Risk Measures

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CHAPTER 1

Risk Measures

Risk measures are quantitative tools used to assess the risk associated with financial investments. They help investors and financial professionals to understand the potential losses and volatility in their portfolios. This document provides a detailed explanation of various risk measures used in mathematical finance, including examples.

1.1. Types of Risk Measures

1.1.1. Value at Risk (VaR). Value at Risk (VaR) is a widely used risk measure that estimates the potential loss in value of a portfolio over a defined period for a given confidence interval. VaR is defined as the maximum loss not exceeded with a certain confidence level.

1.1.1.1. *Definition.* For a portfolio P, the VaR at confidence level α over time horizon t is given by:

$$\mathrm{VaR}_{\alpha,t} = -\inf\{x \in \mathbb{R} \mid P(X \leq x) > 1 - \alpha\}$$

1.1.1.2. *Example.* Consider a portfolio with daily returns following a normal distribution with mean $\mu = 0.001$ and standard deviation $\sigma = 0.02$. To calculate the 1-day VaR at 95

$$\mathrm{VaR}_{0.95,1} = \mu + z_{0.95}\sigma$$

where $z_{0.95} = 1.645$ (the z-score for 95

$$\mathrm{VaR}_{0.95,1} = 0.001 + 1.645 \times 0.02 = 0.0349$$

Thus, the 1-day VaR at 95

1.1.2. Conditional Value at Risk (CVaR). Conditional Value at Risk (CVaR), also known as Expected Shortfall, measures the expected loss given that the loss has exceeded the VaR threshold. It provides information about the tail risk of the distribution.

1.1.2.1. *Definition.* For a portfolio P, the CVaR at confidence level α over time horizon t is given by:

$$\mathrm{CVaR}_{\alpha,t} = E[X \mid X \geq \mathrm{VaR}_{\alpha,t}]$$

1.1.2.2. *Example.* Using the previous example with normal distribution, the CVaR at 95

$$\mathrm{CVaR}_{0.95,1} = \mu + \frac{\phi(z_{0.95})}{1 - 0.95}\sigma$$

where $\phi(z)$ is the standard normal pdf.

$$\mathrm{CVaR}_{0.95,1} = 0.001 + \frac{0.055}{0.05} \times 0.02 = 0.023 + 1.1 \times 0.02 = 0.045$$

Thus, the 1-day CVaR at 95

1.1.3. Standard Deviation (Volatility). Standard deviation, also known as volatility, measures the dispersion of returns around the mean. It is a key indicator of risk in financial markets.

1.1.3.1. *Definition.* For a portfolio with returns $R_1, R_2, \ldots, R_n$, the standard deviation σ is given by:

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2}$$

where $\bar{R}$ is the mean return.

1.1.3.2. *Example.* Consider a portfolio with returns over five days: $0.01, -0.02, 0.015, -0.005, 0.02$. The mean return $\bar{R}$ is:

$$\bar{R} = \frac{0.01 - 0.02 + 0.015 - 0.005 + 0.02}{5} = 0.004$$

The standard deviation σ is:

$$\sigma = \sqrt{\frac{1}{4}[(0.01 - 0.004)^2 + (-0.02 - 0.004)^2 + (0.015 - 0.004)^2 + (-0.005 - 0.004)^2 + (0.02 - 0.004)^2]}$$

$$\sigma = \sqrt{\frac{1}{4}[0.000036 + 0.000576 + 0.000121 + 0.000081 + 0.000256]} = \sqrt{0.0002675} \approx 0.0164$$

Thus, the standard deviation of the portfolio returns is 1.64

1.1.4. Beta. Beta measures the sensitivity of a portfolio's returns to the returns of the market. It indicates the level of systematic risk.

1.1.4.1. *Definition.* For a portfolio with returns R_P and market returns R_M, the beta β is given by:

$$\beta = \frac{\mathrm{Cov}(R_P, R_M)}{\mathrm{Var}(R_M)}$$

1.1.4.2. *Example.* Consider a portfolio with returns R_P and market returns R_M. Suppose the covariance $\mathrm{Cov}(R_P, R_M) = 0.002$ and the variance of the market returns $\mathrm{Var}(R_M) = 0.01$. The beta is:

$$\beta = \frac{0.002}{0.01} = 0.2$$

Thus, the portfolio has a beta of 0.2, indicating that it is less volatile than the market.

1.1.5. Sharpe Ratio. The Sharpe ratio measures the risk-adjusted return of a portfolio. It is the ratio of the excess return of the portfolio over the risk-free rate to the standard deviation of the portfolio returns.

1.1.5.1. *Definition.* For a portfolio with returns R_P, risk-free rate R_f, and standard deviation σ_P, the Sharpe ratio S is given by:

$$S = \frac{R_P - R_f}{\sigma_P}$$

1.1.5.2. *Example.* Consider a portfolio with an annual return $R_P = 0.08$, a risk-free rate $R_f = 0.02$, and a standard deviation $\sigma_P = 0.15$. The Sharpe ratio is:

$$S = \frac{0.08 - 0.02}{0.15} = 0.4$$

Thus, the Sharpe ratio of the portfolio is 0.4, indicating the portfolio's risk-adjusted return.

REMARK 1.1. Risk measures are essential tools in mathematical finance for assessing and managing the risk associated with financial investments. By understanding and applying these risk measures, financial professionals can make more informed decisions, optimize portfolios, and manage risk effectively.

1.2. Exercises

(1) Assume a portfolio has daily returns that are normally distributed with a mean of 0.001 and a standard deviation of 0.02. Calculate the 1-day VaR at the 95% confidence level.

(2) Using the same portfolio from Exercise 1, calculate the 1-day CVaR at the 95% confidence level.

(3) Consider a portfolio with the following daily returns: $0.01, -0.02, 0.015, -0.005, 0.02$. Calculate the standard deviation of the portfolio returns.

(4) A portfolio's returns have a covariance with the market returns of 0.003. The variance of the market returns is 0.015. Calculate the beta of the portfolio.

(5) A portfolio has an annual return of 0.12, a risk-free rate of 0.03, and a standard deviation of 0.2. Calculate the Sharpe ratio of the portfolio.

(6) Given two assets with daily returns that are normally distributed with means of 0.001 and 0.002, and standard deviations of 0.02 and 0.03, respectively. The correlation between the two assets is 0.5. Calculate the 1-day VaR at the 95% confidence level for a portfolio equally weighted between the two assets.

(7) Consider a portfolio with two assets A and B. Asset A has a weight of 40%, a standard deviation of 0.25, and a beta of 1.2. Asset B has a weight of 60%, a standard deviation of 0.15, and a beta of 0.8. Calculate the contribution of each asset to the portfolio's total risk.

(8) Assume a portfolio has daily returns that follow a normal distribution with a mean of 0.002 and a standard deviation of 0.018. Calculate the 1-day expected shortfall at the 99% confidence level.

(9) Given the following historical daily returns of a portfolio: $-0.015, 0.012, -0.007, 0.005, 0.02, -0.01, 0.011, -0.003, 0.01$ Calculate the 1-day historical VaR at the 95% confidence level.

(10) A portfolio has the following asset allocation: 50% in stocks, 30% in bonds, and 20% in real estate. Assume that in a stress scenario, stocks drop by 25%, bonds drop by 10%, and real estate drops by 15%. Calculate the overall portfolio loss in this stress scenario.

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