Principles of Uncertainty

Homework 01

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1 1.12

1.1 (b)

Problem: Show that the standard measurable space on $\Omega = \{0, 1, ..., \infty\}$ equipped with Poisson measure is a discrete probability space.

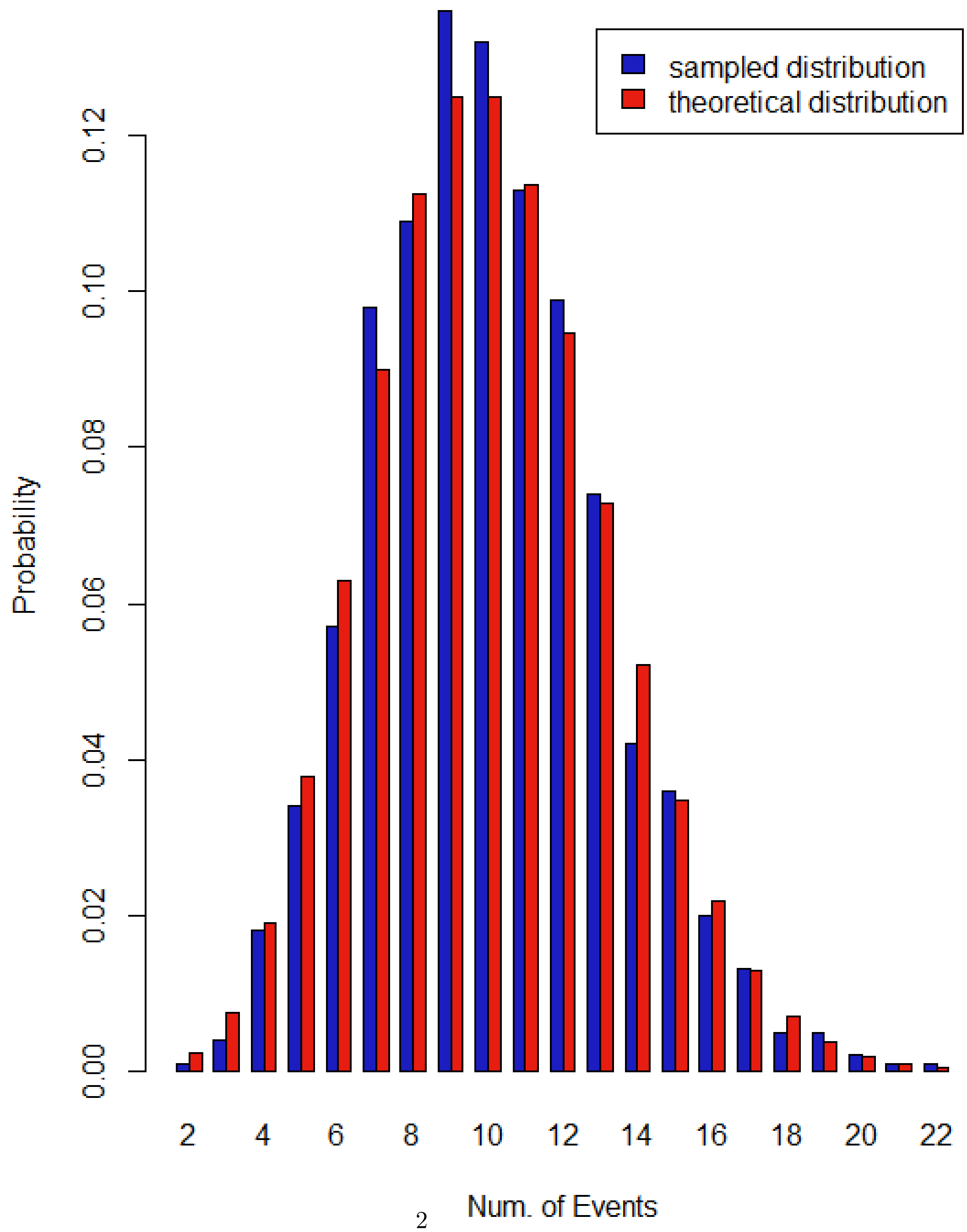
Solution:

$$P(\cup_{k=1}^{\infty} k) = \Sigma_{k=1}^{\infty} P(k) = \Sigma_{k=1}^{\infty} \lambda^k e^{-\lambda}/k! = e^{-\lambda} \Sigma_{k=1}^{\infty} \lambda^k/k!$$

The macLauren expansion of $e^{\lambda} = \Sigma_{k=0}^{\infty} \lambda^k/k!$, therefore

$$P(\cup_{k=1}^{\infty} k) = e^{-\lambda} \Sigma_{k=1}^{\infty} \lambda^k/k! = e^{-\lambda} e^{\lambda} = 1$$

1.2 (d)


sampled distribution
theoretical distribution
Probability
0.00
0.02
0.04
0.06
0.08
0.10
0.12
2
4
6
8
10
12
14
16
18
20
22
Num. of Events

2 2.3

Problem: Let C and D be two collections of subsets on σ such that $C \subset D$. Prove that $\sigma(C) \subseteq \sigma(D)$.

Solution: By definition $\sigma(C)$ is the smallest σ-algebra that contains C. Corollary:

1. $\sigma(C) = \cap_i F_i$, for all σ-algebras F_i that contain C.
2. $D \subset \sigma(D)$

Transitively, we know that C is also a subset of $\sigma(D)$, therefore

$$\sigma(C) = (\cap_i F_i - \sigma(D)) \cap \sigma(D) \subseteq \sigma(D)$$

3 2.6

Problem: Show that the Lebesgue measure of rational numbers on [0,1] is 0.

Solution: The Lebesque measure λ is defined on the borel set $\mathbb{B}_{(0,1]}$. In the previous exercises we have proven that the σ-algebra of $\mathbb{B}_{(0,1]}$ contains all singletons. For any singleton b the Lebesque measure $\lambda(\{b\}) = 0$ We can prove this by using the quality

$$\lambda((a, b]) = \lambda((a, b))$$

By countable additivity

$$\lambda((a, b]) = \lambda((a, b) \cup \{b\}) = \lambda((a, b)) + \lambda(\{b\})\lambda(\{b\}) = \lambda((a, b]) - \lambda((a, b)) = 0$$

$$\lambda(\{b\}) = \lambda((a, b]) - \lambda((a, b)) = 0$$

This proves that λ of a singleton is zero. Considering that the set of all rational elements is a union of rational elements it stand follows from countable additivity that the union of all elements is zero.

$$\lambda(\cup_{q_i \in \mathbb{Q}} q_i) = \Sigma_{q_i \in \mathbb{Q}} \lambda(q_i) = \Sigma_{q_i \in \mathbb{Q}} 0 = 0$$

where q_i is a rational singleton, therefore $\lambda(q_i) = 0$

R code:
(Comment: It is impossible to generate irrational numbers on a computer. As all floats are represented as finite number that can always be represented as rational numbers.)