Stochastic Processes

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Contents

CHAPTER 1

Stochastic Processes

Stochastic processes are fundamental to understanding various phenomena in fields ranging from finance to physics. Among these processes, Brownian motion stands out as a critical model for random movement, often used to represent fluctuating stock prices and physical particles in fluid dynamics. The mathematical treatment of Brownian motion involves stochastic differentiation and integration, where traditional calculus is extended to accommodate the randomness inherent in these processes. Key concepts such as Itô calculus enable the formulation and solution of stochastic differential equations, which describe systems influenced by random noise. Additionally, martingales—a class of stochastic processes that exhibit fair game properties—play a pivotal role in the theory of stochastic processes. They provide a framework for modeling fair betting systems and underpin many financial models, including those for option pricing. This introduction to stochastic processes lays the groundwork for exploring these sophisticated mathematical tools and their applications in various domains.

1.1. Probability measure space

Definition 1.1 (σ-algebra (σ-field)). Let $\Omega \neq \emptyset$. Then a collection Σ of subsets of Ω is a σ-algebra if:

(i). $\emptyset \in \Sigma$

(ii). $\Omega \in \Sigma$

(iii). If $\Omega \supseteq X \in \Sigma$, then $X^c \in \Sigma$

(iv). For any pairwise disjoint family X_n of subsets of Ω, then $\cup_n X_n \in \Sigma$

Definition 1.2 (Measurable space). The pair of a set Ω and a $\sigma-$algebra Σ on Ω is a measurable space.

Definition 1.3 (Measure). Let (Ω, Σ) be a measurable space. Then a measure μ on (Ω, Σ) is a function $\mu : \Sigma \to [0, \infty]$ such that:

(i). $\mu(\emptyset) = 0$

(ii). For any $X \in \Omega$, then $\mu(X) \geq 0$

(iii). For any family X_i of pairwise disjoint sets in Σ, we have $\mu(\cup_i X_i) = \sum_i \mu(X_i)$.

The triple (Ω, Σ, μ) is called a measure space.

Remark 1.4. An example of a measure space is a probability space $(\Omega, \Sigma, \mathbb{P})$ where $\mathbb{P}(\Omega) = 1$. Any space of finite measure can be scaled to a probability space.

Remark 1.5. We say that a function is measurable if it has a measurable pre-image in the source measure space.

Definition 1.6 (Random Variable). A random variable X on probability space $(\Omega, \Sigma, \mathbb{P})$ is a Σ-measurable function from Ω to $\mathbb{R}$.

1.2. Stochastic processes

Definition 1.7. A stochastic process (or a random process, or simply a process) on the probability space (Ω, Σ, P), denoted by $X = \{X_t : t \in J\}$, is a function of two variables with domain $J \times \Omega$ and range $S \subseteq \mathbb{R}$ and is expressed by:

$$X : J \times \Omega \to S, \quad (t, \omega) \mapsto X_t(\omega) \quad (\text{i.e., } X(t, \omega) = X_t(\omega) = X_t.$$

Here, $J \subseteq \mathbb{R}$ is a nonempty set called the index set[1] of the process X, and the range S is called the state space[2] of the process X. In short, a stochastic process is a collection of random variables on the same probability space, representing the evolution of randomness over time.

Definition 1.8 (Filtered Probability Space). A filtered probability space is a quadruple $(\Omega, \Sigma, \{\Sigma_t\}, P)$, where:

- (Ω, Σ, P) is a probability space.
- $\{\Sigma_t\}_{t \geq 0}$ is a filtration.

A filtration is a non-decreasing collection of σ-algebras $\{\Sigma_t \subseteq \Sigma \colon t \geq 0\}$ such that for all $s, t \geq 0$ with $s \leq t$, we have $\Sigma_s \subseteq \Sigma_t$.

Definition 1.9 (Natural Filtration). The filtration $\mathbb{F} = \{F_t \subseteq \mathcal{F} : t \geq 0\}$ is called a natural (or standard) filtration of process X if

$$F_t = \sigma(X_s, 0 \leq s \leq t), \quad t \geq 0, \tag{1.1}$$

where $\sigma(X_s, 0 \leq s \leq t)$ denotes the σ-algebra generated by the random variables X_s for $0 \leq s \leq t$.

We also say that the filtration $\mathbb{F} = \{F_t : t \geq 0\}$ defined by $F_t = \sigma(X_s : s \leq t)$ is the filtration induced by $\{X_t : t \geq 0\}$.

Furthermore, it holds that $\sigma(X_u, 0 \leq u \leq s) \subseteq \sigma(X_u, 0 \leq u \leq t)$, for $0 \leq s \leq t$.

Definition 1.10 (Adapted Stochastic Process). A stochastic process $\{X_t : t \geq 0\}$ defined on a filtered probability space $(\Omega, \Sigma, \{\Sigma_t\}, P)$ is said to be adapted (or non-anticipating) if X_t is Σ_t-measurable for each $t \geq 0$.

Definition 1.11 (Martingale). A stochastic process $\{X_t : t \geq 0\}$ on a filtered probability space $(\Omega, \Sigma, \{\Sigma_t\}, P)$ is said to be a *martingale* with respect to the filtration $\{\Sigma_t\}$ if $\{X_t\}$ is adapted to $\{\Sigma_t\}$ and satisfies the condition $\mathbb{E}(|X_t|) < \infty$ for $\forall t \geq 0$ and the property

$$\mathbb{E}(X_t \mid \Sigma_s) = X_s \quad \text{for } \forall s < t, \ t \geq 0.$$

DEFINITION 1.12 (Martingale with natural filtration). A stochastic sequence $\{X_n : n = 0, 1, 2, \cdots\}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, P)$, where $\mathcal{F}_n = \sigma(X_k, 0 \le k \le n)$, is said to be a martingale with respect to its natural filtration if the sequence $\{X_n\}$ satisfies the conditions:

(1) Integrability: $\mathbb{E}[|X_n|] < \infty$ for all $n \ge 0$.

(2) Fair Game Property: $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ for all $n \ge 0$.

EXAMPLE 1.13 (Random Walk). Let $\{R_n\ n \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean $\mu = 0$ and finite variance σ^2. Define

$$S_n = R_1 + R_2 + \cdots + R_n. \tag{1.2}$$

The sequence $\{S_n\ n \ge 1\}$ is a random walk. For example, if

$$R_1 = \begin{cases} 1 & \text{with probability } p = \frac{1}{2} \\ -1 & \text{with probability } p = \frac{1}{2} \end{cases}$$

and set $S_0 = 0$, the stochastic sequence $\{S_n : n \ge 0\}$ is called the simple random walk on $\mathbb{Z}$. Clearly, the walk starts at 0 and at each step moves either +1 (to the right) or -1 (to the left) with equal probability.

We are interested in showing that $\{S_n : n \ge 1\}$ defined by (6.23) is a martingale.

Proof. We verify the two conditions required by Definition 6.24:

Step 1. Integrability and Mean:

- $\mathbb{E}(S_n) = n\mathbb{E}(R_1) = 0$ (since the mean of R_n is zero).
- $\mathbb{E}(S_n^2) = \frac{n(n+1)}{2}\mathbb{E}(R_iR_j) = n\sigma^2$ (due to independence and finite variance). This implies $\mathbb{E}(\|S_n\|) < \infty$.

Step 2. Fair Game Property:

$$\mathbb{E}(S_{n+1} \mid S_1, S_2, \ldots, S_n) = \mathbb{E}(R_1+R_2+\cdots+R_{n+1} \mid S_1, S_2, \ldots, S_n) = R_1+R_2+\cdots+R_n+\mathbb{E}(R_{n+1}) = S_n+0 = S_n$$

Since we verified both integrability and the fair game property, we can conclude that $\{S_n : n \ge 1\}$ is a martingale with respect to its natural filtration.

DEFINITION 1.14 (Sample path). For each fixed $\omega \in \Omega$, the sample path (or realization, or trajectory, or sample function) of a stochastic process $X = \{X_t : t \in J\}$ on $(\Omega, \mathcal{F}, P)$ is the (graph of) function $X(\omega) : J \to \mathbb{R}, \quad t \mapsto X(t, \omega) \quad$ (i.e., $t \mapsto X_t(\omega)$) and denoted by either $X_t(\omega)$ or $X(t, \omega)$ (where J may be discrete or continuous). Note that for a fixed ω, a continuous sample path of the process $\{X_t : t \ge 0\}$ is defined in the ordinary calculus sense. That is,

$$\lim_{s \to t} X(s, \omega) = X(t, \omega)$$
$$\lim_{s \to 0, s > 0} X(s, \omega) = X(0, \omega)$$

for each $t > 0$, for $t = 0$.

Definition 1.15 (Sample path continuous). A stochastic process $X = \{X_t : t \geq 0\}$ is said to be sample-continuous (or almost surely continuous, or simply continuous) if almost surely all sample paths are continuous. That is, $X(\omega) : [0, \infty) \to \mathbb{R}, \quad t \mapsto X(t, \omega)$ is a continuous sample path for **almost surely (a.s.)** all $\omega \in \Omega$, which means that $P(\{\omega \in \Omega \mid X(\omega) \text{ is not a continuous sample path}\}) = 0$.

1.3. Convergence of Random Variables

Definition 1.16 (Almost everywhere). A sequence $\{X_n\}$ of random variables is said to converge almost surely (or converge almost everywhere or converge with probability 1) to a random variable X, written $X_n \xrightarrow{a.s.} X$, if

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

Definition 1.17 (Convergence in probability). A sequence $\{X_n\}$of random variables is said to converge in probability to a random variable$X$, written $X_n \xrightarrow{P} X$, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0. \tag{1.3}$$

(Equivalently, $X_n \xrightarrow{P} X$ if and only if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1. \tag{1.4}$$

)

Definition 1.18 (Convergence in Mean Square). A sequence $\{X_n\}$ of random variables is said to converge in mean square (or converge in the L^2-norm, or simply converge in L^2) to a random variable X, written $X_n \xrightarrow{m.s.} X$, if

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^2\right] = 0. \tag{1.5}$$

Definition 1.19 (Convergence in Distribution). A sequence $\{X_n\}$ of random variables is said to converge in distribution (or converge in law or converge weakly) to a random variable X, written $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_n(x) = F(x), \tag{1.6}$$

for each x at which F is continuous, where F_n and F are the cumulative distribution functions of random variables X_n and X, respectively.

DEFINITION 1.20 (Noises). (1) The process $\{\varepsilon_t\}$ is said to be a white noise (process) if its mean value function and autocovariance function respectively are

$$m(t) = \mathbb{E}(\varepsilon_t) = 0, \quad \forall t,$$
$$\gamma_\tau(t) = \begin{cases} \sigma_\varepsilon^2 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0, \end{cases}$$

where $\sigma_\varepsilon^2 > 0$ is a constant. We write $\varepsilon_t \sim \text{WN}(0, \sigma_\varepsilon^2)$.

(2) An independent white noise (process) $\{\varepsilon_t\}$ is a white noise process consisting of mutually independent random variables. We write $\varepsilon_t \sim \text{i.WN}(0, \sigma_\varepsilon^2)$.

(3) A strict white noise (process) $\{\varepsilon_t\}$ is a white noise process consisting of independent and identically distributed (i.i.d.) random variables. We denote it by $\varepsilon_t \sim \text{i.i.d.WN}(0, \sigma_\varepsilon^2)$.

(4) A white noise process $\{\varepsilon_t\}$ is said to be a Gaussian white noise process if $\varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$.

A white noise process has no linear prediction value (it is unpredictable) because it is serially uncorrelated.

1.4. Brownian Motion

DEFINITION 1.21 (Normal Distributions). We denote a normally distributed random variable X with mean μ and variance σ^2 by $X \sim \mathcal{N}(\mu, \sigma^2)$. Let a and b be constants. The following properties hold:

(1) Scaling: If $X \sim \mathcal{N}(0, 1)$, then $a + bX \sim \mathcal{N}(a, b^2)$.

(2) Linear Transformation: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$.

(3) Moment Generating Function:** If $X \sim \mathcal{N}(\mu, \sigma^2)$, then its moment generating function, defined as $M_X(t) \equiv \mathbb{E}(e^{tX})$, is $M_X(t) = \exp(\mu t + \frac{\sigma^2 t^2}{2})$.

DEFINITION 1.22 (Brownian Motion). A standard (one-dimensional) Brownian motion or Wiener process, denoted by $\mathbf{B} = \{B(t) : t \geq 0\}$, is a stochastic process on $(\Omega, \mathcal{F}, P)$ satisfying the following properties:

(1) Starting Point: $B(0) = 0$ (almost surely). This means $P(\omega \in \Omega \mid B(0) = 0) = 1$.

(2) Sample Continuity: With probability 1, sample paths of $\mathbf{B}$ are continuous. In other words, $P(\omega \in \Omega \mid B(t, \omega)$ is not continuous for any $t) = 0$.

(3) Independent Increments: For any choice of non-negative real numbers $0 \leq t_1 < t_2 < t_3 < \cdots < t_{n-1} < t_n < \infty$, the increments $B(t_2) - B(t_1), B(t_3) - B(t_2), \ldots, B(t_n) - B(t_{n-1})$ are mutually independent random variables.

(4) Normally Distributed Increments: For each $0 \leq s < t < \infty$, the increment $B(t) - B(s)$ is a normal random variable with mean 0 and variance $t - s$, written $B(t) - B(s) \sim \mathcal{N}(0, t - s)$.

Definition 1.23 (Brownian motion with starting point). A Brownian motion with starting point b is a stochastic process that can be expressed by $b + \mathbf{B}$, where $b \in \mathbb{R}$ is a constant and $\mathbf{B}$ is a standard Brownian motion. In simpler terms, a Brownian motion $X = \{X(t)\}$ with initial value $X(0) = b$ is obtained by adding b to a standard Brownian motion $B = \{B(t)\}$. We can write this as $X = B + b$ or equivalently $X(t) = b + B(t)$, for all $t \geq 0$.

Definition 1.24 (Brownian motion with drift and scaling). A Brownian motion with drift and scaling is a stochastic process that can be expressed by $b + \mu t + \sigma \mathbf{B}$, where $b, \mu \in \mathbb{R}$ are constants, $\sigma > 0$ is another constant, and $\mathbf{B}$ is a standard Brownian motion.

In this process: * b represents the starting point. * μ represents the drift, which determines the average change per unit time (positive for upward drift, negative for downward drift). * σ represents the scaling factor, which affects the volatility of the process (larger sigma leads to larger fluctuations). * $\mathbf{B}$ denotes a standard Brownian motion, providing the random component of the process.

1.5. Stochastic Calculus

Just as the indefinite integral defines a function in the deterministic calculus, the Itó's integral defines a stochastic process.

Definition 1.25. A one-dimensional Itô process is a stochastic process $X = \{X(t)\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with the following expression:

$$X(t) = X(0) + \int_0^t \mu(X(s), s)ds + \int_0^t \sigma(X(s), s)dB(s), \quad 0 \leq t \leq T, \tag{1.7}$$

where:

* $\mu(X(t), t)$ is an adapted drift process, meaning its value at time t depends only on the information available up to time t (typically the past history of the Brownian motion B). * $\sigma(X(t), t)$ is an adapted volatility process, which is also square integrable (i.e., has a finite second moment).

An Itó process $X = \{X(t)\}$ is called an Itó diffusion if both the drift process and the volatility process are functions of $X(t)$ only. In this case, the expression for X becomes:

$$X(t) = X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dB(s), \quad 0 \leq t \leq T, \tag{1.8}$$

where $\mu(X(t))$ and $\sigma(X(t))$ are also known as the drift coefficient and diffusion coefficient of the Itô diffusion X, respectively.

1.6. Exercises

(1) (i). **Ito's Lemma:** Use Ito's Lemma to find the differential $df(X_t)$ if X_t follows the stochastic differential equation (SDE):

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{1.9}$$

where $f(X_t) = \ln(X_t)$.

(ii). **Ito Integral:** Compute the Ito integral

$$\int_0^T W_t \, dW_t, \tag{1.10}$$

where W_t is a standard Brownian motion.

(iii). **SDE Solution:** Solve the SDE

$$dX_t = \alpha X_t dt + \beta dW_t, \tag{1.11}$$

where α and β are constants, and W_t is a standard Brownian motion.

(iv). **Girsanov's Theorem:** Explain Girsanov's Theorem and provide an example of how it can be used to change the measure in a financial context.

(v). **Martingales:** Prove that $e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$ is a martingale with respect to the filtration $\mathcal{F}_t$.

(2) **Brownian Motion**

(i). **Properties:** List and prove the basic properties of standard Brownian motion W_t.

(ii). **Reflection Principle:** State and prove the reflection principle for Brownian motion.

(iii). **Hitting Times:** Calculate the expected hitting time for a Brownian motion starting at 0 to reach a positive level a.

(iv). **Brownian Bridge:** Define a Brownian bridge and derive its covariance function.

(v). **Quadratic Variation:** Show that the quadratic variation of Brownian motion W_t is $[W]_t = t$.

(3) **Applications in Finance**

(i). **Black-Scholes Model:** Derive the Black-Scholes partial differential equation (PDE) for a European call option using the no-arbitrage principle and Ito's Lemma.

(ii). **Geometric Brownian Motion:** Explain how geometric Brownian motion (GBM) is used to model stock prices and derive the solution to the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.12}$$

where S_t represents the stock price, μ is the drift rate, and σ is the volatility.

(iii). **Interest Rate Models:** Discuss the Vasicek model for interest rates, which is given by the SDE

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t, \tag{1.13}$$

and solve for the mean and variance of r_t.

(iv). **Option Greeks:** Compute the delta and gamma for a European call option in the Black-Scholes model.

(v). **Stochastic Volatility:** Briefly explain the concept of stochastic volatility and provide an example of a model that incorporates stochastic volatility.

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