ARMA Variance Function

For an ARMA(1,1) model, the process is defined as:

$$X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1}$$

where ϵ_t is white noise with mean zero and variance σ^2.

To derive the variance of X_t, follow these simplified steps:

1. **Rewrite the ARMA(1,1) model:**

$$X_t - \phi X_{t-1} = \epsilon_t + \theta \epsilon_{t-1}$$

2. **Express** X_t as an infinite MA process:

Start with the ARMA(1,1) model:

$$X_t = \phi X_{t-1} + \epsilon_t + \theta \epsilon_{t-1}$$

Substitute X_{t-1} from the same model (replace t with $t-1$):

$$X_{t-1} = \phi X_{t-2} + \epsilon_{t-1} + \theta \epsilon_{t-2}$$

Substitute X_{t-1} into the first equation:

$$X_t = \phi(\phi X_{t-2} + \epsilon_{t-1} + \theta \epsilon_{t-2}) + \epsilon_t + \theta \epsilon_{t-1}$$

$$X_t = \phi^2 X_{t-2} + \phi \epsilon_{t-1} + \phi \theta \epsilon_{t-2} + \epsilon_t + \theta \epsilon_{t-1}$$

Repeat this process iteratively to express X_t in terms of ϵ_t:

$$X_t = \epsilon_t + \theta \epsilon_{t-1} + \phi \epsilon_{t-1} + \phi \theta \epsilon_{t-2} + \phi^2 \epsilon_{t-2} + \phi^2 \theta \epsilon_{t-3} + \cdots$$

$$X_t = \epsilon_t + (\theta + \phi)\epsilon_{t-1} + (\phi\theta + \phi^2)\epsilon_{t-2} + (\phi^2\theta + \phi^3)\epsilon_{t-3} + \cdots$$

$$X_t = \sum_{k=0}^{\infty} \psi_k \, \epsilon_{t-k}$$

where $\psi_0 = 1, \psi_1 = \theta + \phi, \psi_2 = \phi(\theta + \phi), \psi_3 = \phi^2(\theta + \phi)$, and so on.

3. **Calculate the variance of** X_t:

$$\mathrm{Var}(X_t) = \sigma^2 \sum_{k=0}^{\infty} \psi_k^2$$

Calculate the first few terms to see the pattern:

$$\psi_0 = 1$$

$$\psi_1 = \theta + \phi$$

$$\psi_2 = \phi(\theta + \phi)$$

$$\psi_3 = \phi^2(\theta + \phi)$$

Sum of the squares of the coefficients:

$$\sum_{k=0}^{\infty} \psi_k^2 = 1 + (\theta + \phi)^2 + \left(\phi(\theta + \phi)\right)^2 + \left(\phi^2(\theta + \phi)\right)^2 + \cdots$$

$$\sum_{k=0}^{\infty} \psi_k^2 = 1 + (\theta + \phi)^2 \sum_{k=0}^{\infty} \phi^{2k}$$

Using the geometric series sum formula:

$$\sum_{k=0}^{\infty} \phi^{2k} = \frac{1}{1 - \phi^2}$$

Therefore:

$$\sum_{k=0}^{\infty} \psi_k^2 = 1 + (\theta + \phi)^2 \frac{1}{1 - \phi^2}$$

$$\sum_{k=0}^{\infty} \psi_k^2 = \frac{1 - \phi^2 + (\theta + \phi)^2}{1 - \phi^2}$$

Thus, the variance of X_t is:

$$\mathrm{Var}(X_t) = \sigma^2 \frac{(1 - \phi^2 + (\theta + \phi)^2)}{1 - \phi^2}$$