A random formula with logs of primes

dnzc

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1 Introduction

We will derive the following fact:
$\forall a, b, p, q \in \mathbb{N}$ with p, q distinct primes,

$$ab = \sum_{x=1}^{a} \min(b, \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \min(a, \lfloor x \log_p q \rfloor)$$

This formula was derived by thinking about the following problem:

Let $n = 2^{31}3^{19}$. How many positive divisors of n^2 are less than n but do not divide n?

We will solve a more general version of the above problem in two different ways, which will lead us to the main result arrived at in the Summary section.

2 Solving the Problem

The version of the problem that we will think about is as follows: For some $a, b, p, q \in \mathbb{N}$ with p, q distinct primes, let $n = p^a q^b$. How many positive divisors of n^2 are less than n but do not divide n?

2.1 Computational Way

The factors of n^2 are in the form $p^r q^s,\ r, s \in \mathbb{Z}$ with $0 \leq r \leq 2a, 0 \leq q \leq 2b$
If this factor does not divide n then we must have $r > a$ or $s > b$.
We cannot have both $r > a$ and $s > b$ since that would imply $p^r q^s > n$ and we are counting the factors that are less than n. Thus these cases are disjoint and we can count them separately.

2.1.1 Case 1

We will count the case when $r > a$.
Then more specifically, $a < r \leq 2a$ and $0 \leq s < b$. And:

$$\begin{aligned} p^r q^s &< p^a q^b \\ \implies p^{r-a} &< q^{b-s} \end{aligned}$$

Let $x = r - a, y = b - s$
Then with some inequality manipulations we get $1 \leq x \leq a$ and $1 \leq y \leq b$

$$\begin{aligned} p^x &< q^y \\ \implies x \log_q p &< y \text{ (log rules)} \\ \implies \lfloor x \log_q p \rfloor + 1 \leq y &\leq b \text{ (} \log_q p \text{ is irrational)} \end{aligned}$$

So for any given x, the number of possibilities for y contributed to the count is

$$\max(0, b - \lfloor x \log_q p \rfloor)$$

We use the "max" function here to contribute 0 if $\lfloor x \log_q p \rfloor \geq b$ rather than contributing a negative number (since there are 0 possibilities for y in that case) So, the total number of possibilities counted by Case 1 is the sum of this over all possible values of x:

$$= \sum_{x=1}^{a} \max(0, b - \lfloor x \log_q p \rfloor)$$

2.1.2 Case 2

The case when $s > b$ is symmetric to Case 1, with p,q swapped and a,b swapped but the argument is entirely the same.
Thus similarly to Case 1, the count contributed by this case is

$$\sum_{x=1}^{b} \max(0, a - \lfloor x \log_p q \rfloor)$$

2.1.3 Total

So the overall answer to the problem is the sum of Case 1 and Case 2, which is

$$\begin{aligned} &\sum_{x=1}^{a} \max(0, b - \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \max(0, a - \lfloor x \log_p q \rfloor) \\ &= \sum_{x=1}^{a} \left[b - \min(b, \lfloor x \log_q p \rfloor) \right] + \sum_{x=1}^{b} \left[a - \min(a, \lfloor x \log_p q \rfloor) \right] \\ &= 2ab - \left[\sum_{x=1}^{a} \min(b, \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \min(a, \lfloor x \log_p q \rfloor) \right] \end{aligned}$$

$$\boxed{\therefore \text{answer} = 2ab - \left[\sum_{x=1}^{a} \min(b, \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \min(a, \lfloor x \log_p q \rfloor) \right] \quad (1)}$$

2.2 Combinatorial Way

We will show that the overall answer is simply ab.

$n^2 = p^{2a} q^{2b}$ so there are $(2a + 1)(2b + 1)$ factors of n^2.

We can consider positive factor pairs of n^2 - one number of the pair will $< n$ and the other will be $> n$, except for the case when the factor pair is (n, n). In other words, if a is a positive factor of n^2 and $a < n$ then $\frac{n^2}{a} > n$ so exactly one of the factor pair $(a, \frac{n^2}{a})$ will be less than n.

Thus the number of factors of n^2 that are less than n is

$$\# \text{ factors of } n^2 \text{ less than } n = \frac{(2a+1)(2b+1)-1}{2} = 2ab + a + b$$

where we exclude the (n, n) case and then halve.

Now,

$$\begin{aligned} &\# \text{ factors of } n^2 \text{ less than } n \text{ AND divide } n \\ (\# \text{ factors of } n) - 1 \text{ (since we exclude n as a factor of n)} \\ &= (a+1)(b+1) - 1 = ab + a + b \end{aligned}$$

Finally,

$$\begin{aligned} \text{answer} &= \# \text{ factors of } n^2 \text{ less than } n \text{ AND NOT divide } n \\ &= \# \text{ factors of } n^2 \text{ less than } n - \# \text{ factors of } n^2 \text{ less than } n \text{ AND divide } n \\ &= (2ab + a + b) - (ab + a + b) \\ &= ab \end{aligned}$$

$$\therefore \text{answer} = ab \tag{2}$$

3 Summary

Equating (1) and (2):

$$ab = 2ab - \left[\sum_{x=1}^{a} \min(b, \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \min(a, \lfloor x \log_p q \rfloor) \right]$$

Rearrange to get

$$ab = \sum_{x=1}^{a} \min(b, \lfloor x \log_q p \rfloor) + \sum_{x=1}^{b} \min(a, \lfloor x \log_p q \rfloor)$$
$$\forall a, b, p, q \in \mathbb{N} \text{ with } p, q \text{ distinct primes}$$

We have shown our result.