On the Cauchy-Schwarz Inequality

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Consider a box of water with a width, height, and depth. Let the depth be z.
We separate the box into compartments by placing separators. Let the widths between them be $x_1, x_2, \ldots, x_n$ for some arbitrary n. Then we fill up each compartment to a certain height y_k, so each compartment has a depth z, width x_k and height of water y_k. See here for a rough diagram.

The separators are instantaneously removed, and the water settles. We consider what happens to the total GPE of the water.

$$GPE_{final} \geq GPE_{inital}$$

This is obvious, and I claim that it is nothing but the Cauchy-Schwarz inequality.

1 Proof

1.1 Initial GPE

$$GPE_{inital} = \sum GPE_k$$

Assume the density of the water and gravitational field strength are both 1, then:

$GPE_k = mgh_{COM} = v_k \times \dfrac{y_k}{2}$ where v_k is the volume of water in compartment k

$$= \frac{zx_k y_k^2}{2}$$

$$\therefore GPE_{inital} = \frac{z}{2} \sum x_k y_k^2$$

1.2 Final GPE

The water will all be at the same height after it settles, call this y_{avg}.

$GPE_{final} = mgh_{COM} = v_t \dfrac{y_{avg}}{2}$ where v_t is the total volume of all the water

The volume of water does not change:

$$v_t = y_{avg} \times z \times \sum x_k = \sum x_k y_k z$$

$$\implies y_{avg} = \frac{\sum x_k y_k}{\sum x_k}$$

$$\therefore GPE_{final} = v_t \frac{y_{avg}}{2} = \frac{y_{avg}^2 \times z \times \sum x_k}{2} = \frac{z(\sum x_k y_k)^2}{2 \sum x_k}$$

1.3 Completing the proof

$$GPE_{final} \geq GPE_{inital}$$

Using our results from 1.1 and 1.2:

$$\frac{z(\sum x_k y_k)^2}{2 \sum x_k} \geq \frac{z}{2} \sum x_k y_k^2$$

$$\frac{(\sum x_k y_k)^2}{\sum x_k} \geq \sum x_k y_k^2$$

$$(\sum x_k y_k)^2 \geq (\sum x_k y_k^2)(\sum x_k)$$

Let $x_k y_k^2 = a_k^2$ and $x_k = b_k^2$. Note that $\implies a_k^2 b_k^2 = x_k^2 y_k^2 \implies a_k b_k = x_k y_k$ Substitute this into the result above:

$$(\sum a_k b_k)^2 \geq (\sum a_k^2)(\sum b_k^2)$$

Which is exactly Cauchy-Schwarz.