Exercises-Matrices-Questions

1. Apply Gauss elimination to the following system:

$$(1):\quad 0.0030000x_1 + 59.14x_2 = 59.17$$
$$(2):\quad 5.291x_1 - 6.130x_2 = 46.78,$$

Use 4-digit rounding (i.e. 4 significant numbers):
$\pm 0.d_1 d_2 d_3 d_4 \times 10^n$, with $1 \le d_1 \le 9, 1 \le d_i \le 9$ for $i = 2,3,4.$
Compare the results with the exact solution $x_1 = 10.00$ and $x_2 = 1.000.$

2. The Fibonacci sequence also appears in the study of rabbit populations. Leonardo Fibonacci used the following rules:

- a newly born pair of rabbits, one male, one female, are put in a field and they produce another pair of rabbits
- rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits
- rabbits never die
- a mating pair always produces one new pair every month from the second month on.

The number of pairs of rabbits is equal to the number of new pairs plus the number of pairs alive last month: $F_k = F_{k-1} + F_{k-2}$, with $F_0 = 0$ and $F_1 = 1$. We get the Fibonacci sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

This can be written in matrix form as

$$\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix}.$$

If we define:

$$x_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{then we get } x_k = A^{k-1} x_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

a. Determine the eigenvalues of the A matrix.

b. Determine the 2 eigenvectors of the A matrix.

c. Let $V = \begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}$ contain the eigenvectors. Verify that $V^{-1} = \frac{1}{4\sqrt{5}} \begin{bmatrix} \sqrt{5}+1 & 2 \\ 1-\sqrt{5} & 2 \end{bmatrix}$ is indeed the inverse of V.

d. Verify that $D = V^{-1} A V$.

e. Given that $x_{k+1} = A^k x_1 = V D^k V^{-1} x_1$, calculate in succession:

(i) $V^{-1} x_1$, (ii) $D^k V^{-1} x_1$ and (iii) $V D^k V^{-1} x_1$.

Finally, derive an exact expression for F_k. Note that this is the second component of

$$x_{k+1} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}.$$

f. It is given that $\lambda_1 = \frac{1+\sqrt{5}}{2} = 1.618034$ and $\lambda_2 = \frac{1-\sqrt{5}}{2} = -0.618034.$ How could you approximate F_k? In what way does the rabbit population increase?