On the Eigenvalues of the q-Laplacian Matrix of a Graph

Matt Burnham[1] Steve Butler[1] Mitchell Johnson[2] Deniz Tanaci[3]

[1]Iowa State University [2]Hamilton College [3]Pomona College

The q-Laplacian

The **q-Laplacian** of a simple graph G is

$$L_q(G) = qD(G) + A(G)$$

where

- q - fixed real number,
- $D(G)$ - diagonal degree matrix, and
- $A(G)$ - adjacency matrix.

Two graphs are **cospectral** with respect to the q-Laplacian if they have the same eigenvalues with multiplicity.

Cycles and Trees

Assume $q \neq 0$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the (possibly repeated) eigenvalues of $L_q(G)$.

Theorem 1. *Assume $n \geq 3$. If G is a cycle, then*

$$\prod_{i=1}^{n} \left(\lambda_i + \frac{1-q^2}{q} \right) = q^n + \frac{1}{q^n} + 2(-1)^{n+1}.$$

Theorem 2. *Assume $q \in \mathbb{Q} \setminus \{0, \pm 1\}$. Then G is a forest with k trees if and only if*

$$\prod_{i=1}^{n} \left(\lambda_i + \frac{1-q^2}{q} \right) = \frac{(1-q^2)^k}{q^n}$$

and

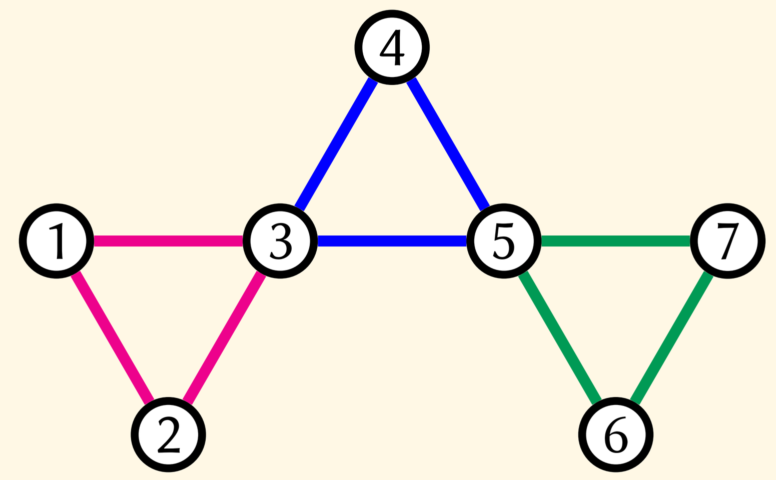
$$\sum_{i=1}^{n} \lambda_i = 2q(n-k).$$

In particular, no tree is cospectral with a graph which is not a tree for $q \in \mathbb{Q} \setminus \{0, \pm 1\}$.

K_n-Decomposable Graphs

- A graph is **K_n-decomposable** if its edges can be partitioned into edge-disjoint copies of K_n, labeled $1, \ldots, \ell$.
- The **K_n-incidence matrix** M has entry $M_{i,j} = 1$ if the K_n labeled i contains the vertex labeled j and 0 otherwise.

Example. The graph below is K_3-decomposable.



$$M = \begin{array}{c} \\ \blacktriangle \\ \blacktriangle \\ \blacktriangle \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- A **K_n-line graph** $\Gamma(G)$ has the K_n subgraphs from a decomposition as the vertices and two K_n's are adjacent if they share a vertex.

Theorem 3. *If M is the K_n-incidence matrix for a K_n-decomposition of G, then*

$$M^T M = L_{\frac{1}{n-1}}(G).$$

In particular, $L_{\frac{1}{n-1}}(G)$ is positive semidefinite.

Furthermore,

$$MM^T = A(\Gamma(G)) + nI,$$

where $A(\Gamma(G))$ is the adjacency matrix of the corresponding K_n-line graph.

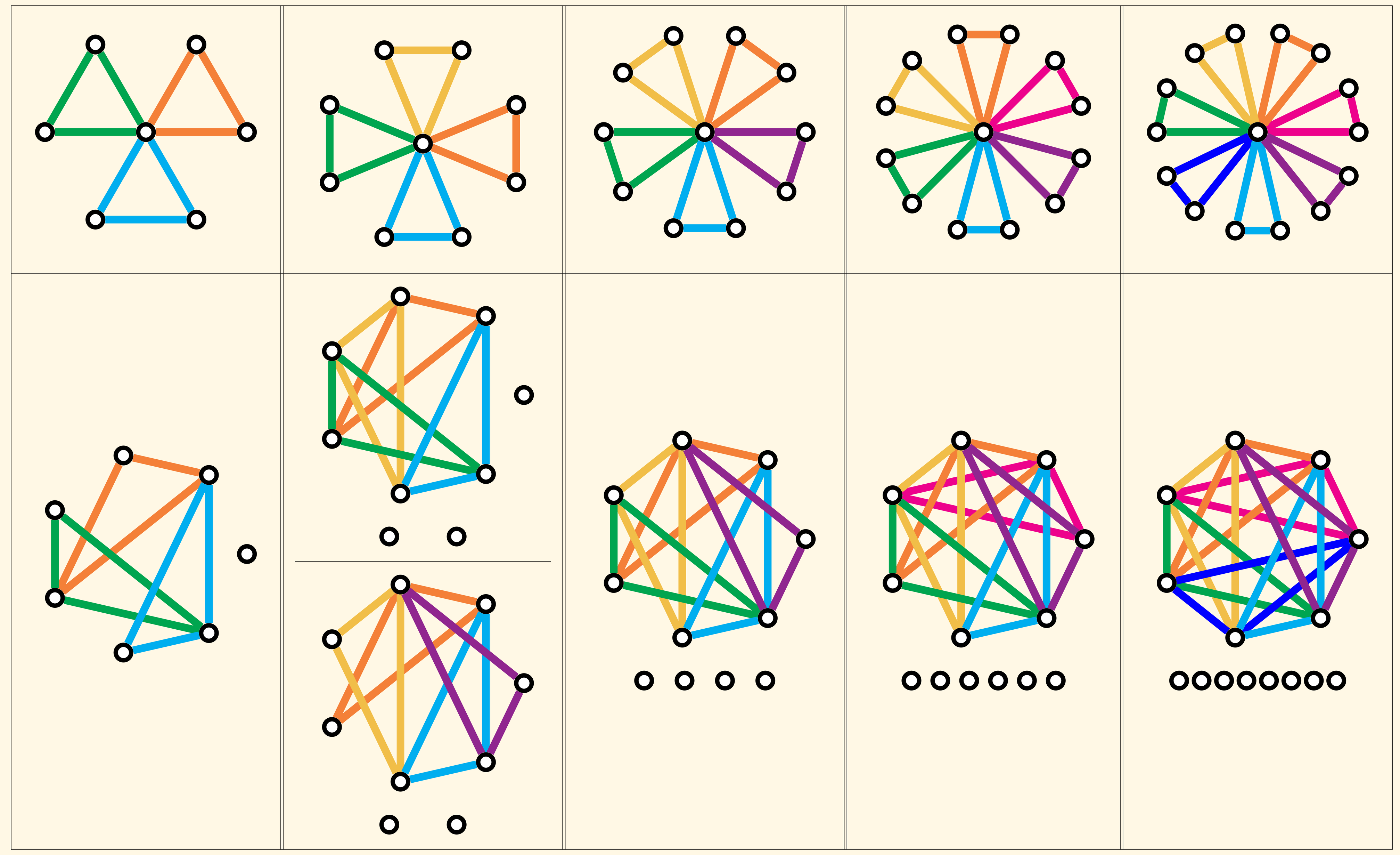
Example. Applying this on the previous example, we obtain the following:

$$M^T M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad MM^T = \begin{array}{c} \\ \blacktriangle \\ \blacktriangle \\ \blacktriangle \end{array} \begin{array}{c} \begin{matrix} \blacktriangle & \blacktriangle & \blacktriangle \end{matrix} \\ \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \end{array}$$

$q = 1/2$ and Friendship Graphs

By the connection between $L_{\frac{1}{n-1}}$ and $A(\Gamma)$, looking at cospectral pairs with respect to $A(\Gamma)$ allows us to construct cospectral pairs with respect to $L_{\frac{1}{n-1}}$.

Example. Each of the following columns have graphs with isomorphic K_n-line graphs. So, graphs in each column are also cospectral themselves with respect to $L_{1/2}$.



The graph composed of p copies of K_3 that all intersect at one central vertex is called the p-th friendship graph, denoted F_p. The top row consists of the graphs $F_3, \ldots, F_7$.

It turns out that these are the only examples of graphs cospectral to F_p.

Theorem 4. *If graph G is cospectral but not isomorphic to F_p, then G is composed of p copies of K_3 that pairwise intersect. In particular, $3 \leq p \leq 7$.*

Future Directions

- What other structures can we detect with the spectrum of L_q?
- Is there a condition on the spectrum that implies the graph is K_n-decomposable?

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