Matrices

- Matrix operations
- Special Matrices
- Properties of Matrices and Systems of Equations
  - Determinants of matrices
- Gauss elimination
- Eigenvalues and Eigenvectors
- Matrix Decomposition
  - $A = VDV^{-1}$
- Characteristic polynomial
  - $p(\lambda) = \det(A - \lambda I)$
- Application: Markov model

Matrix Operations

Addition $C = A + B$ and ***subtraction*** $(D = A - B)$ works in R without special functions [provided that the dimensions agree]

```
> A<-matrix(1:4,2,2,byrow=T); B<-matrix(4:1,2,2,byrow=T)
> C<-A+B
> C
     [,1] [,2]
[1,]    5    5
[2,]    5    5
```

Multiplication and division:

- `C=A/B` yields $c_{i,j} = a_{i,j} / b_{i,j}$
- `C=A*B` yields $c_{i,j} = a_{i,j} b_{i,j}$ (elementwise)
- `C=A%*%B` calculates the matrix multiplication $A \cdot B$

```
> C<-A/B; C
> C<-A*B; C
> C<-A%*%B; C
```

Matrix Operations

Power raising:

- `C=A^n` yields $c_{i,j} = a_{i,j}^n$ (elementwise)
- there is no build-in R function that calculates $A^n = A \cdot \ldots \cdot A$
- the package **matrixcalc** provides `matrix.power(x,k)`

```
> matrix.power(A,2)
```

Many operations can be applied to matrices, like `abs`, `sign`, `cos`, `sin` etc.

Special Matrices:

`diag(n)`: $n \times n$ identity matrix I_n

`diag(x)`: assigns vector `x` on the main diagonal

```
> diag(1:3)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    2    0
[3,]    0    0    3
```

Intermezzo: Special Matrices

- Matrices with $m = n$ are called ***square matrices***
- There are various special kind of square matrices:

Symmetric

$$[A] = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

Diagonal

$$[A] = \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$$

Identity

$$[A] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Upper triangular matrix

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix}$$

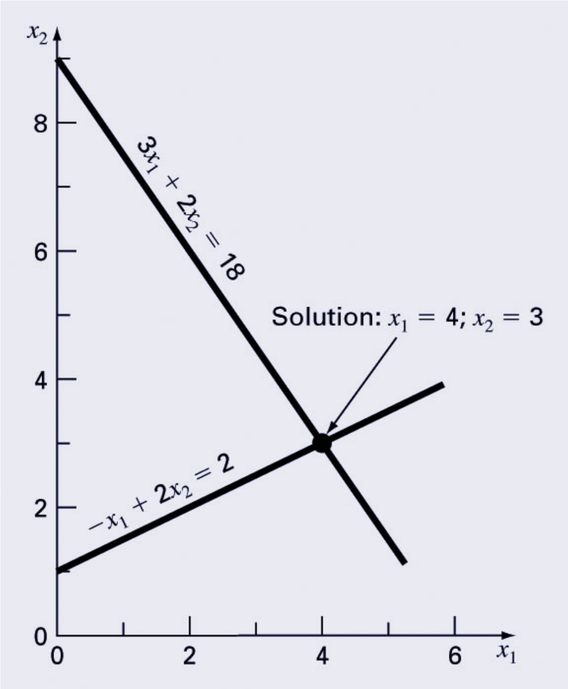
Lower triangular matrix

$$[A] = \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Band (tridiagonal matrix)

$$[A] = \begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

Properties of Matrices and Systems of Equations

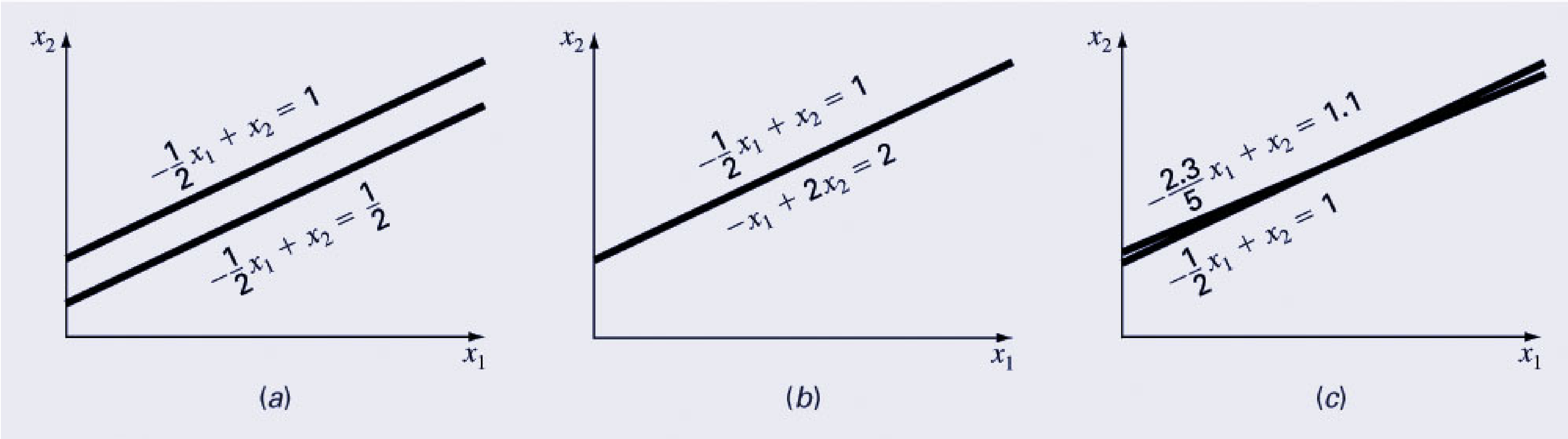


There is not always a unique solution:

a) No solution

b) Infinitely many solutions

c) System is ill conditioned (the rate at which the solution changes with respect to a change in the constant)



Figures 9.1 (top) and 9.2 (bottom) from Chapra, S. C. (2007) Applied Numerical Methods With Matlab: For Engineers And Scientists. McGraw Hill.

Properties of Matrices and Systems of Equations

Unique solution: $\left.\begin{aligned} x - y &= 1 \\ x + y &= 2 \end{aligned}\right\}$ $x = 3/2, y = 1/2$

No solutions: $\left.\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}\right\}$

Infinitely many solutions: $\left.\begin{aligned} x + y &= 1 \\ x + y &= 1 \end{aligned}\right\}$ $x = \alpha, y = 1 - \alpha$

Rank of a matrix: the maximum number of linear independently rows or columns of a matrix

`qr(A)$rank`: determine the rank of matrix A, if rank(A)=dim(A), then the matrix has full rank

If the matrix has full rank, A^{-1} exists, such that $AA^{-1} = A^{-1}A = I$

System of Linear Equations

Matrices offer a concise notation for displaying and solving systems of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$Ax = b$$

Characterization of the solution(s) of linear system Ax = b and the augmented matrix (A|b):

1. If $rank(A) = rank(A\,|\,b)$, then the system has solutions
2. If the rank is equal to the number of variables, i.e. $rank(A) = rank(A\,|\,b) = \dim(A)$, then the system has a unique solution
3. If the rank is less than the number of variables, i.e. $rank(A) = rank(A\,|\,b) < \dim(A)$,then there are an infinite number of solutions. The difference between these numbers gives the number of degrees of freedom.

Determinants of Matrices

A square matrix A has an inverse if and only if $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \qquad [\operatorname{adj}(\mathrm{A}) = C^T, C_{ij} = (-1)^{i+j} M_{ij};\ C \text{ is cofactor}; M \text{ is minor}]$$

Determinant 2×2 matrix: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant 3×3 matrix: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Expand along the 1st row: $\det(A) = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ $\qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Elementary Row Operations

The following 3 operations do ***not*** change a system of equations:

1. Exchange rows: $R_1 \rightleftharpoons R_3$
   - this involves changing the order of the equations
2. Multiply a row by a constant ($\neq 0$): $R_2 \to \lambda R_2$
   - this corresponds to the multiplication of an equation
3. Combining rows: e.g. $R_2 \to R_2 - a_{21}R_1 / a_{11}$
   - This corresponds to adding/subtracting equations

Ex.
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$R_2 \to R_2 - a_{21} / a_{11}R_1$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ \cancel{(a_{21} - \tfrac{a_{21}}{a_{11}}a_{11})x_1} + (a_{22} - \tfrac{a_{21}}{a_{11}}a_{12})x_2 + (a_{23} - \tfrac{a_{21}}{a_{11}}a_{13})x_3 &= b_2 - \tfrac{a_{21}}{a_{11}}b_1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Naive Gauss Elimination

$[A:I]:$
$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - c/aR_1:$
$$\begin{bmatrix} a & b & 1 & 0 \\ 0 & d-\frac{c}{a}b & -\frac{c}{a} & 1 \end{bmatrix}$$

$R_2 \to R_2/(d-\frac{c}{a}b) = \dfrac{aR_2}{ad-bc}:$
$$\begin{bmatrix} a & b & 1 & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$R_1 \to R_1 - bR_2:$
$$\begin{bmatrix} a & 0 & 1+\frac{bc}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$R_1 \to R_1/a:$
$$\begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = [I:A^{-1}]$$

`rref(cbind(A,diag(n)))`: carries out Gauss elimination automatically

Naive Gauss Elimination (more efficient)

- ***Forward elimination***
  - Starting with the first row, + or – multiples of that row such that the first coefficient of the second and higher rows are eliminated
  - Continue this process with the 2nd row and remove the second coefficient of the 3rd and higher rows
  - Stop when a triangle matrix is reached
- ***Back substitution***
  - Starting with the *last* row, solve for the unknown and substitute the value in the nearest rows
  - Because the matrix is upper triangle, each row will contain only one additional unknown

$$\left.\begin{matrix}\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \\ \downarrow \\ \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]\end{matrix}\right\}\text{(a) Forward elimination}$$

$$\downarrow$$

$$\left.\begin{aligned} x_3 &= b''_3/a''_{33} \\ x_2 &= (b'_2 - a'_{23}x_3)/a'_{22} \\ x_1 &= (b_1 - a_{13}x_3 - a_{12}x_2)/a_{11} \end{aligned}\right\}\text{(b) Back substitution}$$

This is the method implemented in R: `solve(A)`

Figure 9.3 from Chapra, S. C. (2007) Applied Numerical Methods With Matlab: For Engineers And Scientists. McGraw Hill.

Eigenvalues and Eigenvectors

An eigenvalue $\lambda(\neq 0)$ and eigenvector $x(\neq 0)$ is such that

$$Ax = \lambda x, \qquad \textbf{\textit{i.e.}}\ (A - \lambda I)x = 0$$

The equation $(A - \lambda I)x = 0$ has non-trivial ($\neq$0) solutions if $\det(A - \lambda I) = 0$

Homogeneous linear system, i.e. $Ax = 0$, has always a trivial solution $x = 0$

Theorem: if the number of equations is the number of unknowns, then the homogenous system $Ax = 0$ has a non-trivial solution if and only if the coefficient matrix A is singular (non-invertible)

Example 1:

$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, then eigenvector ($x_1$) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with eigenvalue (λ_1) 3 and $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $\lambda_2 = -2$

Example 2:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 \cdot 3$$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$$

with solutions: $\lambda = -1, \ \ \lambda = 4$

Eigenvalues and Eigenvectors (cont.)

$$\lambda = 4: \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad \begin{matrix} x + 2y = 4x \\ 3x + 2y = 4y \end{matrix} \Rightarrow \quad \begin{matrix} 2y = 3x \\ 2y = 3x \end{matrix} \rightarrow x = \frac{2}{3} y$$

$$\text{Suppose } y = -3 \rightarrow x = -2 \rightarrow e_{\lambda=4} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}; \;\; \mathrm{R} \quad \hat{e}_{\lambda=4} = \frac{e_{\lambda=4}}{|e_{\lambda=4}|} = \frac{1}{\sqrt{(-2)^2 + (-3)^2}} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{13} \\ -3/\sqrt{13} \end{pmatrix}$$

$$\lambda = -1: \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad \begin{matrix} x + 2y = -x \\ 3x + 2y = -y \end{matrix} \Rightarrow \quad \begin{matrix} x = -y \\ x = -y \end{matrix} \rightarrow x = -y$$

$$\text{Suppose } y = 1 \rightarrow x = -1 \rightarrow e_{\lambda=-1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \;\; \mathrm{R} \quad \hat{e}_{\lambda=-1} = \frac{e_{\lambda=-1}}{|e_{\lambda=-1}|} = \frac{1}{\sqrt{(-1)^2 + 1^2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Eigenvalues and Eigenvectors (cont.)

R $\hat{e}_{\lambda=4} = \dfrac{e_{\lambda=4}}{|e_{\lambda=4}|} = \dfrac{1}{\sqrt{(-2)^2+(-3)^2}}\begin{pmatrix}-2\\-3\end{pmatrix} = \begin{pmatrix}-2/\sqrt{13}\\-3/\sqrt{13}\end{pmatrix}$

R $\hat{e}_{\lambda=-1} = \dfrac{e_{\lambda=-1}}{|e_{\lambda=-1}|} = \dfrac{1}{\sqrt{(-1)^2+1^2}}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

```
> A<-matrix(c(1,3,2,2),2,2)
> r<-eigen(A)
> V<-r$vectors; V
           [,1]       [,2]
[1,] -0.5547002 -0.7071068
[2,] -0.8320503  0.7071068
> lambda<-r$values; lambda
[1]  4 -1
```

Matrix Decomposition

Let A be a square (N×N) matrix with linearly independent vectors (so all eigenvalues are non-zero). Then A can be factorized as $A = VDV^{-1}$ where V is the square (N×N) matrix whose columns contain the eigenvectors of A and D a diagonal matrix whose diagonal elements are the corresponding eigenvalues

Note: $A^2 = AA = (VDV^{-1})(VDV^{-1})$

$$= VDV^{-1}VDV^{-1} = VDDV^{-1} = VD^2V^{-1}$$

It can be proven that $A^n = VD^nV^{-1}$ met $D^n = diag(\lambda_1^n, ..., \lambda_N^n)$

```
> matrix.power(A,5)
     [,1] [,2]
[1,]  409  410
[2,]  615  614
> V%*%diag(lambda^5)%*%solve(V)
     [,1] [,2]
[1,]  409  410
[2,]  615  614
```

Characteristic Polynomials

$$p(\lambda) = \det(A - \lambda I)$$

Ex. let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 \cdot 3$$

$$= 2 - \lambda - 2\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$$

```
> A<-matrix(c(1,3,2,2),2,2)
> pracma::charpoly(A)
[1]  1 -3 -4
```

A motivating example: Unemployment

Unemployment rates change over time as individuals gain or lose their employment. We consider a simple model, called a Markov model that describes the dynamics of unemployment using transitional probabilities.

In this model, we assume:

- If an individual is unemployed in a given week, the probability is p for this individual to be employed the following week, and $1 - p$ for him or her to stay unemployed

- If an individual is employed in a given week, the probability is q for this individual to stay employed the following week, and $1 - q$ for him or her to be unemployed

Markov Model for Unemployment

Let x_t be the ratio of individuals employed in week t, and let y_t be the ratio of individuals unemployed in week t. Then the week-on-week changes are given by these equations:

$$\begin{aligned} x_{t+1} &= qx_t + py_t \\ y_{t+1} &= (1-q)x_t + (1-p)y_t \end{aligned}$$

Note that these equations are linear, and can be written in matrix form as $v_{t+1} = Av_t$, where

$$A = \begin{pmatrix} q & p \\ 1-q & 1-p \end{pmatrix}, \quad v_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

We call A the transition matrix and v_t the state vector of the system.

Question:

- What is the long-term state of the system?
- Are there any equilibrium states? If so, will these equilibrium states be reached?

Long-term State of the System

The state of the system after t weeks is given by:

- $v_1 = Av_0$
- $v_2 = Av_1 = A(Av_0) = A^2 v_0$
- $v_3 = Av_2 = A(A^2 v_0) = A^3 v_0$
- $\Rightarrow v_t = A^t v_0$

For white males in the US in 1966, the probabilities where found to be $p = 0.136$ and $q = 0.998$. If the unemployment rate is 5% at $t = 0$, expressed by $x_0 = 0.95$ and $y_0 = 0.05$, the situation after 100 weeks would be

$$\begin{pmatrix} x_{100} \\ y_{100} \end{pmatrix} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}^{100} \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} = ?$$

We need eigenvalues and eigenvectors to compute A^{100} efficiently.

Steady states

Definition

A ***steady state*** is a state vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \geq 0$ and $x + y = 1$ such that $Av = v$. The last condition is an ***equilibrium condition***

Example

Find the steady state when $A = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}$.

Solution

The equation $Av = v$ is a linear system, since it can be written as

$$\begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0.998-1 & 0.136 \\ 0.002 & 0.864-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Steady states

Solution (Continued)

So, we see that the system has one degree of freedom, and can be written as

$$-0.002x + 0.136y = 0 \Rightarrow \begin{cases} x = 68y \\ y = \text{free variable} \end{cases}$$

The only solution that satisfies $x + y = 1$ is therefore given by

$$x = \frac{68}{69} \approx 0.986, \quad y = \frac{1}{69} \approx 0.014$$

In other words, there is an equilibrium or steady state of the system in which the unemployment is 1.4%. The question if this steady state will be reached is more difficult, but can be solved using ***eigenvalues***

Example: Diagonalization

Solving $\det(A-\lambda I)=0$:

$$|A-I_2\lambda| = \begin{vmatrix} 0.998-\lambda & 0.136 \\ 0.002 & 0.864-\lambda \end{vmatrix}$$

$$= (0.998-\lambda)(0.864-\lambda) - 0.136 \times 0.002$$

$$= \lambda^2 - 1.862\lambda + 0.862 = 0$$

gives

$$\lambda_1 = 1, \lambda_2 = \tfrac{431}{500} = 0.862$$

Eigenvector for $\lambda_1 = 1$:

$$\begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0.998-1 & 0.136 \\ 0.002 & 0.864-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.002 & 0.136 \\ 0.002 & -0.136 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 68y$$

Take e.g. $y=1$, then $x=68$

R: $$\hat{e}_{\lambda=1} = \frac{e_{\lambda=1}}{|e_{\lambda=1}|} = \frac{1}{\sqrt{68^2+1^2}}\begin{pmatrix} 68 \\ 1 \end{pmatrix} = \begin{pmatrix} 68/(5\sqrt{185}) \\ 1/(5\sqrt{185}) \end{pmatrix} \approx \begin{pmatrix} 0.99989189 \\ 0.01470429 \end{pmatrix}$$

Example: Diagonalization (Continued)

Eigenvector for $\lambda_2 = 0.862$:

$$\begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0.998-0.862 & 0.136 \\ 0.002 & 0.864-0.862 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.136 & 0.136 \\ 0.002 & 0.002 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = -y$$

Take e.g. $y = 1$, then $x = -1$

R: $\hat{e}_{\lambda=0.862} = \dfrac{e_{\lambda=0.862}}{|e_{\lambda=0.862}|} = \dfrac{1}{\sqrt{1^2+1^2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \approx \begin{pmatrix} -0.7071068 \\ 0.7071068 \end{pmatrix}$

```
> r<-eigen(A)
> V<-r$vectors; V
           [,1]       [,2]
[1,] 0.99989189 -0.7071068
[2,] 0.01470429  0.7071068

> lambda<-r$values; lambda
[1] 1.000 0.862
```

Example: Diagonalization (Continued)

For white males in the US in 1966, the probabilities where found to be $p = 0.136$ and $q = 0.998$. If the unemployment rate is 5% at $t = 0$, expressed by $x_0 = 0.95$ and $y_0 = 0.05$, the situation after 100 weeks would be $\begin{pmatrix} x_{100} \\ y_{100} \end{pmatrix} = \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix}^{100} \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} = ?$

We need eigenvalues and eigenvectors to compute A^{100} efficiently:

$$A^{100} = VD^{100}V^{-1} = V\begin{pmatrix} 1^{100} & 0 \\ 0 & 0.862^{100} \end{pmatrix}V^{-1} = V\begin{pmatrix} 1 & 0 \\ 0 & 0.0000003554075 \end{pmatrix}V^{-1}$$

```
> V%*%diag(lambda^100)%*%solve(V)
           [,1]      [,2]
[1,] 0.98550725 0.9855069
[2,] 0.01449275 0.0144931
> V%*%diag(lambda^100)%*%solve(V)%*%c(0.95,0.05)
           [,1]
[1,] 0.98550723
[2,] 0.01449277
```

Very close to the steady state!