Ring Pattern Matching Theory

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1 Sequences

Let **Elt** be the set of (concrete) elements, **EV** be the set of element variables and **SV** be the set of sequence variables.

Definition 1 (Sequence Patterns). *The set* **SP** *of sequence patterns is inductively defined as follows:*

1. $\varepsilon \in \mathbf{SP}$ *(the empty sequence);*
2. *For each element* $e \in \mathbf{Elt}$, $e \in \mathbf{SP}$*;*
3. *For each element variable* $E \in \mathbf{EV}$, $E \in \mathbf{SP}$*;*
4. *For each sequence variable* $S \in \mathbf{SV}$, $S \in \mathbf{SP}$*;*
5. *For any sequence patterns* $SP_1, SP_2 \in \mathbf{SP}$, $SP_1\ SP_2 \in \mathbf{SP}$.

The binary juxtaposition operator used in $SP_1\ SP_2$ *is associative, namely that* $(SP_1\ SP_2)\ SP_3 = SP_1\ (SP_2\ SP_3)$ *for any sequence patterns* $SP_1, SP_2, SP_3 \in \mathbf{SP}$. ε *is an identity of the binary juxtaposition operator, namely that* $\varepsilon\ SP = SP$ *and* $SP\ \varepsilon = SP$ *for any sequence patterns* $SP \in \mathbf{SP}$.

Sequence patterns that do not have any variables at all are called sequences. Let $\mathbf{Seq} \subseteq \mathbf{SP}$ be the set of all sequences.

A substitution σ is a function from the disjoint union $\mathbf{EV} \uplus \mathbf{SV}$ of **EV** and **SV** to the disjoint union $\mathbf{Seq} \uplus \mathbf{EV} \uplus \mathbf{SV}$ of **Seq**, **EV** and **SV**. For $E \in \mathbf{EV}$, $\sigma(E)$ is an element $e \in \mathbf{Elt}$ or E and for $S \in \mathbf{SV}$, $\sigma(S)$ is a sequence $seq \in \mathbf{Seq}$ or S. The domain of a substitution σ can be naturally extended to **SP** such that $\sigma(\varepsilon)$ is ε, for an element $e \in \mathbf{Elt}$, $\sigma(e)$ is e and for a sequence pattern $SP_1, SP_2 \in \mathbf{SP}$, $\sigma(SP_1\ SP_2)$ is $\sigma(SP_1)\ \sigma(SP_2)$.

Definition 2 (Sequence pattern match). *Pattern match between* $sp \in \mathbf{SP}$ *&* $seq \in \mathbf{Seq}$ *is to find all substitutions* σ *such that* $\sigma(sp) = seq$*. Let* $sp =?= seq$ *be the set of all such substitutions.*

Elements, element variables and sequence variables used in sequence patterns are called components in the sequence patterns. For $sp \in \mathbf{SP}$, let $|sp|$ be the number of components in it. $sp \in \mathbf{SP}$ can be in the form $ES_1\ ES_2\ \ldots\ ES_{|sp|}$, where each ES_i is an element, an element variable or a sequence variable. For $sp \in \mathbf{SP}$, let $sp(i)$, where $i \in \{1, 2, \ldots, |sp|\}$, be the ith element ES_i

in sp. Let $e, e_1, e_2, \ldots \in \mathbf{Elt}$, $E, E_1, E_2, \ldots \in \mathbf{EV}$, $S, S_1, S_2, \ldots \in \mathbf{SV}$ and $ES, ES_1, ES_2, \ldots \in \mathbf{Elt} \uplus \mathbf{EV} \uplus \mathbf{SV}$. A binary construct $\mathrm{sv}(S, I)$ that is not in $\mathbf{SV}$, where S is a sequence variable and I is either 0 or 1, is used as an extra sequence variable. Let $\mathbf{SSV}$ be $\{\mathrm{sv}(S, I) \mid S \in \mathbf{SV}, I \in \{0, 1\}\}$. $\mathbf{SV} \cup \mathbf{SSV}$ may be used as the set of sequence variables instead of $\mathbf{SV}$.

Definition 3 (Split sequence patterns). *For $sp \in \mathbf{SP}$, $\mathrm{split}(sp)$ is a sequence pattern such that each sequence variable S in sp is replaced with $\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$. $\mathrm{split}(\varepsilon) = \varepsilon$, $\mathrm{split}(e) = e$ for $e \in \mathbf{Elt}$, $\mathrm{split}(E) = E$ for $E \in \mathbf{EV}$, $\mathrm{split}(S) = \mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$ for $S \in \mathbf{SV}$ and $\mathrm{split}(SP_1\ SP_2) = \mathrm{split}(SP_1)\ \mathrm{split}(SP_2)$ for $SP_1, SP_2 \in \mathbf{SP}$.*

Definition 4 (Joining split sequence variables). *For $sp \in \mathbf{SP}$ and $seq \in \mathbf{Seq}$, let σ be in $(\mathrm{split}(sp) =?= seq)$. $\mathrm{join}(\sigma)$ is the substitution σ' such that for each sequence variable S in sp $\sigma'(S) = \sigma(\mathrm{sv}(S, 0))\ \sigma(\mathrm{sv}(S, 1))$ and for any other variables X $\sigma'(X) = \sigma(X)$. The domain of join can be naturally extended to the set of substitutions such that $\mathrm{join}(\mathrm{split}(sp) =?= seq)$ is $\{\mathrm{join}(\sigma) \mid \sigma \in (\mathrm{split}(sp) =?= seq)\}$.*

Proposition 1. *For any sequence pattern $sp \in \mathbf{SP}$, any sequence $seq \in \mathbf{Seq}$ and any substitution $\sigma \in (\mathrm{split}(sp) =?= seq)$, $\mathrm{join}(\sigma)(sp) = seq$.*

Proof. *Let sp be $ES_1 \ldots ES_i \ldots ES_n$. $\mathrm{split}(sp)$ is $\mathrm{split}(ES_1) \ldots \mathrm{split}(ES_i) \ldots \mathrm{split}(ES_n)$. Then, $\sigma(\mathrm{split}(sp))$ is $\sigma(\mathrm{split}(ES_1)) \ldots \sigma(\mathrm{split}(ES_i)) \ldots \sigma(\mathrm{split}(ES_n))$. If ES_i is an element e, $\sigma(\mathrm{split}(ES_i)) = e = \mathrm{join}(\sigma)(ES_i)$. If ES_i is an element variable E, $\sigma(\mathrm{split}(ES_i)) = \sigma(E) = \mathrm{join}(\sigma)(ES_i)$. If ES_i is a sequence variable S, $\sigma(\mathrm{split}(ES_i)) = \sigma(\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)) = \mathrm{join}(\sigma)(ES_i)$. Therefore, $\sigma(\mathrm{split}(sp)) = \mathrm{join}(\sigma)(sp)$ and thus $\mathrm{join}(\sigma)(sp) = seq$.* □

Lemma 1. *For any sequence pattern $sp \in \mathbf{SP}$ and any sequence $seq \in \mathbf{Seq}$, $(sp =?= seq) = \mathrm{join}(\mathrm{split}(sp) =?= seq)$.*

Proof. *Let $\sigma \in (sp =?= seq)$. Let σ' be the substitution such that $\sigma'(\mathrm{sv}(S, 0)) = \sigma(S)$ and $\sigma'(\mathrm{sv}(S, 1)) = \varepsilon$ for each sequence variable S in sp and $\sigma'(X) = \sigma(X)$ for any other variables X. By the construction of σ', $\sigma'(\mathrm{split}(sp)) = seq$ and then $\sigma' \in (\mathrm{split}(sp) =?= seq)$. Moreover, $\mathrm{join}(\sigma') = \sigma$ and hence $\sigma \in \mathrm{join}(\mathrm{split}(sp) =?= seq)$. From Definition 4, $\mathrm{join}(\mathrm{split}(sp) =?= seq)$ is $\{\mathrm{join}(\sigma) \mid \sigma \in (\mathrm{split}(sp) =?= seq)\}$. From Proposition 1, $\mathrm{join}(\sigma)(sp) = seq$. Hence, $\mathrm{join}(\sigma) \in (sp =?= seq)$.* □

rtt (that stands for rotate) takes a sequence pattern sp and returns the sequence pattern obtained by rotating sp clockwise. rev (that stands for reverse) takes a sequence pattern sp and returns the sequence pattern obtained by reversing sp. Let sp be $ES_1\ ES_2 \ldots ES_{|sp|-1}\ ES_{|sp|}$. $\mathrm{rtt}(sp) = ES_{|sp|}\ ES_1\ ES_2 \ldots ES_{|sp|-1}$ and $\mathrm{rev}(sp) = ES_{|sp|}\ ES_{|sp|-1} \ldots ES_2\ ES_1$. Let us suppose that a subscript exp of ES_{exp} used as an element in sp is interpreted as $(exp \bmod |sp|) + 1$.

Definition 5 (Reversing substitutions). *σ_{rev} is defined as follows: for an element $e \in \mathbf{Elt}$ $\sigma_{\mathrm{rev}}(e) = \sigma(e) = e$, for an element variable $E \in (E)$ $\sigma_{\mathrm{rev}}(E) = \sigma(E)$ and for a sequence variable $S \in (SV)$ $\sigma_{\mathrm{rev}}(S) = \mathrm{rev}(\sigma(S))$.*

Proposition 2. *For any substitution σ,* $(\mathrm{join}(\sigma_{\mathrm{rev}}))_{\mathrm{rev}} = \mathrm{join}(\sigma)$.

Proof. For each sequence variable S $(\mathrm{join}(\sigma_{\mathrm{rev}}))_{\mathrm{rev}}(S) = \mathrm{rev}((\mathrm{join}(\sigma_{\mathrm{rev}}))(S)) = \mathrm{rev}(\sigma(\mathrm{sv}(S, 0)))\ \sigma(\mathrm{sv}(S, 1))$ *and for any other variables X $\sigma'(X) = \sigma(X)$.* □

Given two sequence patterns $sp, sp' \in \mathbf{SP}$, $spsp'$ holds if there exists a natural number n such that $sp = \mathrm{rtt}^n(sp')$, namely that sp is obtained by rotating sp' finitely many times; $spsp'$ holds if there exists a natural number n such that $sp = \mathrm{rtt}^n(\mathrm{rev}(sp'))$, namely that sp is obtained by reversing sp' once and rotating it finitely many times. For example, let sp and sp' be $e_1\ S_1\ e_2$ and $e_1\ e_2\ S_1$ $(= \mathrm{rtt}(sp))$ and then $(spsp')$ holds, while $(spsp')$ does not; let sp and sp' be $e_1\ S_1\ e_2$ and $e_2\ e_1\ S_1$ $(= \mathrm{rtt}(\mathrm{rev}(sp)))$ and then $(spsp')$ does not, while $(spsp')$ holds. $e_1\ S_1\ e_2$ and $e_2\ e_1\ S_1$ $(= \mathrm{rtt}(\mathrm{rev}(sp)))$ and then both $(spsp')$ and $(spsp')$ hold.

Definition 6 (Sequence pattern contexts). *A sequence pattern context is a sequence pattern $sp \in \mathbf{SP}$ in which one component (say, ith component, where $1 \leq i \leq |sp|$) is replaced with a special symbol called a hole, denoted $sp_{(i)}\{\}$. A hole is treated as an element. Let sp be $ES_1 \ldots ES_i \ldots ES_{|sp|}$ and then $sp_{(i)}\{\}$ is $ES_1 \ldots \ldots ES_{|sp|}$.*

For a sequence pattern or a sequence pattern context spc and a sequence pattern sp, $spc_{(i)}\{sp\}$ is spc in which the ith component in spc is replaced with sp. $(sp_{(i)}\{\})_{(i)}\{sp(i)\} = sp_{(i)}\{sp(i)\} = sp$.

Definition 7 (Correspondent components). *Let $sp \in \mathbf{SP}$ be $ES_1 \ldots ES_{i-1}$ ES_i ES_{i+1} $\ldots ES_{|sp|}$, where $1 \leq i \leq |sp|$ and $sp' \in \mathbf{SP}$ be $ES'_1 \ldots ES'_{j-1}$ ES'_j $ES_{j+1} \ldots ES'_{|sp'|}$, where $1 \leq j \leq |sp'|$. If $ES_i\ ES_{i+1} \ldots ES_{|sp|}\ ES_1 \ldots ES_{i-1} = ES'_j\ ES'_{j+1} \ldots ES'_{|sp'|}\ ES'_1 \ldots ES'_{j-1}$ or $ES_i\ ES_{i+1} \ldots ES_{|sp|}\ ES_1 \ldots ES_{i-1} = ES'_j\ ES'_{j-1} \ldots ES'_1\ ES'_{|sp'|} \ldots ES'_{j+1}$, then ES'_j is the corresponding component in sp' to ES_i in sp.*

Proposition 3. *For any sequence patterns $sp, sp' \in \mathbf{SP}$ and any natural numbers $i \in \{1, \ldots, |sp|\}$ and $j \in \{1, \ldots, |sp'|\}$ such that the jth component $sp'(j)$ in sp' is the corresponding component in sp' to $sp(i)$ in sp, (1) $(spsp') \Leftrightarrow sp_{(i)}\{\}sp'_{(j)}\{\}$ and (2) $(spsp') \Leftrightarrow sp_{(i)}\{\}sp'_{(j)}\{\}$.*

Proof. (1) ($\Rightarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp)$ is $sp(i)\ sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(sp')$ is $sp'(j)$ $sp'(j+1) \ldots sp'(j-1)$. Because of $spsp'$, $\mathrm{rtt}^m(sp) = \mathrm{rtt}^n(sp')$ and then $(\mathrm{rtt}^m(sp))_{(1)}$ $\{\} = (\mathrm{rtt}^n(sp'))_{(1)}\ \{\}$. $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp))_{(1)}\ \{\}) = sp_{(i)}\ \{\}$ and $\mathrm{rtt}^{-n}\ ((\mathrm{rtt}^n(sp'))_{(1)}$ $\{\}) = sp'_{(j)}\{\}$. Therefore, $sp_{(i)}\{\}sp'_{(j)}\{\}$. ($\Leftarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp_{(i)}\{\})$ is $sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(sp'_{(j)}\{\})$ is $sp'(j+1) \ldots sp'(j-1)$. Because of $sp_{(i)}\{\}$ $sp'_{(j)}\ \{\}$, $\mathrm{rtt}^m(sp_{(i)}\{\}) = \mathrm{rtt}^n(sp'_{(j)}\ \{\})$ and then $(\mathrm{rtt}^m(sp_{(i)}\{\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(sp'_{(j)}\{\}))_{(1)}\ \{sp(i)\}$. Because $sp'(j)$ in sp' is the corresponding component to $sp(i)$ in sp, $sp'(j) = sp(i)$. Therefore, $(\mathrm{rtt}^m(sp_{(i)}\{\}))_{(1)}\ \{sp(i)\} =$

$(\mathrm{rtt}^n(sp'_{(j)}\{\}))_{(1)}\{sp'(j)\}$ and then $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp_{(i)}\ \{\}))_{(1)}\{sp(i)\}) = sp$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(sp'_{(j)}\{\}))_{(1)}\{sp'(j)\}) = sp'$. Thus, $spsp'$.

(2) ($\Rightarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp)$ is $sp(i)\ sp(i+1)\ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(\mathrm{rev}(sp'))$ is $sp'(j)\ sp'(j-1)\ldots sp'(j+1)$. Because of $spsp'$, $\mathrm{rtt}^m(sp) = \mathrm{rtt}^n(\mathrm{rev}(sp'))$ and then $(\mathrm{rtt}^m(sp))_{(1)}\ \{\} = (\mathrm{rtt}^n(\mathrm{rev}(sp')))_{(1)}\{\}$. $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp))_{(1)}\{\}) = sp_{(i)}\{\}$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(\mathrm{rev}(sp')))_{(1)}\{\}) = (\mathrm{rev}(sp'))_{(j)}\{\}$. Therefore, $sp_{(i)}\ \{\}\quad sp'_{(j)}\ \{\}$. ($\Leftarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp_{(i)}\{\})$ is $sp(i+1)\ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(\mathrm{rev}\ (sp'_{(j)}\ \{\}))$ is $sp'(j-1)\ \ldots\ sp'(j+1)$. Because of $sp_{(i)}\{\}sp'_{(j)}\{\}$, $\mathrm{rtt}^m(sp_{(i)}\{\}) = \mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\ \{\}))$ and then $(\mathrm{rtt}^m(sp_{(i)}\ \{\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\})))_{(1)}\ \{sp(i)\}$. Because $sp'(j)$ in sp' is the corresponding component to $sp(i)$ in sp, $sp'(j) = sp(i)$. Therefore, $(\mathrm{rtt}^m(sp_{(i)}\{\}))_{(1)}\ \{sp(i)\} = (\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\ \{\})))_{(1)}\{sp'(j)\}$ and then $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp_{(i)}\{\}))_{(1)}\{sp(i)\}) = sp$ and $\mathrm{rtt}^{-n}\ ((\mathrm{rtt}^n\ (\mathrm{rev}\ (sp'_{(j)}\ \{\}\ )))_{(1)}\ \{sp'(j)\}) = \mathrm{rev}(sp')$. Thus, $spsp'$. □

2 Rings

Definition 8 (Rings). *For $sp \in \mathbf{SP}$, $[sp]$ is called a ring pattern and satisfies (1) the rotative law ($[sp] = [\mathrm{rtt}(sp)]$) and (2) the reversible law ($[sp] = [\mathrm{rev}(sp)]$). When sp is a sequence $seq \in \mathbf{Seq}$, $[seq]$ is called a ring.*

Proposition 4. *For any sequence patterns $sp, sp' \in \mathbf{SP}$ and natural numbers m, n, if $[sp] = [sp']$, then (1) $[sp] = [\mathrm{rtt}^m(sp')]$ and (2) $[sp] = [\mathrm{rev}^n(sp')]$.*

Proof. *Let us suppose $[sp] = [sp']$. (1) By induction on m. (1.1) Base case ($m = 0$) can be discharged from the assumption $[sp] = [sp']$. (1.2) Induction case ($m = k + 1$). From Definition 8, $[\mathrm{rtt}^k(sp')] = [\mathrm{rtt}^{k+1}(sp')]$. From this and the induction hypothesis $[sp] = [\mathrm{rtt}^k(sp')]$, $[sp] = [\mathrm{rtt}^{k+1}(sp')]$. (2) By induction on n. (2.1) Base case ($n = 0$) can be discharged from the assumption $[sp] = [sp']$. (2.2) Induction case ($n = k + 1$). From Definition 8, $[\mathrm{rev}^k(sp')] = [\mathrm{rev}^{k+1}(sp')]$. From this and the induction hypothesis $[sp] = [\mathrm{rev}^k(sp')]$, $[sp] = [\mathrm{rev}^{k+1}(sp')]$.* □

For any sequence patterns $sp, sp' \in \mathbf{SP}$, if $([sp] = [sp']) \Rightarrow [sp] = [\mathrm{rtt}(sp')] \wedge [sp] = [\mathrm{rev}(sp')]$, then $[sp] = [\mathrm{rtt}(sp)]$ and $[sp] = [\mathrm{rev}(sp)]$ because the equivalence relation is reflexive, namely $[sp] = [sp]$. Therefore, Definition 8 can be rephrased as follows:

Definition 9 (Another definition of rings). *For $sp, sp' \in \mathbf{SP}$, $[sp] = [sp']$ is inductively defined as follows: (1) $[sp] = [sp]$ and (2) if $[sp] = [sp']$, then $[sp] = [\mathrm{rtt}(sp')]$ and $[sp] = [\mathrm{rev}(sp')]$.*

Let sp be $ES_1\ ES_2\ \ldots\ ES_{|sp|-1}\ ES_{|sp|}$. $\mathrm{rtt}^{-1}(sp)$ is $ES_2\ \ldots\ ES_{|sp|-1}\ ES_{|sp|}\ ES_1$ and $\mathrm{rev}^{-1}(sp)$ is $ES_{|sp|}\ ES_{|sp|-1}\ldots ES_1\ ES_2$. Therefore, $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$ and $\mathrm{rev}^{-1} = \mathrm{rev}$.

Proposition 5. *For any sequence patterns* $sp, sp' \in \mathbf{SP}$*, if* $[sp] = [sp']$*, then* $[sp] = [\mathrm{rtt}^{-1}(sp')]$ *and* $[sp] = [\mathrm{rev}^{-1}(sp')]$.

Proof. *This is derived from* $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$, $\mathrm{rev}^{-1} = \mathrm{rev}$ *and Proposition 4.* □

Definition 10 (Ring pattern match). *For* $sp \in \mathbf{SP}$ *and* $seq \in \mathbf{Seq}$*, pattern match between* $[sp]$ *and* $[seq]$ *is to find all substitutions* σ *such that* $[\sigma(sp)] = [seq]$*. Let* $[sp] =?= [seq]$ *be the set of all such substitutions.*

Definition 11 (Sequences rotated and/or reversed). *For* $sp \in \mathbf{SP}$, $[[sp]]$ *is the set of sequences inductively defined as follows: (1)* $sp \in [[sp]]$ *and (2) if* $sp' \in [[sp]]$*, then* $\mathrm{rtt}(sp') \in [[sp]]$ *and* $\mathrm{rev}(sp') \in [[sp]]$.

Proposition 6. *For any sequence patterns* $sp, sp' \in \mathbf{SP}$*, if* $sp' \in [[sp]]$*, then* $\mathrm{rtt}^{-1}(sp') \in [[sp]]$ *and* $\mathrm{rev}^{-1}(sp') \in [[sp]]$.

Proof. *This is derived from* $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$, $\mathrm{rev}^{-1} = \mathrm{rev}$ *and Definition 11.* □

Proposition 7. *For any sequences* $seq, seq', seq'' \in \mathbf{SP}$ *and any natural number* $i \in \{1, \ldots, |seq|\}$ *and* $j \in \{1, \ldots, |seq'|\}$ *such that* $seq'(j)$ *is the correspond component to* $seq(i)$*,* seq *is* $e_1 \ldots e_{i-1}\ e_i\ e_{i+1} \ldots e_{|sp|}$ *and* seq' *is* $e'_1 \ldots e'_{j-1}\ e'_j\ e'_{i+1} \ldots e'_{|sp'|}$*, (1) if* $seq_{(i)}\{\}seq'_{(j)}\{\}$*, then* $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\{seq''\}]$*, and (2) if* $seq_{(i)}\ \{\}\quad seq'_{(j)}\ \{\}$*, then* $[seq_{(i)}\ \{seq''\}] = [seq'_{(j)}\ \{\mathrm{rev}(seq'')\}]$.

Proof. *(1) Let* seq *be* $e_1 \ldots e_{i-1}\ e_i\ e_{i+1} \ldots e_{|seq|}$ *and* seq' *be* $e'_1 \ldots e'_{j-1}\ e'_j\ e'_{j+1} \ldots e'_{|seq'|}$*. (1) Because* $seq_{(i)}\ \{\}\quad seq'_{(j)}\{\}$, $e_{i+1} \ldots e_{|seq|}\ e_1 \ldots e_{i-1} = e'_{j+1} \ldots e'_{|seq'|}\ e'_1 \ldots e'_{j-1}$ *and then* $seq''\ e_{i+1} \ldots e_{|seq|}\ e_1 \ldots e_{i-1} = seq''\ e'_{j+1} \ldots e'_{|seq'|}\ e'_1 \ldots e'_{j-1}$*. Therefore,* $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\ \{seq''\}]$*. (2) Because* $seq_{(i)}\{\}seq'_{(j)}\{\}$, $e_{i+1} \ldots e_{|seq|}\ e_1 \ldots e_{i-1} = e'_{j-1} \ldots e'_1\ e'_{|seq'|} \ldots e'_{j+1}$ *and then* $seq''\ e_{i+1} \ldots e_{|seq|}\ e_1 \ldots e_{i-1} = seq''\ e'_{j-1} \ldots e'_1\ e'_{|seq'|} \ldots e'_{j+1}$. $\mathrm{rev}\ (seq''\ e'_{j-1} \ldots e'_1\ e'_{|seq'|} \ldots e'_{j+1})$ *is* $e'_{j+1} \ldots e'_{|seq'|}\ e'_1 \ldots e'_{j-1}\ \mathrm{rev}(seq'')$*. Thus,* $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\ \{\mathrm{rev}(seq'')\}]$. □

Lemma 2. *For sequence patterns* $sp, sp' \in \mathbf{SP}$, $(sp' \in [[sp]]) \Leftrightarrow ([sp] = [sp'])$.

Proof. $(sp' \in [[sp]]) \Rightarrow ([sp] = [sp'])$ *is proved by induction on Definition 11. (1) Base case in which* $sp \in [[sp]]$ *holds.* $[sp] = [sp]$ *holds because of Definition 6. (2) Induction case in which* $\mathrm{rtt}(sp') \in [[sp]]$ *and* $\mathrm{rev}(sp') \in [[sp]]$ *hold.* $sp' \in [[sp]]$ *holds from Proposition 6. From the induction hypothesis (*$[sp] = [sp']$*) and Definition 9, therefore,* $[sp] = [\mathrm{rtt}(sp')]$ *and* $[sp] = [\mathrm{rev}(sp')]$ *hold.*

$(sp' \in [[sp]]) \Leftarrow ([sp] = [sp'])$ *is proved by induction on Definition 9. (1) Base case in which* $[sp] = [sp]$ *holds.* $sp \in [[sp]]$ *holds because of Definition 11. (2) Induction case in which* $[sp] = [\mathrm{rtt}(sp')]$ *and* $[sp] = [\mathrm{rev}(sp')]$ *hold.* $[sp] = [sp']$ *holds from Proposition 5. From the induction hypothesis (*$sp' \in [[sp]]$*) and Definition 11, therefore,* $\mathrm{rtt}(sp') \in [[sp]]$ *and* $\mathrm{rev}(sp') \in [[sp]]$ *hold.* □

Let sp be $e_1\ S_1\ e_4\ S_2$ and sp' be $\mathrm{rev}(sp)$, namely $S_2\ e_4\ S_1\ e_1$. Clearly, $sp' \in [[sp]]$ and $[sp] = [sp']$. Let us consider a substitution σ such that $\sigma(S_1) = e_2\ e_3$, $\sigma(S_2) = e_5\ e_6$ and $\sigma(X) = X$ for any other variable X. $\sigma(sp)$ is $e_1\ e_2\ e_3\ e_4\ e_5\ e_6$ and $\sigma(sp')$ is $e_5\ e_6\ e_4\ e_2\ e_3\ e_1$. Clearly, $\sigma(sp') \notin [[\sigma(sp)]]$ and $[\sigma(sp)] \neq [\sigma(sp')]$. If $spsp'$ does not hold but $spsp'$ holds, we need to reverse the sequence that replaces each sequence variable. $\sigma_{\mathrm{rev}}(sp')$ is $e_6\ e_5\ e_4\ e_3\ e_2\ e_1$. Therefore, $\sigma_{\mathrm{rev}}(sp') \in [[\sigma(sp)]]$ and $[\sigma(sp)] = [\sigma_{\mathrm{rev}}(sp')]$.

Lemma 3. *For any sequence pattern $sp \in \mathbf{SP}$ and any substitution σ, for each $sp' \in [[sp]]$ if $spsp'$, then $[\sigma(sp)] = [\sigma(sp')]$; if $spsp'$, then $[\sigma(sp)] = [\sigma_{\mathrm{rev}}(sp')]$.*

Proof. *By induction on the number of element and sequence variable occurrences in sp.*

(1) Base case in which the number is 0. Because sp does not have any variables, $\sigma(sp) = sp$, $\sigma(sp') = sp'$ and $\sigma_{\mathrm{rev}}(sp') = sp'$. From Lemma 2, $[sp] = [sp']$.

(2) Induction case in which the number is $k+1$. Let us arbitrarily choose a component that is a variable in sp and the component be the ith component $sp(i)$ in sp. Let sp be $sp_1\ sp(i)\ sp_2$. sp' can be obtained by rotating and/or reversing sp and then must have the correspondent component in sp' to $sp(i)$ in sp. Then, sp' can be $sp'_1\ sp(i)\ sp'_2$.

(2.1) Let us suppose that $spsp'$ holds. From Proposition 3, $(sp_1\ \ sp_2)(sp'_1\ \ sp'_2)$. By induction hypothesis, $[\sigma(sp_1\ \ sp_2)] = [\sigma(sp'_1\ \ sp'_2)]$ and then $[\sigma(sp_1)\ \ \sigma(sp_2)] = [\sigma(sp'_1)\ \ \ \sigma(sp'_2)]$. From Lemma 7, $[\sigma(sp_1)\ \ \sigma(sp(i))\ \ \sigma(sp_2)] = [\sigma(sp'_1)\ \ \sigma(sp(i))\ \ \sigma(sp'_2)]$. Hence, $[\sigma(sp_1\ \ sp(i)\ \ sp_2)] = [\sigma(sp'_1\ \ sp(i)\ \ sp'_2)]$.

(2.2) Let us suppose that $spsp'$ holds. From Proposition 3, $(sp_1\ \ sp_2)\ \ (sp'_1\ \ sp'_2)$. By induction hypothesis, $[\sigma(sp_1\ \ sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1\ \ sp'_2)]$ and then $[\sigma(sp_1)\ \ \sigma(sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1)\ \ \ \sigma_{\mathrm{rev}}(sp'_2)]$. From Lemma 7, $[\sigma(sp_1)\ \ \sigma(sp(i))\ \ \sigma(sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1)\ \mathrm{rev}(\sigma(sp(i)))\ \ \sigma_{\mathrm{rev}}(sp'_2)]$. Beause $\mathrm{rev}(\sigma(sp(i))) = \sigma_{\mathrm{rev}}(sp(i))$, $[\sigma(sp_1\ \ sp(i))\ \ sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1\ \ sp(i)\ \ sp'_2)]$. □

Definition 12 (Ring pattern match simulated (1)). *For $sp \in \mathbf{SP}$ and $seq \in \mathbf{Seq}$, pattern match between sp and $[[seq]]$ is to find all substitutions σ such that $\sigma(sp) = seq'$ for some $seq' \in [[seq]]$. Let sp =?= $[[seq]]$ be the set of all such substitutions.*

Lemma 4. *For any sequence pattern $sp \in \mathbf{SP}$ and any sequence $seq \in \mathbf{Seq}$, $([sp]$ =?= $[seq]) = (sp$ =?= $[[seq]])$.*

Proof. *Let $\sigma \in ([sp]$ =?= $[seq])$. $[\sigma(sp)] = [seq]$ by Definition 10. $\sigma(sp) \in [[seq]]$ due to Lemma 2. Thus, $\sigma \in (sp$ =?= $[[seq]])$.*

Let $\sigma \in (sp$ =?= $[[seq]])$. Let $seq' \in [[seq]]$ such that $\sigma(sp) = seq'$. $[seq'] = [seq]$ due to Lemma 2 and then $[\sigma(sp)] = [seq]$. Hence, $\sigma \in ([sp]$ =?= $[seq])$. □

Definition 13 (Ring pattern match simulated (2)). *For $sp \in \mathbf{SP}$ and $seq \in \mathbf{Seq}$, pattern match between $[[sp]]$ and seq is to find all substitutions σ such that $\sigma'(sp') = seq$ for some substitution σ' and some $sp' \in [[sp]]$ and if $spsp'$, then $\sigma = \sigma'$ and if $spsp'$, then $\sigma = \sigma'_{\mathrm{rev}}$. Let $[[sp]]$ =?= seq be the set of all such substitutions.*

Note that $([[sp]] =?= seq) \subset ([sp] =?= [seq])$ but $([sp] =?= [seq]) \not\subset ([[sp]] = ?= seq)$.

Lemma 5. *For any sequence pattern $sp \in \mathbf{SP}$, any sequence $seq \in \mathbf{Seq}$ and any substitution $\sigma \in (sp =?= [[seq]])$, there exist σ' and a sequence $seq' \in [[seq]]$ such that $\sigma = \mathrm{join}(\sigma')$, $\sigma'(\mathrm{split}(sp)) = seq'$ and there exists $sp' \in [[\mathrm{split}(sp)]]$ such that $\sigma'(sp') = seq$. Besides, $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$.*

Proof. *Let sp be $ES_1\ ES_2 \ldots ES_m$ and seq be $e_1\ e_2 \ldots e_n$.*

If there exists $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is $\ldots e_n\ e_1 \ldots$ or $\ldots e_1\ e_n \ldots$, ES_i is a sequence variable S that is replaced with $\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$ in $\mathrm{split}(sp)$.

If $\sigma(S)$ is $\ldots e_n\ e_1 \ldots$, then $\sigma'(\mathrm{sv}(S, 0))$ is $\ldots e_n$, $\sigma'(\mathrm{sv}(S, 1))$ is $e_1 \ldots$ and $\sigma'(\mathrm{sv}(S', 0))$ is $\sigma(S')$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any other sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma')$ and $\sigma'(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{sv}(S, 1)\ \mathrm{split}(ES_{i+1}) \ldots\ \mathrm{split}(ES_{i-1})\ \mathrm{sv}(S, 0)$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'(sp') = seq$. Therefore, $\sigma' \in ([[\mathrm{split}(sp)]] =?= seq)$ because of spsp′ from Definition 13. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 4.

If $\sigma(S)$ is $\ldots e_1\ \ e_n \ldots$, then $\sigma'(\mathrm{sv}(S, 0))$ is $\mathrm{rev}(\ldots e_1)$, $\sigma'(\mathrm{sv}(S, 1))$ is $\mathrm{rev}(e_n \ldots)$ and $\sigma'(\mathrm{sv}(S', 0))$ is $\mathrm{rev}(\sigma(S'))$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any other sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma'_{\mathrm{rev}})$ and $\sigma'_{\mathrm{rev}}(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{rev}(\mathrm{sv}(S, 1)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})\ \mathrm{sv}(S, 0))$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'_{\mathrm{rev}}(sp') = seq$. Therefore, $\sigma'_{\mathrm{rev}} \in ([[\mathrm{split}(sp)]] =?= seq)$ because of spsp′ from Definition 13. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 4.

If there exists no $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is $\ldots e_n\ e_1 \ldots$ or $\ldots e_1\ e_n \ldots$, there must be $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is e_1, $e_1 \ldots$ or $\ldots e_1$. If $\sigma(ES_i)$ is e_1, there are two possible cases: (1) $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = seq$ and (2) $\sigma(ES_i\ ES_{i-1} \ldots ES_{i+1}) = seq$.

Case (1) can be treated in the same way as the case in which $\sigma(ES_i)$ is $e_1 \ldots$. In either case, $\sigma'(\mathrm{sv}(S', 0))$ is $\sigma(S')$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma')$ and $\sigma'(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{split}(ES_i)\ \mathrm{split}(ES_{i+1}) \ldots\ \mathrm{split}(ES_{i-1})$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'(sp') = seq$. Therefore, $\sigma' \in ([[\mathrm{split}(sp)]] =?= seq)$ because of spsp′ from Definition 13. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 4.

Case (2) can be treated in the same way as the case in which $\sigma(ES_i)$ is $\ldots e_1$. In either case, $\sigma'(\mathrm{sv}(S', 0))$ is $\mathrm{rev}(\sigma(S'))$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma'_{\mathrm{rev}})$ and $\sigma'_{\mathrm{rev}}(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{rev}(\mathrm{split}(ES_{i+1}) \ldots\ \mathrm{split}(ES_{i-1})\ \mathrm{split}(ES_i))$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'_{\mathrm{rev}}(sp') = seq$. Therefore, $\sigma'_{\mathrm{rev}} \in ([[\mathrm{split}(sp)]] =?= seq)$ because of spsp′ from Definition 13. □

Lemma 6. *For any sequence pattern $sp \in \mathbf{SP}$, any sequence $seq \in \mathbf{Seq}$ and any substitution $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$, $\sigma \in ([sp] =?= [seq])$.*

Proof. *Let sp be* $ES_1\ ES_2 \ldots ES_m$ *and seq be* $e_1\ e_2 \ldots e_n$. *Let* σ' *be an arbitrary substituion in* $([[\mathrm{split}(sp)]] =?= seq)$ *from which* σ *is constructed, namely that* $\sigma = \mathrm{join}(\sigma')$. *Let* $sp' \in [[\mathrm{split}(sp)]]$ *such that* $\sigma''(sp') = seq$, *if* $\mathrm{split}(sp) sp'$, *then* $\sigma' = \sigma''$ *and if* $\mathrm{split}(sp) sp'$, *then* $\sigma' = \sigma''_{\mathrm{rev}}$. *There are four possible cases: (1)* sp' *is* $\mathrm{split}(E_i)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})$, *(2)* sp' *is* $\mathrm{rev}(\mathrm{split}(E_i))$ $\mathrm{rev}\ (\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}\ (\mathrm{split}\ (ES_{i+1}))$. *(3)* sp' *is* $\mathrm{sv}(S,1)\ \mathrm{split}(ES_{i+1}) \ldots$ $\mathrm{split}(ES_{i-1})$
$\mathrm{sv}(S,0)$ *and (4)* sp' *is* $\mathrm{sv}(S,0)\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))\ \mathrm{sv}(S,1)$.

(1) For each ES_j *for* $j = 1, 2, \ldots, m$, *we calculate* $\sigma''(\mathrm{split}(\ ES_j))$ *and* $\sigma(ES_j)$. *There are three possible cases: (1.1)* ES_j *is an element* e, *(1.2)* ES_j *is an element variable* E *and (1.3)* ES_j *is a sequence variable* S. *(1.1)* $\sigma''(\mathrm{split}(e)) = e$ *and* $\sigma(e) = e = \sigma''(\mathrm{split}(e))$. *(1.2)* $\sigma''(\mathrm{split}(E)) = \sigma''(E)$ *and* $\sigma(E) = (\mathrm{join}\ (\sigma''))(E) = \sigma''(E) = \sigma''(\mathrm{split}(E))$. *(1.3)* $\sigma''(\mathrm{split}(S))$ *and* $\sigma(S)$ *are calculated as follows:*

$$\sigma''(\mathrm{split}(S)) = \sigma''(\mathrm{sv}(S,0)\ \mathrm{sv}(S,1))$$

$$\begin{aligned}\sigma(S) &= (\mathrm{join}(\sigma''))(S) = \sigma''(\mathrm{sv}(S,0))\ \sigma''(\mathrm{sv}(S,1))\\ &= \sigma''(\mathrm{split}(S))\end{aligned}$$

Therefore, $\sigma(ES_j) = \sigma''((\mathrm{split}(ES_j)))$ *and then* $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1})$ *is calculated as follows:*

$$\begin{aligned}\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) &= \sigma(ES_i)\ \sigma(ES_{i+1}) \ldots \sigma(ES_{i-1})\\ &= \sigma''((\mathrm{split}(ES_i)))\ \sigma''((\mathrm{split}(ES_{i+1}))) \ldots \sigma''((\mathrm{split}(ES_{i-1})))\\ &= \sigma''(\mathrm{split}(ES_i)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1}))\\ &= \sigma''(sp')\end{aligned}$$

Because $\sigma''(sp') = seq$ *from the assumption,* $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = seq$. *Because* $sp ES_i\ ES_{i+1} \ldots ES_{i-1}$, $[\sigma(sp)] = [\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1})]$ *from Lemma 3. Thus,* $[\sigma(sp)] = [seq]$ *and then* $\sigma \in ([sp] =?= [seq])$.

(2) For each ES_j *for* $j = 1, 2, \ldots, m$, *we calculate* $\sigma''(\mathrm{rev}(\ \mathrm{split}(ES_j)))$ *and* $\sigma(ES_j)$. *There are three possible cases: (2.1)* ES_j *is an element* e, *(2.2)* ES_j *is an element variable* E *and (2.3)* ES_j *is a sequence variable* S. *(2.1)* $\sigma''(\mathrm{rev}(\mathrm{split}(e))) = e$ *and* $\sigma(e) = e = \sigma''(\mathrm{rev}(\mathrm{split}(e))) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(e))))$. *(2.2)* $\sigma''(\mathrm{rev}(\mathrm{split}(E))) = \sigma''(E)$ *and* $\sigma(E) = (\mathrm{join}(\sigma''_{\mathrm{rev}}))(E) = \sigma''(E) = \sigma''(\mathrm{rev}\ (\mathrm{split}(E))) = \mathrm{rev}(\sigma''(\mathrm{rev}\ (\mathrm{split}(E))))$. *(2.3)* $\sigma''(\mathrm{rev}(\mathrm{split}(S)))$ *and* $\sigma(S)$ *are calculated as follows:*

$$\begin{aligned}\sigma''(\mathrm{rev}(\mathrm{split}(S))) &= \sigma''(\mathrm{rev}(\mathrm{sv}(S,0)\ \mathrm{sv}(S,1)))\\ &= \sigma''(\mathrm{sv}(S,1)\ \mathrm{sv}(S,1))\end{aligned}$$

$$\begin{aligned}\sigma(S) &= (\mathrm{join}(\sigma''_{\mathrm{rev}}))(S) = \sigma''_{\mathrm{rev}}(\mathrm{sv}(S,0))\ \sigma''_{\mathrm{rev}}(\mathrm{sv}(S,1))\\ &= \mathrm{rev}(\sigma''(\mathrm{sv}(S,0)))\ \mathrm{rev}(\sigma''(\mathrm{sv}(S,1)))\\ &= \mathrm{rev}(\sigma''(\mathrm{sv}(S,1))\ \sigma''(\mathrm{sv}(S,0))) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(S))))\end{aligned}$$

Therefore, $\sigma(ES_j) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_j))))$ *and then* $\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i)$ *is calculated as follows:*

$\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i) = \sigma(ES_{i+1}) \ldots \sigma(ES_{i-1})\ \sigma(ES_i)$
$= \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_{i+1})))) \ldots$
$\mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_{i-1})))) \ \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_i))))$
$= \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_i)))$
$\sigma''(\mathrm{rev}(\mathrm{split}(ES_{i-1}))) \ldots \sigma''(\mathrm{rev}(\mathrm{split}(ES_{i+1}))))$
$= \mathrm{rev}(\sigma''(sp'))$

Because $\sigma''(sp') = seq$ from the assumption, $\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i) = \mathrm{rev}(seq)$. Because $spES_{i+1} \ \ldots \ ES_{i-1}\ ES_i$, $[\sigma(sp)] = [\sigma(ES_{i+1} \ \ldots \ ES_{i-1}\ ES_i)]$ from Lemma 3. Moreover, $[seq] = [\mathrm{rev}(seq)]$ from Proposition 4. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$.

(3) $\mathrm{rtt}(sp')$ is calculated as follows:

$\mathrm{rtt}(sp')$
$= \mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)\ \mathrm{rev}(\mathrm{split}(ES_{i+1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i-1}))$
$= \mathrm{split}(ES_i)\ \mathrm{rev}(\mathrm{split}(ES_{i+1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i-1}))$

Because $\sigma''(sp') = seq$, there exists a natural number k such that $\sigma''(\mathrm{rtt}(sp')) = \mathrm{rtt}^k(seq)$. As what has been done for case (1), we have $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = \mathrm{rtt}^k(seq)$. Because $spES_i\ ES_{i+1} \ldots ES_{i-1}$, $[\sigma(sp)] = [\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}]$ from Lemma 3. Moreover, $[seq] = [\mathrm{rtt}^k(seq)]$ from Proposition 4. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$.

(4) $\mathrm{rtt}(sp')$ is calculated as follows:

$\mathrm{rtt}(sp')$
$= \mathrm{sv}(S, 1)\ \mathrm{sv}(S, 0)\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))$
$= \mathrm{rev}(\mathrm{split}(ES_i))\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))$

Because $\sigma''(sp') = seq$, there exists a natural number k such that $\sigma''(\mathrm{rtt}(sp')) = \mathrm{rtt}^k(seq)$. As what has been done for case (2), we have $\sigma(ES_{i+1} \ \ldots \ ES_{i-1}\ ES_i) = \mathrm{rev}(\mathrm{rtt}^k(seq))$. Because $sp \quad ES_{i+1} \ \ldots ES_{i-1}\ ES_i$, $[\sigma(sp)] = [\sigma(ES_{i+1} \ \ldots \ ES_{i-1}\ ES_i)]$ from Lemma 3. Moreover, $[seq] = [\mathrm{rev}(\mathrm{rtt}^k(seq))]$ from Proposition 4. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$. □

Lemma 7. *For any sequence pattern $sp \in \mathbf{SP}$ and any sequence $seq \in \mathbf{Seq}$, $\mathrm{join}([[\mathrm{split}(sp)]] =?= seq) = (sp =?= [[seq]]))$.*

Proof. Let $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$. Let σ' be an arbitrary substitution in $([[\mathrm{split}(sp)]] =?= seq)$ from which σ is constructed, namely that $\sigma = \mathrm{join}(\sigma')$. Let $\mathrm{split}(sp') \in [[\mathrm{split}(sp)]]$ such that if $\mathrm{split}(sp')\mathrm{split}(sp)$, $\sigma'(\mathrm{split}(sp')) = seq$ and if $\mathrm{split}(sp')\mathrm{split}(sp)$, $\sigma''(\mathrm{split}(sp')) = seq$ such that $\sigma' = \sigma''_{\mathrm{rev}}$. . Then, $\sigma(sp') = seq$. From Lemma 3, $[\sigma(sp)] = [\sigma(sp')]$. Therefore, $[\sigma(sp)] = [seq]$. From Lemma 4, $\sigma \in (sp =?= [[seq]]))$. Let $\sigma \in (sp =?= [[seq]]))$. From Lemma 5, $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$. □

Theorem 1. *For any sequence pattern $sp \in \mathbf{SP}$ and any sequence $seq \in \mathbf{Seq}$, $\mathrm{join}([[\mathrm{split}(sp)]] =?= seq) = ([sp] =?= [seq])$.*

Proof. It is derived from Lemma 4, Lemma 5 and Lemma 6. □