Infinite summation for perimeter of an ellipse

dnzc

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1 Introduction

Using an arc length integral it is possible to show that the circumference, C, of an ellipse with height 2b and width 2a is

$$C = 4a \int_0^{\pi/2} \sqrt{1 + (\frac{a^2}{b^2} - 1)\sin^2 x}\ dx$$

This is derived in the appendix. In section 1.1 we setup the integral to be turned into an infinite sigma summation. In later sections I go through each case established in 1.1 to derive two infinite sums equal to the circumference, then equate them.

1.1 Problem setup

Let $c = \frac{a^2}{b^2} - 1$. Then the integral will be

$$C = 4a \int_0^{\pi/2} (1 + c\sin^2 x)^{\frac{1}{2}}\ dx$$

We will use the binomial expansion for nonintegers on $(1 + c\sin^2 x)^{\frac{1}{2}}$, then integrate term by term.
In the next sections we consider the 2 cases of c: $|c| \leq 1$, and $|c| \geq 1$.

2 Case 1

We solve the case of $|c| \geq 1$.

$$\begin{aligned} C &= 4a \int_0^{\pi/2} (1 + c\sin^2 x)^{\frac{1}{2}}\ dx \\ &= 4a\sqrt{c+1} \int_0^{\pi/2} (1 - \frac{c}{c+1}\cos^2 x)^{\frac{1}{2}}\ dx \end{aligned} \tag{1}$$

Note that $\because c = \frac{a^2}{b^2} - 1 > -1, |c| \geq 1 \implies c \geq 1$
$\therefore |\frac{c}{c+1}cos^2 x| < 1 \forall x \in \mathbb{R}$
So now we can use the binomial expansion.

$$\begin{aligned} \left(1 - \frac{c}{c+1}\cos^2 x\right)^{\frac{1}{2}} &= 1 + \sum_{k\geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \cdots \times (\frac{1}{2} - k + 1)}{k!} \left(-\frac{c}{c+1}\cos^2 x\right)^k \\ &= 1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)}{2^k k!} \left(\frac{c}{c+1}\right)^k \cos^{2k} x \end{aligned}$$

Substitute this back into (1):

$$C = 4a\sqrt{c+1}\int_0^{\pi/2}\left(1+\sum_{k\geq 1}\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\cos^{2k}x\right)dx$$
$$= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\int_0^{\pi/2}\sum_{k\geq 1}\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\cos^{2k}x\,dx$$

Integrating term by term (sum and integral are interchanged):

$$= 2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\int_0^{\pi/2}\cos^{2k}x\,dx\right) \quad (2)$$

It is proved in the appendix that $\int_0^{\pi/2}\cos^{2k}(x)\ dx = \frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}$
Substitute this back into (2):

$$C = 2\pi a\sqrt{c+1} + 4a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)}{2^k k!}\left(\frac{c}{c+1}\right)^k\frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}\right)$$
$$= 2\pi a\sqrt{c+1} + 2\pi a\sqrt{c+1}\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)\prod_{n=1}^{k}(2n-1)}{2^k k!\prod_{n=1}^{k}(2n)}\left(\frac{c}{c+1}\right)^k\right) \quad (3)$$
$$= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{4^k (k!)^2}\left(\frac{c}{c+1}\right)^k\right)\right)$$

Note

$$\prod_{n=1}^{k}(2n-3)(2n-1) = (-1\times 1)\times(1\times 3)\times\cdots\times[(2k-3)\times(2k-1)]$$
$$= -1\times 1^2\times 3^2\times\cdots\times(2k-3)^2\times(2k-1)$$
$$= (1-2k)\left[1\times 3\times\cdots\times(2k-3)\right]^2 \quad (4)$$
$$= (1-2k)\left[\frac{1\times 2\times 3\times\cdots\times(2k-3)\times(2k-2)}{2\times 4\times\cdots\times(2k-2)}\right]^2$$
$$= \frac{1-2k}{4^{k-1}}\left[\frac{(2k-2)!}{(k-1)!}\right]^2$$

Substitute back into (3):

$$C = 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{(1-2k)\left[\frac{(2k-2)!}{(k-1)!}\right]^2}{4^{k-1}4^k(k!)^2}\left(\frac{c}{c+1}\right)^k\right)\right)$$

$$= 2\pi a\sqrt{c+1}\left(1+\sum_{k\geq 1}\left(\frac{-1}{4^{2k-1}(2k-1)}\left[\frac{(2k-1)(2k-2)!}{k!(k-1)!}\right]^2\left(\frac{c}{c+1}\right)^k\right)\right)$$

$$= 2\pi a\sqrt{c+1}\left(1-\sum_{k\geq 1}\frac{4^{1-2k}}{2k-1}\binom{2k-1}{k-1}^2\left(\frac{c}{c+1}\right)^k\right)$$

Reindex:

$$= 2\pi a\sqrt{c+1}\left(1-\sum_{k\geq 0}\frac{4^{-1-2k}}{2k+1}\binom{2k+1}{k}^2\left(\frac{c}{c+1}\right)^{k+1}\right)$$

$$\boxed{C = \frac{\pi a\sqrt{c+1}}{2}\left(4-\sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{c}{c+1}\right)^{k+1}\right)}$$

3 Case 2

Now we solve the case $|c| \leq 1$.

$$C = 4a\int_0^{\pi/2}(1+c\sin^2 x)^{\frac{1}{2}}\ dx$$

Note that $|csin^2x| < 1 \forall x, 0 \leq x < \frac{\pi}{2}$
So we can use the binomial expansion.

$$= 4a\int_0^{\pi/2}\left(1+\sum_{k\geq 1}\frac{\frac{1}{2}\times -\frac{1}{2}\times\cdots\times(\frac{1}{2}-k+1)}{k!}(c\sin^2 x)^k\right)\ dx$$

Integrate term by term:

$$= 2\pi a + 4a\sum_{k\geq 1}\frac{\frac{1}{2}\times\cdots\times(\frac{3}{2}-k)}{k!}c^k\int_0^{\pi/2}\sin^{2k} x\ dx \tag{5}$$

It is proved in the appendix that $\int_0^{\pi/2}\sin^{2k} x\ dx = \frac{\pi}{2}\prod_{n=1}^{k}\frac{2n-1}{2n}$

Substitute back into (5):

$$C = 2\pi a + 4a \sum_{k\geq 1} \frac{\frac{1}{2} \times \cdots \times (\frac{3}{2} - k)}{k!} c^k \frac{\pi}{2} \prod_{n=1}^{k} \frac{2n-1}{2n}$$

$$= 2\pi a + 2\pi a \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(3-2n) \prod_{n=1}^{k}(2n-1)}{2^k k! \prod_{n=1}^{k}(2n)} c^k$$

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{\prod_{n=1}^{k}(2n-3)(2n-1)}{4^k (k!)^2} (-c)^k \right)$$

Substitute (4) in:

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{(1-2k)\left[\frac{(2k-2)!}{(k-1)!}\right]^2}{4^{k-1} 4^k (k!)^2} (-c)^k \right)$$

$$= 2\pi a \left(1 + \sum_{k\geq 1} \frac{-1}{4^{2k-1}(2k-1)} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!}\right]^2 (-c)^k \right)$$

$$= 2\pi a \left(1 - \sum_{k\geq 1} \frac{4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 (-c)^k \right)$$

Reindex:

$$= 2\pi a \left(1 - \sum_{k\geq 0} \frac{4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 (-c)^{k+1} \right)$$

$$\boxed{C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} (-c)^{k+1} \right)}$$

Note when c=0 (a=b so the ellipse is a circle) the whole summation vanishes so C becomes $2\pi a$.

4 Summary

I have derived 2 infinite sums for the circumference, C, of an ellipse depending on $|c|$ Where 2a is the width, 2b is the height and $c = \frac{a^2}{b^2} - 1$.

If $|c| \geq 1$:

$$C = \frac{\pi a\sqrt{c+1}}{2}\left(4 - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{c}{c+1}\right)^{k+1}\right)$$

If $|c| \leq 1$:

$$C = \frac{\pi a}{2}\left(4 - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}(-c)^{k+1}\right)$$

4.1 Discovery

I realized if you plug c=1 into both sums, they must be equal, therefore:

$$\frac{\pi a\sqrt{2}}{2}\left(4 - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{1}{2}\right)^{k+1}\right) = \frac{\pi a}{2}\left(4 - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}(-1)^{k+1}\right)$$

Now simplify:

$$\begin{aligned}
4\sqrt{2} - \sqrt{2}\sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{1}{2}\right)^{k+1} &= 4 - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}(-1)^{k+1} \\
\sqrt{2}\sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}\left(\frac{1}{2}\right)^{k+1} - \sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}(-1)^{k+1} &= 4\sqrt{2} - 4 \\
\sum_{k=0}^{\infty}\left(\frac{16^{-k}\binom{2k+1}{k}^2\sqrt{2}}{(2k+1)2^{k+1}} - \frac{16^{-k}\binom{2k+1}{k}^2}{2k+1}(-1)^{k+1}\right) &= 4\sqrt{2} - 4 \\
\sum_{k=0}^{\infty}\frac{16^{-k}\binom{2k+1}{k}^2}{(2k+1)}\left(\frac{\sqrt{2}}{2^{k+1}} - (-1)^{k+1}\right) &= 4\sqrt{2} - 4 \\
\sum_{k=0}^{\infty}\frac{16^{-k}2^{-k}\binom{2k+1}{k}^2}{(2k+1)}\left(\frac{\sqrt{2}}{2} + (-2)^{k}\right) &= 4\sqrt{2} - 4
\end{aligned}$$

$$\boxed{\sum_{k=0}^{\infty}\frac{\binom{2k+1}{k}^2\left(1+\sqrt{2}(-2)^k\right)}{32^k(2k+1)} = 8 - 4\sqrt{2}}$$

Nice.

5 Appendix