Mathematics 2, Part 4

Homework 1

Polona Oblak

The homework consists of three problems. The solutions are to be submitted to the appropriate mailbox on ucilnica before the exam, but preferrably in a week. The solutions should contain a clear and well explained proofs, procedures, explanations, etc.

(1) Let $p_1(x) = x^2 - 1$, $p_2(x) = x^2 + x + 1$ and $p_3(x) = x^2 + x$.
   (a) Prove that $\{p_1, p_2, p_3\}$ form a basis for $\mathbb{R}_2[x]$.
   (b) Find the dual basis for $\{p_1, p_2, p_3\}$.

(2) Let $f_1(p) = \int_{-1}^{1} p(x)\,dx$, $f_2(p) = \int_0^1 p(x)\,dx$ and $f_3(p) = \int_0^2 p(x)\,dx$.
   (a) Prove that $\{f_1, f_2, f_3\}$ form a basis for $(\mathbb{R}_2[x])^*$.
   (b) Find the basis $\mathcal{B} = \{p_1, p_2, p_3\}$ for $\mathbb{R}_2[x]$ such that $\mathcal{B}^* = \{f_1, f_2, f_3\}$.

(3) For $p, q \in \mathbb{R}_3[x]$ let us define

$$g(p,q) = \int_{-1}^{1} p(x)q(x)\,dx \quad \text{and} \quad h(p,q) = \int_0^1 p(x)q(x)\,dx.$$

   (a) Prove g and h are inner products on $\mathbb{R}_3[x]$. (You can use a word 'similarly' for the second one.)
   (b) Find polynomials $p, q \in \mathbb{R}_3[x]$ that are orthogonal in g but not in h.
   (c) Let $V = \mathcal{L}\{1, x, x^2\}$. Find the orthonormal basis for V with respect to g.
   (d) Find the orthogonal complement $V^\perp$ with respect to g.