

Infinite summation for perimeter of an ellipse

dnzc

Contents

1	Introduction	2
1.1	Problem setup	2
2	Algebra crunching	2
3	A specific case	3
4	Summary	6
5	Appendix	7
5.1	Lemma	7
5.2	Lemma	8

1 Introduction

Using an arc length integral it is possible to show that the circumference, C , of an ellipse with eccentricity e and semi-major axis a is

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 x} \, dx$$

In Section 1.1 we setup the integral to be turned into an infinite sigma summation. In Section 2 this is evaluated. In Section 3 we consider a specific case of e , using a trigonometric identity to derive a closed form for a nice infinite sum.

1.1 Problem setup

The integral is

$$C = 4a \int_0^{\pi/2} (1 - e^2 \sin^2 x)^{\frac{1}{2}} \, dx \tag{1}$$

We will use the binomial expansion for nonintegers on $(1 - e^2 \sin^2 x)^{\frac{1}{2}}$, then integrate term by term. Lemmas 5.1 and 5.2 will be referenced and can be found in the appendix.

2 Algebra crunching

Recalling (1), note that $0 \leq e < 1 \implies e^2 \sin^2 x < 1$ so we can use the binomial expansion.

$$C = 4a \int_0^{\pi/2} \left(1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \times \cdots \times (\frac{1}{2} - k + 1)}{k!} (-e^2 \sin^2 x)^k \right) \, dx$$

Integrate term by term (sum and integral are interchanged):

$$= 2\pi a + 4a \sum_{k \geq 1} \frac{\frac{1}{2} \times \cdots \times (\frac{3}{2} - k)}{k!} (-1)^k e^{2k} \int_0^{\pi/2} \sin^{2k} x \, dx$$

Substitute Lemma 5.1 in:

$$\begin{aligned}
&= 2\pi a + 4a \sum_{k \geq 1} \frac{\frac{1}{2} \times \cdots \times (\frac{3}{2} - k)}{k!} (-1)^k e^{2k} \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n} \\
&= 2\pi a + 2\pi a \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)}{2^k k!} e^{2k} \prod_{n=1}^k \frac{2n-1}{2n} \\
&= 2\pi a + 2\pi a \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3) \prod_{n=1}^k (2n-1)}{2^k k! \prod_{n=1}^k (2n)} e^{2k} \\
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{\prod_{n=1}^k (2n-3)(2n-1)}{4^k (k!)^2} e^{2k} \right)
\end{aligned}$$

Substitute Lemma 5.2 in:

$$\begin{aligned}
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{(1-2k) \left[\frac{(2k-2)!}{(k-1)!} \right]^2}{4^{k-1} 4^k (k!)^2} e^{2k} \right) \\
&= 2\pi a \left(1 + \sum_{k \geq 1} \frac{-1}{4^{2k-1} (2k-1)} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!} \right]^2 e^{2k} \right) \\
&= 2\pi a \left(1 - \sum_{k \geq 1} \frac{4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 e^{2k} \right)
\end{aligned}$$

Reindex:

$$= 2\pi a \left(1 - \sum_{k \geq 0} \frac{4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 e^{2k+2} \right)$$

$$C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 e^{2k+2}}{16^k (2k+1)} \right) \quad (2)$$

Note when $e = 0$ (the ellipse is a circle) the whole summation vanishes so C becomes $2\pi a$, as expected.

3 A specific case

We will evaluate (1) for a specific value of e , then equate the result with (2). Recall (1) and set $e^2 = \frac{1}{2}$:

$$C \Big|_{e^2=1/2} = 4a \int_0^{\pi/2} \left(1 - \frac{1}{2} \sin^2 x \right)^{\frac{1}{2}} dx$$

Using the identity $\sin^2 x + \cos^2 x \equiv 1$:

$$\begin{aligned} &= 4a \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2} \cos^2 x \right)^{\frac{1}{2}} dx \\ &= 2a\sqrt{2} \int_0^{\pi/2} (1 + \cos^2 x)^{\frac{1}{2}} dx \end{aligned}$$

Note that $0 \leq x < \pi/2 \implies \cos^2 x < 1$

So now we can use the binomial expansion.

$$= 2a\sqrt{2} \int_0^{\pi/2} \left(1 + \sum_{k \geq 1} \frac{\frac{1}{2} \times -\frac{1}{2} \cdots \times (\frac{1}{2} - k + 1)}{k!} (\cos^2 x)^k \right) dx$$

Integrate term by term (sum and integral are interchanged):

$$= \pi a\sqrt{2} + 2a\sqrt{2} \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (3 - 2n)}{2^k k!} \int_0^{\pi/2} \cos^{2k} x dx \right)$$

Substitute Lemma 5.1 in:

$$\begin{aligned} &= \pi a\sqrt{2} + 2a\sqrt{2} \sum_{k \geq 1} \left(\frac{\prod_{n=1}^k (3 - 2n)}{2^k k!} \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n} \right) \\ &= \pi a\sqrt{2} + \pi a\sqrt{2} \sum_{k \geq 1} \left((-1)^k \frac{\prod_{n=1}^k (2n-3) \prod_{n=1}^k (2n-1)}{2^k k! \prod_{n=1}^k 2n} \right) \\ &= \pi a\sqrt{2} \left(1 + \sum_{k \geq 1} \left((-1)^k \frac{\prod_{n=1}^k (2n-3)(2n-1)}{4^k (k!)^2} \right) \right) \end{aligned}$$

Substitute Lemma 5.2 in:

$$\begin{aligned} &= \pi a\sqrt{2} \left(1 + \sum_{k \geq 1} \left((-1)^k \frac{(1-2k) \left[\frac{(2k-2)!}{(k-1)!} \right]^2}{4^{k-1} 4^k (k!)^2} \right) \right) \\ &= \pi a\sqrt{2} \left(1 + \sum_{k \geq 1} \left((-1)^{k+1} \frac{1}{(2k-1) 4^{2k-1}} \left[\frac{(2k-1)(2k-2)!}{k!(k-1)!} \right]^2 \right) \right) \\ &= \pi a\sqrt{2} \left(1 + \sum_{k \geq 1} \left(\frac{(-1)^{k+1} 4^{1-2k}}{2k-1} \binom{2k-1}{k-1}^2 \right) \right) \end{aligned}$$

Reindex:

$$= \pi a \sqrt{2} \left(1 + \sum_{k \geq 0} \frac{(-1)^k 4^{-1-2k}}{2k+1} \binom{2k+1}{k}^2 \right)$$

$$C \Big|_{e^2=1/2} = \frac{\pi a \sqrt{2}}{4} \left(4 + \sum_{k \geq 0} \frac{(-1)^k 16^{-k}}{2k+1} \binom{2k+1}{k}^2 \right)$$

But note, if we evaluate C when $e^2 = \frac{1}{2}$ using (2) we get:

$$C \Big|_{e^2=1/2} = \frac{\pi a}{2} \left(4 - \sum_{k \geq 0} \frac{\binom{2k+1}{k}^2}{(2k+1)16^k} \cdot \frac{1}{2^{k+1}} \right)$$

Thus:

$$\frac{\pi a \sqrt{2}}{4} \left(4 + \sum_{k \geq 0} \frac{(-1)^k 16^{-k}}{2k+1} \binom{2k+1}{k}^2 \right) = \frac{\pi a}{2} \left(4 - \sum_{k \geq 0} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \frac{1}{2^{k+1}} \right)$$

Now simplify:

$$\sqrt{2} \left(4 + \sum_{k \geq 0} \frac{(-1)^k 16^{-k}}{2k+1} \binom{2k+1}{k}^2 \right) = 2 \left(4 - \sum_{k \geq 0} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \frac{1}{2^{k+1}} \right)$$

$$4\sqrt{2} + \sum_{k \geq 0} \sqrt{2} \frac{(-1)^k 16^{-k}}{2k+1} \binom{2k+1}{k}^2 = 8 - \sum_{k \geq 0} \frac{16^{-k} \binom{2k+1}{k}^2}{(2k+1)2^k}$$

$$\sum_{k \geq 0} \frac{16^{-k} \binom{2k+1}{k}^2}{(2k+1)2^k} + \sum_{k \geq 0} \sqrt{2} \frac{(-1)^k 16^{-k}}{2k+1} \binom{2k+1}{k}^2 = 8 - 4\sqrt{2}$$

$$\sum_{k \geq 0} \frac{16^{-k} \binom{2k+1}{k}^2}{2k+1} \left(\frac{1}{2^k} + \sqrt{2}(-1)^k \right) = 8 - 4\sqrt{2}$$

$$\sum_{k \geq 0} \frac{32^{-k} \binom{2k+1}{k}^2}{2k+1} \left(1 + \sqrt{2}(-2)^k \right) = 8 - 4\sqrt{2}$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 (1 + \sqrt{2}(-2)^k)}{(2k+1)32^k} = 8 - 4\sqrt{2} \quad (3)$$

Nice.

4 Summary

I have derived two infinite sums, (2) and (3). One is a sum that I have found a closed form for:

$$\sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 (1 + \sqrt{2}(-2)^k)}{(2k+1)32^k} = 8 - 4\sqrt{2}$$

And the other is the circumference, C , of an ellipse with eccentricity e and semi-major axis a :

$$C = \frac{\pi a}{2} \left(4 - \sum_{k=0}^{\infty} \frac{\binom{2k+1}{k}^2 e^{2k+2}}{16^k (2k+1)} \right)$$

Note that we may calculate the first few coefficients of e in the sum, say up to $k=5$:

$$C \approx 2\pi a - \frac{\pi a e^2}{2} - \frac{3\pi a e^4}{32} - \frac{5\pi a e^6}{128} - \frac{175\pi a e^8}{8192} - \frac{441\pi a e^{10}}{32768}$$

This approximation should be more accurate when the eccentricity is small.

5 Appendix

5.1 Lemma

Claim: $\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k} x \, dx = \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n}$
 The proof is below.

Firstly, note

$$\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k} \left(\frac{\pi}{2} - x \right) dx$$

so $\int_0^{\pi/2} \sin^{2k} x \, dx = \int_0^{\pi/2} \cos^{2k} x \, dx$ can be verified by doing a substitution with $u = \frac{\pi}{2} - x$.

It remains to show that $I_k := \int_0^{\pi/2} \sin^{2k} x \, dx = \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n}$

$$I_k = \int_0^{\pi/2} \sin^{2k-1} x \sin(x) \, dx$$

Using IBP:

$$\begin{aligned} &= -\sin^{2k-1} x \cos(x) \Big|_0^{\pi/2} + (2k-1) \int_0^{\pi/2} \sin^{2k-2} x \cos^2 x \, dx \\ &= (2k-1) \int_0^{\pi/2} (\sin^{2k-2} x - \sin^{2k} x) \, dx \\ &= (2k-1) \int_0^{\pi/2} \sin^{2k-2} x \, dx - (2k-1) \int_0^{\pi/2} \sin^{2k} x \, dx \\ &\quad \therefore I_k + (2k-1)I_k = (2k-1)I_{k-1} \\ &\quad I_k = \frac{2k-1}{2k} I_{k-1} \end{aligned}$$

Note that $I_0 = \frac{\pi}{2}$

$$\therefore I_k = \frac{2k-1}{2k} \times \frac{2k-3}{2k-2} \times \cdots \times \frac{1}{2} I_0 = \frac{\pi}{2} \prod_{n=1}^k \frac{2n-1}{2n}$$

Done.

5.2 Lemma

Claim: $\prod_{n=1}^k (2n-3)(2n-1) = \frac{1-2k}{4^{k-1}} \left[\frac{(2k-2)!}{(k-1)!} \right]^2$

Proof:

$$\begin{aligned} \prod_{n=1}^k (2n-3)(2n-1) &= (-1 \times 1) \times (1 \times 3) \times \cdots \times [(2k-3) \times (2k-1)] \\ &= -1 \times 1^2 \times 3^2 \times \cdots \times (2k-3)^2 \times (2k-1) \\ &= (1-2k) [1 \times 3 \times \cdots \times (2k-3)]^2 \\ &= (1-2k) \left[\frac{1 \times 2 \times 3 \times \cdots \times (2k-3) \times (2k-2)}{2 \times 4 \times \cdots \times (2k-2)} \right]^2 \\ &= \frac{1-2k}{4^{k-1}} \left[\frac{(2k-2)!}{(k-1)!} \right]^2 \end{aligned}$$