On the q-Laplacian Matrix of a Graph

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Abstract

We study the eigenvalues of the q-Laplacian matrix of a graph G, defined as $L_q(G) = qD(G) + A(G)$, where q is a fixed real number, $A(G)$ is the adjacency matrix, and $D(G)$ is the diagonal degree matrix. This generalizes the adjacency, Laplacian, and signless Laplacian matrices. We consider a shifted determinant that works nicely with forests and unicyclic graphs. In particular, for many q, we prove a necessary and sufficient condition on the eigenvalues for a graph to be a forest with a specific number of components. We also generalize the line-graph connection of the signless Laplacian and show how L_q behaves particularly well with K_n-decomposable graphs. In particular, we develop a method to construct pairs of graphs with the same eigenvalues (called cospectral mates) and apply it to construct cospectral mates of friendship graphs. Additionally, we prove that the graphs constructed by our method are the only cospectral mates of friendship graphs.

Contents

1 Introduction

Let G be a simple finite graph. We let D be the matrix with the degree sequence of G on the diagonal and A be the adjacency matrix of G. The adjacency matrix A, alongside the Laplacian Matrix $L = D - A$ and the Signless Laplacian Matrix $Q = D + A$, is among the most studied matrices in Spectral Graph Theory. In this paper, we generalize these three matrices into a class of matrices called q-Laplacians, $L_q = qD + A$, and establish some properties of it. (This does not generalize the Laplacian L but the negative of the Laplacian.)

In Section 2, we prove some results for when G is unicyclic or a forest. We also introduce a necessary and sufficient condition for G being a forest with a specific number of connected components when $q \in \mathbb{Q} \setminus \{0, \pm 1\}$. In Section 3, we examine how letting $q = \frac{1}{n-1}$ can be helpful for investigating K_n-decomposable graphs. We also present a method for constructing cospectral graphs with respect to L_q by using K_n-decomposable graphs. In Section 4, we examine the case $q = 1/2$ and we apply our cospectral construction method on friendship graphs. Later, we also prove that all graphs that are cospectral to some friendship graph must have from this method. Lastly, in section Section 5, we discuss how the tools we built could be helpful for tackling open problems about K_n-decomposable graphs.

1.1 Preliminaries

For graph G, we denote the q-Laplacian $qD + A$ by $L_q(G)$. We may also just write L_q when the graph we are working with is clear from the context.

The spectrum of G with respect to L_q, or the q-spectrum of G, refers to the multiset of eigenvalues of $L_q(G)$. When the choice of q is clear from context, we may also just say the spectrum of G.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of L_q. Then

$$\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(L_q) = \sum_{i=1}^{n} q d_i = 2q|E|.$$

This means that when $q \neq 0$, we can obtain the number of edges in G from the spectrum of L_q.

2 Some Properties of Unicyclic Graphs and Forests

For $q \neq 0$, we let

$$\sigma_q(G) := \det\left(qD + A + \frac{1-q^2}{q} I\right) = \prod_{i=1}^{n} \left(\lambda_i + \frac{1-q^2}{q}\right),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues with multiplicity of $L_q(G)$.

This is a way to generalize the determinant of the Laplacian ($q = -1$) and the Signless Laplacian ($q = 1$) matrices. Also, notice that we can obtain $\sigma_q(G)$ from the spectrum of G.

Some other things to note are $\sigma_q(K_1) = \frac{1-q^2}{q}$ and $\sigma_q(G \cup H) = \sigma_q(G)\sigma_q(H)$.

Proposition 2.1. *For $q \neq 0$ and $n \geq 3$, we have*

$$\sigma_q(C_n) = q^n + \frac{1}{q^n} + 2(-1)^{n+1}.$$

Proof. Consider the following $n \times n$ matrix M:

$$\begin{bmatrix} \sqrt{q} & 1/\sqrt{q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{q} & 1/\sqrt{q} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{q} & 1/\sqrt{q} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q} & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 1/\sqrt{q} & 0 & 0 & 0 & \cdots & 0 & \sqrt{q} \end{bmatrix}.$$

When $q < 0$, $\sqrt{q}$ denotes $\sqrt{|q|} \cdot i$. It is easy to see that

$$\det(M) = (\sqrt{q})^n + (-1)^{n+1} \frac{1}{(\sqrt{q})^n}.$$

Also, notice that

$$MM^T = \begin{bmatrix} q+\frac{1}{q} & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & q+\frac{1}{q} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & q+\frac{1}{q} & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & q+\frac{1}{q} & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 1 & q+\frac{1}{q} \end{bmatrix}$$

$$= \begin{bmatrix} 2q+\frac{1-q^2}{q} & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 2q+\frac{1-q^2}{q} & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2q+\frac{1-q^2}{q} & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2q+\frac{1-q^2}{q} & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 1 & 2q+\frac{1-q^2}{q} \end{bmatrix} = L_q(C_n) + \frac{1-q^2}{q} I.$$

So,

$$\begin{aligned} \sigma_q(C_n) = \det\left(L_q(C_n) + \frac{1-q^2}{q} I\right) &= \det(MM^T) = \det(M)^2 \\ &= \left((\sqrt{q})^n + (-1)^{n+1}(1/\sqrt{q})^n\right)^2 \\ &= q^n + \frac{1}{q^n} + 2(-1)^{n+1}. \end{aligned}$$

□

We will generalize this result to all unicyclic graphs, but we need the following lemma first.

Lemma 2.2. *Assume $q \neq 0$ and G' is the graph obtained by adding a leaf to somewhere in G. Then,*

$$\sigma_q(G') = \frac{\sigma_q(G)}{q}.$$

Proof. Assume the added leaf of G' is represented by the first row and column of $L_q(G')$ and the vertex it is connected to is represented by the second row and column.

See that the $(1,1)$ entry of $L_q(G') = qD + A$ will have $q \cdot (\text{degree of the leaf}) = q$, and so in $L_q(G') + \frac{1-q^2}{q} I$, the $(1,1)$ will have $q + \frac{1-q^2}{q} = \frac{1}{q}$. The entries right next to and below it will have 1 representing the edge incident to the leaf. The remaining entries of the first row and column will be 0 since the leaf is not connected to anything else. The rest of the matrix will be identical to $L_q(G') + \frac{1-q^2}{q} I$ except for the $(2,2)$ entry, which has an extra q added since the leaf is contributing to its degree. So,

$$\sigma_q(G') = \det\left(L_q(G') + \frac{1-q^2}{q}I\right) = \det\begin{pmatrix} \frac{1}{q} & 1 \quad\quad 0 \quad 0 \quad \cdots \\ 1 & \text{extra } q \\ 0 & \\ 0 & L_q(G) + \frac{1-q^2}{q}I \\ \vdots & \end{pmatrix}$$

When we subtract q times the first row from the second, the 1 and the extra q goes away. Then, the determinant above becomes

$$\det\begin{pmatrix} \frac{1}{q} & 1 \quad 0 \quad 0 \quad \cdots \\ 0 & \\ 0 & \\ 0 & L_q(G) + \frac{1-q^2}{q}I \\ \vdots & \end{pmatrix} = \frac{1}{q} \cdot \det\left(L_q(G) + \frac{1-q^2}{q}I\right) = \frac{\sigma_q(G)}{q}.$$

$\square$

This implies the following result about unicyclic graphs.

Corollary 2.3. *For $q \neq 0$, if G is a unicyclic graph containing a cycle of length $k \geq 3$ and t vertices outside the cycle, then we have*

$$\sigma_q(G) = \left(q^k + \frac{1}{q^k} + 2(-1)^{k+1}\right)\frac{1}{q^t}.$$

Proof. A unicyclic graph containing a k-cycle will always have a vertex of degree 1 unless it is isomorphic to C_k. So, at every step, we can remove a vertex of degree 1 until we reach C_k. This will take t steps and will multiply the σ_q value of the graph by $1/q$ each time by Theorem 2.2. Combining this with Theorem 2.1, we obtain,

$$\sigma_q(G) = \sigma_q(C_k) \cdot \frac{1}{q^t} = \left(q^k + \frac{1}{q^k} + 2(-1)^{k+1}\right)\frac{1}{q^t}.$$

$\square$

Corollary 2.4. *For $q \neq 0$, if G is a tree with n vertices, then we have*

$$\sigma_q(G) = \frac{1-q^2}{q^n}.$$

Proof. If G is a tree with n vertices, we can take away a leaf at each step until we reach an isolated vertex. This will take $n-1$ steps, so by Theorem 2.2, we have

$$\sigma_q(G) = \frac{1}{q^{n-1}}\sigma_q(K_1).$$

Since K_1 has no edges, $L_q(K_1)$ is just the scalar 0. Thus,

$$\sigma_q(K_1) = \det\left(L_q(K_1) + \frac{1-q^2}{q}I\right) = \frac{1-q^2}{q}.$$

And so,

$$\sigma_q(G) = \frac{1}{q^{n-1}}\sigma_q(K_1) = \frac{1}{q^{n-1}}\frac{1-q^2}{q} = \frac{1-q^2}{q^n}.$$

□

Turns out, we can go even further.

Proposition 2.5. *Let $q \in \mathbb{Q} \setminus \{0, \pm 1\}$. If G has n vertices and $n-1$ edges, then G is a tree if and only if*

$$\sigma_q(G) = \frac{(1-q^2)}{q^n}.$$

Proof. The only if direction is already proved as Theorem 2.4, so we will prove the if direction.

Let $q = a/b$ where a, b are coprime integers. They are also non-zero and satisfy $|a| \neq |b|$ because $a/b = q \in \mathbb{Q} \setminus \{0, \pm 1\}$.

Assume that G has n vertices and $n-1$ edges and its satisfies

$$\sigma_q(G) = \frac{(1-q^2)}{q^n}.$$

We will show G is a tree. Assume for the sake of contradiction it is not a tree. Further assume G has no leaves, as taking off a leaf will reduce the vertex and edge counts by exactly 1, while dividing $\sigma_q(G)$ by exactly $1/q$, so the hypothesis would still hold.

Notice that some connected component of G must be a tree. Since we removed all leaves, that tree will be an isolated vertex. So $G = K_1 \cup S$ for some graph S where S has $n-1$ vertices and $n-1$ edges. We will now consider two cases:

- Case 1: Some connected component of S is a tree.

  Since we removed leaves, it must be that $S = K_1 \cup P$ for some graph P.

  Then, we have $G = K_1 \cup K_1 \cup P$, which implies

$$\frac{(1-q^2)}{q^n} = \sigma_q(G) = \sigma_q(K_1)\sigma_q(K_1)\sigma_q(P) = \frac{(1-q^2)}{q}\frac{(1-q^2)}{q}\sigma_q(P).$$

  Since $q \neq \pm 1$, this means

$$\sigma_q(P) = \frac{1}{q^{n-2}}\frac{1}{(1-q^2)} = \frac{b^{n-2}}{a^{n-2}}\frac{b^2}{(b^2-a^2)}.$$

  However, notice that if

$$\sigma_q(P) = \det\left(qD + A + \frac{1-q^2}{q}I\right) = \det\left(\frac{a}{b}D + A + \frac{b^2-a^2}{ab}I\right) = k/m$$

  for coprime integers k, m, then the prime divisors of m can only be the prime divisors of ab.

But, if $|b^2 - a^2| \geq 2$, this is not possible, because the gcd of $b^2 - a^2$ with a and with b are both 1. So, $b^2 - a^2$ has a prime divisor that does not divide ab. This gives us a contradiction.

So $|b^2 - a^2| < 2$. But then, this would imply $a = \pm b$ or $a, b \in \{0, \pm 1\}$, which means $q \in \{0, \pm 1\}$. Thus, we obtain a contradiction.

- Case 2: No connected component of S is a tree.

  This means S is a disjoint union of unicyclic graphs. In our case, we removed all leaves, so S must be a disjoint union of cycles of at least 3 vertices. Assume $S = \bigcup C_{a_i}$ with $a_i \geq 3$.

  Then, notice
$$\frac{(1-q^2)}{q^n} = \sigma_q(G) = \sigma_q(K_1)\sigma_q(S) = \frac{(1-q^2)}{q}\sigma_q(S)$$

  Since $q \neq \pm 1$, this means
$$\begin{aligned}
\frac{b^{n-1}}{a^{n-1}} = \frac{1}{q^{n-1}} &= \sigma_q(S) = \sigma_q\left(\bigcup C_{a_i}\right) = \prod \sigma_q\left(C_{a_i}\right) \\
&= \prod \left(q^{a_i} + 1/q^{a_i} + 2(-1)^{a_i+1}\right) && \text{(Theorem 2.1)} \\
&= \prod \left(\frac{a^{a_i}}{b^{a_i}} + \frac{b^{a_i}}{a^{a_i}} + 2(-1)^{a_i+1}\right) \\
&= \prod \frac{a^{2a_i} + b^{2a_i} + (-1)^{a_i+1}2a^{a_i}b^{a_i}}{(ab)^{a_i}} \\
&= \prod \frac{(a^{a_i} + (-1)^{a_i+1}b^{a_i})^2}{(ab)^{a_i}}.
\end{aligned}$$

  We know that $|a| \neq |b|$, so $a^{a_i} + (-1)^{a_i+1}b^{a_i} \neq 0$. Also, $ab \neq 0$ and $a_i \geq 3$, so, $a^{a_i} + (-1)^{a_i+1}b^{a_i} \neq \pm 1$.

  So, $a^{a_i} + (-1)^{a_i+1}b^{a_i}$ has some prime divisor p. But, p will be relatively prime to ab as otherwise it must divide both a and b, contradicting their coprimeness.

  This means that the numerator on the RHS is divisible by p while the numerator on the LHS is not. This is a contradiction.

We get a contradiction either way, so G must be a tree.

□

See that the above theorem does not hold when $q \notin \mathbb{Q}$.

Example 2.6. Consider $q = 2^{1/4}$ and $G = C_4 \cup K_1$, a graph with 5 vertices and 4 edges. Then,

$$\sigma_q(G) = \sigma_q(C_4)\sigma_q(K_1) = \left((2^{1/4})^4 + (2^{-1/4})^4 + 2(-1)^5\right)\left(\frac{1-(2^{1/4})^2}{2^{1/4}}\right) = \left(\frac{1-(2^{1/4})^2}{2^{5/4}}\right) = \frac{1-q^2}{q^5}.$$

But $G = C_4 \cup K_1$ is not a tree.

Combining that result with the fact that we can obtain the number of edges in G from its spectrum, we obtain the necessary and sufficient condition for G being a tree given the spectrum of $L_q(G)$ when $q \in \mathbb{Q} \setminus \{0, \pm 1\}$.

Corollary 2.7. *Let $q \in \mathbb{Q} \setminus \{0, \pm 1\}$, G be a graph and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $L_q(G)$. Then G is a tree if and only if*

$$\sum_{i=1}^{n} \lambda_i = 2q(n-1),$$

and

$$\prod_{i=1}^{n} \left(\lambda_i + \frac{1-q^2}{q} \right) = \frac{1-q^2}{q^n}.$$

In particular, if G is q-cospectral to a tree T, then G must also be a tree.

This provides a necessary and sufficient condition to check whether a graph is a tree just by looking at its spectrum with respect to L_q.

We have shown that when $q \in \mathbb{Q} \setminus \{0, \pm 1\}$, there is no tree that is q-cospectral to a non-tree. Butler et. al. [1] investigated the same question for $q \in \{0, \pm 1\}$, and found that such pairs exist when $q = 0$ and $q = 1$ but not when $q = -1$. So, combining our results with theirs, we obtain that for all rational q except $q = 0$ and $q = 1$, there is no tree and non-tree cospectral pair with respect to L_q.

The result about trees can be generalized to forests as well.

Theorem 2.8. *For $q \neq 0$, if G is a forest with n vertices and k connected components, then we have*

$$\sigma_q(G) = \frac{(1-q^2)^k}{q^n}.$$

For $q \in \mathbb{Q} \setminus \{0, \pm 1\}$, if G is a graph with n vertices, $n-k$ edges and

$$\sigma_q(G) = \frac{(1-q^2)^k}{q^n},$$

then G is a forest with k connected components.

Proof. The first part follows from Theorem 2.4.

For the second part, we induct over k, where $k = 1$ case is Theorem 2.5.

For the inductive step, assume the result holds for $k-1$ and consider the case for $k \geq 2$. Since G has $n-k \leq n-2$ edges, it will have some connected component T that is a tree. Assume T has m vertices and that $G = T \cup H$ for some graph H. Then, H will have $n-m$ vertices and $(n-k)-(m-1) = (n-m)-(k-1)$ edges. Also, since $\sigma_q(G) = \sigma_q(T)\sigma_q(H)$, we have

$$\sigma_q(H) = \frac{\sigma_q(G)}{\sigma_q(T)} = \frac{(1-q^2)^k}{q^n} \Big/ \frac{1-q^2}{q^m} = \frac{(1-q^2)^{k-1}}{q^{n-m}}.$$

So, H must be a forest with $k-1$ components, and so $G = T \cup H$ is a forest with k components. $\square$

This means that when $q \in \mathbb{Q} \setminus \{0, \pm 1\}$, the q-Laplacian can not only detect a forest, but it can also detect how many connected components it has. In particular, no forest is cospectral to a graph with a cycle or another forest with different number of connected components.

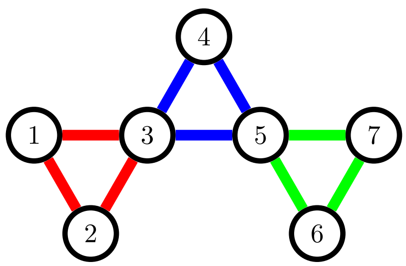
3 $q = \frac{1}{n-1}$ Case and K_n-Decompositions

We have shown that L_q acts nicely with unicyclic graphs and forests. We now discuss how it helps in the case of K_n-decomposable graphs.

For the Signless Laplacian ($q = 1$) a well known result is its relationship with line graphs. One generalization of line graphs can be found by looking at the line graph of a n uniform linear hypergraph. Rather than looking at n uniform linear hypergraphs, we can instead replace each hyperedge with a n-clique and look at the K_n-decomposition. Let G be a simple graph, G is said to have a K_n-decomposition if there exists an edge disjoint set of K_n subsets of G such that every edge of G is within one of those copies of K_n.

For a specific K_n-decomposition of G, we define the K_n-to-vertex incidence matrix M to have entry $M_{i,j} = 1$ if the K_n labeled i contains the vertex labeled j and 0 otherwise.

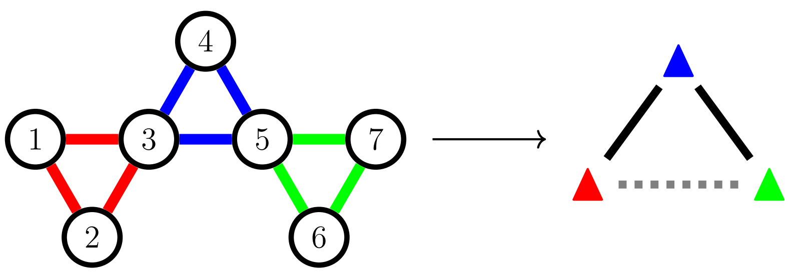
Example 3.1. The graph below is K_3 decomposable.



$$M = \begin{array}{c} \\ {\color{red}\blacktriangle} \\ {\color{blue}\blacktriangle} \\ {\color{green}\blacktriangle} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

We also define a K_n-decomposition graph $D(G)$ for a specific K_n-decomposition of G as follows: We let the nodes of $D(G)$ be the copies of K_n that we partition the edges of G into. If these copies of K_n intersect at some node of G, we add an edge between the corresponding nodes in $D(G)$. This is a generalization of a line graph.

Example 3.2. If we look back at the graph from Theorem 3.1, the following is the corresponding K_3-decomposition graph:



Theorem 3.3. *If G is K_n-decomposable and M is the K_n-to-vertex incidence matrix for some specific K_n-decomposition, then*

$$M^T M = L_{\frac{1}{n-1}}(G)$$

and hence $L_{\frac{1}{n-1}}(G)$ is positive semidefinite.

Furthermore, if $D(G)$ is the K_n-decomposition graph of G for the same decomposition, then

$$MM^T = A(D(G)) + nI.$$

where $A(D(G))$ denotes the adjacency matrix of the K_n-decomposition graph.

Example 3.4. Applying this theorem on Theorem 3.1, we have

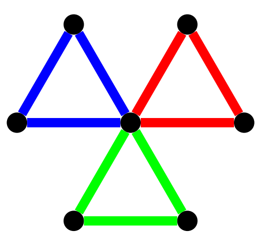
Theorem 3.5. *If a graph G has a K_n-decomposition, then $(\frac{1}{n-1}D + A)(G)$ is positive semi definite.*

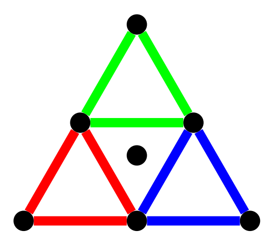
$$
M^TM = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}
= L_{\frac{1}{2}} = \frac{1}{2}D + A
\qquad
MM^T = \begin{array}{c} \\ {\color{red}\blacktriangle} \\ {\color{blue}\blacktriangle} \\ {\color{green}\blacktriangle} \end{array}
\begin{array}{c}
\begin{array}{ccc} {\color{red}\blacktriangle} & {\color{blue}\blacktriangle} & {\color{green}\blacktriangle} \end{array} \\
\begin{bmatrix}
3 & 1 & 0 \\
1 & 3 & 1 \\
0 & 1 & 3
\end{bmatrix}
\end{array}
$$

Proof. Let G be a graph with r vertices with a K_n-decomposition into $F_1, F_2, \ldots, F_k$. Now, define the matrix M to be a $r \times k$ matrix such that $M_{i,j} = 1$ if vertex $j \in F_i$, and 0 otherwise. Now, consider the matrix MM^T. Firstly, note that $(MM^T)_{i,i}$ gives the number of F_j which vertex i is in. Notably, because G is a simple graph, no two F_j's can intersect in more than 1 vertex. Thus, every K_n that a vertex is in contributes $n-1$ to its degree. Thus, $(MM^T)_{i,i} = \frac{d_i}{n-1}$. Additionally, $(MM^T)_{i,j} = 1$ if vertices i and j share a F_ℓ and 0 otherwise. Notably, two vertices are adjacent if and only if they share a F_ℓ. Thus, $MM^T = \frac{1}{n-1}D + A$. Because $\frac{1}{n-1}D + A$ can be factored, it is positive semi definite. $\square$

By a well-known linear algebra fact, the eigenvalues with multiplicity of $MM^T = A(D(G)) + nI$ and $M^TM = L_q(G)$ are the same except for some 0s. We will use this to construct cospectral graphs. If we can find two K_n-decomposable graphs G_1 and G_2 such that $D(G_1)$ and $D(G_2)$ are cospectral with respect to the adjacency matrix, we would obtain that the spectrums of G_1 and G_2 with respect to $L_{\frac{1}{n-1}}$ are the same except for some 0s. And we can make up for this by adding as many isolated vertices as we need, which will each increase the multiplicity of 0 by 1.

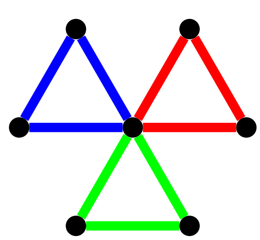
Example 3.6. The following K_3-decomposable graphs G_1 and G_2 have isomorphic (and so cospectral) decomposition graphs $D(G_1) \cong D(G_2) \cong K_3$. That is, each pair of K_3's in the K_3-decomposition has an intersection. Thus, G_1 and G_2 are also cospectral with respect to $L_{1/2}$.





4 $q = 1/2$ Case and Friendship Graphs

A Friendship graph, F_p is the graph with p triangle which meet at a central vertex. F_3 is shown below.



All Friendship graphs will contain $3p$ edges, and $2p+1$ vertices. Additionally, because K_3 decomposable into p triangles all of which share a vertex, $D_3(F_p) = K_p$. Thus, the eigenvalues of F_p

are

$$p + 2^{(1)}, 2^{(p-1)}, 0^{(p+1)}.$$

4.1 Cospectral Constructions

An interest of spectral graph theory is to come up with ways of constructings cospectral graphs. In this subsection, we will use the tools built in the previous section about K_n-decomposable graphs to construct cospectral graphs to the first few friendship graphs.

Lemma 4.1. *If G is a K_3 decomposable graph which is cospectral with a friendship graph F_p, then $D_3(G) = K_p$.*

Proof. Because F_p is cospectral to G, then $D(F_p) = K_p$ is cospectral to $D(K_p)$ under the adjacency matrix. Additionally, there is no graph which is cospectral to any complete graph for the adjacency matrix, so $D_3(G) = K_p$. □

This means that all of the edge disjoin triangles must share a vertex with every other graph.

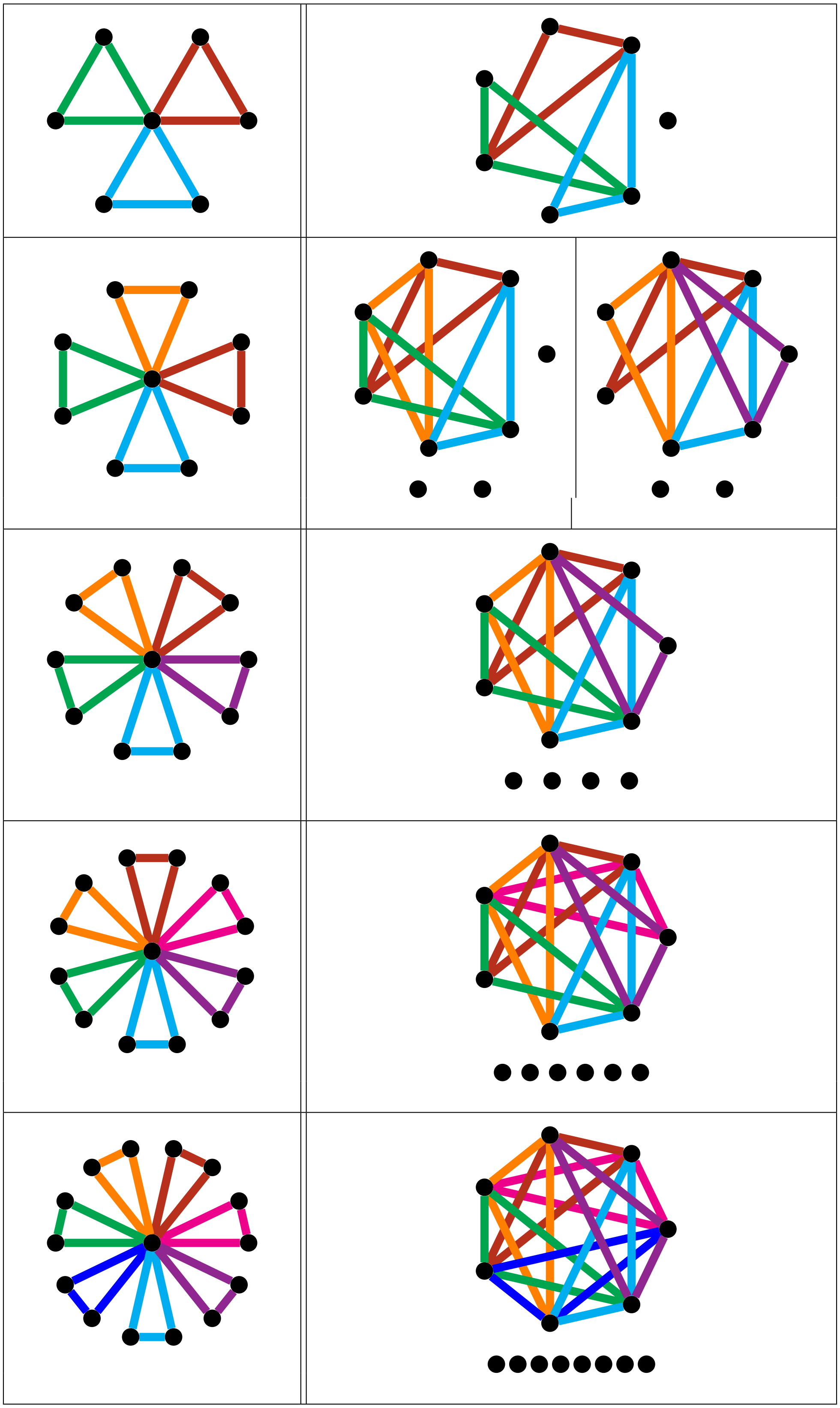
Lemma 4.2. *If G is not a friendship graph, and $D_3(G) = K_p$, then $p \leq 7$.*

Proof. Let G be a K_3 decomposable graph which is not a friendship graph, but $D(G) = K_p$ for $p \geq 8$. Let T be a set of edges that forms on of the triangles in the decomposition of G. Then, $G - T$ is still K_3 decomposable, and $D(G - T) = K_{p-1}$.

Firstly, if $G - T$ is a friendship graph, then T must intersect with the central vertex, or else it cannot intersect with all $p - 1 \geq 7$ triangles, so G is a friendship graph which is a contradiction. Thus, $G - T$ must be a non-friendship graph K_3 decomposable graph. For $p = 8$, then $G - T$ must be K_7 as it is the only graph such that $D_3(H) = K_7$, and $H \neq F_7$ (this can be shown using a simple brute force check). Note that T can only contain one vertex in K_7, and the remaining two vertices must be isolated vertices in F_7. Because no vertex is incident to every triangle, then T cannot be adjacent to every triangle which contradicts that $D_3(G) = K_8$.

Note that for $p > 8$, it must contain a non friendship graph subgraph such that $D_3(H) = K_8$, but there is none, so $p \leq 7$. □

The following table shows all of the K_3 decomposable graphs which are cospectral with a friendship graph. The left column has friendship graphs while the right has all cospectral pairs that we can construct with this method.



This method does produce graphs that are cospectral to the friendship graph, but it only works until the 7-th friendship graph, and can generate only finitely many graphs that are cospectral but not isomorphic to a friendship graph.

While this sounds disappointing at first, in the next subsection we prove that no further examples exist for $p > 7$, and with a complete computer check for $p \leq 7$, we conclude that our construction method in fact produces all possible non-isomorphic cospectral mates of friendship graphs, K_3-decomposable or not. This means that our straightforward but finite method is actually not that bad.

4.2 Classifying All Faux Friendship Graphs

In this chapter, we show that our constructions from the previous subsection are all possible faux friendship graphs.

Proposition 4.3. *Assume G is not isomorphic but cospectral to the p-th friendship graph. Then it must have at least one vertex whose degree is 0 or 1.*

Proof. Assume otherwise that all the vertices of G has degree at least 2. Then, we can express the degrees of G as $2 + a_1, 2 + a_2, \cdots, 2 + a_{2p+1}$ where $a_i \geq 0$.

We know that the sum of the degrees as well as the sum of their squares will be same over cospectral graphs. So, we have

$$\sum(2 + a_i) = (\text{sum of the degrees of the } p\text{-th Friendship Graph}) = 6p$$

and

$$\sum(2 + a_i)^2 = (\text{sum of the squares of the degrees of the } p\text{-th Friendship Graph}) = 4p^2 + 8p.$$

Then, after some simplification, it is easy to obtain $\sum a_i = 2p - 2$ and $\sum a_i^2 = (2p-2)^2$. This means

$$0 = \left(\sum a_i\right)^2 - \sum a_i^2 = \sum_{i \neq j} 2a_i a_j.$$

Since $a_i \geq 0$, this implies at most one a_i can be non-zero. So, we conclude that the degree sequence of G is the same as the friendship graph. Then G has to be the friendship graph.

□

Proposition 4.4. *Assume G is not isomorphic but cospectral to the p-th friendship graph. Then it cannot be connected.*

Proof. Assume for the sake of contradiction that it was connected.

Then, it must have a vertex with degree 1 from the previous result. But the diameter of this graph is 2, so the support of that leaf must have degree 2p.

Let the degrees of the vertices of G be $1 = d_1 \leq d_2 \leq \cdots \leq d_{2p} \leq d_{2p+1} = 2p$.

Then, we know that

$$\sum_{i=1}^{2p+1} d_i = (\text{sum of the degrees of the } p\text{-th Friendship Graph}) = 6p$$

and

$$\sum_{i=1}^{2p+1} d_i^2 = (\text{sum of the squares of the degrees of the } p\text{-th Friendship Graph}) = 4p^2 + 8p.$$

But then, if we subtract the last term in both of these sums, that is d_{2p+1} from the first and d_{2p+1}^2 from the second, we obtain

$$\sum_{i=1}^{2p} d_i = 4p$$

and

$$\sum_{i=1}^{2p} d_i^2 = 8p.$$

We obtained

$$\left(\sum_{i=1}^{2p} d_i\right)^2 = 16p^2 = 2p \cdot \sum_{i=1}^{2p} d_i^2.$$

By Cauchy-Schwarz, this can only be satisfied if $d_1 = d_2 = \cdots = d_{2p} = 2$. This contradicts $d_1 = 1$. □

Now, we will find all faux friendship graphs. Before that, we need to prove the following lemma.

Lemma 4.5. *If G is cospectral to F_p, we have*

$$(A^3)_{i,i} = -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}\sum_{j\sim i} d_j$$

for every vertex i of G.

Proof. Let the halfian of G be H. H has spectrum $\{0^{(p+1)}, 2^{(p-1)}, p+2\}$. So, we have

$$0 = H(H-2I)(H-(p+2)I) = H^3 - (p+4)H^2 + (2p+4)H.$$

Substituting $H = \frac{1}{2}D + A$, we get

$$\begin{aligned}
0 &= H^3 - (p+4)H^2 + (2p+4)H \\
&= \left(\frac{1}{2}D + A\right)^3 - (p+4)\left(\frac{1}{2}D + A\right)^2 + (2p+4)\left(\frac{1}{2}D + A\right) \\
&= \frac{1}{8}D^3 + \frac{1}{4}(D^2A + DAD + AD^2) + \frac{1}{2}(DA^2 + ADA + A^2D) + A^3 \\
&- (p+4)\left(\frac{1}{4}D^2 + DA + AD + A^2\right) + (2p+4)\left(\frac{1}{2}D + A\right).
\end{aligned}$$

We will focus on the (i,i) diagonal entry. The terms colored red have 0s on the diagonal, so we can ignore them. After rearranging the terms, we obtain,

$$
\begin{aligned}
(A^3)_{i,i} &= -\frac{1}{8}(D^3)_{i,i} - \frac{1}{2}(DA^2 + ADA + A^2D)_{i,i} \\
&\quad + (p+4)\left(\frac{1}{4}D^2 + A^2\right)_{i,i} - (2p+4)\left(\frac{1}{2}D\right)_{i,i} \\
&= -\frac{1}{8}d_i^3 - \frac{1}{2}(d_i^2 + (ADA)_{i,i} + d_i^2) \\
&\quad + (p+4)\left(\frac{1}{4}d_i^2 + d_i\right) - (2p+4)\left(\frac{1}{2}d_i\right) \\
&= -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}(ADA)_{i,i} \\
&= -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}\sum_{j\sim i} d_j.
\end{aligned}
$$

□

We are now ready to prove the main theorem of this section.

Theorem 4.6. *If G is cospectral but not isomorphic to F_p, then either $p \leq 6$, or $p = 7$ and $G = K_7$.*

Proof. Assume G is cospectral but not isomorphic to F_p and H is the halfian of G, and $p \geq 6$. We will show that this is impossible.

We have

$$
(A^3)_{i,i} = -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}\sum_{j\sim i} d_j \tag{$\star$}
$$

for every vertex i of G by Theorem 4.5.

Notice that $\sum_{j\sim i} d_j \leq \sum_{j\neq i} d_j = 6p - d_i$, and $(A^3)_{i,i} \leq d_i(d_i-1)$. So, we have

$$
\begin{aligned}
d_i(d_i-1) \geq (A^3)_{i,i} &= -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}\sum_{j\sim i} d_j \\
&\geq -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}(6p - d_i).
\end{aligned}
$$

If $d_i \geq 6$, comparing the LHS and RHS and rearranging gives us

$$
p \leq \frac{d_i^3 + 8d_i^2 - 28d_i}{2d_i^2 - 24} = \frac{1}{2}d_i + 4 + \frac{-16d_i + 96}{2d_i^2 - 24} \leq \frac{1}{2}d_i + 4.
$$

That is, if $d_i \geq 6$, then $d_i \geq 2p - 8$. In other words, either $d_i \leq 4$ or $d_i \geq 2p - 8$. Now, we consider the following cases:

1. There does not exist an i such that $d_i = 2$ or $d_i = 4$.
2. There exists an i such that $d_i = 4$.
3. There does not exist an i such that $d_i = 4$ but there does exist an i such that $d_i = 2$.
   (a) There exist exactly 1 i such that $d_i \geq 2p - 8$.
   (b) There exist exactly 2 distinct i's such that $d_i \geq 2p - 8$.

(c) There exist exactly 3 distinct i's such that $d_i \geq 2p - 8$.

(d) There exist at least 4 distinct i's such that $d_i \geq 2p - 8$.

- Case 1: There does not exist an i such that $d_i = 2$ or $d_i = 4$.

  We know that the total degree of the connected component is $6p$ and there is at least p vertices in that component. So, the average degree is at most 6. If we want every degree to be at least 6, it must be that every degree is exactly 6, and there is exactly p vertices in the connected component. This means the spectrum of the connected component is $\{2^{(p-1)}, p+2\}$, ie. no 0s. This means the connected component has diameter 1, ie. is complete. The only 6-regular complete graph is K_7.

- Case 2: There exists an i such that $d_i = 4$.

  Let d_1, d_2, d_3, d_4 be the degrees of the neighbors of i.

  Then, substituting this i at $(\star)$, we get

  $$4 \cdot 3 \leq (A^3)_{i,i} = -\frac{1}{8}d_i^3 + \frac{p}{4}d_i^2 + 2d_i - \frac{1}{2}(d_1 + d_2 + d_3 + d_4).$$

  This implies

  $$d_1 + d_2 + d_3 + d_4 \geq 8p - 24.$$

  Now, let's estimate the number of edges of G that at least one of these 4 vertices is incident to. See that there are $3p$ edges total, so

  $$\begin{aligned} 3p &\geq (\#\text{ of edges incident to at least 1 of } 1,2,3,4) \\ &= d_1 + d_2 + d_3 + d_4 - (\#\text{ of edges incident to 2 of } 1,2,3,4) \\ &\geq d_1 + d_2 + d_3 + d_4 - \binom{4}{2} \\ &\geq (8p - 24) - \binom{4}{2}. \end{aligned}$$

  Comparing the LHS and RHS, we obtain $p \leq 6$.

  (And equality only if there are 4 vertices with degree exactly $2p - 6 = 6$ each, they are all pairwise adjacent, and every edge in the graph is incident to at least one of these 4 vertices.)

- Case 3: There does not exist an i such that $d_i = 4$ but there does exist an i such that $d_i = 2$.

  Assume there exists an i such that $d_i = 2$, and assume its neighbors are labeled 1 and 2.

  Substituting i in $(\star)$ shows us that if 1 and 2 are adjacent, then $d_1 + d_2 = 2p + 2$ and if they are not, $d_1 + d_2 = 2p + 6$. Now, we start going into the subcases.

  - Subcase 3.(a): There exist exactly 1 i such that $d_i \geq 2p - 8$.

    We have $2p + 2 \leq d_1 + d_2$. But only one of d_1 and d_2 can be greater than 2. But the highest degree a vertex of G can be is $2p - 1$ as G is not connected. So, $d_1 + d_2 \leq (2p - 1) + 2 < 2p + 2$. We get a contradiction.

  - Subcase 3.(b): There exist exactly 2 i's such that $d_i \geq 2p - 8$.

    Let these be vertices 1 and 2.

    We have seen above that it must be that these vertices 1 and 2 with high degree must be the neighbors of any degree 2 vertex. Assume there are m vertices of degree 2. Then our big connected component in G is either $K_2 + mK_1$ or $2K_1 + mK_1$. In other words, either 1 and 2 are adjacent or not.

In the first case, there are $2m + 1 = 3p$ edges. And we also have $(m + 1) + (m + 1) = d_1 + d_2 = 2p + 2$. This implies $p = 1$, and $G \cong K_3 \cong F_1$.

In the second case, there are $2m = 3p$ edges. And we also have $(m) + (m) = d_1 + d_2 = 2p + 6$. This implies $p = 6$, and $G \cong 2K_1 + 3K_1$. Manually checking shows this is not possible.

– Subcase 3.(c): There exist exactly 3 i's such that $d_i \geq 2p - 8$.

Let these be vertices 1, 2, 3.

It must be that

$$\begin{aligned} d_1 + d_2 &\geq 2p + 2 \\ d_2 + d_3 &\geq 2p + 2 \\ d_3 + d_1 &\geq 2p + 2 \end{aligned}$$

and thus $d_1 + d_2 + d_3 \geq 3p + 3$. By an argument similar to that in Case 2, considering the number of edges incident to at least one of $1, 2, 3$ gives

$$3p \geq d_1 + d_2 + d_3 - 3 \geq 3p.$$

Thus, equality is only possible when $1, 2, 3$ are adjacent to each other, and $d_1 = d_2 = d_3 = p + 1$.

By our observation from Case 3.(a), it must be that every vertex of degree 2 must be adjacent to two of 1, 2, 3. Let a, b, c be the number of vertices of degree 2 adjacent to both 2 and 3, to both 1 and 3 and to both 1 and 2 respectively.

This means that $d_1 = b+c+2$, $d_2 = c+a+2$, $d_3 = a+b+2$. Since $d_1 = d_2 = d_3 = p+1$, we obtain $a = b = c = \frac{p-1}{2}$.

We know that there are $\frac{3(p-1)}{2}$ vertices of degree 2 and 3 vertices of degree $p + 1$. We know that the sum of squares of degrees must be $4p^2 + 8p$, which means

$$4p^2 + 8p = \frac{3(p-1)}{2} \cdot 4 + 3 \cdot (p + 1)^2.$$

This leads to $p = 1$ and $p = 6$. $p = 1$ case is the triforce graph, while $p = 6$ is impossible as $(p - 1)/2$ must be an integer.

– Subcase 3.(d): There exist at least 4 i's such that $d_i \geq 2p - 8$.

Let the degrees of these four vertices be d_1, d_2, d_3, d_4. At least two of these will be adjancent to a common degree 2 vertex. Wlog, let these have degrees d_1 and d_2. Then $d_1 + d_2 \geq 2p + 2$.

By a similar argument as Case 2, we have

$$\begin{aligned} 3p &\geq (d_1 + d_2) + d_3 + d_4 - \binom{4}{2} \\ &\geq (2p + 2) + 2(2p - 8) - 6. \end{aligned}$$

This leads to $p \leq 6$.

$\square$

We can do a computer check to find all faux friendship graphs for $p \leq 6$. Turns out, the examples in the previous subsection are all examples.

Thus the $L_{1/2}$-spectrum not only determines whether a graph is cospectral with a friendship graph, but also forces it to share the same K_3-decomposition structure.

This result motivates our questions in the next section about using $L_{1/(n-1)}$-spectrum to reverse-engineer K_n-decomposition information.

There are examples of cospectral pairs where they have two different decomposition structures. There also are examples of cospectral pairs where one of the graphs is K_n-decomposable and the other is not. So, we know that the spectrum cannot detect everything about K_n-decompositions.

But we are optimistic that some conditions on the spectrum can be found that guarantee that every graph with that spectrum has a specific decomposition structure. Our friendship classification result is one example of such a result.

5 The Nash-Williams Conjecture and Future Work

Notably, it is not true that all positive semi definite graphs are K_n decomposable with respect to $L_{\frac{1}{n-1}}$ as for $n = 3$, K_4 is positive semi definite, but does not have a K_3 decomposition. Additionally, there exists a pair of graphs which are cospectral for $q = \frac{1}{2}$, but one of them has a K_3 decomposition, and the other does not. Despite this, this does create another necessary condition for a graph to have a K_n-decomposition. There is an open problem from Nash-Williams that for any graph such that $\delta(G) \geq \frac{3}{4}n$ and G is K_3 divisible, then G has a K_3 decomposition.

Theorem 5.1. *If G is a graph with n vertices, and $\delta(G) \geq \frac{3}{4}n$, then $H(G)$ is positive definite, and has minimum eigenvalue greater than or equal to $\frac{1}{2}$.*

Proof. Let F be an $n \times n$ real, symmetric matrix with $\frac{3}{8}n$ along the diagonal, and row sums $\frac{9}{8}n$. Now, define the matrix $E = J + \frac{n-3}{2}I - F$ where J is the all ones matrix and I is the identity matrix. Then, E has $\frac{1}{8}n - \frac{1}{2}$ along the diagonal, and row sum of $\frac{3}{8}n - \frac{3}{2}$. Note that the eigenvalues of F are $\lambda_1 = \frac{9}{8}n$, and $\lambda_i = \frac{n-3}{2} - \lambda_i(E)$ for $i \neq 1$. Thus, the smallest eigenvalue of F will be $\frac{n-3}{2} - \lambda_2(E)$ where $\lambda_2(E)$ is the second largest eigenvalue of E. The largest eigenvalue of E with eigenvector $\nVdash$ is $\frac{3}{8}n - \frac{3}{2}$. Thus,

$$\lambda_n(F) \geq \frac{n-3}{2} - \left(\frac{3}{8}n - \frac{3}{2}\right) = \frac{1}{8}n.$$

Now, let G be a graph such that $\delta(G) \geq \frac{3}{4}n$. Now, consider the matrix $M = H(G) - F$. Note that the off diagonal entries of $H(G)$ is exactly twice the diagonal entry, and the off diagonal entries of F is exactly twice the diagonal entry. Thus, the off diagonal entry in M is exactly twice the diagonal entry. Additionally, because the diagonal entries of $H(G) \geq \frac{3n}{8}$, then the diagonal entries of M are nonnegative. Therefore, by the Gershgorin circle theorem, the smallest possible eigenvalue of M will be $d_i - 2d_i = -d_i$ where d_i is the diagonal entry row i such that row i has the largest possible diagonal entry. Note that the largest possible diagonal entry of $H(G)$ is $\frac{n-1}{2}$. Thus, the largest possible diagonal entry of M is $\frac{n-1}{2} - \frac{3n}{8} = \frac{n}{8} - \frac{1}{2}$. Therefore, the smallest possible eigenvalue of M is $-\frac{n}{8}+\frac{1}{2}$. Finally, by Weyl's inequality,

$$\begin{aligned}\lambda_{min}(H(G)) &\geq \lambda_{min}(M)+\lambda_{min}(F)\\ &= -\frac{n}{8}+\frac{1}{2}+\frac{n}{8}\\ &= \frac{1}{2}.\end{aligned}$$

□

While this result says nothing about if Nash-Williams's conjecture is true, it does provide evidence towards the conjecture being true. Notably, there are examples of graphs which are K_3 divisible and have a minimum eigenvalue $\frac{1}{2}$ or greater, and are not fractional K_3 decomposable.

Note that this bound is sharp as $\delta(K_4)=3=\frac{3}{4}n$, and the minimum eigenvalue of K_4 is exactly $\frac{1}{2}$, but this bound may be able to be improved in terms of n.

Conjecture 5.1. If G is a graph with n vertices and $\delta(G)\geq\frac{3}{4}n$, then the minimum eigenvalue of $H(G)$ is greater than or equal to $\frac{1}{2}\lfloor\frac{3}{4}n\rfloor+\frac{1}{2}(n \mod 4)$.

This bound has been observed when looking at examples with few vertices. So far this bound has been verified to be strict for $n\leq 15$. Every observed example of a graph we have which is fractionally K_3 decomposable has a minimum eigenvalue far smaller than this bound which leads us to conjecture the following.

Conjecture 5.2. If G is a graph with n vertices and a minimum eigenvalue greater than or equal to $\frac{1}{2}\lfloor\frac{3}{4}n\rfloor+\frac{1}{2}(n \mod 4)$, then G is fractionally K_3 decomposable.

The previous two conjectures together would be stronger than Nash-William's conjecture.

Conjecture 5.3. Every graph with $\delta(G)\geq n-\lfloor\frac{n}{3}\rfloor$, then $H(G)$ is positive semi definite.

It is known that every unbalanced tripartite graph has a negative eigenvalue (this can be shown using the same method as the minimum eigenvalue for a bipartite graph), and the unbalanced tripartite graph with maximal minimum degree would have $\delta(G)=n-1-\lfloor\frac{n}{3}\rfloor$. It is conjectured that this would be optimal.

Similar to how the Signless Laplcian can be related to a line graph, we can relate a K_n-decomposable graph to a "K_n line graph" in which every copy of K_n is replaced with a vertex, and two vertices are adjacent if and only if the copies of K_n shared a vertex.

While we can find some properties for a K_n-decomposable graph, and its resulting K_n line graph, determining if a graph actually is K_n-decomposable is still challenging, but we can determine if a graph is K_n divisible from the eigenvalues for some values of n.

Lemma 5.2. *If $n-1$ is prime, then the characteristic polynomial of $(\frac{1}{n-1}D+A)(G)$ has integer coefficients if and only if every degree of G is divisible by $n-1$.*

Proof. Let G be a graph in which every degree is divisible by $n-1$. Then, $H(G)$ has an integer at every entry, and thus the characteristic polynomial of G has integer coefficients.

Now, let G be a graph with at least one degree not divisible by $n-1$. Then, note that for all i such that d_i is not divisible by $n-1$, then

$$\lambda^k \prod \frac{d_i}{n-1} = \lambda^k \frac{\prod d_i}{(n-1)^\ell}$$

is part of the sum which will go into the minimum polynomial. Additionally, because $n-1$ is prime, $\prod d_i$ is not divisible by $n-1$. Note that any other term which is the coefficient to λ^k must be of the form $\frac{x}{(n-1)^p}$ for some x not divisible by $n-1$, and $p < \ell$. Thus,

$$\frac{x}{(n-1)^p} + \frac{\prod d_i}{(n-1)^\ell}$$

cannot be an integer. Thus, the characteristic polynomial does not have integer coefficients. □

Thus, it follows that

Theorem 5.3. *if $n-1$ is prime, then G is K_n divisible if and only if $\frac{n-1}{2}\sum \lambda_i$ is divisible by n, and $\prod \lambda - \lambda_i$ is a polynomial with integer coefficients.*

Proof. This result directly follows from the definition of K_n divisible, the previous lemma, and the sum of the eigenvalues. □

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References

[1] Steve Butler et al. "Spectral faux trees". In: *Involve, a Journal of Mathematics* 17.4 (Oct. 2024), pp. 651–668. DOI: 10.2140/involve.2024.17.651.