# Infinite summation for perimeter of an ellipse

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# 1 Introduction

Using an arc length integral it is possible to show that the circumference of an ellipse with height 2b and width 2a is

$$4a \int_0^{\pi/2} \sqrt{1 + (\frac{a^2}{b^2} - 1)\sin^2 x} \ dx$$

This can be proved with an arc length integral on an ellipse. In 1.1 the integral is setup to be turned into an infinite summation. In later sections I go through each case established in 1.1.

## 1.1 Problem setup

Let $c = (\frac{a^2}{b^2} - 1)$. Then the integral will be

$$4a \int_0^{\pi/2} (1 + c\sin^2 x)^{\frac{1}{2}} \ dx$$

We will use the binomial expansion for nonintegers on $(1 + c\sin^2 x)^{\frac{1}{2}}$, then integrate term by term.
In the next sections we consider the 2 cases of c: $|c| \leq 1$, and $|c| \geq 1$.

# 2 Case 1

We solve the case of $|c| \geq 1$.

$$\begin{aligned}
4a \int_0^{\pi/2} (1 + c\sin^2 x)^{\frac{1}{2}} \ dx &= 4a \int_0^{\pi/2} (1 + c - c\cos^2 x)^{\frac{1}{2}} \ dx \\
&= 4a \int_0^{\pi/2} \left( (c+1)(1 - \frac{c}{c+1}\cos^2 x) \right)^{\frac{1}{2}} \ dx \qquad (1) \\
&= 4a\sqrt{c+1} \int_0^{\pi/2} (1 - \frac{c}{c+1}\cos^2 x)^{\frac{1}{2}} \ dx
\end{aligned}$$

Note that if $|c| \geq 1$ then $c \geq 1$ since $\frac{a^2}{b^2} - 1$ must be $> -1$. $\therefore |\frac{c}{c+1} cos^2 x| < 1 \forall x \in \mathbb{R}$
So now we can use the binomial expansion.

$$\left(1 - \frac{c}{c+1}\cos^2 x\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\left(-\frac{c}{c+1}\cos^2 x\right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}\left(-\frac{c}{c+1}\cos^2 x\right)^2 \dots$$

$$= 1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \cdots \times (\frac{1}{2} - k + 1)}{k!} \left(-\frac{c}{c+1}\cos^2 x\right)^k = 1 + \sum_{k \geq 1} \frac{\prod_{n=1}^{k}(3-2n)}{2^k k!} \left(-\frac{c}{c+1}\right)^k \cos^{2k} x$$

$$= 1 + \sum_{k \geq 1} \frac{\prod_{n=1}^{k}(2n-3)}{2^k k!} \left(\frac{c}{c+1}\right)^k \cos^{2k} x$$

Substitute this back into (1):

$$
\begin{aligned}
&4a\sqrt{c+1}\int_0^{\pi/2}\left(1+\sum_{k\geq 1}\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\cos^{2k}x\right)dx \\
&= 4a\sqrt{c+1}\int_0^{\pi/2}1dx + 4a\sqrt{c+1}\int_0^{\pi/2}\sum_{k\geq 1}\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\cos^{2k}x\,dx \\
&= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\int_0^{\pi/2}\sum_{k\geq 1}\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\cos^{2k}x\,dx
\end{aligned}
$$

Integrating term by term (sum and integral are interchanged):

$$
= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\int_0^{\pi/2}\cos^{2k}x\,dx\right) \quad (2)
$$

We now focus on $\int_0^{\pi/2}\cos^{2k}x\ dx$, let this be $I_k$.

$$
I_k = \int_0^{\pi/2}\cos^{2k-1}(x)\cos x\ dx
$$

Using IBP:

$$
\begin{aligned}
&= \cancelto{0}{\sin x\cos^{2k-1}x\Big|_0^{\pi/2}} - \int_0^{\pi/2}(2k-1)\times -\sin^2 x\cos^{2k-2}x\ dx \\
&= (2k-1)\int_0^{\pi/2}\cos^{2k-2}x(1-\cos^2 x)\ dx \\
&= (2k-1)\left(\cancelto{I_{k-1}}{\int_0^{\pi/2}\cos^{2k-2}x\ dx}\right) - (2k-1)\cancelto{I_k}{\int_0^{\pi/2}\cos^2 x\ dx} \\
&\therefore I_k + (2k-1)I_k = (2k-1)I_{k-1} \\
&I_k = \frac{2k-1}{2k}I_{k-1}
\end{aligned}
$$

Note that $I_0 = \frac{\pi}{2}$

$$
\therefore I_k = \frac{\pi}{2}\left(\frac{1}{2}\times\frac{3}{4}\times\cdots\times\frac{2k-1}{2k}\right)
$$

$$
= \int_0^{\pi/2}\cos^{2k}(x)\ dx = \frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}
$$

Substitute this back into (2):

$$
\begin{aligned}
&2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k \frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}\right) \\
&= 2\pi a\sqrt{c+1} + 2\pi a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)\prod_{n=1}^{k}(2n-1)}{2^k k!\prod_{n=1}^{k}(2n)}\left(\frac{c}{c+1}\right)^k\right) \qquad (3) \\
&= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{2^k k! 2^k k!}\left(\frac{c}{c+1}\right)^k\right)\right)
\end{aligned}
$$

(Here I spent a lot of time trying to use the Wallis product because it looked similar but it never worked)

Note

$$
\begin{aligned}
\prod_{n=1}^{k}(2n-3)(2n-1) &= (-1\times 1)\times(1\times 3)\times\cdots\times[(2k-3)\times(2k-1)] \\
&= -1\times 1^2\times 3^2\times\cdots\times(2k-3)^2\times(2k-1) \\
&= (1-2k)\left[1\times 3\times\cdots\times(2k-3)\right]^2 \qquad (4) \\
&= (1-2k)\left[\frac{1\times 2\times 3\times\cdots\times(2k-3)\times(2k-2)}{2\times 4\times\cdots\times(2k-2)}\right]^2 \\
&= (1-2k)\left[\frac{(2k-2)!}{2^{k-1}(k-1)!}\right]^2 = (1-2k)\frac{[(2k-2)!]^2}{4^{k-1}[(k-1)!]^2}
\end{aligned}
$$

Substitute back into (3):

$$
\begin{aligned}
&= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{(1-2k)[(2k-2)!]^2}{2^k k! 2^k k! 4^{k-1}[(k-1)!]^2}\left(\frac{c}{c+1}\right)^k\right)\right) \\
&= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{(1-2k)[(2k-2)!]^2}{4^{2k-1}(k!)^2[(k-1)!]^2}\left(\frac{c}{c+1}\right)^k\right)\right) \\
&= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{(1-2k)}{4^{2k-1}}\left[\frac{(2k-2)!}{k!(k-1)!}\right]^2\left(\frac{c}{c+1}\right)^k\right)\right) \\
&= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{(1-2k)}{4^{2k-1}}\left[\frac{1}{k}\binom{2k-2}{k-1}\right]^2\left(\frac{c}{c+1}\right)^k\right)\right) \\
&= 2\pi a\sqrt{c+1}\left(1-\sum_{k\geq 1}\left(\frac{(2k-1)}{4^{2k-1}k^2}\binom{2k-2}{k-1}^2\left(\frac{c}{c+1}\right)^k\right)\right)
\end{aligned}
$$

Reindex:

$$= 2\pi a\sqrt{c+1}\left[1-\sum_{k\geq 0}\left(\frac{(2k+1)}{4^{2k+1}(k+1)^2}\binom{2k}{k}^2\left(\frac{c}{c+1}\right)^{k+1}\right)\right]$$

$$= 2\pi a\sqrt{c+1}\left[1-\frac{1}{4}\sum_{k\geq 0}\left(\frac{(2k+1)}{4^{2k}(k+1)^2}\binom{2k}{k}^2\left(\frac{c}{c+1}\right)^{k+1}\right)\right]$$

$$= \frac{\pi a\sqrt{c+1}}{2}\left[4-\sum_{k\geq 0}\left(\frac{(2k+1)}{16^{k}(k+1)^2}\binom{2k}{k}^2\left(\frac{c}{c+1}\right)^{k+1}\right)\right]$$

$$\boxed{= \frac{\pi a\sqrt{c+1}}{2}\left[4-\sum_{k=0}^{\infty}\frac{(2k+1)\binom{2k}{k}^2 c^{k+1}}{16^{k}(k+1)^2(c+1)^{k+1}}\right]}$$

Note the integral approximation equals the summation approximation so it should be valid.

# 3 Case 2

Now we solve the case $|c| \leq 1$.

$$4a\int_0^{\pi/2}(1+c\sin^2 x)^{\frac{1}{2}}\ dx$$

Note that $|csin^2x| < 1\forall x, 0 \leq x < \frac{\pi}{2}$
So we can use the binomial expansion for $|c| \leq 1$ since the values of x only range from 0 to $\frac{\pi}{2}$ in the integral.

$$= 4a\int_0^{\pi/2}\left(1+\frac{1}{2}c\sin^2 x+\frac{\frac{1}{2}\times -\frac{1}{2}}{2!}(c\sin^2 x)^2+\cdots\right)\ dx$$

$$= 4a\int_0^{\pi/2}\left(1+\sum_{k\geq 1}\frac{\frac{1}{2}\times -\frac{1}{2}\times\cdots\times(\frac{1}{2}-k+1)}{k!}(c\sin^2 x)^k\right)\ dx$$

Integrate term by term:

$$= 4a\int_0^{\pi/2}1\,dx+\sum_{k\geq 1}4a\int_0^{\pi/2}\frac{\frac{1}{2}\times\cdots\times(\frac{1}{2}-k+1)}{k!}(c\sin^2 x)^k\ dx$$

$$= 2\pi a+4a\sum_{k\geq 1}\frac{\frac{1}{2}\times\cdots\times(\frac{3}{2}-k)}{k!}c^k\overbrace{\int_0^{\pi/2}\sin^{2k}x\ dx}^{\text{Set }=I_k} \tag{5}$$

Now we solve $I_k$.

$$I_k = \int_0^{\pi/2} \sin^{2k-1} x \sin(x)\, dx$$

Using IBP:

$$= -\cancelto{0}{\sin^{2k-1} x \cos(x)\Big|_0^{\pi/2}} + (2k-1)\int_0^{\pi/2} \sin^{2k-2} x \cos^2 x\, dx$$

$$= (2k-1)\int_0^{\pi/2} \left(\sin^{2k-2} x - \sin^{2k} x\right)\, dx$$

$$= (2k-1)\cancelto{I_{k-1}}{\int_0^{\pi/2} \sin^{2k-2} x\, dx} - (2k-1)\cancelto{I_k}{\int_0^{\pi/2} \sin^{2k}\, dx}$$

$$\therefore I_k + (2k-1)I_k = (2k-1)I_{k-1}$$

$$I_k = \frac{2k-1}{2k} I_{k-1}$$

Note that $I_0 = \frac{\pi}{2}$

$$\therefore I_k = \frac{2k-1}{2k} \times \frac{2k-3}{2k-2} \times \cdots \times \frac{1}{2} I_0 = \frac{\pi}{2}\prod_{n=1}^{k} \frac{2n-1}{2n}$$

Substitute back into (5):

$$2a\pi + 4a\sum_{k\geq 1} \frac{\frac{1}{2}\times\cdots\times(\frac{3}{2}-k)}{k!} c^k I_k$$

$$= 2\pi a + 4a\sum_{k\geq 1} \frac{\frac{1}{2}\times\cdots\times(\frac{3}{2}-k)}{k!} c^k \frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}$$

$$= 2\pi a + 2a\pi\sum_{k\geq 1} \frac{\prod_{n=1}^{k}(3-2m)}{2^k k!} c^k \prod_{n=1}^{k}\frac{2n-1}{2n}$$

$$= 2\pi a\left(1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(3-2n)\prod_{n=1}^{k}(2n-1)}{2^k k!\prod_{n=1}^{k}(2n)} c^k\right)$$

$$= 2\pi a\left(1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(3-2n)(2n-1)}{2^k k! 2^k k!} c^k\right)$$

$$= 2\pi a\left(1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{4^k (k!)^2} c^k (-1)^k\right)$$

Substitute (4) in:

$$2\pi a\left(1+\sum_{k\geq1}\frac{(1-2k)\frac{[(2k-2)!]^2}{4^{k-1}[(k-1)!]^2}}{4^k(k!)^2}c^k(-1)^k\right)$$

$$=2\pi a\left(1+\sum_{k\geq1}\frac{(1-2k)[(2k-2)!]^2}{4^{2k-1}(k!)^2[(k-1)!]^2}c^k(-1)^k\right)$$

$$=2\pi a\left(1+\sum_{k\geq1}\frac{(1-2k)}{4^{2k-1}}c^k(-1)^k\left(\frac{(2k-2)!}{k!(k-1)!}\right)^2\right)$$

$$=2\pi a\left(1-\sum_{k\geq1}\frac{(2k-1)}{4^{2k-1}}(-c)^k\left(\frac{(2k-2)!}{k[(k-1)!]^2}\right)^2\right)$$

$$=2\pi a\left(1-\sum_{k\geq1}\frac{(2k-1)}{4^{2k-1}}(-c)^k\left(\frac{1}{k}\binom{2k-2}{k-1}\right)^2\right)$$

$$=2\pi a\left(1-\sum_{k\geq1}\frac{(2k-1)}{4^{2k-1}k^2}(-c)^k\binom{2k-2}{k-1}^2\right)$$

Reindex:

$$=2\pi a\left(1-\sum_{k\geq0}\frac{(2k+1)}{4^{2k+1}(k+1)^2}(-c)^{k+1}\binom{2k}{k}^2\right)$$

$$=2\pi a\left(1-\frac{1}{4}\sum_{k\geq0}\frac{(2k+1)\binom{2k}{k}^2}{4^{2k}(k+1)^2}(-c)^{k+1}\right)$$

$$\boxed{=\frac{\pi a}{2}\left(4-\sum_{k=0}^{\infty}\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-c)^{k+1}\right)}$$

The approximation of this summation also equals the integral approximation so it should be valid. Also note when c=0 (a=b so the ellipse is a circle) the whole summation vanishes so the expression becomes $2\pi a$ which is expected.

# 4 Summary

I have derived 2 infinite sums for the perimeter of an ellipse depending on $|c|$ Where 2a is the width, 2b is the height and $c=\frac{a^2}{b^2}-1$. I can't find them anywhere on the internet so they might be unique?

If $|c| \leq 1$:

$$\text{Perimeter} = \frac{\pi a}{2}\left(4 - \sum_{k=0}^{\infty} \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-c)^{k+1}\right)$$

If $|c| \geq 1$:

$$\text{Perimeter} = \frac{\pi a\sqrt{c+1}}{2}\left[4 - \sum_{k=0}^{\infty} \frac{(2k+1)\binom{2k}{k}^2 c^{k+1}}{16^k(k+1)^2(c+1)^{k+1}}\right]$$

I noticed that the formula for $|c| \geq 1$ is just the other formula but with a and b flipped around, which is very nice - so you could probably make this just one formula by defining $a > b$.

# 5 Discovery

I realized if you plug c=1 into both formulae, they must be equal, therefore:

$$\frac{\pi a}{2}\left(4 - \sum_{k\geq 0} \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-1)^{k+1}\right) = \frac{\pi a\sqrt{2}}{2}\left[4 - \sum_{k\geq 0} \frac{(2k+1)\binom{2k}{k}^2 1^{k+1}}{16^k(k+1)^2 2^{k+1}}\right]$$

Now simplify:

$$4 - \sum_{k\geq 0} \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-1)^{k+1} = 4\sqrt{2} - \sqrt{2}\sum_{k\geq 0} \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2 2^{k+1}}$$

$$\sum_{k\geq 0} \sqrt{2}\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2 2^{k+1}} - \sum_{k\geq 0} \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-1)^{k+1} = 4\sqrt{2} - 4$$

$$\sum_{k\geq 0}\left[\sqrt{2}\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2 2^{k+1}} - \frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}(-1)^{k+1}\right] = 4\sqrt{2} - 4$$

$$\sum_{k\geq 0}\left[\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}\left(\frac{\sqrt{2}}{2^{k+1}} - (-1)^{k+1}\right)\right] = 4\sqrt{2} - 4$$

$$\sum_{k\geq 0}\left[\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}\left(\frac{2}{2^{k+1}} - \sqrt{2}(-1)^{k+1}\right)\right] = 8 - 4\sqrt{2}$$

$$\sum_{k\geq 0}\left[\frac{(2k+1)\binom{2k}{k}^2}{16^k(k+1)^2}\frac{1}{2^k}\left(1 + \sqrt{2}(-1)^k 2^k\right)\right] = 8 - 4\sqrt{2}$$

$$\sum_{k\geq 0}\left[\frac{(2k+1)\binom{2k}{k}^2}{16^k 2^k(k+1)^2}\left(1 + \sqrt{2}(-2)^k\right)\right] = 8 - 4\sqrt{2}$$

$$\sum_{k=0}^{\infty}\left[\frac{(2k+1)\binom{2k}{k}^2}{32^k(k+1)^2}\left(1+\sqrt{2}(-2)^k\right)\right]=8-4\sqrt{2}$$

Python and WolframAlpha approximations agree - WolframAlpha doesn't know the exact form and I do :)