Derivatives Pricing

Joseph Owuor Owino

United States International University Africa

Email address: josephowuorowino@gmail.com

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CHAPTER 1

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The world of finance thrives on efficiency,

and a key concept that ensures this is the no-arbitrage principle. This principle dictates that there shouldn't be risk-free ways to make money in the market. Understanding this principle is crucial for exploring derivative instruments like options. Options contracts allow investors to leverage the future price movement (forward price) of an underlying asset, like a stock. To manage the inherent risk associated with options, traders use a concept called hedging. This involves creating a portfolio that offsets potential losses from one holding with gains from another. One important relationship in option pricing is put-call parity. It establishes a fair value connection between call and put options with identical characteristics. Finally, the cornerstone of modern option pricing is the Black-Scholes equation. This complex formula, derived under the assumption of no-arbitrage, helps determine the theoretical value of options. Let's delve deeper into these concepts and explore how they shape the landscape of options trading.

1.1. No-Arbitrage Principle

Definition 1.1 (No-Arbitrage Principle). The no-arbitrage principle is a fundamental concept in finance which states that there are no opportunities to make a risk-free profit with zero net investment. In a market that allows for arbitrage, prices would adjust until the arbitrage opportunity is eliminated.

The no-arbitrage principle ensures that the prices of financial instruments are fair and consistent. It leads to the Law of One Price, which states that identical assets must have the same price.

Definition 1.2 (Forward Price). The forward price F_0 is the agreed-upon price for a transaction that will occur at a future date. It is determined at the inception of the forward contract and remains fixed for the contract's duration. For an asset with no dividends, the forward price is given by:

$$F_0 = S_0 e^{rT}, \tag{1.1}$$

where S_0 is the current spot price, r is the risk-free interest rate, and T is the time to maturity.

Definition 1.3 (Put-Call Parity). Put-call parity is a relationship between the prices of European put and call options with the same strike price and expiration date. It ensures that arbitrage opportunities do not exist between these options. The put-call parity relationship is given by:

$$C - P = S_0 - Ke^{-rT}, \tag{1.2}$$

where C is the price of the European call option, P is the price of the European put option, S_0 is the current spot price of the underlying asset, K is the strike price, r is the risk-free interest rate, and T is the time to maturity.

DEFINITION 1.4 (Hedging). Hedging is the practice of reducing or eliminating financial risk by taking offsetting positions in related securities. It is commonly used to protect against adverse price movements.

EXAMPLE 1.5 (Delta Hedging). Delta hedging involves creating a portfolio with a delta (sensitivity to the underlying asset price) of zero. For a call option, the delta (Δ) is given by:

$$\Delta = \frac{\partial C}{\partial S}, \tag{1.3}$$

where C is the price of the call option and S is the price of the underlying asset. To hedge a long position in a call option, an investor would short Δ shares of the underlying asset.

1.2. Derivation of the Black-Scholes Equation

The Black-Scholes model is based on several key assumptions:

(i). The stock price follows a geometric Brownian motion with constant drift and volatility.

(ii). There are no arbitrage opportunities.

(iii). The market is frictionless (no transaction costs or taxes).

(iv). The risk-free interest rate is constant.

(v). The stock does not pay dividends during the option's life.

1.2.1. Stock Price Dynamics. The stock price S_t follows the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.4}$$

where μ is the drift rate, σ is the volatility, and W_t is a Wiener process.

1.2.2. Derivation Using Ito's Lemma. Consider a European call option with price $C(S,t)$ as a function of the stock price S and time t. Using Ito's Lemma, we have:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} dS^2. \tag{1.5}$$

Substituting $dS = \mu S dt + \sigma S dW$ and $dS^2 = \sigma^2 S^2 dt$, we get:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S}(\mu S dt + \sigma S dW) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt. \tag{1.6}$$

Grouping the terms:

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dW. \tag{1.7}$$

1.2.3. Risk-Neutral Valuation. To eliminate the stochastic term, we construct a risk-free portfolio consisting of a long position in the option and a short position in $\Delta = \frac{\partial C}{\partial S}$ shares of the stock. The portfolio value Π is:

$$\Pi = C - \Delta S. \tag{1.8}$$

The change in the portfolio value is:

$$d\Pi = dC - \Delta dS. \tag{1.9}$$

Substituting the expressions for dC and dS, and setting the stochastic term to zero, we get:

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt. \tag{1.10}$$

Since the portfolio is risk-free, it must earn the risk-free rate r:

$$d\Pi = r\Pi dt. \tag{1.11}$$

Equating the two expressions for $d\Pi$:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r(C - S\frac{\partial C}{\partial S}). \tag{1.12}$$

Rearranging terms, we obtain the Black-Scholes partial differential equation (PDE):

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0. \tag{1.13}$$

The no-arbitrage principle, forward price, put-call parity, hedging strategies, and the Black-Scholes equation are fundamental concepts in mathematical finance. Understanding these concepts is crucial for pricing derivatives, managing financial risks, and developing trading strategies.

1.3. Exercises

(1) **No-Arbitrage Principle**

(2) (i). **No-Arbitrage Condition:** Explain why the no-arbitrage principle is essential in financial markets. Provide an example of an arbitrage opportunity and describe how the market would correct it.

(ii). **Law of One Price:** Consider two identical assets that are priced differently in two markets. Describe how you would take advantage of this arbitrage opportunity and explain how the prices would adjust to eliminate the arbitrage.

(iii). **Interest Rate Parity:** Given the spot exchange rate between two currencies and their respective risk-free interest rates, derive the no-arbitrage condition for the forward exchange rate.

(3) **Forward Price**

(4) (i). **Forward Price Calculation:** A stock currently trades at \$50. The risk-free interest rate is 5% per annum. Calculate the forward price for a contract expiring in 6 months.

(ii). **Dividends and Forward Price:** Suppose the stock mentioned above pays a dividend of \$2 in 3 months. How does this affect the forward price for the contract expiring in 6 months?

(iii). **Commodity Forward Price:** Explain how storage costs and convenience yield affect the forward price of a commodity. Provide an example calculation.

(5) **Put-Call Parity**

(i). **Derivation of Put-Call Parity:** Derive the put-call parity relationship for European options. Use a portfolio argument involving a long call, a short put, and a risk-free bond.

(ii). **Put-Call Parity Application:** Given the following market data: $S_0 = 100$, $K = 100$, $r = 0.05$, $T = 1$ year, $C = 10$. Calculate the price of the European put option using put-call parity.

(iii). **Arbitrage from Put-Call Parity:** Suppose you find that the call option price is \$10, the put option price is \$5, the stock price is \$100, and the present value of the strike price is \$95. Identify the arbitrage opportunity and describe the strategy to exploit it.

(6) **Hedging**

(i). **Delta Hedging:** A trader holds a long position in a European call option on a stock. The option's delta is 0.6, and the trader holds 100 options. How many shares of the stock should the trader sell to delta-hedge the position?

(ii). **Gamma Hedging:** Explain the concept of gamma hedging. How does it differ from delta hedging? Why is gamma hedging important for managing the risk of an options portfolio?

(iii). **Hedging with Futures:** A portfolio manager wants to hedge a stock portfolio using index futures. Describe the steps involved in creating an effective hedge. How do you determine the number of futures contracts to use?

(7) **Derivation of the Black-Scholes Equation**

(i). **Black-Scholes PDE:** Starting from the stochastic differential equation for the stock price $dS_t = \mu S_t dt + \sigma S_t dW_t$, derive the Black-Scholes partial differential equation for a European call option.

(ii). **Boundary Conditions:** What are the boundary conditions for the Black-Scholes PDE when pricing a European call option? Explain why these conditions are necessary.

(iii). **Solution to the Black-Scholes PDE:** Solve the Black-Scholes PDE to find the formula for the price of a European call option. State any assumptions made during the derivation.

(iv). **Greeks:** Derive the expressions for delta, gamma, and theta for a European call option under the Black-Scholes framework. Explain the economic interpretation of each Greek.

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