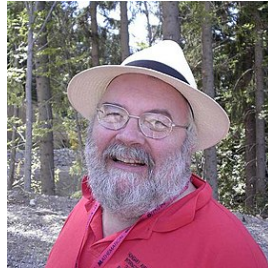


Triathlon - Maths

Olympiad Maths

May 2022



Instructions

Read these before attempting the questions.

Time: 4 hours

Format

This paper has 6 available questions each worth 10 marks.
Make sure to write full written proofs where possible.

Tools

Rulers and compasses are allowed, but no protractors or calculators.

Submission

Submit your paper by dm to `e=pi=3#5257` before **23:59 on Sunday 22nd May**.
Please scan in then send as a **single PDF**. Online tools are available for this.
On the front page, write your **discord ID** (right click username -> copy ID).
After submission, you can be added to a channel to discuss the questions. Please don't discuss them outside of that channel.

The questions begin on the next page.

Good luck, and most importantly have fun!

Questions

- Two numbers a and b satisfy the equations $a + b = 2$ and $a^2 + b^2 = 6$.
 - What is the value of $a^3 + b^3$?
 - What is the value of $a^6 + b^6$?
- In triangle ABC , M is the midpoint of BC . Let P be a point on AM , inside triangle ABC . Rays BP and CP meet segments AC and AB at D and E respectively. Let the circles with diameters BD and CD meet at X and Y .
Show that A, X, Y are collinear.
- There are n sweets in a pile. Alice and Bob take turns eating the sweets (Alice starts). Each turn, they eat either one sweet, or a prime number of sweets that divides the current number of sweets in the pile.
For what values of n can Bob eat the last sweet?

- Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y$$

- The six members of the illuminati have designed a world-ending robot. They wish to give it ample safeguarding though, so they lock up the robot inside an indestructible case, using multiple locks that each have different fitting keys. The group wishes to distribute lock keys among themselves so that the robot can be released when and only when three or more members are present. What is the minimum possible number of locks needed? What is the minimum possible number of keys each member must carry?
- Define the sequences x_n, y_n as follows:
 - $x_0 = 1, y_0 = 2$
 - $x_{n+1} = \frac{x_n + y_n}{2}$
 - $y_{n+1} = \sqrt{x_{n+1}y_n}$

Show that $\lim_{n \rightarrow \infty} y_n$ exists, and find its value.

END OF PAPER