Isomorphism between inner product spaces of $\mathbb{R}^n$

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Introduction

In usual Linear Algebra course it is not rare task, to calculate coordinates of some vector y relative to basis $U = \{u_1, u_2, ...\}$. If the U is orthogonal basis of $\mathbb{R}^n$, then the formula for calculating coordinates are computationally simple:

$$([y]_U)_i = \frac{y \cdot u_i}{u_i \cdot u_i}.$$

Where, $([y]_U)_i$ is i-th coordinate of y relative to basis U.

However, if the basis is not orthogonal, it is required to solve:

$$[y]_U = U^{-1}y.$$

Which is unpleasant task to do manually, can't we use another inner product space where the U becomes orthogonal basis and use our simple orthogonal projection formula? In fact we can do it and here is some motivating example.

Motivating Example

Let $V = \mathbb{R}^2$, and take basis vectors:

$$u_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This basis is **not orthogonal** with respect to the standard inner product. So, we redefine the inner product as:

$$\langle u, v \rangle_U := u_1 v_1 + 2u_2 v_2.$$

It is easy to check that it is indeed inner product, so it is left to reader. And in this inner product space we have:

$$\langle u_1, u_2 \rangle_U = (-1)(2) + 2(1)(1) = -2 + 2 = 0,$$

so $u_1 \perp_U u_2$.

Now, for any vector $y \in V$, we can compute its coordinates in the basis $U = \{u_1, u_2\}$ using:

$$[y]_U = \left(\frac{\langle y, u_1 \rangle_U}{\langle u_1, u_1 \rangle_U}, \frac{\langle y, u_2 \rangle_U}{\langle u_2, u_2 \rangle_U} \right).$$

Proof:

Let

$$y = \begin{pmatrix} a \\ b \end{pmatrix}, \quad C_1 = \frac{\langle y, u_1 \rangle_U}{\langle u_1, u_1 \rangle_U}, \quad C_2 = \frac{\langle y, u_2 \rangle_U}{\langle u_2, u_2 \rangle_U}.$$

We compute:

$$\langle u_1, u_1 \rangle_U = (-1)^2 + 2(1)^2 = 3, \quad \langle y, u_1 \rangle_U = (-1)(a) + 2(b) = -a + 2b,$$
$$\Rightarrow C_1 = \frac{2b - a}{3}.$$

Also,

$$\langle u_2, u_2 \rangle_U = 2^2 + 2(1)^2 = 4 + 2 = 6, \quad \langle y, u_2 \rangle_U = 2a + 2b = 2a + 2b,$$
$$\Rightarrow C_2 = \frac{2a + 2b}{6} = \frac{a + b}{3}.$$

Now verify the decomposition:

$$U[y]_U = C_1 u_1 + C_2 u_2 = \frac{-a + 2b}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{a + b}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

So C_1, C_2 are indeed the coordinates of y relative to basis U. □

After motivating example, it is tempting to generalize further, to make any basis of $\mathbb{R}^n$ orthogonal. So we proceed with generalization.

Orthogonalization via Inner Product Construction

Suppose we have a basis $u_1, \ldots, u_n \in \mathbb{R}^n$. Is it possible to define an inner product such that $u_1, \ldots, u_n$ become orthogonal? Apparently, yes — with the following construction:

Define:

$$\langle x, y \rangle := (A^{-1}x) \cdot (A^{-1}y) = (A^{-1}x)^T (A^{-1}y) = x(A^{-1})^T (A^{-1})y, \quad \text{where } A = [u_1 \ \cdots \ u_n].$$

We verify the four inner product axioms:

1. **Symmetry:**
$$\langle x, y \rangle = (A^{-1}x)^T (A^{-1}y) = (A^{-1}y)^T (A^{-1}x) = \langle y, x \rangle.$$

2. **Linearity in the first argument:**
$$\langle u + v, w \rangle = (A^{-1}(u + v))^T A^{-1}w = (A^{-1}u + A^{-1}v)^T A^{-1}w = \langle u, w \rangle + \langle v, w \rangle.$$

3. **Homogeneity:**
$$\langle cu, w \rangle = (A^{-1}(cu))^T A^{-1}w = c(A^{-1}u)^T A^{-1}w = c\langle u, w \rangle.$$

4. **Positive Definiteness:**
$$\langle x, x \rangle = (A^{-1}x)^T (A^{-1}x) = \|A^{-1}x\|^2 > 0 \quad \text{for } x \neq 0.$$

   Since A is invertible, $A^{-1}x = 0 \iff x = 0$.

Now we proceed that under such construction basis U becomes orthogonal and orthogonal decomposition in such inner product space works.

Orthogonality of Basis

To show that $u_i \perp u_j$ for $i \neq j$, compute:

$$\langle u_i, u_j \rangle = (A^{-1}u_i)^T(A^{-1}u_j) = e_i^T e_j = 0.$$

This holds since $A^{-1}u_i = e_i$, the standard basis vector.

Orthogonal Decomposition

Let $y \in \mathbb{R}^n$. Then we can decompose:

$$y = \sum_{i=1}^{n} \frac{\langle y, u_i \rangle}{\langle u_i, u_i \rangle} u_i.$$

We verify:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\langle y, u_i \rangle}{\langle u_i, u_i \rangle} u_i &= \sum_{i=1}^{n} \left(\frac{(A^{-1}y)^T e_i}{e_i^T e_i} \right) u_i \\
&= \sum_{i=1}^{n} ((A^{-1}y)^T e_i) u_i \\
&= A(A^{-1}y) = y.
\end{aligned}$$

Hence, even though the basis is not orthogonal under the standard dot product, we can define an inner product under which it is orthogonal and still retain decomposition. □

Our construction gave us huge class of inner products. Natural question to ask, whether all inner product of $\mathbb{R}^n$ have such form or not? Interestingly all inner products are actually of such form. And here is proof.

Isomorphism between inner product spaces of $\mathbb{R}^n$

Let $U = [u_1, \dots, u_n]$, where $u_1, \dots, u_n$ is an orthonormal basis with respect to the inner product $\langle \cdot, \cdot \rangle$. Such a basis exists via Gram–Schmidt orthonormalization process. It is classical result, so proof omitted.

Let $x, y \in \mathbb{R}^n$. Then, since $u_1, \dots, u_n$ is a basis,

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n, \quad y = \beta_1 u_1 + \cdots + \beta_n u_n, \quad \text{for some } \alpha_i, \beta_i \in \mathbb{R}.$$

Then:

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \beta_j u_j \right\rangle \\
&= \sum_{i,j} \alpha_i \beta_j \langle u_i, u_j \rangle \\
&= \sum_{i=1}^{n} \alpha_i \beta_i \quad (\text{since } u_i \perp u_j \text{ for } i \neq j, \text{ and } \|u_i\| = 1).
\end{aligned}$$

On the other hand, since $U^{-1}x = (\alpha_1, \ldots, \alpha_n)^T$, and $U^{-1}y = (\beta_1, \ldots, \beta_n)^T$, we have:

$$(U^{-1}x)^T(U^{-1}y) = \sum_{i=1}^{n} \alpha_i \beta_i.$$

Therefore,

$$\langle x, y \rangle = (U^{-1}x)^T(U^{-1}y).$$

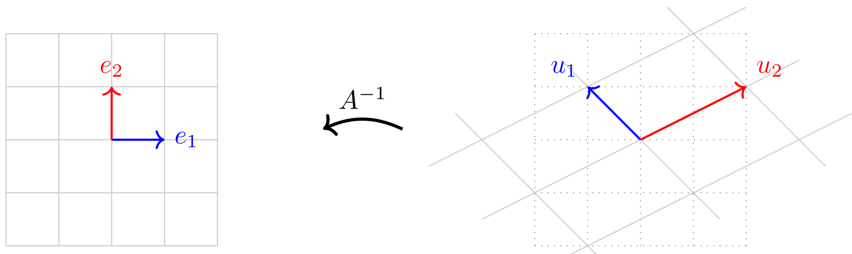
□

Intuitive interpretation of this result is that – any inner product space $\mathbb{R}^n$ is isomorphic to standard Euclidean space, under a change of basis. The inner product is essentially the dot product in transformed coordinates. Such visualization can aid intuitive understanding:

As before, let

$$u_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad A = [\, u_1 \; u_2 \,].$$

Under the new inner product $\langle x, y \rangle = (A^{-1}x)^\top (A^{-1}y)$, the skewed axes u_1, u_2 become orthonormal. Here's how the integer grid in the standard coordinates $\{(i, j) : i, j \in \mathbb{Z}\}$ gets mapped by A:



Standard integer grid

Skewed integer grid

Under this mapping, each square cell becomes a parallelogram spanned by u_1 and u_2. In the new inner product, those axes behave exactly like standard orthonormal axes under the dot product.