Infinite summation for perimeter of an ellipse

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Contents

1 Introduction

Using an arc length integral it is possible to show that the circumference, C, of an ellipse with eccentricity e and semi-major axis a is

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 x}\ dx$$

In Section 1.1 we setup the integral to be turned into an infinite sigma summation. In Section 2 this is evaluated. In Section 3 we consider a specific case of e, using a trigonometric identity to derive a closed form for a nice infinite sum.

1.1 Problem setup

The integral is

$$C = 4a \int_0^{\pi/2} (1 - e^2 \sin^2 x)^{\frac{1}{2}}\ dx \tag{1}$$

We will use the binomial expansion for nonintegers on $(1 - e^2 \sin^2 x)^{\frac{1}{2}}$, then integrate term by term. Lemmas 5.1 and 5.2 will be referenced and can be found in the appendix.

2 Algebra crunching

Recalling (1), note that $0 \leq e < 1 \implies e^2 \sin^2 x < 1$ so we can use the binomial expansion.

$$C = 4a \int_0^{\pi/2} \left(1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \times \cdots \times (\frac{1}{2} - k + 1)}{k!} (-e^2 \sin^2 x)^k \right)\ dx$$

Integrate term by term (sum and integral are interchanged):

$$= 2\pi a + 4a \sum_{k \geq 1} \frac{\frac{1}{2} \times \cdots \times (\frac{3}{2} - k)}{k!} (-1)^k e^{2k} \int_0^{\pi/2} \sin^{2k} x\ dx$$

Substitute Lemma 5.1 in:

$$= 2\pi a + 4a \sum_{k\geq 1} \frac{\frac{1}{2} \times \cdots \times (\frac{3}{2} - k)}{k!} (-1)^k e^{2k} \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n}$$

$$= 2\pi a + 2\pi a \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)}{2^k k!} e^{2k} \prod_{n=1}^{k} \frac{2n-1}{2n}$$

$$= 2\pi a + 2\pi a \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)\prod_{n=1}^{k}(2n-1)}{2^k k! \prod_{n=1}^{k}(2n)} e^{2k}$$

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{4^k (k!)^2} e^{2k} \right)$$

Substitute Lemma 5.2 in:

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{(1-2k)\left[\frac{(2k-2)!}{(k-1)!}\right]^2}{4^{k-1}4^k (k!)^2} e^{2k} \right)$$

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{-1}{4^{2k-1}(2k-1)} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!}\right]^2 e^{2k} \right)$$

$$= 2\pi a \left(1 - \sum_{k\geq 1} \frac{4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 e^{2k} \right)$$

Reindex:

$$= 2\pi a \left(1 - \sum_{k\geq 0} \frac{4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 e^{2k+2} \right)$$

$$\boxed{C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 e^{2k+2}}{16^k (2k+1)} \right) \qquad (2)}$$

Note when $e = 0$ (the ellipse is a circle) the whole summation vanishes so C becomes $2\pi a$, as expected.

3 A specific case

We will evaluate (1) for a specific value of e, then equate the result with (2). Recall (1) and set $e^2 = \frac{1}{2}$:

$$C\Big|_{e^2=1/2} = 4a \int_0^{\pi/2} (1 - \frac{1}{2}\sin^2 x)^{\frac{1}{2}}\ dx$$

Using the identity $sin^2 x + cos^2 x \equiv 1$:

$$= 4a \int_0^{\pi/2} (\frac{1}{2} + \frac{1}{2}\cos^2 x)^{\frac{1}{2}}\ dx$$
$$= 2a\sqrt{2} \int_0^{\pi/2} (1 + \cos^2 x)^{\frac{1}{2}}\ dx$$

Note that $0 \le x < \pi/2 \implies \cos^2 x < 1$
So now we can use the binomial expansion.

$$= 2a\sqrt{2} \int_0^{\pi/2} \left(1 + \sum_{k\ge1} \frac{\frac{1}{2} \times -\frac{1}{2} \cdots \times (\frac{1}{2} - k + 1)}{k!} \left(\cos^2 x\right)^k \right)\ dx$$

Integrate term by term (sum and integral are interchanged):

$$= \pi a\sqrt{2} + 2a\sqrt{2} \sum_{k\ge1} \left(\frac{\prod_{n=1}^{k}(3-2n)}{2^k k!} \int_0^{\pi/2} \cos^{2k} x\, dx \right)$$

Substitute Lemma 5.1 in:

$$= \pi a\sqrt{2} + 2a\sqrt{2} \sum_{k\ge1} \left(\frac{\prod_{n=1}^{k}(3-2n)}{2^k k!} \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n} \right)$$
$$= \pi a\sqrt{2} + \pi a\sqrt{2} \sum_{k\ge1} \left((-1)^k \frac{\prod_{n=1}^{k}(2n-3) \prod_{n=1}^{k}(2n-1)}{2^k k! \prod_{n=1}^{k} 2n} \right)$$
$$= \pi a\sqrt{2} \left(1 + \sum_{k\ge1} \left((-1)^k \frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{4^k (k!)^2} \right) \right)$$

Substitute Lemma 5.2 in:

$$= \pi a\sqrt{2} \left(1 + \sum_{k\ge1} \left((-1)^k \frac{(1-2k) \left[\frac{(2k-2)!}{(k-1)!}\right]^2}{4^{k-1} 4^k (k!)^2} \right) \right)$$
$$= \pi a\sqrt{2} \left(1 + \sum_{k\ge1} \left((-1)^{k+1} \frac{1}{(2k-1)4^{2k-1}} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!} \right]^2 \right) \right)$$
$$= \pi a\sqrt{2} \left(1 + \sum_{k\ge1} \left(\frac{(-1)^{k+1} 4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 \right) \right)$$

Reindex:

$$= \pi a\sqrt{2}\left(1 + \sum_{k\geq 0} \frac{(-1)^k 4^{-1-2k}}{2k+1}\binom{2k+1}{k}^2\right)$$

$$C\bigg|_{e^2=1/2} = \frac{\pi a\sqrt{2}}{4}\left(4 + \sum_{k\geq 0} \frac{(-1)^k 16^{-k}}{2k+1}\binom{2k+1}{k}^2\right)$$

But note, if we evaluate C when $e^2 = \frac{1}{2}$ using (2) we get:

$$C\bigg|_{e^2=1/2} = \frac{\pi a}{2}\left(4 - \sum_{k\geq 0} \frac{\binom{2k+1}{k}^2}{(2k+1)16^k} \cdot \frac{1}{2^{k+1}}\right)$$

Thus:

$$\frac{\pi a\sqrt{2}}{4}\left(4 + \sum_{k\geq 0} \frac{(-1)^k 16^{-k}}{2k+1}\binom{2k+1}{k}^2\right) = \frac{\pi a}{2}\left(4 - \sum_{k\geq 0} \frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\frac{1}{2^{k+1}}\right)$$

Now simplify:

$$\sqrt{2}\left(4 + \sum_{k\geq 0} \frac{(-1)^k 16^{-k}}{2k+1}\binom{2k+1}{k}^2\right) = 2\left(4 - \sum_{k\geq 0} \frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\frac{1}{2^{k+1}}\right)$$

$$4\sqrt{2} + \sum_{k\geq 0}\sqrt{2}\frac{(-1)^k 16^{-k}}{2k+1}\binom{2k+1}{k}^2 = 8 - \sum_{k\geq 0}\frac{16^{-k}\binom{2k+1}{k}^2}{(2k+1)2^k}$$

$$\sum_{k\geq 0}\frac{16^{-k}\binom{2k+1}{k}^2}{(2k+1)2^k} + \sum_{k\geq 0}\sqrt{2}\frac{(-1)^k 16^{-k}}{2k+1}\binom{2k+1}{k}^2 = 8 - 4\sqrt{2}$$

$$\sum_{k\geq 0}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{1}{2^k} + \sqrt{2}(-1)^k\right) = 8 - 4\sqrt{2}$$

$$\sum_{k\geq 0}\frac{32^{-k}\binom{2k+1}{k}^2}{2k+1}\left(1 + \sqrt{2}(-2)^k\right) = 8 - 4\sqrt{2}$$

$$\boxed{\sum_{k=0}^{\infty}\frac{\binom{2k+1}{k}^2\left(1+\sqrt{2}(-2)^k\right)}{(2k+1)32^k} = 8 - 4\sqrt{2} \qquad (3)}$$

Nice.

4 Summary

I have derived two infinite sums, (2) and (3). One is a sum that I have found a closed form for:

$$\sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 \left(1+\sqrt{2}(-2)^k\right)}{(2k+1)32^k} = 8 - 4\sqrt{2}$$

And the other is the circumference, C, of an ellipse with eccentricity e and semi-major axis a:

$$C = \frac{\pi a}{2}\left(4 - \sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 e^{2k+2}}{16^k(2k+1)}\right)$$

5 Appendix

5.1 Lemma

Claim: $\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k} x \, dx = \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n}$
The proof is below.

Firstly, note

$$\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k}(\frac{\pi}{2} - x) \, dx$$

so $\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k} x \, dx$ can be verified by doing a substitution with $u = \frac{\pi}{2} - x$.
It remains to show that $I_k := \int_0^{\pi/2} \sin^{2k} x \, dx = \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n}$

$$\begin{aligned}
I_k &= \int_0^{\pi/2} \sin^{2k-1} x \sin(x) \, dx \\
&\text{Using IBP:} \\
&= \cancelto{0}{- \sin^{2k-1} x \cos(x) \Big|_0^{\pi/2}} + (2k-1) \int_0^{\pi/2} \sin^{2k-2} x \cos^2 x \, dx \\
&= (2k-1) \int_0^{\pi/2} \left(\sin^{2k-2} x - \sin^{2k} x \right) \, dx \\
&= (2k-1) \cancelto{I_{k-1}}{\int_0^{\pi/2} \sin^{2k-2} x \, dx} - (2k-1) \cancelto{I_k}{\int_0^{\pi/2} \sin^{2k} \, dx} \\
\therefore I_k + (2k-1) I_k &= (2k-1) I_{k-1} \\
I_k &= \frac{2k-1}{2k} I_{k-1}
\end{aligned}$$

Note that $I_0 = \frac{\pi}{2}$

$$\therefore I_k = \frac{2k-1}{2k} \times \frac{2k-3}{2k-2} \times \cdots \times \frac{1}{2} I_0 = \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n}$$

Done.

5.2 Lemma

Claim: $\prod_{n=1}^{k}(2n-3)(2n-1) = \frac{1-2k}{4^{k-1}}\left[\frac{(2k-2)!}{(k-1)!}\right]^2$
Proof:

$$\begin{aligned}
\prod_{n=1}^{k}(2n-3)(2n-1) &= (-1\times 1)\times(1\times 3)\times\cdots\times[(2k-3)\times(2k-1)] \\
&= -1\times 1^2\times 3^2\times\cdots\times(2k-3)^2\times(2k-1) \\
&= (1-2k)\left[1\times 3\times\cdots\times(2k-3)\right]^2 \\
&= (1-2k)\left[\frac{1\times 2\times 3\times\cdots\times(2k-3)\times(2k-2)}{2\times 4\times\cdots\times(2k-2)}\right]^2 \\
&= \frac{1-2k}{4^{k-1}}\left[\frac{(2k-2)!}{(k-1)!}\right]^2
\end{aligned}$$