Quiz 1

Nzambuli Daniel

2024-06-14

1. Define expectation and variance in the context of probability theory. Explain their significance in financial mathematics.

Expectation is a measure of the central value or average of a random variable and provides a summary of the overall behavior of the random variable. it is expressed as

$$E(x) = \sum(random\ variable * Its\ probability\ of\ occurence)$$
$$= \int_{-\infty}^{\infty} x * f(x)dx$$

Variance is a measure of how far a random variable differs from its expected value and measures the degree of dispersion of a random variable

$$Var(x) = \sum_{i=1}^{n} p_i(x_i - E(x))^2$$
$$= \int_{-\infty}^{\infty} (x - E(x))^2 f(x)dx$$

2. Suppose you have historical data for three stocks:A,B,and C.

Day	Stock A Return	Stock A Return	Market Return
1	0.0073	0.0089	0.0058
2	0.0117	0.0102	0.0150
3	0.0054	0.0065	0.0120
4	0.0238	0.0032	0.0141
5	0.0022	0.0137	0.0078
6	0.0174	0.0161	0.0193
7	0.0139	0.0113	0.0087
8	0.0047	0.0098	0.0134
9	0.0088	0.0110	0.0046
10	0.0160	0.0123	0.0061

Calculate the expected return and variance of a portfolio consisting of 30% of stock A, 50% of stock B,and 20% of stock C.

$$expected\ returns = \frac{\sum_{i=a}^{a,b,c} x \in i}{n_i}$$
$$E(A) = \frac{0.1112}{10}$$
$$E(B) = \frac{0.103}{10}$$
$$E(C) = \frac{0.1068}{10}$$

$$portfolio\ return = 0.3E(A) + 0.5E(B) + 0.2E(c)$$
$$= 0.3 * 0.01112 + 0.5 * 0.0103 + 0.2 * 0.01068$$
$$= 0.010622$$

the expected return is $\underline{0.0106}$

```
A = data.frame(a = c(0.0073, 0.0117, 0.0054, 0.0238, 0.0022, 0.0174, 0.0139, 0.0047, 0.0088, 0.0160))
A$mean_var = A$a - mean(A$a)
A$var = A$mean_var ^2
var_A = sum(A$var)
var_A
```

```
## [1] 0.000405576
```

```
B = data.frame(b = c(0.0089, 0.0102, 0.0065, 0.0032, 0.0137, 0.0161, 0.0113, 0.0098, 0.0110, 0.0123))
B$mean_var = B$b - mean(B$b)
B$var = B$mean_var ^2
var_B = sum(B$var)
var_B
```

```
## [1] 0.00011776
```

```
C = data.frame(c = c(0.0058, 0.0150, 0.0120, 0.0141, 0.0078, 0.0193, 0.0087, 0.0134, 0.0046, 0.0061))
C$mean_var = C$c - mean(C$c)
C$var = C$mean_var ^2
var_C = sum(C$var)
var_C
```

```
## [1] 0.000207776
```

$$variance\ A = \sum_{i=1}^{n}(A_i - E(A))^2$$
$$= 4.05576e - 05$$
$$Var\ B = 1.1776e - 05$$
$$Var\ C = 2.07776e - 05$$

covariance

```
cov_tb = data.frame(AB = 1/9 * A$mean_var * B$mean_var,
                    AC = 1/9 * A$mean_var * C$mean_var,
                    BC = 1/9 * B$mean_var * C$mean_var)
cov_tb
```

```
##              AB            AC            BC
## 1   5.942222e-07  2.071289e-06  7.591111e-07
## 2  -6.444444e-09  2.784000e-07 -4.800000e-08
## 3   2.415111e-06 -8.389333e-07 -5.573333e-07
## 4  -1.000311e-05  4.818400e-06 -2.698000e-06
## 5  -3.369778e-06  2.854400e-06 -1.088000e-06
## 6   4.047111e-06  6.014844e-06  5.555111e-06
## 7   3.088889e-07 -6.116000e-07 -2.200000e-07
```

```
## 8   3.566667e-07 -1.940267e-06 -1.511111e-07
## 9  -1.804444e-07  1.567289e-06 -4.728889e-07
## 10  1.084444e-06 -2.483378e-06 -1.017778e-06
```

```
colSums(cov_tb)
```

```
##            AB            AC            BC 
## -4.753333e-06  1.173044e-05  6.111111e-08
```

$$
\begin{aligned}
cov(A,B) &= \frac{1}{n-1}\sum_{i=1}^{n}(A_i - E(A))(B_i - E(B)) \\
&= -4.753333e-06 \\
cov(A,C) &= 1.173044e-05 \\
coc(B,C) &= 6.111111e-08
\end{aligned}
$$

portfolio variance

```
(0.3^2 * 0.000405576 )+ (0.5^2  * 0.00011776) + (0.2^2 * 0.000207776) + (2* 0.3 * 0.5 * -4.753333e-06) +
(2 * 0.3 * 0.2 * 1.173044e-05) + (2 * 0.5 * 0.2 * 6.111111e-08)
```

```
## [1] 7.424676e-05
```

$$
\begin{aligned}
portfolio\ variance &= w_a^2 Var(A) + w_b^2 Var(B) + w_c^2 Var(C) + 2w_a w_b Cov(A,B) + 2w_a w_c Cov(A,C) + 2w_b w_c Cov(B,C) \\
&= 7.424676e-05
\end{aligned}
$$

portfolio variance is $\underline{7.424676e-05}$

3. Write a script in python to solve 2.

```
import numpy as np

def portfolio_expected_val(returns: np.ndarray, weights: np.ndarray):
    """
    Calculate the expected return of a portfolio given the historical returns of individual stocks and th
eir weights.

    Parameters:
    returns (np.ndarray): A 2D numpy array where each row represents the returns of a stock over time.
    weights (np.ndarray): A 1D numpy array representing the weights of the stocks in the portfolio.

    Returns:
    numeric type: Expected return of the portfolio.
    """
    # expected returns for each stock
    expected_returns = np.mean(returns, axis=1)

    # expected return portfolio
    expected_return_portfolio = np.dot(weights, expected_returns)



    return expected_return_portfolio
```

```
returns = np.array([
    [0.0073, 0.0117, 0.0054, 0.0238, 0.0022, 0.0174, 0.0139, 0.0047, 0.0088, 0.0160],  # Stock A
    [0.0089, 0.0102, 0.0065, 0.0032, 0.0137, 0.0161, 0.0113, 0.0098, 0.0110, 0.0123],  # Stock B
    [0.0058, 0.0150, 0.0120, 0.0141, 0.0078, 0.0193, 0.0087, 0.0134, 0.0046, 0.0061]   # Stock C
])

weights = np.array([0.3, 0.5, 0.2])
```

```
expected_return_portfolio = portfolio_expected_val(returns, weights)
print(f"Expected Return of the Portfolio: {expected_return_portfolio}")
```

```
## Expected Return of the Portfolio: 0.010622
```

```
def portfolio_var(returns: np.ndarray, weights: np.ndarray):
    """
    Calculate the expected return of a portfolio given the historical returns of individual stocks and th
eir weights.

    Parameters:
    returns (np.ndarray): A 2D numpy array where each row represents the returns of a stock over time.
    weights (np.ndarray): A 1D numpy array representing the weights of the stocks in the portfolio.

    Returns:
    numeric type: Expected return of the portfolio.
    """
    cov_matrix = np.cov(returns)

    # Calculate the variance of the portfolio
    portfolio_variance = np.dot(weights.T, np.dot(cov_matrix, weights))


    return portfolio_variance
```

```
portfolio_variance = portfolio_var(returns, weights)
print(f"Variance of the Portfolio: {portfolio_variance}")
```

```
## Variance of the Portfolio: 8.244195555555557e-06
```

4. Give the definition of a martingale and explain its role in modeling the evolution of stock prices.

Martingale assuming all prior information of a random variable in continuous time sequence, the best prediction of the next future random variable in the sequence is the current random variable

given a sequence of random variables with $\{X_t\}$ the current variable at time **t** and a sequence $\{F_t\}$ representing prior information up to a time **t.** $\{X_t\}$ is a martingale if

$$E(X_{t+1}|F_t) = X_t$$

5. Let $X_t \sim N(\mu, \sigma^2)$ represent stockprices. Give the statement of Ito's lemma. Hence use Ito's Lemma to find the differential $df(X_t)$ if $X_t$follows the stochastic differential equation(SDE):

$$d(X_t) = \mu X_t dt + \sigma X_t dW_t$$

where $f(X_t) = \ln X_t$. Furthermore, compute the mean and variance of the resulting probability distribution for $f(X_t)$

ito’s lemma

$$df(X_t,t) = (\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X_t} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X_t^2})dt + \sigma(\frac{\partial f}{\partial X_t})dW_t$$

SDE

$$d(X_t) = \mu X_t dt + \sigma X_t dW_t$$

function

$$f(X_t) = \ln X_t$$

derivatives

$$\frac{\partial f}{\partial t} = 0$$
$$\frac{\partial f}{\partial X_t} = \frac{1}{X_t}$$
$$\frac{\partial^2 f}{\partial X_t^2} = -\frac{1}{X_t^2}$$

fit to Ito’s lemma

$$(\mu\frac{1}{X_t} + \frac{1}{2}\sigma^2 - \frac{1}{X_t^2})dt + \sigma X_t\frac{1}{X_t}dW_t$$

6. Prove that a Brownian motion is a martingale

1. Given prior information of a series of random varibles the expected value of the next future random variable is the current random variable

for Brownian motion

$$w_0 = 0$$

and a future event W_t is predicted by the current event as it is independent of all past events in the Brownian motion. This is because Brownian motion increases from time 0 in a continuous path

$$E(W_t|prior_s)$$
$$E(W_t) = E(W_{current}) + E(W_t - W_{current})$$
$$E(W_t) = E(W_{current}) + 0$$
$$\therefore E(W_t|prior_s) = E(W_{current})$$