Introduction to Time Series Analysis

STA 3050A: Time Series and Forecasting

EXPONENTIAL SMOOTHING

A popular forecasting method in business is exponential smoothing. Its adaptability, simplicity in automation, low cost, and high performance are the main reasons for its popularity. Simple exponential smoothing is similar to the moving average, except that instead of taking a simple average over the m most recent values, we take a weighted average of all past values so that the weights decrease exponentially into the past. The decay rate of the observation weights is set by the smoothing parameter of the model α ($0 < \alpha < 1$) and called the **exponential smoothing constant**.

The idea is to give more weight to recent information, but previous information should not be completely ignored. Similar to the moving average, simple exponential smoothing can be used for forecasting, but the main assumption is that the series stays at the same level (that is, the local mean of the series is constant) over time, and therefore, this method is suitable for series with neither trend nor seasonal components. As mentioned earlier, such a series can be obtained by removing trend and/or seasonality from the original time series and then applying exponential smoothing to the series of residuals (which are assumed to contain no trend or seasonality). If y_1 ,y_2 ...,y_t are the observations of a time series, then the smoothed value at time t is given by

$$y'_t = \alpha y_t + \alpha(1-\alpha) y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \dots$$

We can also write, the above expression as:

$$y'_t = \alpha y_t + (1-\alpha) y'_{t-1}$$

where α is called the exponential smoothing constant and lies between 0 and 1. It controls the rate at which weights decrease. This method consists of the following steps:

Step 1: We take the first given value as the first smoothed value, i.e.,

$$y'_1 = y_1$$

Step 2: We compute the second smoothed value using the first smoothed value. We compute the second smoothed value as:

$$y'_2 = \alpha y_2 + (1-\alpha) y'_1$$

Step 3: We repeat this process until all data are exhausted. We compute the t^{th} smoothed value as follows:

$$y'_t = \alpha y_t + (1-\alpha) y'_{t-1}$$

The popular choice of the smoother constant is α = 0.2. For this, we assign a weight of 0.2 on the most recent observation and a weight of 1 − 0.2 = 0.8 on the most recent forecast value.

Let us look at an example which helps you to understand how to compute exponentially smooth values.

Example 3: Consider the data of the number of fire insurance claims received by an insurance company given in Example 1. Smooth the given time series data using smoothing factors 0.2, 0.4 and 0.8. Compute and compare the forecast errors produced by using the different exponential smoothing constants.

Solution: In the exponential smoother method, we take the first smooth(forecast) value as the first given value, i.e.,

$$y_1' = y_1 = 17$$

We can compute the forecast error as

$$e_1 = y_1 - y_1' = 17 - 17 = 0$$

We can compute the second smoothed value using the first smoothed value and α = 0.2 as

$$y_2' = \alpha y_2 + (1-\alpha) y_1' = 0.2 \times 13 + (1-0.2) \times 17 = 16.20$$

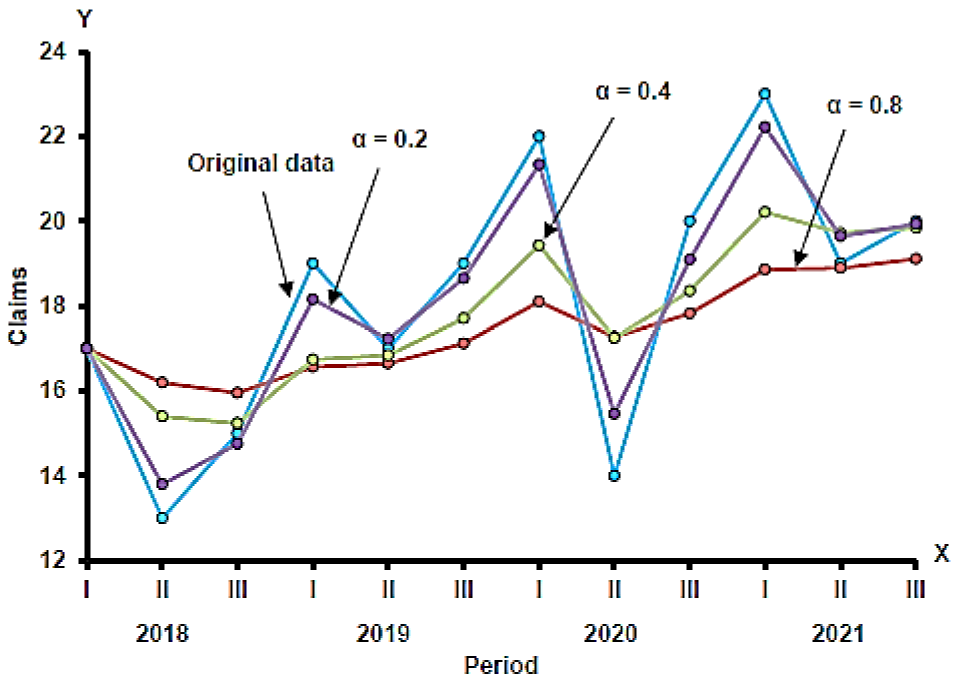
We can compute the forecast error as

$$e_1 = y_2 - y_2' = 13 - 16.20 = 3.80$$

Similarly, you can compute the rest values in the same manner and also for α = 0.4 and 0.8

Year	Period	No. of Claims	Exponential Smoothing			Forecast Error (α = 0.2)	Forecast Error (α = 0.4)	Forecast Error (α = 0.8)
			α = 0.2	α = 0.4	α = 0.8			
2018	I	17	17.00	17.00	17.00	0	0	0
	II	13	16.20	15.40	13.80	–3.20	–2.40	–0.80
	III	15	15.96	15.24	14.76	0.96	–0.24	0.24
2019	I	19	16.57	16.74	18.15	2.43	2.26	0.85
	II	17	16.65	16.85	17.23	0.35	0.15	–0.23
	III	19	17.12	17.71	18.65	1.88	1.29	0.35
2020	I	22	18.10	19.42	21.33	3.90	2.58	0.67
	II	14	17.28	17.25	15.47	–3.28	–3.25	–1.47
	III	20	17.82	18.35	19.09	2.18	1.65	0.91
2021	I	23	18.86	20.21	22.22	4.14	2.79	0.78
	II	19	18.89	19.73	19.64	0.11	–0.73	–0.64
	III	20	19.11	19.84	19.93	0.89	0.16	0.07

From the above table, we observe that as we increase the value of the smoother constant then the forecast error decreases. We now demonstrate the impact of the smoothing factor α using the time series graph as shown below



From the above figure, we observed that as we increase the smoothing constant α, the series is smoother. Therefore, α plays the same role in exponential smoothing as m in the moving average.

After understanding the exponential smoothing and forecast technique, we now discuss the merits and demerits of this method.

Merits

1. It is very simple in concept and very easy to understand.
2. The primary merit of the exponential method over the moving average is that there is no loss of information (data values) as in the case of the moving average.
3. If we forecast using the moving averages method, then m prior values are required. If we have to forecast many values, then this is time-consuming. Whereas the exponential method uses only two pieces of data.

Demerits

1. The method is not flexible in the sense that if some figures are added to the data, then we have to do all calculations again.
2. This method gives good results in the absence of seasonal or cyclical variations. As a result, forecasts are not accurate when data with cyclical or seasonal variations are present.

After understanding the exponential smoothing method, you may be interested in doing the same yourself.

EXERCISE

The annual expenditure levels (in millions) to promote products and services for the financial services sector such as banks, insurance, investments, etc. from 2015 to 2022 are shown in the following table:

Year	2015	2016	2017	2018	2019	2020	2021	2022
Expenditure	5.5	7.2	8.0	9.6	10.2	11.0	12.5	14.0

Use exponential smoothing to obtain filtered values by taking $\alpha = 0.5$, $\alpha = 0.7$, $\alpha = 0.9$ and calculate the forecast errors. Also, plot the original and smoothed values.

ESTIMATION OF TREND COMPONENT USING METHOD OF LEAST SQUARES

There are several ways to determine trend effects in time series data and one of the more prominent is the method of least squares. This method is one of the most common methods for identifying and quantifying the relationship between a dependent variable and single or multiple independent variables. It can also be used to fit a trend. We can also use fitted trend for forecasting. To create a trend model that captures a time series with a global trend, the dependent/ response/output variable (Y) is set as the time series measurement or some function of it, and the independent/predictor variable (X) is set as a time period. In this method, we fit a curve in such a way that the squares of the forecast errors should be minimum.

Many possible trends can be explored with time series data. In this section, we examine only the linear model, the quadratic model and the exponential model because they are the easiest to understand and simplest to compute. Because seasonal effects can confound trend analysis, it is assumed here that no seasonal effects occur in the time series data, or they were removed prior to determining the trend.

A linear trend means that the values of the series increase or decrease linearly in time, whereas an exponential trend captures an exponential increase or decrease.

Linear Trend

When the values of the time series increase or decrease linearly with time then we use linear trend.

In the simplest case, the linear trend model allows for a linear relationship between the forecast variable Y and a single predictor variable time t. In this case, the linear trend line equation is as follows:

$$Y_t = \beta_0 + \beta_1 t$$

The coefficients β_0 and β_1 denote the intercept and the slope of the trend line, respectively. The intercept β_0 represents the predicted value of Y when t = 0 and the slope represents the average predicted change in Y resulting from a one-unit change in t.

We can estimate the values of the constants β_0 and $\beta_1$using the following normal equations:

$$\sum Y_t = n\beta_0 + \beta_1 \sum t$$

$$\sum tY_t = \beta_0 \sum t + \beta_1 \sum t^2$$

where n is the number of observations in the given time series. We obtain the values of ΣY_t , Σt, Σt^2 and ΣtYt from the given time series data and solve these normal equations for the values of β_0 and β_1

Generally, the time t is given in years, therefore, to calculate the values of Σt, ΣtY_t and Σt^2 manually becomes very cumbersome. Therefore, to simplify the calculations, we may make the following transformations in t:

$$X_t = \begin{cases} \dfrac{t - \text{middle value}}{\text{interval in t values}} & (\text{when n is odd}) \\[2ex] \dfrac{t - \text{average of two middle value}}{\text{half of interval in t values}} & (\text{when n is even}) \end{cases}$$

Therefore, the normal equations become:

$$\sum Y_t = n\beta_0 + \beta_1 \sum X_t$$

$$\sum X_t Y_t = \beta_0 \sum X_t + \beta_1 \sum X_t^2$$

The fitted trend line for estimating or forecasting the trend values is given as follows:

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t$$

After that, we put the value of X_t in terms of t to find the final trend line. Let us take an example to understand how to fit a linear trend line for real-life time series data.

Example 4: The sales director of a real estate company wants to study the general direction (trend) of future housing sales. For that, he/she recorded the number of houses sold from 2010 to 2018 as given in the following table:

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018
Sales	52	54	48	60	61	66	70	80	92

i. Construct a simple trend line for the house sales data for the real estate company.
ii. Find the trend values for the given data and find forecast errors.
iii. Plot the given data with trend values.
iv. Use the trend line of best fit to estimate the level of house sales for the year 2022.

Solution: The linear trend line equation is given by

$$Y_t = \beta_0 + \beta_1 t$$

Since n (number of years) = 9 is odd and the middle value is 2014, therefore, we make the following transformation in time t as $X_t = t - 2014$. Therefore, the normal equations for estimating the constants are:

$$\sum Y_t = n\beta_0 + \beta_1 \sum X_t$$

$$\sum X_t Y_t = \beta_0 \sum X_t + \beta_1 \sum X_t^2$$

In the following table, we calculate the values of:

$$\sum Y_t, \sum X_t, \sum X_t Y_t \text{ and } \sum X_t^2$$

Year (t)	Sales (Y_t)	X_t = t–2014	X_tY_t	X_t^2	Trend Value	Forecast Error
2010	52	–4	–208	16	45.58	6.42
2011	54	–3	–162	9	50.38	3.62
2012	48	–2	–96	4	55.18	–7.18
2013	60	–1	–60	1	59.98	0.02
2014	61	0	0	0	64.78	–3.78
2015	66	1	66	1	69.58	–3.58
2016	70	2	140	4	74.38	–4.38
2017	80	3	240	9	79.18	0.82
2018	92	4	368	16	83.98	8.02
Total	583	0	288	60		

Therefore, we find the values of 0 1 β and β using the normal equations as

$$583 = 9 \times \beta_0 + 0 \times \beta_1 \Rightarrow \beta_0 = \frac{583}{9} = 64.78$$

$$288 = 0 \times \beta_0 + 60 \times \beta_1 \Rightarrow \beta_1 = \frac{288}{60} = 4.8$$

Thus, the final linear trend line is given by

$$\hat{Y}_t = 64.78 + 4.8X_t$$

$$\hat{Y}_t = 64.78 + 4.8(t - 2014)$$

We can find the trend values using the above trend line by putting values of t. For example, t = 2010.

$$\hat{Y}_t = 64.78 + 4.8(2010 - 2014) = 45.58$$

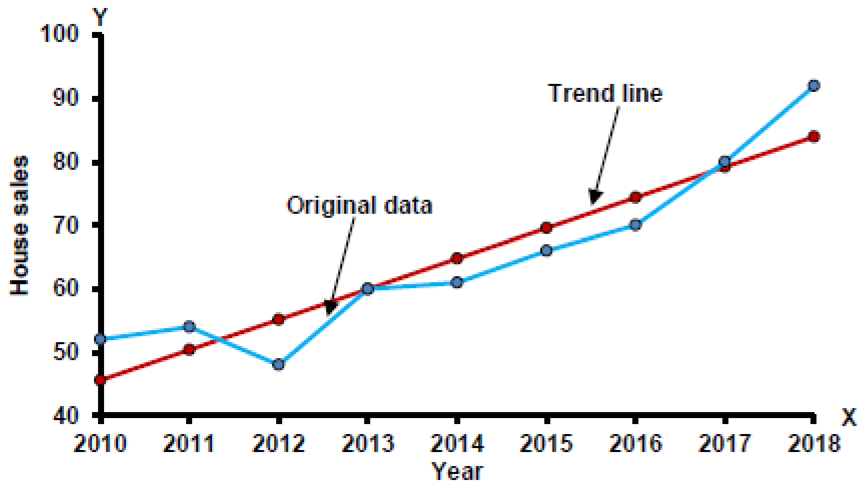
We can compute the forecast error as

$$e_1 = y_1 - \hat{y}_1 = 52 - 45.58 = 6.42$$

You can calculate the rest of the values in a similar manner. We have calculated the same in the above table. We now plot the time series data and trend line by taking years on the X-axis and the house sales and trend values on the Y-axis. We get the time series plot as shown below.

We can estimate the trend value of house sales for 2022 by putting t = 2022 in the above linear trend line as follows:

$$\hat{Y}_t = 64.78 + 4.8(2022 - 2014) = 103.18 \approx 103$$



Quadratic Trend

Sometimes the trend is not linear and shows some curvature. The simplest curvilinear form is a second-degree polynomial. In this case, the quadratic trend equation is given below:

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

We proceed same as in the case of the trend line, the normal equations for estimating β_0, β_1 and β_2 after the transform of the data are given as follows:

$$\sum Y_t = n\beta_0 + \beta_1 \sum X_t + \beta_2 \sum X_t^2$$

$$\sum X_t Y_t = \beta_0 \sum X_t + \beta_1 \sum X_t^2 + \beta_2 \sum X_t^3$$

$$\sum X_t^2 Y_t = \beta_0 \sum X_t^2 + \beta_1 \sum X_t^3 + \beta_2 \sum X_t^4$$

The values of $\sum Y_t$, $\sum X_t$, $\sum X_t^2$, $\sum X_t Y_t$, $\sum X_t^3$, $\sum X_t^2 Y_t$ and $\sum X_t^4$ are obtained from the given data and we solve the normal equations for the constants β_0, β_1 and β_2.

If $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$ represent the estimated values of β_0, β_1 and β_2, respectively then the fitted quadratic trend equation for estimating or forecasting the trend values is given as follows:

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t + \hat{\beta}_2 X_t^2$$

After that, we put the value of X_t in terms of t to find the final quadratic trend.

To illustrate this, let us take an example to fit a quadratic trend.

Example 5: Fit a quadratic trend equation for the house sales data of the real estate company given in Example 4. Also

i. Find forecast errors.
ii. Plot the given data with trend values.
iii. Use the quadratic trend equation, to estimate the level of house sales for year 2022.

Solution: The quadratic trend equation is given as

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

Proceeding in the same way as in the case of Example 4, we make the transform $X_t = t - 2014$, then the normal equations are given as:

$$\sum Y_t = n\beta_0 + \beta_1 \sum X_t + \beta_2 \sum X_t^2$$

$$\sum X_t Y_t = \beta_0 \sum X_t + \beta_1 \sum X_t^2 + \beta_2 \sum X_t^3$$

$$\sum X_t^2 Y_t = \beta_0 \sum X_t^2 + \beta_1 \sum X_t^3 + \beta_2 \sum X_t^4$$

We calculate the values of $\sum Y_t, \sum X_t, \sum X_tY_t, \sum X_t^2Y_t, \sum X_t^2, \sum X_t^3$ and $\sum X_t^4$ in the following table:

Year (t)	Sales (Y_t)	X_t = t – 2014	X_tY_t	X_t^2	X_tY_t	X_t^3	X_t^4	Trend Value	Forecast Error
2010	52	–4	–208	16	832	–64	256	52.31	–0.31
2011	54	–3	–162	9	486	–27	81	52.07	1.93
2012	48	–2	–96	4	192	–8	16	53.27	–5.27
2013	60	–1	–60	1	60	–1	1	55.91	4.09
2014	61	0	0	0	0	0	0	59.99	1.01
2015	66	1	66	1	66	1	1	65.51	0.49
2016	70	2	140	4	280	8	16	72.47	–2.47
2017	80	3	240	9	720	27	81	80.87	–0.87
2018	92	4	368	16	1472	64	256	90.71	1.29
Total	**583**	**0**	**288**	**60**	**4108**	**0**	**708**		

By putting the values from the table in the normal equations, we get

$$583 = 9 \times \beta_0 + 0 \times \beta_1 + \beta_2 \times 60 \Rightarrow 9\beta_0 + 60\beta_2 = 583$$

$$288 = \beta_0 \times 0 + \beta_1 \times 60 + \beta_2 \times 0 \Rightarrow \beta_1 = \frac{288}{60} = 4.8$$

$$4108 = \beta_0 \times 60 + \beta_1 \times 0 + \beta_2 \times 708 \Rightarrow 60\beta_0 + 708\beta_2 = 4108$$

After solving the above equation for ${}_{0\,1\,2}\beta$, β and β get the estimate of these as

$$\hat{\beta}_0 = 59.99,\ \hat{\beta}_1 = 4.8,\ \hat{\beta}_2 = 0.72$$

Thus, the final quadratic trend equation is given by

$$\hat{Y}_t = 59.99 + 4.8X_t + 0.72X_t^2$$

After putting the value of t X in terms of t, we get the desired quadratic trend equation as follows:

$$\hat{Y}_t = 59.99 + 4.8(t - 2014) + 0.72(t - 2014)^2$$

We can find the trend values using the above quadratic trend equation by putting t values. For example, for t = 2010.

$$\hat{Y}_t = 59.99 + 4.8(2010 - 2014) + 0.72(2010 - 2014)^2 = 52.31$$

We can compute the forecast error as

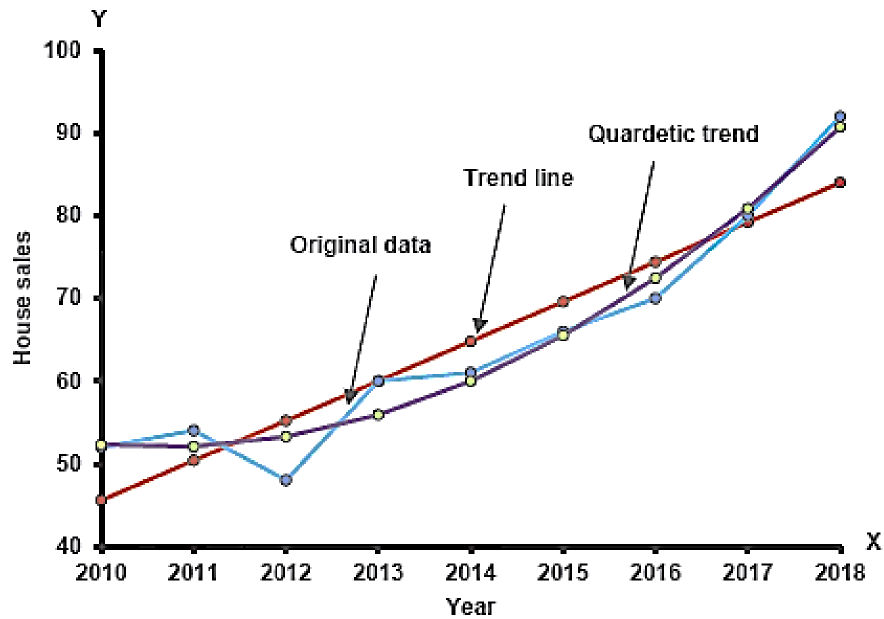
$$e_1 = y_1 - \hat{y}_1 = 52 - 52.31 = -0.31$$

You can calculate the rest of the values in a similar manner. We have calculated the same in the table.

We can estimate the trend value of house sales for 2022 by putting t = 2022 in the above linear trend line as follows:

$$\hat{Y}_t = 59.99 + 4.8(2024 - 2014) + 0.72(2024 - 2014)^2 = 179.99 \approx 180$$

We now plot the time series data and trend line by taking years on the X-axis and the house sales and trend values on the Y-axis. We get the time series plot as shown



If we compare the forecast errors that occurred in both linear and quadratic form, then we observe that these are less in quadratic in comparison to the linear so we can say that the quadratic trend fits better than linear on the number of houses sold by the company.

EXERCISE

The following table gives the gross domestic product (GDP) in 100 million for a certain country from 2010 to 2020:

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
GDP	35	37	51	54	62	64	74	71	83	80

i. Fit a trend line for GDP data and find trend values with the help of trendline.
ii. Find forecast errors.
iii. Use best-fit trend model to predict the country's GDP for 2022.

Exponential Trend

In many situations, the time series data relating to business and economic activities show constant initial growth instead of annual increase as in the case of linear trend. In such situations, the trend can be described best by exponential function rather than linear or quadratic. This can be represented by an exponential form as given below:

$$Y_t = \beta_0 e^{\beta_1 t}$$

We can transform this model to a linear trend model by taking the natural logarithm (base e) on both sides of the above model as follows:

$$\log(Y_t) = \log(\beta_0) + \beta_1 t \log(e)$$

$$Z_t = a + \beta_1 t \qquad [\log(e) = 1]$$

$$\text{where } Z_t = \log(Y_t) \text{ and } a = \log(\beta_0).$$

This is the equation of linear trend. Therefore, we proceed in the same way as in the case of the linear trend line and find the estimate of a and β_1

Once, the estimate of a and β_1 are obtained, then we can obtain an estimate of β_0

$$\hat{\beta}_0 = e^{\hat{a}}$$

After that, we put the values of estimates of β_0 and β_1 in the exponential form to get the equation of best fit.

EXERCISE

The gross revenue of a company from 2015 to 2022 is given in the following table:

Year	2015	2016	2017	2018	2019	2020	2021	2022
Gross Revenue	15	45	52	75	106	158	241	314

i. Fit the exponential trend.
ii. Find the forecast errors.
iii. Use the exponential trend model to predict the company’s gross revenue for 2025.