

Infinite summation for perimeter of an ellipse

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1 Introduction

Using an arc length integral it is possible to show that the circumference, C , of an ellipse with height $2b$ and width $2a$ is

$$C = 4a \int_0^{\pi/2} \sqrt{1 + \left(\frac{a^2}{b^2} - 1\right) \sin^2 x} \, dx$$

This is derived in the appendix. In section 1.1 we setup the integral to be turned into an infinite sigma summation. In later sections I go through each case established in 1.1 to derive two infinite sums equal to the circumference, then equate them.

1.1 Problem setup

Let $c = \frac{a^2}{b^2} - 1$. Then the integral will be

$$C = 4a \int_0^{\pi/2} (1 + c \sin^2 x)^{\frac{1}{2}} \, dx$$

We will use the binomial expansion for nonintegers on $(1 + c \sin^2 x)^{\frac{1}{2}}$, then integrate term by term.

In the next sections we consider the 2 cases of c : $|c| \leq 1$, and $|c| \geq 1$.

2 Case 1

We solve the case of $|c| \geq 1$.

$$\begin{aligned} C &= 4a \int_0^{\pi/2} (1 + c \sin^2 x)^{\frac{1}{2}} \, dx \\ &= 4a\sqrt{c+1} \int_0^{\pi/2} \left(1 - \frac{c}{c+1} \cos^2 x\right)^{\frac{1}{2}} \, dx \end{aligned} \tag{1}$$

Note that $\because c = \frac{a^2}{b^2} - 1 > -1, |c| \geq 1 \implies c \geq 1$

$\therefore \left| \frac{c}{c+1} \cos^2 x \right| < 1 \forall x \in \mathbb{R}$

So now we can use the binomial expansion.

$$\begin{aligned} \left(1 - \frac{c}{c+1} \cos^2 x\right)^{\frac{1}{2}} &= 1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \cdots \times (\frac{1}{2} - k + 1)}{k!} \left(-\frac{c}{c+1} \cos^2 x\right)^k \\ &= 1 + \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)}{2^k k!} \left(\frac{c}{c+1}\right)^k \cos^{2k} x \end{aligned}$$

Substitute this back into (1):

$$\begin{aligned} C &= 4a\sqrt{c+1} \int_0^{\pi/2} \left(1 + \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)}{2^k k!} \left(\frac{c}{c+1} \right)^k \cos^{2k} x \right) dx \\ &= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1} \int_0^{\pi/2} \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)}{2^k k!} \left(\frac{c}{c+1} \right)^k \cos^{2k} x dx \end{aligned}$$

Integrating term by term (sum and integral are interchanged):

$$= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1} \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (2n-3)}{2^k k!} \left(\frac{c}{c+1} \right)^k \int_0^{\pi/2} \cos^{2k} x dx \right) \quad (2)$$

It is proved in the appendix that $\int_0^{\pi/2} \cos^{2k}(x) dx = \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n}$
Substitute this back into (2):

$$\begin{aligned} C &= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1} \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (2n-3)}{2^k k!} \left(\frac{c}{c+1} \right)^k \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n} \right) \\ &= 2\pi a\sqrt{c+1} + 2\pi a\sqrt{c+1} \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (2n-3) \prod_{n=1}^k (2n-1)}{2^k k! \prod_{n=1}^k (2n)} \left(\frac{c}{c+1} \right)^k \right) \quad (3) \\ &= 2\pi a\sqrt{c+1} \left(1 + \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (2n-3)(2n-1)}{4^k (k!)^2} \left(\frac{c}{c+1} \right)^k \right) \right) \end{aligned}$$

Note

$$\begin{aligned} \prod_{n=1}^k (2n-3)(2n-1) &= (-1 \times 1) \times (1 \times 3) \times \cdots \times [(2k-3) \times (2k-1)] \\ &= -1 \times 1^2 \times 3^2 \times \cdots \times (2k-3)^2 \times (2k-1) \\ &= (1-2k) [1 \times 3 \times \cdots \times (2k-3)]^2 \quad (4) \\ &= (1-2k) \left[\frac{1 \times 2 \times 3 \times \cdots \times (2k-3) \times (2k-2)}{2 \times 4 \times \cdots \times (2k-2)} \right]^2 \\ &= \frac{1-2k}{4^{k-1}} \left[\frac{(2k-2)!}{(k-1)!} \right]^2 \end{aligned}$$

Substitute back into (3):

$$\begin{aligned}
C &= 2\pi a\sqrt{c+1} \left(1 + \sum_{k \geq 1} \left(\frac{(1-2k) \left[\frac{(2k-2)!}{(k-1)!} \right]^2}{4^{k-1} 4^k (k!)^2} \left(\frac{c}{c+1} \right)^k \right) \right) \\
&= 2\pi a\sqrt{c+1} \left(1 + \sum_{k \geq 1} \left(\frac{-1}{4^{2k-1} (2k-1)} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!} \right]^2 \left(\frac{c}{c+1} \right)^k \right) \right) \\
&= 2\pi a\sqrt{c+1} \left(1 - \sum_{k \geq 1} \frac{4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 \left(\frac{c}{c+1} \right)^k \right)
\end{aligned}$$

Reindex:

$$= 2\pi a\sqrt{c+1} \left(1 - \sum_{k \geq 0} \frac{4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 \left(\frac{c}{c+1} \right)^{k+1} \right)$$

$$C = \frac{\pi a\sqrt{c+1}}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{c}{c+1} \right)^{k+1} \right)$$

3 Case 2

Now we solve the case $|c| \leq 1$.

$$C = 4a \int_0^{\pi/2} (1 + c \sin^2 x)^{\frac{1}{2}} dx$$

Note that $|c \sin^2 x| < 1 \forall x, 0 \leq x < \frac{\pi}{2}$

So we can use the binomial expansion.

$$= 4a \int_0^{\pi/2} \left(1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \times \dots \times (\frac{1}{2} - k + 1)}{k!} (c \sin^2 x)^k \right) dx$$

Integrate term by term:

$$= 2\pi a + 4a \sum_{k \geq 1} \frac{\frac{1}{2} \times \dots \times (\frac{3}{2} - k)}{k!} c^k \int_0^{\pi/2} \sin^{2k} x dx \quad (5)$$

It is proved in the appendix that $\int_0^{\pi/2} \sin^{2k} x dx = \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n}$

Substitute back into (5):

$$\begin{aligned}
C &= 2\pi a + 4a \sum_{k \geq 1} \frac{\frac{1}{2} \times \dots \times (\frac{3}{2} - k)}{k!} c^k \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n} \\
&= 2\pi a + 2\pi a \sum_{k \geq 1} \frac{\prod_{n=1}^k (3-2n) \prod_{n=1}^k (2n-1)}{2^k k! \prod_{n=1}^k (2n)} c^k \\
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)(2n-1)}{4^k (k!)^2} (-c)^k \right)
\end{aligned}$$

Substitute (4) in:

$$\begin{aligned}
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{(1-2k) \left[\frac{(2k-2)!}{(k-1)!} \right]^2}{4^{k-1} 4^k (k!)^2} (-c)^k \right) \\
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{-1}{4^{2k-1} (2k-1)} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!} \right]^2 (-c)^k \right) \\
&= 2\pi a \left(1 - \sum_{k \geq 1} \frac{4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 (-c)^k \right)
\end{aligned}$$

Reindex:

$$= 2\pi a \left(1 - \sum_{k \geq 0} \frac{4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 (-c)^{k+1} \right)$$

$$C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-c)^{k+1} \right)$$

Note when $c=0$ ($a=b$ so the ellipse is a circle) the whole summation vanishes so C becomes $2\pi a$.

4 Summary

I have derived 2 infinite sums for the circumference, C , of an ellipse depending on $|c|$ Where $2a$ is the width, $2b$ is the height and $c = \frac{a^2}{b^2} - 1$.

If $|c| \geq 1$:

$$C = \frac{\pi a \sqrt{c+1}}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{c}{c+1} \right)^{k+1} \right)$$

If $|c| \leq 1$:

$$C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-c)^{k+1} \right)$$

5 Discovery

I realized if you plug $c=1$ into both sums, they must be equal, therefore:

$$\frac{\pi a \sqrt{2}}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{1}{2} \right)^{k+1} \right) = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-1)^{k+1} \right)$$

Now simplify:

$$\begin{aligned} 4\sqrt{2} - \sqrt{2} \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{1}{2} \right)^{k+1} &= 4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-1)^{k+1} \\ \sqrt{2} \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{1}{2} \right)^{k+1} - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-1)^{k+1} &= 4\sqrt{2} - 4 \\ \sum_{k=0}^{\infty} \left(\frac{16^{-k} \binom{2k+1}{k}^2 \sqrt{2}}{(2k+1)2^{k+1}} - \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-1)^{k+1} \right) &= 4\sqrt{2} - 4 \\ \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{(2k+1)} \left(\frac{\sqrt{2}}{2^{k+1}} - (-1)^{k+1} \right) &= 4\sqrt{2} - 4 \\ \sum_{k=0}^{\infty} \frac{16^{-k} 2^{-k} \binom{2k+1}{k}^2}{(2k+1)} \left(\frac{\sqrt{2}}{2} + (-2)^k \right) &= 4\sqrt{2} - 4 \end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 (1 + \sqrt{2}(-2)^k)}{32^k (2k+1)} = 8 - 4\sqrt{2}$$

Nice.