Exercises-Interpolation-Solutions

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1. Use the table below for $\sqrt{x}$ and approximate the value for $x = 0.15$ using a 3rd degree polynomial and Newton forwards differences based on $x \in \{0.0, 0.1, 0.2, 0.3\}$:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0.0	0.000000	0.316228	-0.185242	0.154765	-0.140064
0.1	0.316228	0.130986	-0.030477	0.014701	
0.2	0.447214	0.100509	-0.015776		
0.3	0.547723	0.084733			
0.4	0.632456				

$$f(x) \approx 0.0 + \frac{0.316228}{0.1}x + \frac{-0.185242}{0.1^2 2!}x(x-0.1) + \frac{0.154765}{0.1^3 3!}x(x-0.1)(x-0.2)$$

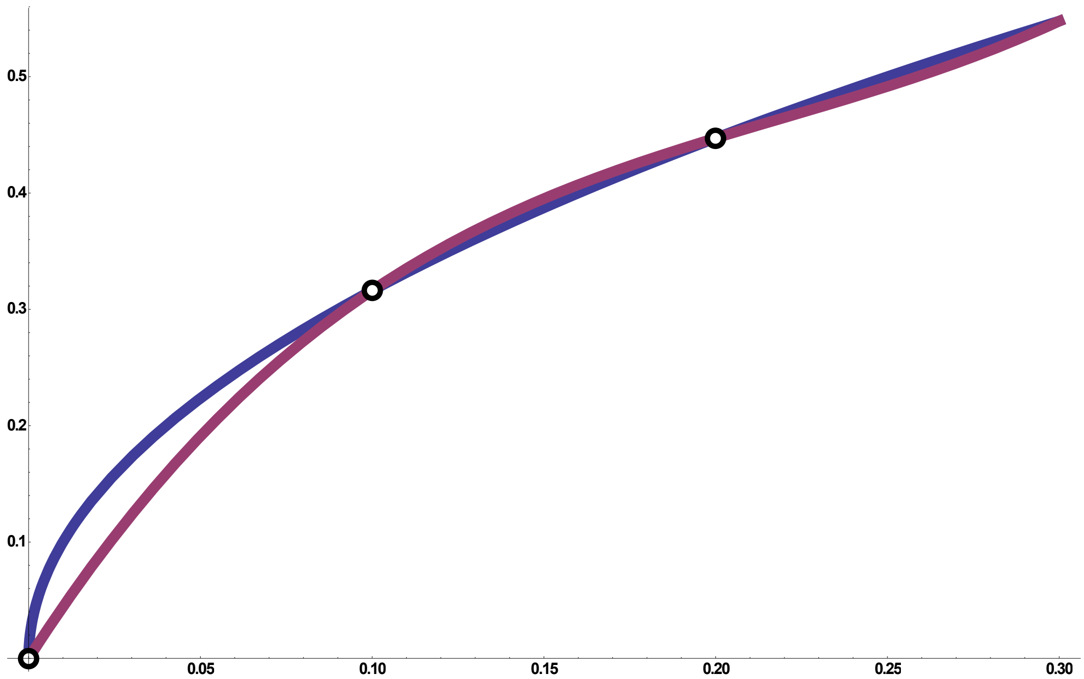
$$= 25.7942 \cdot x^3 - 17.0004 \cdot x^2 + 4.60437 \cdot x$$

$$f(0.15) \approx 3.16228 \cdot 0.15 - 9.2621 \cdot 0.15 \cdot 0.05 + 25.7941667 \cdot 0.15 \cdot 0.05 \cdot -0.05$$

$$= 25.7942 \cdot 0.15^3 - 17.0004 \cdot 0.15^2 + 4.60437 \cdot 0.15 = 0.39520$$

```
> library(Ryacas)
> x <- ysym("x")      # treat x as a symbolic variable
> f <- 0.316228/0.1*x-0.185242/(0.1^2*2)*x*(x-0.1)+0.154765/(0
.1^3*6)*x*(x-0.1)*(x-0.2)
> # execute the command Expand(f) in yacas
> y_fn(f,"Expand")
y: 0.2579416666e2*x^3-17.00035*x^2+4.6043733333*x
```

Relative error: $\dfrac{|0.39520 - \sqrt{0.15}|}{\sqrt{0.15}} = 0.0204$



Note that the polynomial coincides with the function $\sqrt{x}$ in the points $x \in \{0.0, 0.1, 0.2\}$.

2. Determine the 3rd degree Lagrange polynomial based on $x_i \in \{0.0, 0.1, 0.2, 0.3\}$ shown in question 1 and approximate the value for $x = 0.15$.

$$
\begin{aligned}
f_3(x) &= \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) \\
&\quad + \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4) \\
&= \frac{(x-0.1)(x-0.2)(x-0.3)}{(0.0-0.1)(0.0-0.2)(0.0-0.3)} 0.0 + \frac{(x-0.0)(x-0.2)(x-0.3)}{(0.1-0.0)(0.1-0.2)(0.1-0.3)} 0.316228 \\
&\quad + \frac{(x-0.0)(x-0.1)(x-0.3)}{(0.2-0.0)(0.2-0.1)(0.2-0.3)} 0.447214 + \frac{(x-0.0)(x-0.1)(x-0.2)}{(0.3-0.0)(0.3-0.1)(0.3-0.2)} 0.547723 \\
&= 158.114 \cdot x(x-0.2)(x-0.3) - 223.607 \cdot x(x-0.1)(x-0.3) + 91.2871 \cdot x(x-0.1)(x-0.2) \\
&= 25.7942 \cdot x^3 - 17.0004 \cdot x^2 + 4.60437 \cdot x
\end{aligned}
$$

Since we find the same 3rd degree polynomial as in question 1, the approximation is the same!

```
> f <- x*(x-0.2)*(x-0.3)/(0.1*-0.1*-0.2)*0.316228+x*(x-0.1)*(x
-0.3)/(0.2*0.1*-0.1)*0.447214+x*(x-0.1)*(x-0.2)/(0.3*0.2*0.1)*
0.547723
> fexpand<-y_fn(f,"Expand")
> coef<-y_fn(fexpand,"Coef","x","0 .. 3")
> coefNum<-as_r(y_fn(coef,"N","6"))
> formatC(coefNum[4:1],digits=6,format="f")
[1] "25.794167"  "-17.000350" "4.604367"   "0.000000"
```

3. Suppose we want to apply quadratic splines based on 3 points: (x_1, y_1), (x_2, y_2) and (x_3, y_3). Hence, we have to determine the coefficients of the following two spline polynomials ($i = 1, 2$):

$$s_i(x) = a_i + (x - x_i)b_i + (x - x_i)^2 c_i.$$

a. Explain the idea behind (quadratic) splines.

The splines method fits polynomials that go through the data points. So, one polynomial for the interval (x_1, x_2) and a second one for the interval (x_2, x_3). Moreover, it is required that the derivate(s) is/are the same for interior points. In this exercise, there is only one interior point, i.e. x_2. A quadratic spline has non-trivial derivatives up to the second order, i.e.

$$\frac{d^n}{dx^n} s_i(x) = 0 \text{ for } n > 2.$$

b. How many unknown parameters are there in total?

State the system of equations that determine the coefficients. Suppose that the x-points are equidistant, i.e. $x_3 - x_2 = x_2 - x_1$, so $h = x_{i+1} - x_i$ can be used. Furthermore, impose the restriction that the first and second-order derivatives match in the interior points.

There are 6 unknown coefficients: $a_1, b_1, c_1, a_2, b_2, c_2$.

$$s_1(x) = a_1 + (x - x_1)b_1 + (x - x_1)^2 c_1 \rightarrow s_1'(x) = b_1 + 2(x - x_1)c_1 \rightarrow s_1''(x) = 2c_1$$

$$s_2(x) = a_2 + (x - x_2)b_2 + (x - x_2)^2 c_2 \rightarrow s_2'(x) = b_2 + 2(x - x_2)c_2 \rightarrow s_2''(x) = 2c_2$$

There are 6 equations:

(i) $s_1(x_1) = y_1 \rightarrow a_1 = y_1$

(ii) $s_1(x_2) = y_2 \rightarrow a_1 + hb_1 + h^2 c_1 = y_2$

(iii) $s_2(x_2) = y_2 \rightarrow a_2 = y_2$

(iv) $s_2(x_3) = y_3 \rightarrow a_2 + hb_2 + h^2 c_2 = y_3$

(v) $s_1^{'}(x_2) = s_2^{'}(x_2) \rightarrow b_1 + 2hc_1 = b_2$

(vi) $s_1^{''}(x_2) = s_2^{''}(x_2) \rightarrow 2c_1 = 2c_2$

4. Suppose we want to estimate the parameter *a* according to the least squares method in the model:

$$f_L(x) = a\sin(x).$$

Based on the data points $\{(y_i, x_i), i = 1,...,n\}$, the sum of squared residuals can be defined:

$$SSR(a) = \sum_{i=1}^{N} (y_i - a\sin(x_i))^2 .$$

a. Determine the Normal equation, i.e. $\dfrac{\partial SSR(a)}{\partial a} = 0.$

$$\frac{\partial SSR(a)}{\partial a} = \sum_{i=1}^{N} 2(y_i - a\sin(x_i)) \cdot -\sin(x_i) = 0$$

$$\rightarrow -\sum y_i \sin(x_i) + a\sum \sin^2(x_i) = 0 \rightarrow a = \frac{\sum y_i \sin(x_i)}{\sum \sin^2(x_i)}$$

b. Suppose we have the following observations:

$$\{(1.26221,1),(1.36395,2),(0.21168,3)\}.$$

Determine an estimate for the parameter *a*.

$$a = \frac{\sum y_i \sin(x_i)}{\sum \sin^2(x_i)} = \frac{1.26221 \cdot \sin(1) + 1.36395 \cdot \sin(2) + 0.21168 \cdot \sin(3)}{\sin^2(1) + \sin^2(2) + \sin^2(3)} = \frac{2.33222}{1.55481} = 1.5000$$

5. Suppose we want to estimate the parameters *a* and *b* according to the least squares method in the model:

$$f_L(x) = a\sin(bx).$$

Based on the data points $\{(y_i, x_i), i = 1,...,n\}$, the sum of squared residuals can be defined:

$$SSR(a,b) = \sum_{i=1}^{N} (y_i - a\sin(bx_i))^2 .$$

a. Determine the Normal equations, i.e.

$$\frac{\partial SSR(a,b)}{\partial a} = 0 \quad \text{and} \quad \frac{\partial SSR(a,b)}{\partial b} = 0.$$

$$\frac{\partial SSR(a,b)}{\partial a} = \sum_{i=1}^{N} 2(y_i - a\sin(bx_i)) \cdot -\sin(bx_i) = 0$$

$$\rightarrow -\sum y_i \sin(bx_i) + a\sum \sin^2(bx_i) = 0$$

$$\frac{\partial SSR(a,b)}{\partial b} = \sum_{i=1}^{N} 2(y_i - a\sin(bx_i)) \cdot -ax_i\cos(bx_i) = 0$$

$$\rightarrow -a\sum y_i x_i \cos(bx_i) + a^2 \sum x_i \sin(bx_i)\cos(bx_i) = 0$$

b. Given a sample $\{(y_i, x_i), i = 1, \ldots, n\}$, how would you solve for a and b numerically?

For instance, Newton-Raphson:

$$F(a,b) = \begin{pmatrix} -\sum y_i \sin(bx_i) + a \sum \sin^2(bx_i) \\ -a \sum y_i x_i \cos(bx_i) + a^2 \sum x_i \sin(bx_i)\cos(bx_i) \end{pmatrix}$$

Derivative: $G(a,b) = \begin{pmatrix} \dfrac{\partial f_1(a,b)}{\partial a} & \dfrac{\partial f_1(a,b)}{\partial b} \\ \dfrac{\partial f_2(a,b)}{\partial a} & \dfrac{\partial f_2(a,b)}{\partial b} \end{pmatrix}$

$$\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} - G(a_i, b_i)^{-1} F(a_i, b_i)$$