A DSL for Specifying and Model Checking Mobile Ring Robot Algorithms

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1 Appendix

This appendix gives fully the theory including proofs of the propositions and lemmas used in the paper.

1.1 Sequences

Let **Elt** be the set of (concrete) elements, **EV** be the set of element variables and **SV** be the set of sequence variables.

Definition 1.1 (Sequence Patterns)**.** The set **SP** of sequence patterns are inductively defined as follows:

1. $\varepsilon \in \mathbf{SP}$ (the empty sequence);
2. For each element $e \in \mathbf{Elt}$, $e \in \mathbf{SP}$;
3. For each element variable $E \in \mathbf{EV}$, $E \in \mathbf{SP}$;
4. For each sequence variable $S \in \mathbf{SV}$, $S \in \mathbf{SP}$;
5. For any sequence patterns $SP_1, SP_2 \in \mathbf{SP}$, $SP_1\ SP_2 \in \mathbf{SP}$.

The binary juxtaposition operator used in $SP_1\ SP_2$ is associative, namely that $(SP_1\ SP_2)\ SP_3 = SP_1\ (SP_2\ SP_3)$ for any sequence patterns $SP_1, SP_2, SP_3 \in \mathbf{SP}$. ε is an identity of the binary juxtaposition operator, namely that $\varepsilon\ SP = SP$ and $SP\ \varepsilon = SP$ for any sequence patterns $SP \in \mathbf{SP}$.

Sequence patterns that do not have any variables at all are called sequences. Let $\mathbf{Seq} \subseteq \mathbf{SP}$ be the set of all sequences.

A substitution σ is a function from the disjoint union $\mathbf{EV} \uplus \mathbf{SV}$ of **EV** and **SV** to the disjoint union $\mathbf{Seq} \uplus \mathbf{EV} \uplus \mathbf{SV}$ of **Seq**, **EV** and **SV**. For $E \in \mathbf{EV}$, $\sigma(E)$ is an element $e \in \mathbf{Elt}$ or E and for $S \in \mathbf{SV}$, $\sigma(S)$ is a sequence $seq \in \mathbf{Seq}$ or S. The domain of a substitution σ can be naturally extended to **SP** such that $\sigma(\varepsilon)$ is ε, for an element $e \in \mathbf{Elt}$, $\sigma(e)$ is e and for a sequence pattern $SP_1, SP_2 \in \mathbf{SP}$, $\sigma(SP_1\ SP_2)$ is $\sigma(SP_1)\ \sigma(SP_2)$.

Definition 1.2 (Sequence pattern match)**.** Pattern match between $sp \in \mathbf{SP}$ & $seq \in \mathbf{Seq}$ is to find all substitutions σ such that $\sigma(sp) = seq$. Let $sp = ?= seq$ be the set of all such substitutions.

Elements, element variables and sequence variables used in sequence patterns are called components in the sequence patterns. For $sp \in \mathbf{SP}$, let $|sp|$ be the number of components in it. $sp \in \mathbf{SP}$ can be in the form $ES_1\ ES_2\ \ldots\ ES_{|sp|}$, where each ES_i is an element, an element variable or a sequence variable. For $sp \in \mathbf{SP}$, let $sp(i)$, where $i \in \{1, 2, \ldots, |sp|\}$, be the ith element ES_i in sp. Let $e, e_1, e_2, \ldots \in \mathbf{Elt}$, $E, E_1, E_2, \ldots \in \mathbf{EV}$, $S, S_1, S_2, \ldots \in \mathbf{SV}$ and $ES, ES_1, ES_2, \ldots \in \mathbf{Elt} \uplus \mathbf{EV} \uplus \mathbf{SV}$. A binary construct $\mathrm{sv}(S, I)$ that is not in **SV**, where S is a sequence variable and I is either 0 or 1, is used as an extra sequence variable. Let **SSV** be $\{\mathrm{sv}(S, I) \mid S \in \mathbf{SV}, I \in \{0, 1\}\}$. $\mathbf{SV} \cup \mathbf{SSV}$ may be used as the set of sequence variables instead of **SV**.

Definition 1.3 (Split sequence patterns)**.** For $sp \in \mathbf{SP}$, $\mathrm{split}(sp)$ is a sequence pattern such that each sequence variable S in sp is replaced with $\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$. $\mathrm{split}(\varepsilon) = \varepsilon$, $\mathrm{split}(e) = e$ for $e \in \mathbf{Elt}$, $\mathrm{split}(E) = E$ for $E \in \mathbf{EV}$, $\mathrm{split}(S) = \mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$ for $S \in \mathbf{SV}$ and $\mathrm{split}(SP_1\ SP_2) = \mathrm{split}(SP_1)\ \mathrm{split}(SP_2)$ for $SP_1, SP_2 \in \mathbf{SP}$.

Definition 1.4 (Joining split sequence variables)**.** For $sp \in \mathbf{SP}$ and $seq \in \mathbf{Seq}$, let σ be in $(\mathrm{split}(sp) = ?= seq)$. $\mathrm{join}(\sigma)$ is the substitution σ' such that for each sequence variable S in sp $\sigma'(S) = \sigma(\mathrm{sv}(S, 0))\ \sigma(\mathrm{sv}(S, 1))$ and for any other variables X $\sigma'(X) = \sigma(X)$. The domain of join can be naturally extended to the set of substitutions such that $\mathrm{join}(\mathrm{split}(sp) = ?= seq)$ is $\{\mathrm{join}(\sigma) \mid \sigma \in (\mathrm{split}(sp) = ?= seq)\}$.

rtt (that stands for rotate) takes a sequence pattern sp and returns the sequence pattern obtained by rotating sp clockwise. rev (that stands for reverse) takes a sequence pattern sp and returns the sequence pattern obtained by reversing sp. Let sp be $ES_1\ ES_2 \ldots ES_{|sp|-1}\ ES_{|sp|}$. $\mathrm{rtt}(sp) = ES_{|sp|}\ ES_1\ ES_2 \ldots ES_{|sp|-1}$ and $\mathrm{rev}(sp) = ES_{|sp|}\ ES_{|sp|-1} \ldots ES_2\ ES_1$. Let us suppose that a subscript exp of ES_{exp} used as an element in sp is interpreted as $(exp \bmod |sp|) + 1$.

Definition 1.5 (Reversing substitutions)**.** σ_{rev} is defined as follows: for an element $e \in \mathbf{Elt}$ $\sigma_{\mathrm{rev}}(e) = \sigma(e) = e$, for an element variable $E \in (E)$ $\sigma_{\mathrm{rev}}(E) = \sigma(E)$ and for a sequence variable $S \in (SV)$ $\sigma_{\mathrm{rev}}(S) = \mathrm{rev}(\sigma(S))$.

Given two sequence patterns $sp, sp' \in \mathbf{SP}$, $sp \circlearrowright sp'$ holds if there exists a natural number n such that $sp = \mathrm{rtt}^n(sp')$, namely that sp is obtained by rotating sp' finitely many times; $sp \circlearrowleft sp'$ holds if there exists a natural number n such that $sp = \mathrm{rtt}^n(\mathrm{rev}(sp'))$, namely that sp is obtained by reversing sp' once and rotating it finitely many times. For example, let sp and sp' be $e_1\ S_1\ e_2$ and $e_1\ e_2\ S_1$ ($= \mathrm{rtt}(sp)$) and then $(sp \circlearrowright sp')$ holds, while $(sp \circlearrowleft sp')$ does not; let sp and sp' be $e_1\ S_1\ e_2$ and $e_2\ e_1\ S_1$ ($= \mathrm{rtt}(\mathrm{rev}(sp))$) and then $(sp \circlearrowright sp')$ does not, while $(sp \circlearrowleft sp')$ holds; let sp and sp' be $e_1\ S_1\ e_2$ and $e_2\ e_1\ S_1$ ($= \mathrm{rtt}(\mathrm{rev}(sp))$) and then both $(sp \circlearrowright sp')$ and $(sp \circlearrowleft sp')$ hold.

Definition 1.6 (Sequence pattern contexts)**.** A sequence pattern context is a sequence pattern $sp \in \mathbf{SP}$ in which one component (say, ith component, where $1 \leq i \leq |sp|$) is replaced with a special symbol $\square$ called a hole, denoted $sp_{(i)}\{\square\}$. A

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hole $\square$ is treated as an element. Let sp be $ES_1 \ldots ES_i \ldots ES_{|sp|}$ and then $sp_{(i)}\{\square\}$ is $ES_1 \ldots \square \ldots ES_{|sp|}$.

For a sequence pattern or a sequence pattern context spc and a sequence pattern sp, $spc_{(i)}\{sp\}$ is spc in which the ith component in spc is replaced with sp. $(sp_{(i)}\{\square\})_{(i)}\{sp(i)\} = sp_{(i)}\{sp(i)\} = sp$.

Definition 1.7 (Correspondent components)**.** Let $sp \in \mathbf{SP}$ be $ES_1 \ldots ES_{i-1}\ ES_i\ ES_{i+1}\ \ldots ES_{|sp|}$, where $1 \le i \le |sp|$ and $sp' \in \mathbf{SP}$ be $ES'_1 \ldots ES'_{j-1}\ ES'_j\ ES_{j+1}\ \ldots ES'_{|sp'|}$, where $1 \le j \le |sp'|$. If $ES_i\ ES_{i+1}\ \ldots ES_{|sp|}\ ES_1 \ldots ES_{i-1} = ES'_j\ ES'_{j+1}\ \ldots ES'_{|sp'|}\ ES'_1 \ldots ES'_{j-1}$ or $ES_i\ ES_{i+1}\ \ldots ES_{|sp|}\ ES_1 \ldots ES_{i-1} = ES'_j\ ES'_{j-1}\ \ldots ES'_1\ ES'_{|sp'|} \ldots ES'_{j+1}$, then ES'_j is the corresponding component in sp' to ES_i in sp.

Proposition 1.8. *For any sequence patterns $sp, sp' \in \mathbf{SP}$ and any natural numbers $i \in \{1, \ldots, |sp|\}$ and $j \in \{1, \ldots, |sp'|\}$ such that the jth component $sp'(j)$ in sp' is the corresponding component in sp' to $sp(i)$ in sp, (1) $(sp \circlearrowright sp') \Leftrightarrow sp_{(i)}\{\square\} \circlearrowright sp'_{(j)}\{\square\}$ and (2) $(sp \circlearrowleft sp') \Leftrightarrow sp_{(i)}\{\square\} \circlearrowleft sp'_{(j)}\{\square\}$.*

Proof. (1) ($\Rightarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp)$ is $sp(i)\ sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(sp')$ is $sp'(j)\ sp'(j+1) \ldots sp'(j-1)$. Because of $sp \circlearrowright sp'$, $\mathrm{rtt}^m(sp) = \mathrm{rtt}^n(sp')$ and then $(\mathrm{rtt}^m(sp))_{(1)}\{\square\} = (\mathrm{rtt}^n(sp'))_{(1)}\{\square\}$. $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp))_{(1)}\{\square\}) = sp_{(i)}\{\square\}$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(sp'))_{(1)}\{\square\}) = sp'_{(j)}\{\square\}$. Therefore, $sp_{(i)}\{\square\} \circlearrowright sp'_{(j)}\{\square\}$. ($\Leftarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp_{(i)}\{\square\})$ is $\square\ sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(sp'_{(j)}\{\square\})$ is $\square\ sp'(j+1) \ldots sp'(j-1)$. Because of $sp_{(i)}\{\square\} \circlearrowright sp'_{(j)}\{\square\}$, $\mathrm{rtt}^m(sp_{(i)}\{\square\}) = \mathrm{rtt}^n(sp'_{(j)}\{\square\})$ and then $(\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(sp'_{(j)}\{\square\}))_{(1)}\{sp(i)\}$. Because $sp'(j)$ in sp' is the corresponding component to $sp(i)$ in sp, $sp'(j) = sp(i)$. Therefore, $(\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(sp'_{(j)}\{\square\}))_{(1)}\{sp'(j)\}$ and then $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\}) = sp$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(sp'_{(j)}\{\square\}))_{(1)}\{sp'(j)\}) = sp'$. Thus, $sp \circlearrowright sp'$.

(2) ($\Rightarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp)$ is $sp(i)\ sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(\mathrm{rev}(sp'))$ is $sp'(j)\ sp'(j-1) \ldots sp'(j+1)$. Because of $sp \circlearrowleft sp'$, $\mathrm{rtt}^m(sp) = \mathrm{rtt}^n(\mathrm{rev}(sp'))$ and then $(\mathrm{rtt}^m(sp))_{(1)}\{\square\} = (\mathrm{rtt}^n(\mathrm{rev}(sp')))_{(1)}\{\square\}$. $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp))_{(1)}\{\square\}) = sp_{(i)}\{\square\}$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(\mathrm{rev}(sp')))_{(1)}\{\square\}) = (\mathrm{rev}(sp'))_{(j)}\{\square\}$. Therefore, $sp_{(i)}\{\square\} \circlearrowleft sp'_{(j)}\{\square\}$. ($\Leftarrow$) There exists a natural number m such that $\mathrm{rtt}^m(sp_{(i)}\{\square\})$ is $\square\ sp(i+1) \ldots sp(i-1)$ and there exits a natural number n such that $\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\square\}))$ is $\square\ sp'(j-1) \ldots sp'(j+1)$. Because of $sp_{(i)}\{\square\} \circlearrowleft sp'_{(j)}\{\square\}$, $\mathrm{rtt}^m(sp_{(i)}\{\square\}) = \mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\square\}))$ and then $(\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\square\})))_{(1)}\{sp(i)\}$. Because $sp'(j)$ in sp' is the corresponding component to $sp(i)$ in sp, $sp'(j) = sp(i)$. Therefore, $(\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\} = (\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\square\})))_{(1)}\{sp'(j)\}$ and then $\mathrm{rtt}^{-m}((\mathrm{rtt}^m(sp_{(i)}\{\square\}))_{(1)}\{sp(i)\}) = sp$ and $\mathrm{rtt}^{-n}((\mathrm{rtt}^n(\mathrm{rev}(sp'_{(j)}\{\square\})))_{(1)}\{sp'(j)\}) = \mathrm{rev}(sp')$. Thus, $sp \circlearrowleft sp'$. $\square$

1.2 Rings

Definition 1.9 (Rings)**.** For $sp \in \mathbf{SP}$, $[sp]$ is called a ring pattern and satisfies (1) the rotative law ($[sp] = [\mathrm{rtt}(sp)]$) and (2) the reversible law ($[sp] = [\mathrm{rev}(sp)]$). When sp is a sequence $seq \in \mathbf{Seq}$, $[seq]$ is called a ring.

Proposition 1.10. *For any sequence patterns $sp, sp' \in \mathbf{SP}$ and natural numbers m, n, if $[sp] = [sp']$, then (1) $[sp] = [\mathrm{rtt}^m(sp')]$ and (2) $[sp] = [\mathrm{rev}^n(sp')]$.*

Proof. Let us suppose $[sp] = [sp']$. (1) By induction on m. (1.1) Base case ($m = 0$) can be discharged from the assumption $[sp] = [sp']$. (1.2) Induction case ($m = k + 1$). From Definition 1.9, $[\mathrm{rtt}^k(sp')] = [\mathrm{rtt}^{k+1}(sp')]$. From this and the induction hypothesis $[sp] = [\mathrm{rtt}^k(sp')]$, $[sp] = [\mathrm{rtt}^{k+1}(sp')]$. (2) By induction on n. (2.1) Base case ($n = 0$) can be discharged from the assumption $[sp] = [sp']$. (2.2) Induction case ($n = k + 1$). From Definition 1.9, $[\mathrm{rev}^k(sp')] = [\mathrm{rev}^{k+1}(sp')]$. From this and the induction hypothesis $[sp] = [\mathrm{rev}^k(sp')]$, $[sp] = [\mathrm{rev}^{k+1}(sp')]$. $\square$

For any sequence patterns $sp, sp' \in \mathbf{SP}$, if $([sp] = [sp']) \Rightarrow [sp] = [\mathrm{rtt}(sp')] \wedge [sp] = [\mathrm{rev}(sp')]$, then $[sp] = [\mathrm{rtt}(sp)]$ and $[sp] = [\mathrm{rev}(sp)]$ because the equivalence relation is reflexive, namely $[sp] = [sp]$. Therefore, Definition 1.9 can be rephrased as follows:

Definition 1.11 (Another definition of rings)**.** For $sp, sp' \in \mathbf{SP}$, $[sp] = [sp']$ is inductively defined as follows: (1) $[sp] = [sp]$ and (2) if $[sp] = [sp']$, then $[sp] = [\mathrm{rtt}(sp')]$ and $[sp] = [\mathrm{rev}(sp')]$.

Let sp be $ES_1\ ES_2 \ldots\ ES_{|sp|-1}\ ES_{|sp|}$. $\mathrm{rtt}^{-1}(sp)$ is $ES_2 \ldots\ ES_{|sp|-1}\ ES_{|sp|}\ ES_1$ and $\mathrm{rev}^{-1}(sp)$ is $ES_{|sp|}\ ES_{|sp|-1} \ldots ES_1\ ES_2$. Therefore, $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$ and $\mathrm{rev}^{-1} = \mathrm{rev}$.

Proposition 1.12. *For any sequence patterns $sp, sp' \in \mathbf{SP}$, if $[sp] = [sp']$, then $[sp] = [\mathrm{rtt}^{-1}(sp')]$ and $[sp] = [\mathrm{rev}^{-1}(sp')]$.*

Proof. This is derived from $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$, $\mathrm{rev}^{-1} = \mathrm{rev}$ and Proposition 1.10, $\square$

Definition 1.13 (Ring pattern match)**.** For $sp \in \mathbf{SP}$ and $seq \in \mathbf{Seq}$, pattern match between $[sp]$ and $[seq]$ is to find all substitutions σ such that $[\sigma(sp)] = [seq]$. Let $[sp] =?= [seq]$ be the set of all such substitutions.

Definition 1.14 (Sequences rotated and/or reversed)**.** For $sp \in \mathbf{SP}$, $[[sp]]$ is the set of sequences inductively defined as follows: (1) $sp \in [[sp]]$ and (2) if $sp' \in [[sp]]$, then $\mathrm{rtt}(sp') \in [[sp]]$ and $\mathrm{rev}(sp') \in [[sp]]$.

Proposition 1.15. *For any sequence patterns $sp, sp' \in \mathbf{SP}$, if $sp' \in [[sp]]$, then $\mathrm{rtt}^{-1}(sp') \in [[sp]]$ and $\mathrm{rev}^{-1}(sp') \in [[sp]]$.*

Proof. This is derived from $\mathrm{rtt}^{-1} = \mathrm{rev} \circ \mathrm{rtt} \circ \mathrm{rev}$, $\mathrm{rev}^{-1} = \mathrm{rev}$ and Definition 1.14. $\square$

Proposition 1.16. *For any sequences $seq, seq', seq'' \in$ **SP** and any natural number $i \in \{1, \dots, |seq|\}$ and $j \in \{1, \dots, |seq'|\}$ such that $seq'(j)$ is the correspond component to $seq(i)$, seq is $e_1 \dots e_{i-1}\ e_i\ e_{i+1} \dots e_{|sp|}$ and seq' is $e'_1 \dots e'_{j-1}\ e'_j\ e'_{i+1} \dots e'_{|sp'|}$, (1) if $seq_{(i)}\{\square\} \circlearrowleft seq'_{(j)}\{\square\}$, then $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\{seq''\}]$, and (2) if $seq_{(i)}\{\square\} \circlearrowright seq'_{(j)}\{\square\}$, then $[seq_{(i)}\ \{seq''\}] = [seq'_{(j)}\{\mathrm{rev}(seq'')\}]$.*

Proof. (1) Let seq be $e_1 \dots e_{i-1}\ e_i\ e_{i+1} \dots e_{|seq|}$ and seq' be $e'_1 \dots e'_{j-1}\ e'_j\ e'_{j+1} \dots e'_{|seq'|}$. (1) Because $seq_{(i)}\{\square\} \circlearrowleft seq'_{(j)}\{\square\}$, $\square\ e_{i+1} \dots e_{|seq|}\ e_1 \dots e_{i-1} = \square\ e'_{j+1} \dots e'_{|seq'|}\ e'_1 \dots e'_{j-1}$ and then $seq''\ e_{i+1} \dots e_{|seq|}\ e_1 \dots e_{i-1} = seq''\ e'_{j+1} \dots e'_{|seq'|}\ e'_1 \dots e'_{j-1}$. Therefore, $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\ \{seq''\}]$. (2) Because $seq_{(i)}\{\square\} \circlearrowright seq'_{(j)}\{\square\}$, $\square\ e_{i+1} \dots e_{|seq|}\ e_1 \dots e_{i-1} = \square\ e'_{j-1} \dots e'_1\ e'_{|seq'|} \dots e'_{j+1}$ and then $seq''\ e_{i+1} \dots e_{|seq|}\ e_1 \dots e_{i-1} = seq''\ e'_{j-1} \dots e'_1\ e'_{|seq'|} \dots e'_{j+1}$. $\mathrm{rev}(seq''\ e'_{j-1} \dots e'_1\ e'_{|seq'|} \dots e'_{j+1})$ is $e'_{j+1} \dots e'_{|seq'|}\ e'_1 \dots e'_{j-1}\ \mathrm{rev}(seq'')$. Thus, $[seq_{(i)}\{seq''\}] = [seq'_{(j)}\{\mathrm{rev}(seq'')\}]$. $\square$

Lemma 1.17. *For sequence patterns $sp, sp' \in$ **SP**, $(sp' \in [[sp]]) \Leftrightarrow ([sp] = [sp'])$.*

Proof. $(sp' \in [[sp]]) \Rightarrow ([sp] = [sp'])$ is proved by induction on Definition 1.14. (1) Base case in which $sp \in [[sp]]$ holds. $[sp] = [sp]$ holds because of Definition 1.15. (2) Induction case in which $\mathrm{rtt}(sp') \in [[sp]]$ and $\mathrm{rev}(sp') \in [[sp]]$ hold. $sp' \in [[sp]]$ holds from Proposition 1.15. From the induction hypothesis $([sp] = [sp'])$ and Definition 1.11, therefore, $[sp] = [\mathrm{rtt}(sp')]$ and $[sp] = [\mathrm{rev}(sp')]$ hold.

$(sp' \in [[sp]]) \Leftarrow ([sp] = [sp'])$ is proved by induction on Definition 1.11. (1) Base case in which $[sp] = [sp]$ holds. $sp \in [[sp]]$ holds because of Definition 1.14. (2) Induction case in which $[sp] = [\mathrm{rtt}(sp')]$ and $[sp] = [\mathrm{rev}(sp')]$ hold. $[sp] = [sp']$ holds from Proposition 1.12. From the induction hypothesis $(sp' \in [[sp]])$ and Definition 1.14, therefore, $\mathrm{rtt}(sp') \in [[sp]]$ and $\mathrm{rev}(sp') \in [[sp]]$ hold. $\square$

Let sp be $e_1\ S_1\ e_4\ S_2$ and sp' be $\mathrm{rev}(sp)$, namely $S_2\ e_4\ S_1\ e_1$. Clearly, $sp' \in [[sp]]$ and $[sp] = [sp']$. Let us consider a substitution σ such that $\sigma(S_1) = e_2\ e_3$, $\sigma(S_2) = e_5\ e_6$ and $\sigma(X) = X$ for any other variable X. $\sigma(sp)$ is $e_1\ e_2\ e_3\ e_4\ e_5\ e_6$ and $\sigma(sp')$ is $e_5\ e_6\ e_4\ e_2\ e_3\ e_1$. Clearly, $\sigma(sp') \notin [[\sigma(sp)]]$ and $[\sigma(sp)] \neq [\sigma(sp')]$. If $sp \circlearrowleft sp'$ does not hold but $sp \circlearrowright sp'$ holds, we need to reverse the sequence that replaces each sequence variable. $\sigma_{\mathrm{rev}}(sp')$ is $e_6\ e_5\ e_4\ e_3\ e_2\ e_1$. Therefore, $\sigma_{\mathrm{rev}}(sp') \in [[\sigma(sp)]]$ and $[\sigma(sp)] = [\sigma_{\mathrm{rev}}(sp')]$.

Lemma 1.18. *For any sequence pattern $sp \in$ **SP** and any substitution σ, for each $sp' \in [[sp]]$ if $sp \circlearrowleft sp'$, then $[\sigma(sp)] = [\sigma(sp')]$; if $sp \circlearrowright sp'$, then $[\sigma(sp)] = [\sigma_{\mathrm{rev}}(sp')]$.*

Proof. By induction on the number of element and sequence variable occurrences in sp.

(1) Base case in which the number is 0. Because sp does not have any variables, $\sigma(sp) = sp$, $\sigma(sp') = sp'$ and $\sigma_{\mathrm{rev}}(sp') = sp'$. From Lemma 1.17, $[sp] = [sp']$.

(2) Induction case in which the number is $k + 1$. Let us arbitrarily choose a component that is a variable in sp and the component be the ith component $sp(i)$ in sp. Let sp be $sp_1\ sp(i)\ sp_2$. sp' can be obtained by rotating and/or reversing sp and then must have the correspondent component in sp' to $sp(i)$ in sp. Then, sp' can be $sp'_1\ sp(i)\ sp'_2$.

(2.1) Let us suppose that $sp \circlearrowleft sp'$ holds. From Proposition 1.8, $(sp_1 \square sp_2) \circlearrowleft (sp'_1 \square sp'_2)$. By induction hypothesis, $[\sigma(sp_1 \square sp_2)] = [\sigma(sp'_1 \square sp'_2)]$ and then $[\sigma(sp_1) \square \sigma(sp_2)] = [\sigma(sp'_1) \square \sigma(sp'_2)]$. From Lemma 1.16, $[\sigma(sp_1)\ \sigma(sp(i))\ \sigma(sp_2)] = [\sigma(sp'_1)\ \sigma(sp(i))\ \sigma(sp'_2)]$. Hence, $[\sigma(sp_1\ sp(i)\ sp_2)] = [\sigma(sp'_1\ sp(i)\ sp'_2)]$.

(2.2) Let us suppose that $sp \circlearrowright sp'$ holds. From Proposition 1.8, $(sp_1 \square sp_2) \circlearrowright (sp'_1 \square sp'_2)$. By induction hypothesis, $[\sigma(sp_1 \square sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1 \square sp'_2)]$ and then $[\sigma(sp_1) \square \sigma(sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1) \square \sigma_{\mathrm{rev}}(sp'_2)]$. From Lemma 1.16, $[\sigma(sp_1)\ \sigma(sp(i))\ \sigma(sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1)\ \mathrm{rev}(\sigma(sp(i)))\ \sigma_{\mathrm{rev}}(sp'_2)]$. Beause $\mathrm{rev}(\sigma(sp(i))) = \sigma_{\mathrm{rev}}(sp(i))$, $[\sigma(sp_1\ sp(i))\ sp_2)] = [\sigma_{\mathrm{rev}}(sp'_1\ sp(i)\ sp'_2)]$. $\square$

Definition 1.19 (Ring pattern match simulated (1))**.** For $sp \in$ **SP** and $seq \in$ **Seq**, pattern match between sp and $[[seq]]$ is to find all substitutions σ such that $\sigma(sp) = seq'$ for some $seq' \in$ **Seq**. Let $sp =?= [[seq]]$ be the set of all such substitutions.

Lemma 1.20. *For any sequence pattern $sp \in$ **SP** and any sequence $seq \in$ **Seq**, $([sp] =?= [seq]) = (sp =?= [[seq]])$.*

Proof. Let $\sigma \in ([sp] =?= [seq])$. $[\sigma(sp)] = [seq]$ by Definition 1.13. $\sigma(sp) \in [[seq]]$ due to Lemma 1.17. Thus, $\sigma \in (sp =?= [[seq]])$.

Let $\sigma \in (sp =?= [[seq]])$. Let $seq' \in [[seq]]$ such that $\sigma(sp) = seq'$. $[seq'] = [seq]$ due to Lemma 1.17 and then $[\sigma(sp)] = [seq]$. Hence, $\sigma \in ([sp] =?= [seq])$ $\square$

Definition 1.21 (Ring pattern match simulated (2))**.** For $sp \in$ **SP** and $seq \in$ **Seq**, pattern match between $[[sp]]$ and seq is to find all substitutions σ such that $\sigma'(sp') = seq$ for some substitution σ' and some $sp' \in [[sp]]$ and if $sp \circlearrowleft sp'$, then $\sigma = \sigma'$ and if $sp \circlearrowright sp'$, then $\sigma = \sigma'_{\mathrm{rev}}$. Let $[[sp]] =?= seq$ be the set of all such substitutions.

Note that $([[sp]] =?= seq) \subset ([sp] =?= [seq])$ but $([sp] =?= [seq]) \not\subset ([[sp]] =?= seq)$.

Lemma 1.22. *For any sequence pattern $sp \in$ **SP**, any sequence $seq \in$ **Seq** and any substitution $\sigma \in (sp =?= [[seq]])$, there exist σ' and a sequence $seq' \in [[seq]]$ such that $\sigma = \mathrm{join}(\sigma')$, $\sigma'(\mathrm{split}(sp)) = seq'$ and there exists $sp' \in [[\mathrm{split}(sp)]]$ such that $\sigma'(sp') = seq$. Besides, $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$.*

Proof. Let sp be $ES_1\ ES_2 \dots ES_m$ and seq be $e_1\ e_2 \dots e_n$.

If there exists $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is $\ldots e_n\ e_1 \ldots$ or $\ldots e_1\ e_n \ldots$, ES_i is a sequence variable S that is replaced with $\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)$ in $\mathrm{split}(sp)$.

If $\sigma(S)$ is $\ldots e_n\ e_1 \ldots$, then $\sigma'(\mathrm{sv}(S, 0))$ is $\ldots e_n$, $\sigma'(\mathrm{sv}(S, 1))$ is $e_1 \ldots$ and $\sigma'(\mathrm{sv}(S', 0))$ is $\sigma(S')$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any other sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma')$ and $\sigma'(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{sv}(S, 1)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})\ \mathrm{sv}(S, 0)$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'(sp') = seq$. Therefore, $\sigma' \in ([[\mathrm{split}(sp)]] =?= seq)$ because of $sp \circlearrowleft sp'$ from Definition 1.21. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 1.4.

If $\sigma(S)$ is $\ldots e_1\ e_n \ldots$, then $\sigma'(\mathrm{sv}(S, 0))$ is $\mathrm{rev}(\ldots e_1)$, $\sigma'(\mathrm{sv}(S, 1))$ is $\mathrm{rev}(e_n \ldots)$ and $\sigma'(\mathrm{sv}(S', 0))$ is $\mathrm{rev}(\sigma(S'))$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any other sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma'_{\mathrm{rev}})$ and $\sigma'_{\mathrm{rev}}(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{rev}(\mathrm{sv}(S, 1)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})\ \mathrm{sv}(S, 0))$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'_{\mathrm{rev}}(sp') = seq$. Therefore, $\sigma'_{\mathrm{rev}} \in ([[\mathrm{split}(sp)]] =?= seq)$ because of $sp \circlearrowleft sp'$ from Definition 1.21. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 1.4.

If there exists no $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is $\ldots e_n\ e_1 \ldots$ or $\ldots e_1\ e_n \ldots$, there must be $i \in \{1, \ldots, m\}$ such that $\sigma(ES_i)$ is e_1, $e_1 \ldots$ or $\ldots e_1$. If $\sigma(ES_i)$ is e_1, there are two possible cases: (1) $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = seq$ and (2) $\sigma(ES_i\ ES_{i-1} \ldots ES_{i+1}) = seq$.

Case (1) can be treated in the same way as the case in which $\sigma(ES_i)$ is $e_1 \ldots$. In either case, $\sigma'(\mathrm{sv}(S', 0))$ is $\sigma(S')$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma')$ and $\sigma'(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{split}(ES_i)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'(sp') = seq$. Therefore, $\sigma' \in ([[\mathrm{split}(sp)]] =?= seq)$ because of $sp \circlearrowleft sp'$ from Definition 1.21. Hence $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$ from Definition 1.4.

Case (2) can be treated in the same way as the case in which $\sigma(ES_i)$ is $\ldots e_1$. In either case, $\sigma'(\mathrm{sv}(S', 0))$ is $\mathrm{rev}(\sigma(S'))$ and $\sigma'(\mathrm{sv}(S', 1))$ is ε for any sequence variable S' in sp, and $\sigma'(E) = \sigma(E)$ for any element variable E in sp. By the construction of σ', $\sigma = \mathrm{join}(\sigma'_{\mathrm{rev}})$ and $\sigma'_{\mathrm{rev}}(\mathrm{split}(sp)) = \sigma(sp)$, where $\sigma(sp) \in [[seq]]$. Let sp' be $\mathrm{rev}(\mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})\ \mathrm{split}(ES_i))$. Then, $sp' \in [[\mathrm{split}(sp)]]$ and $\sigma'_{\mathrm{rev}}(sp') = seq$. Therefore, $\sigma'_{\mathrm{rev}} \in ([[\mathrm{split}(sp)]] =?= seq)$ because of $sp \circlearrowleft sp'$ from Definition 1.21. □

Lemma 1.23. *For any sequence pattern $sp \in \mathbf{SP}$, any sequence $seq \in \mathbf{Seq}$ and any substitution $\sigma \in \mathrm{join}([[\mathrm{split}(sp)]] =?= seq)$, $\sigma \in ([sp] =?= [seq])$.*

Proof. Let sp be $ES_1\ ES_2 \ldots ES_m$ and seq be $e_1\ e_2 \ldots e_n$. Let σ' be an arbitrary substituion in $([[\mathrm{split}(sp)]] =?= seq)$ from which σ is constructed, namely that $\sigma = \mathrm{join}(\sigma')$. Let $sp' \in [[\mathrm{split}(sp)]]$ such that $\sigma''(sp') = seq$, if $\mathrm{split}(sp) \circlearrowleft sp'$, then $\sigma' = \sigma''$ and if $\mathrm{split}(sp) \circlearrowleft sp'$, then $\sigma' = \sigma''_{\mathrm{rev}}$. There are four possible cases: (1) sp' is $\mathrm{split}(E_i)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})$, (2) sp' is $\mathrm{rev}(\mathrm{split}(E_i))\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))$. (3) sp' is $\mathrm{sv}(S, 1)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1})\ \mathrm{sv}(S, 0)$ and (4) sp' is $\mathrm{sv}(S, 0)\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))\ \mathrm{sv}(S, 1)$.

(1) For each ES_j for $j = 1, 2, \ldots, m$, we calculate $\sigma''(\mathrm{split}(ES_j))$ and $\sigma(ES_j)$. There are three possible cases: (1.1) ES_j is an element e, (1.2) ES_j is an element variable E and (1.3) ES_j is a sequence variable S. (1.1) $\sigma''(\mathrm{split}(e)) = e$ and $\sigma(e) = e = \sigma''(\mathrm{split}(e))$. (1.2) $\sigma''(\mathrm{split}(E)) = \sigma''(E)$ and $\sigma(E) = (\mathrm{join}(\sigma''))(E) = \sigma''(E) = \sigma''(\mathrm{split}(E))$. (1.3) $\sigma''(\mathrm{split}(S))$ and $\sigma(S)$ are calculated as follows:

$$\sigma''(\mathrm{split}(S)) = \sigma''(\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1))$$

$$\begin{aligned}&\sigma(S) = (\mathrm{join}(\sigma''))(S) = \sigma''(\mathrm{sv}(S, 0))\ \sigma''(\mathrm{sv}(S, 1))\\&= \sigma''(\mathrm{split}(S))\end{aligned}$$

Therefore, $\sigma(ES_j) = \sigma''((\mathrm{split}(ES_j)))$ and then $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1})$ is calculated as follows:

$$\begin{aligned}&\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = \sigma(ES_i)\ \sigma(ES_{i+1}) \ldots \sigma(ES_{i-1})\\&= \sigma''((\mathrm{split}(ES_i)))\ \sigma''((\mathrm{split}(ES_{i+1}))) \ldots \sigma''((\mathrm{split}(ES_{i-1})))\\&= \sigma''(\mathrm{split}(ES_i)\ \mathrm{split}(ES_{i+1}) \ldots \mathrm{split}(ES_{i-1}))\\&= \sigma''(sp')\end{aligned}$$

Because $\sigma''(sp') = seq$ from the assumption, $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = seq$. Because $sp \circlearrowleft ES_i\ ES_{i+1} \ldots ES_{i-1}$, $[\sigma(sp)] = [\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1})]$ from Lemma 1.18. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$.

(2) For each ES_j for $j = 1, 2, \ldots, m$, we calculate $\sigma''(\mathrm{rev}(\mathrm{split}(ES_j)))$ and $\sigma(ES_j)$. There are three possible cases: (2.1) ES_j is an element e, (2.2) ES_j is an element variable E and (2.3) ES_j is a sequence variable S. (2.1) $\sigma''(\mathrm{rev}(\mathrm{split}(e))) = e$ and $\sigma(e) = e = \sigma''(\mathrm{rev}(\mathrm{split}(e))) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(e))))$. (2.2) $\sigma''(\mathrm{rev}(\mathrm{split}(E))) = \sigma''(E)$ and $\sigma(E) = (\mathrm{join}(\sigma''_{\mathrm{rev}}))(E) = \sigma''(E) = \sigma''(\mathrm{rev}(\mathrm{split}(E))) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(E))))$. (2.3) $\sigma''(\mathrm{rev}(\mathrm{split}(S)))$ and $\sigma(S)$ are calculated as follows:

$$\begin{aligned}&\sigma''(\mathrm{rev}(\mathrm{split}(S))) = \sigma''(\mathrm{rev}(\mathrm{sv}(S, 0)\ \mathrm{sv}(S, 1)))\\&= \sigma''(\mathrm{sv}(S, 1)\ \mathrm{sv}(S, 1))\end{aligned}$$

$$\begin{aligned}&\sigma(S) = (\mathrm{join}(\sigma''_{\mathrm{rev}}))(S) = \sigma''_{\mathrm{rev}}(\mathrm{sv}(S, 0))\ \sigma''_{\mathrm{rev}}(\mathrm{sv}(S, 1))\\&= \mathrm{rev}(\sigma''(\mathrm{sv}(S, 0)))\ \mathrm{rev}(\sigma''(\mathrm{sv}(S, 1)))\\&= \mathrm{rev}(\sigma''(\mathrm{sv}(S, 1))\ \sigma''(\mathrm{sv}(S, 0))) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(S))))\end{aligned}$$

Therefore, $\sigma(ES_j) = \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_j))))$ and then $\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i)$ is calculated as follows:

$$\begin{aligned}&\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i) = \sigma(ES_{i+1}) \ldots \sigma(ES_{i-1})\ \sigma(ES_i)\\&= \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_{i+1})))) \ldots\\&\quad \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_{i-1}))))\ \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_i))))\\&= \mathrm{rev}(\sigma''(\mathrm{rev}(\mathrm{split}(ES_i)))\\&\qquad \sigma''(\mathrm{rev}(\mathrm{split}(ES_{i-1}))) \ldots \sigma''(\mathrm{rev}(\mathrm{split}(ES_{i+1}))))\\&= \mathrm{rev}(\sigma''(sp'))\end{aligned}$$

Because $\sigma''(sp') = seq$ from the assumption, $\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i) = \mathrm{rev}(seq)$. Because $sp \circlearrowleft ES_{i+1} \ldots ES_{i-1}\ ES_i$, $[\sigma(sp)] = [\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i)]$ from Lemma 1.18. Moreover, $[seq] = [\mathrm{rev}(seq)]$ from Proposition 1.10. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$.

(3) $\mathrm{rtt}(sp')$ is calculated as follows:

$$\begin{aligned}&\mathrm{rtt}(sp')\\ &= \mathrm{sv}(S,0)\ \mathrm{sv}(S,1)\ \mathrm{rev}(\mathrm{split}(ES_{i+1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i-1}))\\ &= \mathrm{split}(ES_i)\ \mathrm{rev}(\mathrm{split}(ES_{i+1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i-1}))\end{aligned}$$

Because $\sigma''(sp') = seq$, there exists a natural number k such that $\sigma''(\mathrm{rtt}(sp')) = \mathrm{rtt}^k(seq)$. As what has been done for case (1), we have $\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}) = \mathrm{rtt}^k(seq)$. Because $sp \circlearrowleft ES_i\ ES_{i+1} \ldots ES_{i-1}$, $[\sigma(sp)] = [\sigma(ES_i\ ES_{i+1} \ldots ES_{i-1}]$ from Lemma 1.18. Moreover, $[seq] = [\mathrm{rtt}^k(seq)]$ from Proposition 1.10. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] =?= [seq])$.

(4) $\mathrm{rtt}(sp')$ is calculated as follows:

$$\begin{aligned}&\mathrm{rtt}(sp')\\ &= \mathrm{sv}(S,1)\ \mathrm{sv}(S,0)\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))\\ &= \mathrm{rev}(\mathrm{split}(ES_i))\ \mathrm{rev}(\mathrm{split}(ES_{i-1})) \ldots \mathrm{rev}(\mathrm{split}(ES_{i+1}))\end{aligned}$$

Because $\sigma''(sp') = seq$, there exists a natural number k such that $\sigma''(\mathrm{rtt}(sp')) = \mathrm{rtt}^k(seq)$. As what has been done for case (2), we have $\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i) = \mathrm{rev}(\mathrm{rtt}^k(seq))$. Because $sp \circlearrowleft ES_{i+1} \ldots ES_{i-1}\ ES_i$, $[\sigma(sp)] = [\sigma(ES_{i+1} \ldots ES_{i-1}\ ES_i)]$ from Lemma 1.18. Moreover, $[seq] = [\mathrm{rev}(\mathrm{rtt}^k(seq))]$ from Proposition 1.10. Thus, $[\sigma(sp)] = [seq]$ and then $\sigma \in ([sp] = ?= [seq])$. ◻

Theorem 1.24. *For any sequence pattern $sp \in$ **SP** and any sequence $seq \in$ **Seq**, $\mathrm{join}([[\mathrm{split}(sp)]] =?= seq) = ([sp] =?= [seq])$.*

Proof. It is derived from Lemma 1.20, Lemma 1.22 and Lemma 1.23. ◻