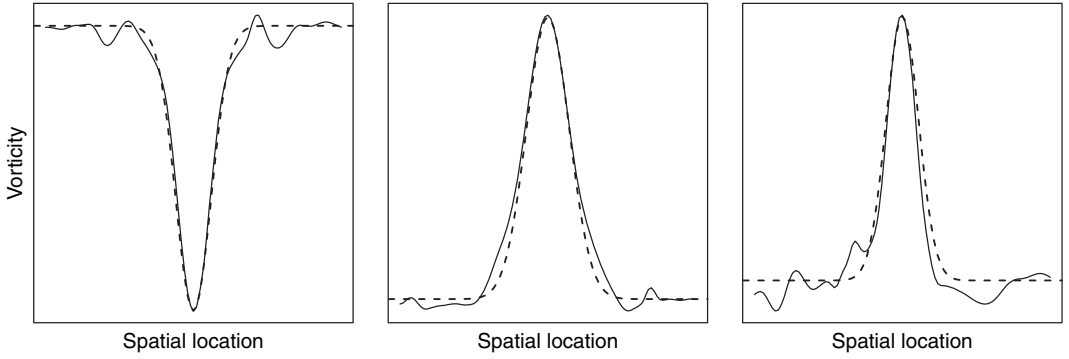
Recall that a vortex may be loosely described as a concentration of anomalously high (in absolute value) vorticity. As a simple outline of a vortex we consider the Gaussian kernel. If we translate a single vortex to the origin, then  $\tau(\mathbf{x}) = \eta \exp(-\|\mathbf{x}\|^2/\sigma^2)$  is our vortex template function where  $\eta$  is the maximum vorticity at its centre. This choice for  $\tau(\mathbf{x})$  visually appears to capture the relevant features of an idealized vortex even when the observed vortices decay back to zero at a different rate from that of Gaussian tails. For illustration in Fig. 4,  $\eta$  and  $\sigma^2$  were



**Fig. 3.** Two-dimensional MRA ( $j = 4, 5, 6$ ) of a  $100 \times 100$  pixel section of the vorticity field centred at  $(x, y) = (399, 101)$  by using the Daubechies least asymmetric wavelet filter ( $L = 8$ ) (LH corresponds to horizontal, HL to vertical and HH to diagonal wavelet coefficients): (a) LH4; (b) HL4; (c) HH4; (d) LH5; (e) HL5; (f) HH5; (g) LH6; (h) HL6; (i) HH6

chosen to coincide with the peak value and spread of the specific coherent structure. However, when used in the multi-resolution census algorithm the template function will have a fixed magnitude and spread—any modifications to fit the vortex will be induced by the multiscale representation of  $\tau(\mathbf{x})$ . Thus, a perfect fit is not required since deviations from Gaussianity will be captured through the model fitting procedure.

Figs 5 and 6 show an MRA ( $J = 6$ ) of  $\tau(\mathbf{x})$  that is derived from the outer product of two zero-mean Gaussian kernels with the same variance. The rows correspond to wavelet scales and the columns correspond to spatial directions within each scale. Although the Gaussian kernel is



**Fig. 4.** Vortex cross-sections (—) and normalized Gaussian kernels (-----) from the observed vorticity field at time  $t = 15$

too simple for modelling an observed vortex directly, the components of its MRA are strikingly similar to the MRA of individual coherent structures in the observed vorticity field (Figs 2 and 3). This observation identifies the basic components for building a template basis and is key to our methodology. We propose to represent each matrix of vorticity coefficients as a linear combination of the matrices that are derived from the MRA of the Gaussian kernel. A straightforward way to relate the MRA of an observed coherent structure to the template function is through a series of simple linear regression models relating the multi-resolution coefficients from the observed vorticity field  $\alpha$  with the multi-resolution coefficients from the template function  $\mathbf{z}$  via

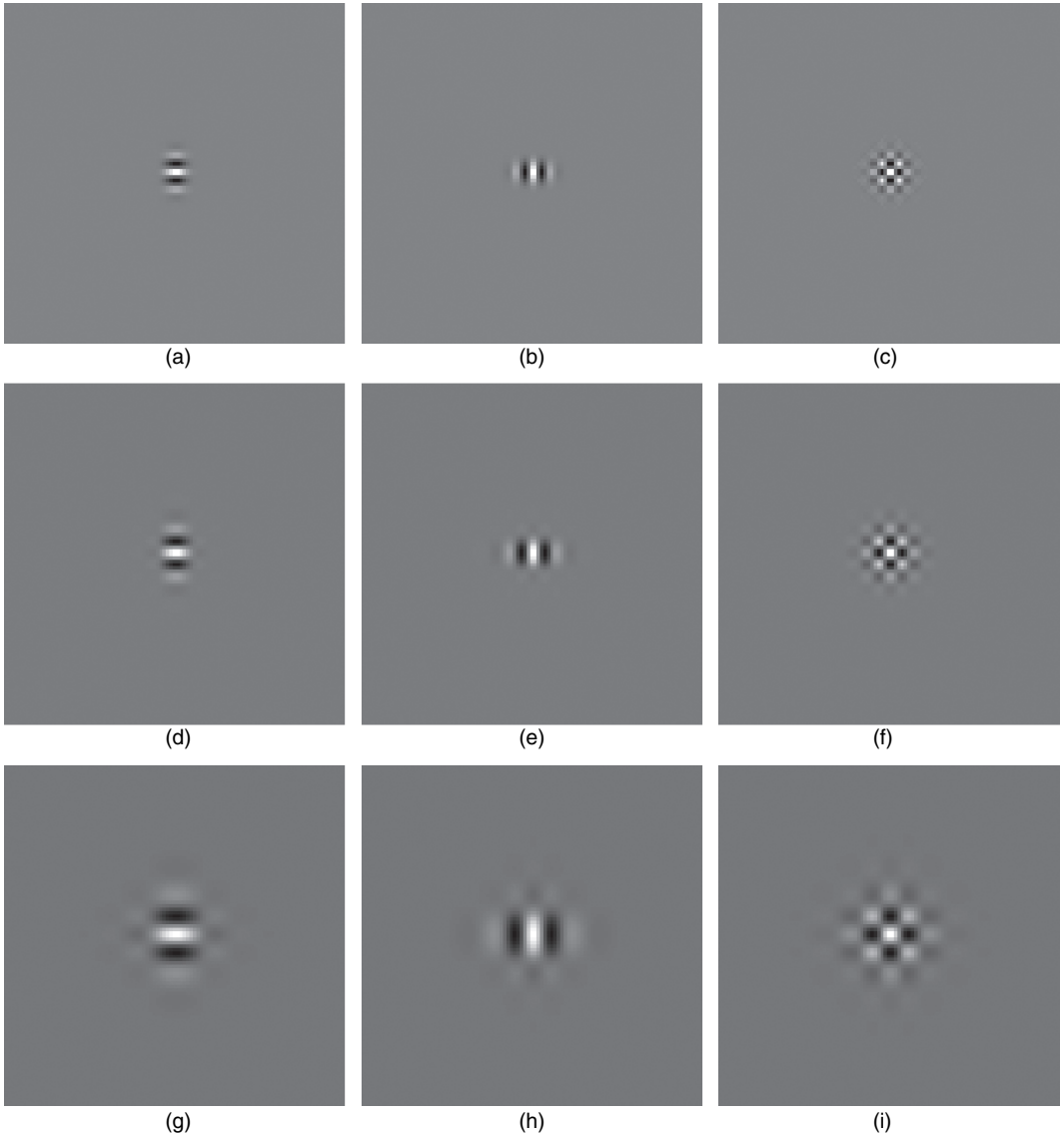
$$\begin{aligned}\alpha_{j,k,l}^D &= \beta_j^D z_{j,k,l}^D + \varepsilon_{j,k,l}^D, & k=1, \dots, M, \quad l=1, \dots, N, \quad \text{for all } j \text{ and } D, \\ \alpha_{j,k,l}^\phi &= \xi_j^\phi + \beta_j^\phi z_{j,k,l}^\phi + \varepsilon_{j,k,l}^\phi, & k=1, \dots, M, \quad l=1, \dots, N.\end{aligned}\quad (4)$$

At each spatial scale and direction there is only one final parameter to be fitted, since each image from the MRA is guaranteed to be mean 0 except for the smoothed field  $\alpha^\phi$ . The implied linear relationship between the multi-resolution components of the data and template function allows for differences in the magnitude and direction of vorticity through the regression coefficients, i.e. once  $\tau(\mathbf{x})$  has been defined (by using fixed parameters  $\eta$  and  $\sigma^2$ ) we shall use it to model all possible coherent structures in the vorticity field. The MRA decomposes the template function into spatial scales and directions, but the model in equation (4) is limited because each spatial scale and direction from  $\zeta(\mathbf{x})$  is tied to the same spatial scale and direction of  $\tau(\mathbf{x})$ . To make full use of the MRA, we propose to estimate every spatial scale and direction of the observed vorticity field by using all possible spatial scales and directions from the template function. This suggests the multiple linear regression model

$$\begin{aligned}\alpha_{j,k,l}^D &= \beta_1^h z_{1,k,l}^h + \beta_1^v z_{1,k,l}^v + \beta_1^d z_{1,k,l}^d + \dots + \beta_J^h z_{J,k,l}^h \\ &\quad + \beta_J^v z_{J,k,l}^v + \beta_J^d z_{J,k,l}^d + \beta_J^\phi z_{J,k,l}^\phi + \varepsilon_{j,k,l}^D,\end{aligned}\quad (5)$$

$k=1, \dots, M, l=1, \dots, N$ , for all  $j$  and  $D$ . The linear regression model for the field of wavelet smooth coefficients in equation (4) does not change. The intercept is included to account for potential low frequency oscillations that may not be provided by the template function. At each spatial scale and direction from the observed vorticity field there are  $3J+1$  parameters to be fitted. The full multivariate multiple linear regression model for a single vortex may now be formulated as

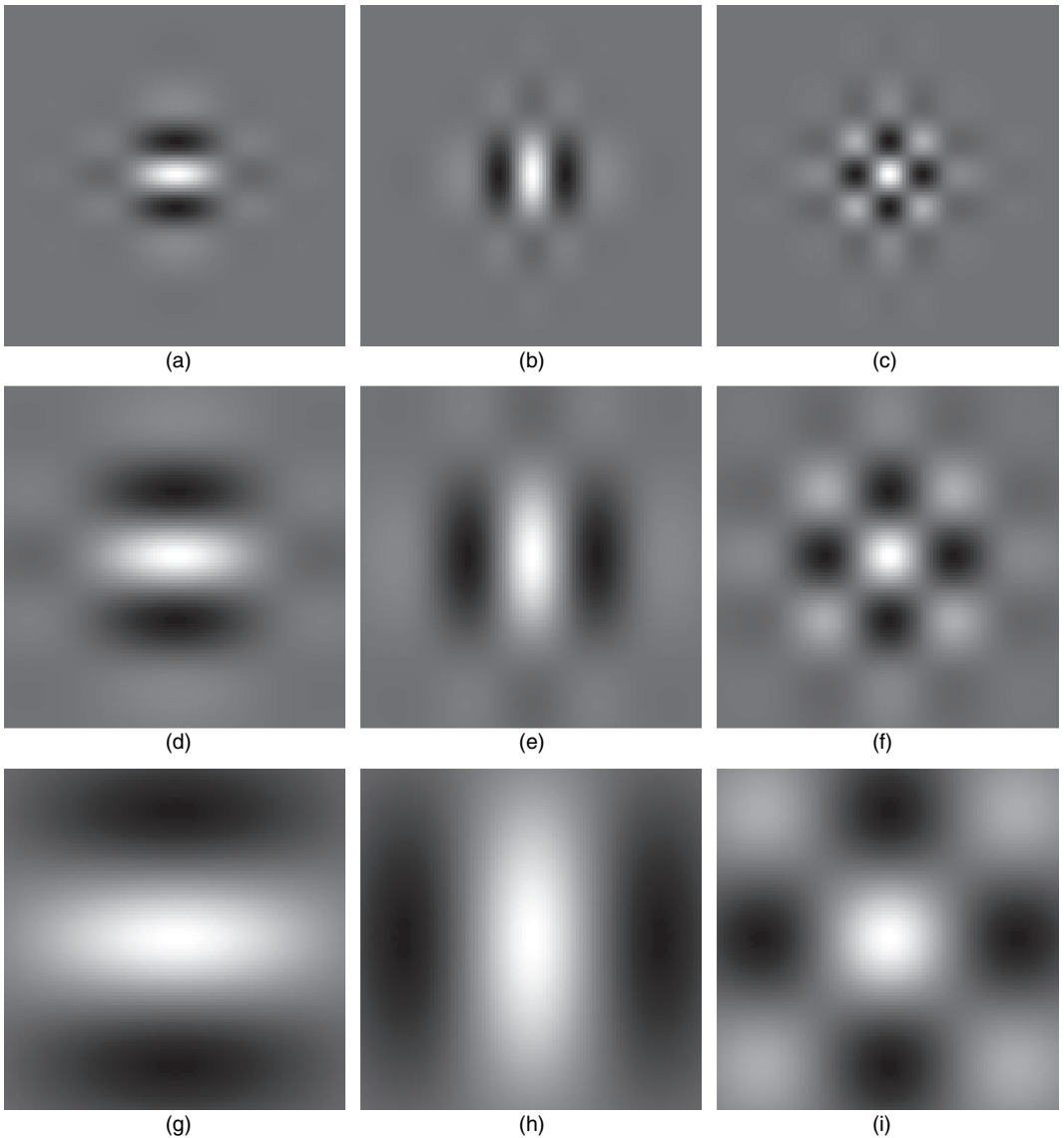
$$\mathbf{Y} = \mathbf{Z}\beta + \varepsilon, \quad (6)$$



**Fig. 5.** Two-dimensional MRA ( $j = 1, 2, 3$ ) of a  $100 \times 100$  pixel section of the template function (the rows correspond to the scale  $j$  and the columns correspond to spatial directions; LH is horizontal, HL is vertical and HH is diagonal): (a) LH1; (b) HL1; (c) HH1; (d) LH2; (e) HL2; (f) HH2; (g) LH3; (h) HL3; (i) HH3

where  $\mathbf{Y}$  is the  $MN \times (3J + 1)$  response matrix from the MRA of the observed vorticity field,  $\mathbf{Z}$  is the  $MN \times (3J + 1)$  design matrix whose columns consist of the MRA of the template function that is centred at the location  $\boldsymbol{\mu}$  and  $\boldsymbol{\beta}$  is the  $(3J + 1) \times (3J + 1)$  regression coefficient matrix for a given coherent structure.

There are several differences between the Gaussian kernel  $\tau(\mathbf{x})$  and the observed vorticity field  $\zeta(\mathbf{x})$  that are handled automatically through the multivariate multiple linear regression model in equation (5); these include fixed spatial size, amplitude and radial symmetry. The fixed spatial size of  $\tau(\mathbf{x})$  is taken care of by the fact that all spatial scales from its MRA are associated with



**Fig. 6.** Two-dimensional MRA ( $j = 4, 5, 6$ ) of a  $100 \times 100$  pixel section of the template function (the rows correspond to the scale  $j$  and the columns correspond to spatial directions; LH is horizontal; HL is vertical and HH is diagonal): (a) LH4; (b) HL4; (c) HH4; (d) LH5; (e) HL5; (f) HH5; (g) LH6; (h) HL6; (i) HH6

each spatial scale in the observed vorticity field. The scale-dependent multiple linear regression, equation (5), may represent any particular spatial scale of  $\zeta(\mathbf{x})$  and associate it with any spatial scale of  $\tau(\mathbf{x})$ . Thus, larger spatial scales may be either favoured or penalized through the individual regression coefficients. The amplitude of the template function is similarly handled through the magnitude of these regression coefficients.

An initial impression may be that the radial symmetry of  $\tau(\mathbf{x})$  will restrict the model to radially symmetric vortices. However, asymmetry is accommodated through the multivariate multiple linear regression by decoupling the three unique spatial directions. Each spatial direction is

manipulated through its own regression coefficient, thus allowing for eccentric vortex shapes by favouring one or two of the three possible spatial directions at each scale (Figs 5 and 6).

The level of association that is needed between the template function and feature of interest to model the vorticity field is currently unknown. We have found that small pixel shapes, such as a Gaussian or triangular function, work well in the problem that is presented here. However, we believe that this algorithm is quite adaptive through the user-defined template function  $\tau(\mathbf{x})$ . For example, if the filament structures were deemed interesting in this problem, an effective template function to pick out these features could consist of nothing more than a collection of concentric circles.

## 4. Model estimation

### 4.1. A linear model framework

On the basis of the previous development of the template function and the final step of discretization we are led to the following model for a vortex field at a single time point:

$$\mathbf{Y} = \mathbf{Z}_1\beta_1 + \mathbf{Z}_2\beta_2 + \dots + \mathbf{Z}_S\beta_S + \varepsilon, \quad (7)$$

where  $\mathbf{Y}$  is the  $MN \times (3J + 1)$  response matrix from the MRA of the observed vorticity field (the columns index the spatial scales and direction where each column vector is the result of unwrapping the  $M \times N$  image matrix into a vector of length  $MN$ ),  $\mathbf{Z}_s$  is the  $MN \times (3J + 1)$  design matrix whose columns consist of the MRA of the template function centred at the location  $\mu_s$  and  $\beta_s$  is the  $(3J + 1) \times (3J + 1)$  regression coefficient matrix for the  $k$ th coherent structure. Note that this model parallels the continuous versions in equations (1) and (2), but for fixed  $S$  and  $\{\mu_s\}$  is a linear model. Given this framework, the main statistical challenge is model selection, determining the number of coherent structures  $S$  and their locations  $\{\mu_s\}$ .

### 4.2. Coherent feature extraction

We choose to implement our procedure for coherent structure extraction in a modular fashion so that we may modify specific steps without jeopardizing the integrity of the entire method. For this, the first major step is to produce a set of vortex candidate points  $\mathcal{C}$  (co-ordinates in two dimensions), i.e. a subset of all possible spatial locations in the observed vorticity field. Our goal at this point is not to generate the exact locations of all coherent structures in the field, but we want  $\mathcal{C}$  to contain all possible coherent structures and additional spatial locations that are not vortices; i.e.  $MN \gg \#\mathcal{C} > S$ . Although the number of elements in  $\mathcal{C}$  is large, their refinement is amenable to more conventional statistical analysis. Vortices are selected from  $\mathcal{C}$  by using classic likelihood procedures to obtain a final model. The first step is a rough screening of the model space and should be done with computational efficiency in mind. The second step devotes more computing resources on a much smaller set of models.

#### 4.2.1. Candidate point selection

Instead of relying on specific features in the vorticity field that are present in our current data set and may or may not be present in future applications, we propose a model-based approach to extracting the set of vortex candidate points  $\mathcal{C}$ . After selecting an appropriate template function  $\tau(\mathbf{x})$ , we fit a single-vortex model (Section 3.2) to every  $M \times N$  spatial location. This is done in a computationally efficient manner by first performing an orthogonal decomposition on the design matrix  $\mathbf{Z}$  and using a discrete Fourier transform to perform the matrix multiplications (Whitcher *et al.*, 2003). The result is that the multivariate multiple linear regression model in

equation (7) can be fitted to all  $M \times N$  grid points in the image without the need for a massive computing environment. The reason for fitting the single-vortex model everywhere is that we may now use the regression coefficient matrix  $\hat{\beta}_s$  to indicate the presence or absence of a coherent structure at  $\mu_s$ . This technique has the inherent flexibility of the template function. If a different feature were to be extracted from the vorticity field, we would simply fit a different template function and use its regression coefficients.

With a regression coefficient matrix for each location, a distance measure is computed between  $\hat{\beta}$  and the identity matrix  $\mathbf{I}_J$ , of dimension  $3J + 1$ , by using

$$\Lambda = \min[\text{tr}\{(\hat{\beta} - \mathbf{I}_J)^2\}, \text{tr}\{(\hat{\beta} + \mathbf{I}_J)^2\}].$$

The idea is that  $\Lambda$  measures the feasibility that a vortex is centred at this location. The value of  $\Lambda$  is small when the regression matrix  $\hat{\beta}$  is similar to the identity matrix and, thus, the single-vortex model is similar to the template function. Two comparisons are made, one to the positive identity matrix and one to the negative identity matrix, so that positive and negative spinning vortices are favoured equally. The parametric image of  $\Lambda$ -values is smoothed by using a nine-point nearest neighbour kernel with weights that are based on Euclidean distance. A local minimum search is performed on the smoothed image to obtain the set of candidate points  $\mathcal{C}$ . We found that this technique captures all coherent structures that we can identify visually in the vorticity field along with additional features that may or may not be vortices.

#### 4.2.2. Model selection for the vortex field

A final selection of coherent structures from the set of candidate points  $\mathcal{C}$  is achieved through forward subset selection by minimizing the generalized cross-validation function

$$\text{GCV}(S) = \frac{\text{RSS}(S)/MN}{\{1 - p(S)/MN\}^2},$$

for a given choice of  $S$  vortex locations. Here  $\text{RSS}(S)$  is the residual sum of squares and  $p(S)$  is the total number of effective parameters in equation (7). The following simple method was adopted to count the total number of effective parameters. For each estimated  $\beta_s$  in equation (7), which are denoted as  $\hat{\beta}_s = (\hat{\beta}_1, \dots, \hat{\beta}_{(3J+1) \times (3J+1)})$ , its effective number of parameters is defined as the number of  $\hat{\beta}_{j,s}$ s whose absolute value is greater than twice the sample standard deviation of the  $\hat{\beta}_{j,s}$ s. Then the overall total effective number of parameters in equation (7) is calculated as the sum of the effective number of parameters of  $\hat{\beta}_s$  for all  $s$ .

Because of the size of this problem, it is not possible to compute  $\hat{\beta}$  via exact linear algebra. For an observed vorticity field with  $M = N = 512$  and  $J = 6$ , model fitting will produce approximately 95 million regression coefficients. Instead we use an iterative method, *backfitting* (Friedman and Stuetzle, 1981), to find an approximate solution to the  $S(3J + 1)^2$  simultaneous linear equations that are associated with equation (7). Convergence for this algorithm is achieved by looking at the absolute difference between each regression coefficient matrix  $\hat{\beta}$  from step  $i - 1$  to  $i$ . The number of iterations was found to be small across a wide range of  $S$ : usually 2–4 iterations sufficed. We attribute the small number of iterations to the fact that most coherent structures in  $\zeta(\mathbf{x})$  are spatially isolated; see, for example, Fig. 1.

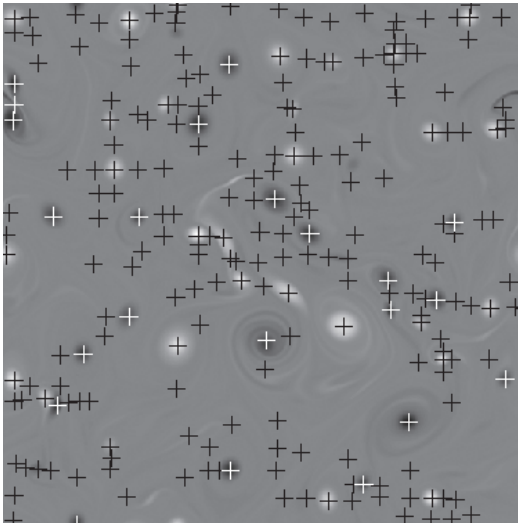
## 5. Application to two-dimensional turbulent flows

Now we return to the simulated vorticity field in Fig. 1 and compute the vortex statistics by using our multi-resolution census algorithm that was outlined in Section 4.2.

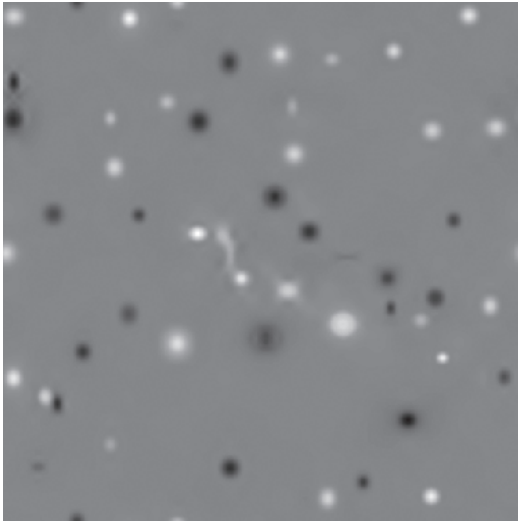
### 5.1. Isolated and multiple coherent structures

The vorticity field at time step  $t = 15$  from the numerical simulation provides a reasonable example of multiple coherent structures that are embedded within a quiescent background. Summary statistics for this vorticity field include an average circulation given by  $\bar{\zeta} = 384.10$ , an average enstrophy of  $\bar{\mathcal{E}} = 397.4 \times 10^3$  and an average maximum peak amplitude of 25.63.

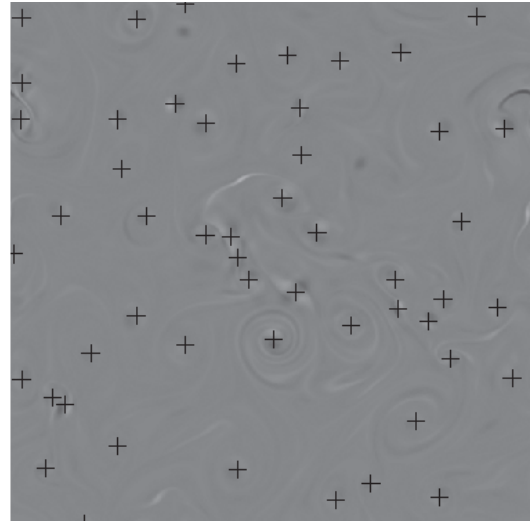
To illustrate the ability of our procedure to find coherent structures of varying sizes and shapes, Fig. 7 shows the results from the two stages of our census algorithm: coherent structures that were identified from our candidate procedure, the vortex field model fitted and the residual



(a)



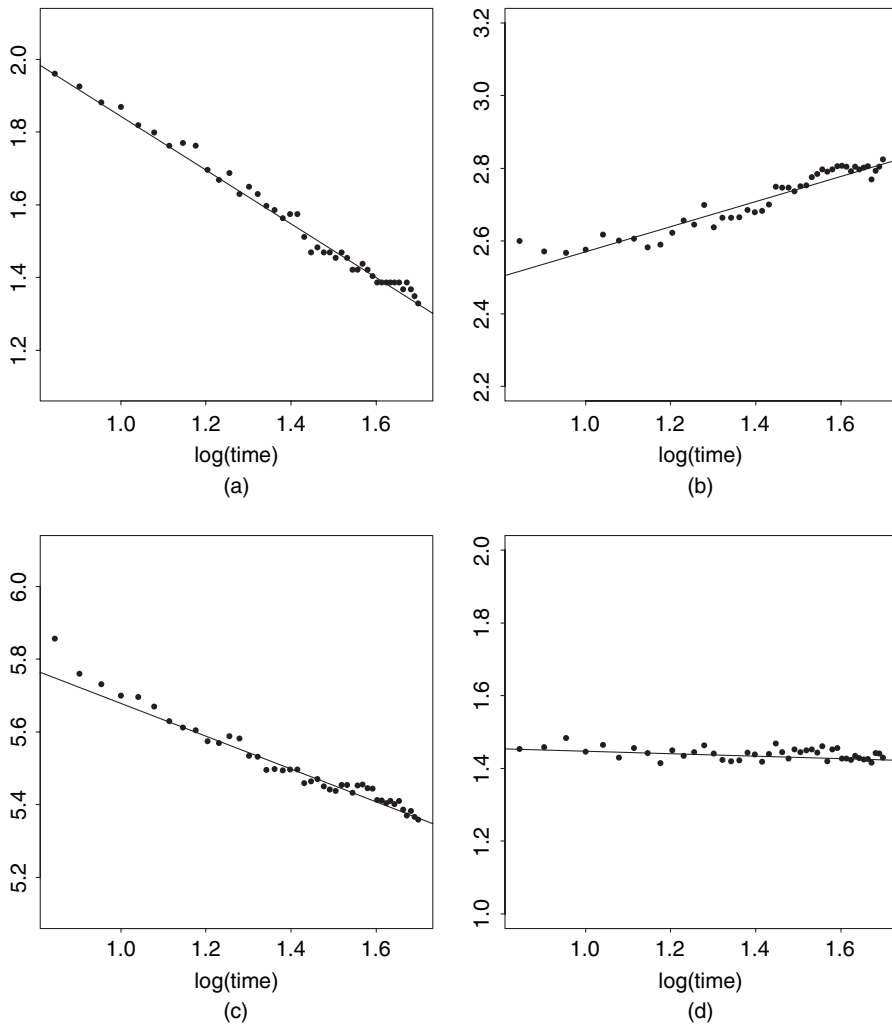
(b)



(c)

**Fig. 7.** (a) Candidate vortex locations, (b) vortex field model and (c) residual field for the observed vorticity field at time  $t = 15$ , plotted on the same grey scale: the spatial locations of the estimated coherent structures from the vortex field model are plotted on the residual field for comparison; changes in colour in the locations of the candidate points are artificial and meant only to visualize points in areas of large negative vorticity





**Fig. 8.** Temporal scaling of vortex statistics from the multi-resolution-based census algorithm (all four quantities are expected to follow a scaling relationship in time and are thus plotted on a log-log scale; —, best fitting regression line): (a) number of vortices; (b) average circulation; (c) average enstrophy; (d) average maximum amplitude

field ( $t = 15$ ). The set of candidate points, roughly 200, appears to capture all potential vortices that are apparent through visual inspection along with other locations that do not appear to contain a substantial accumulation of vorticity. After model selection, a total number of 54 coherent structures were identified. Although the template function has fixed spatial size and amplitude, the coherent structures fitted are distinct from it and from each other, and exhibit a wide variation in both size and amplitude.

A limitation of the current vortex field model is that elliptical structures, such as structures being warped by others in their local neighbourhood, are not well modelled. Although the DWT decomposes two-dimensional structures into horizontal, vertical and diagonal features, warped vortices do not follow both diagonal directions simultaneously. Hence, the diagonal elements from the template function are not heavily utilized in the modelling process. Alternative wavelet

transforms, such as the complex wavelet transform (Kingsbury, 1999, 2001), produce coefficients that are associated with several directions and have been more successful in representing structures in images when compared with common orthogonal wavelet filters. One could also apply other types of multiscale transforms that are more efficient in representing and extracting other features of interest, e.g. the curvelet transform of Starck *et al.* (2002).

### 5.2. Application to temporal scaling

One scientific end point for studying a turbulence experiment is quantifying how coherent structures and their related statistics scale over time. Some examples of the analysis of two-dimensional turbulent flow are Siegel and Weiss (1997), who used a wavelet-based procedure, and the ‘objective observer’ approach of McWilliams (1990). We applied our new method to identify the vortices in all the vorticity fields at time  $t \geq 7$ . Fig. 8 displays the number of vortices, average circulation, average enstrophy and average maximum peak amplitude.

Weiss and McWilliams (1993) investigated the temporal scaling of vortex statistics by using two techniques: long time integration of the fluid equations and from a modified point vortex model that describes turbulence as a set of interacting coherent structures. The two systems gave the same scaling exponent  $\xi \approx 0.72$ . The scaling exponent measures the rate of decay of the turbulence and is governed by the dynamics of the vortex population. In addition, it provides a quantitative measure to test our algorithm. Bracco *et al.* (2000) studied the evolution of vortex statistics from very high resolution numerical simulations. The results of the numerical simulation at low Reynolds number found  $\xi \approx 0.72$ , whereas the vortex decay rate at high Reynolds number was  $\xi \approx 0.76$ . The Reynolds number is the ratio of inertial forces to viscous forces. Low Reynolds numbers correspond to laminar flow where viscous forces dominate and is characterized by smooth fluid motion and high Reynolds numbers are dominated by inertial forces and contain vortices and jets. The evolution of other vortex statistics can be expressed in terms of  $\xi$  (Carnevale *et al.*, 1991). The slopes with standard errors of the best fitting regression lines in Fig. 8 are  $-0.739$  (0.0127),  $0.346$  (0.0242),  $-0.452$  (0.0232) and  $-0.0337$  (0.0112) respectively for Figs 8(a)–8(d). These values agree with those given in Weiss and McWilliams (1993).

## 6. Discussion

Using a multi-resolution multivariate regression model, we have been able to model accurately and to extract coherent structures from a simulated two-dimensional fluid flow at high Reynolds number. The vortex statistics from analysing each time step individually reproduces well-known empirical scaling behaviour. This technique allows for an efficient, objective analysis of observed turbulent fields. Although these results have been achieved by previous algorithms, our method is not limited to simple accumulations of positive or negative vorticity. Through straightforward specification of the template function a variety of features may be extracted from turbulent flow. The modular implementation of our algorithm ensures flexibility across applications and precision of the results is guaranteed by using sound statistical techniques.

The methodology that is presented here can be directly transferred to a three-dimensional setting. For freely decaying, homogeneous geostrophic turbulence, coherent structures are compact regions of large vorticity that is organized in the vertical direction (so-called tubes); see, for example, Farge *et al.* (2001a, b). Previous studies have only implemented a vortex census algorithm by adapting the ‘objective observer’ approach of McWilliams (1990). For the multi-resolution approach, software implementations for the DWT and maximal overlap DWT are already available that extend computations to three dimensions. Further collaboration with

experts in turbulence would be necessary, but a reasonable template function to try would be a three-dimensional Gaussian kernel or possibly a cylinder of fixed radius in  $(x, y)$  and fixed height in the vertical direction.

Storlie *et al.* (2004) recently proposed a novel statistical model for object tracking using an elliptical model for coherent structures. A distinct feature of this tracking model is that it allows splitting and merging of objects, and at the same time it also allows imperfect detection of objects at each time step (e.g. false positive results). This tracking model has been, with preliminary success, applied to track the time evolution of coherent structures in turbulence. We see an opportunity to merge this technique with the methodology that is presented here to track the time evolution of the turbulent fluid by using the regression coefficient matrix as a concise description of the coherent structure.

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