Exercises-Matrices-Solutions

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1. If we want to eliminate x_1 from the first equation, we have to multiply the 1st equation with

$m_{21} = \dfrac{5.291}{0.003000} = 1763.6\overline{6}$. Rounded this equals 1764.

Now, we get:

(1): $0.0030x_1 + 59.14x_2 \approx 59.17$

(2): $-104300x_2 \approx 104400$,

Calculations:

Coef. x_1: $5.291 - 1764 \cdot 0.003000 \overset{\text{4d-round}}{\approx} 5.291 - 5.292 = -0.001$ [set to 0]

Coef. x_2: $-6.130 - 1764 \cdot 59.14 \overset{\text{4d-round}}{\approx} -6.130 - 104300 \overset{\text{4d-round}}{\approx} -104300$

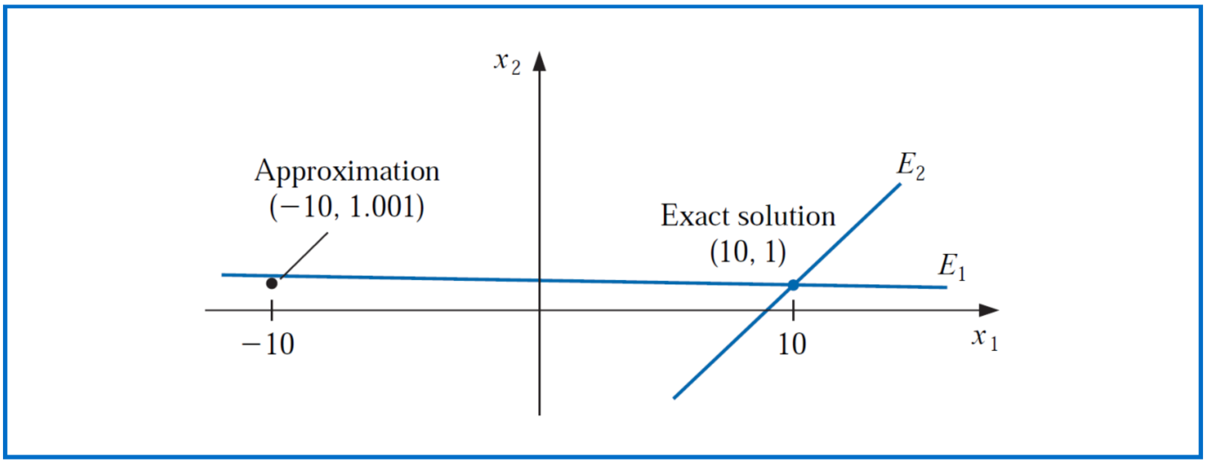
Coef. intercept: $46.78 - 1764 \cdot 59.17 \overset{\text{4d-round}}{\approx} 46.78 - 104400 \overset{\text{4d-round}}{\approx} -104400$

Solving the second equation gives: $x_2 = \dfrac{104400}{104300} \overset{\text{4d-round}}{\approx} 1.001$, which is near the exact solution.

Substituting this in the 1st equation yields:

$$0.0030x_1 = 59.17 - 59.14x_2$$
$$= 59.17 - 59.14 \cdot 1.001 \overset{\text{4d-round}}{\approx} 59.17 - 59.20 = -0.030 \to x_1 = \frac{-0.030}{0.003} = -10.$$

Due to rounding we get a huge difference with the exact solution for x_1.



2.

a. Determine the eigenvalues of the A matrix.

$$\det[A - \lambda I] = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)\cdot -\lambda - 1 = \lambda^2 - \lambda - 1 = 0$$

The ABC-formula gives: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$

b. Determine the 2 eigenvectors of the A matrix.

$$A - \lambda_1 I = 0$$

$$\left[\begin{array}{cc|c} 1-(\frac{1+\sqrt{5}}{2}) & 1 & 0 \\ 1 & -(\frac{1+\sqrt{5}}{2}) & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{-1-\sqrt{5}}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to v_1 = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\tfrac{1-\sqrt{5}}{2} v_x + v_y = 0 \to v_y = -\tfrac{1-\sqrt{5}}{2} v_x \to \text{suppose } v_x = 2 \to v_y = \sqrt{5} - 1 \to v_1 = \begin{pmatrix} 2 \\ \sqrt{5}-1 \end{pmatrix}$$

$$A - \lambda_2 I = 0$$

$$\left[\begin{array}{cc|c} 1-(\frac{1-\sqrt{5}}{2}) & 1 & 0 \\ 1 & -(\frac{1-\sqrt{5}}{2}) & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{1+\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{-1+\sqrt{5}}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \frac{1+\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to v_2 = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\tfrac{1+\sqrt{5}}{2} v_x + v_y = 0 \to v_y = -\tfrac{1+\sqrt{5}}{2} v_x \to \text{suppose } v_x = -2 \to v_y = 1 + \sqrt{5} \to v_2 = \begin{pmatrix} -2 \\ \sqrt{5}+1 \end{pmatrix}$$

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> # let check these calculations by R
> A <- matrix(c(1,1,1,0),2,2)
> r <- eigen(A)
> # V contains the matrix with normalized eigen vectors
> V <- r$vectors; V
           [,1]       [,2]
[1,] -0.8506508  0.5257311
[2,] -0.5257311 -0.8506508
> # lambda contains the vector with eigen values
> lambda <- r$values; lambda
[1]  1.618034 -0.618034
> v1 <- matrix(c(2,sqrt(5)-1),2,1)
> # length of the vector v1
> sqrt((2)^2+(sqrt(5)-1)^2)
[1] 2.351141
> # this is the same as the Euclidean norm
> norm(v2,"F")
[1] 3.804226
> # the normalized vector is minus the 1st column of V
> v1/norm(v1,"F")
          [,1]
[1,] 0.8506508
[2,] 0.5257311
> # the second normalized vector is minus the 2nd column of V
> v2 <- matrix(c(-2,sqrt(5)+1),2,1); v2
          [,1]
[1,] -2.000000
[2,]  3.236068
> v2/norm(v2,"F")
           [,1]
[1,] -0.5257311
[2,]  0.8506508
```

c. $$V^{-1}V = \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}+1 & 2 \\ 1-\sqrt{5} & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2(\sqrt{5}+1)+2(\sqrt{5}-1) & -2(\sqrt{5}+1)-2(\sqrt{5}+1) \\ 2(1-\sqrt{5})+2(\sqrt{5}-1) & -2(1-\sqrt{5})+2(\sqrt{5}+1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d.

$$V^{-1}AV = \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}+1 & 2 \\ 1-\sqrt{5} & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}+3 & \sqrt{5}+1 \\ 3-\sqrt{5} & 1-\sqrt{5} \end{bmatrix}\begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2(\sqrt{5}+3)+(\sqrt{5}+1)(\sqrt{5}-1) & -2(\sqrt{5}+3)+(\sqrt{5}+1)^2 \\ 2(3-\sqrt{5})+(1-\sqrt{5})(\sqrt{5}-1) & -2(3-\sqrt{5})+(1-\sqrt{5})(\sqrt{5}+1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

e. $$V^{-1}x_1 = \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}+1 & 2 \\ 1-\sqrt{5} & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4\sqrt{5}}\begin{bmatrix} \sqrt{5}+1 \\ 1-\sqrt{5} \end{bmatrix}$$

$$D^kV^{-1}x_1 = \frac{1}{4\sqrt{5}}\begin{bmatrix} (\frac{1+\sqrt{5}}{2})^k & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^k \end{bmatrix}\begin{bmatrix} \sqrt{5}+1 \\ 1-\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}}\begin{bmatrix} (\frac{1+\sqrt{5}}{2})^k(\sqrt{5}+1) \\ (\frac{1-\sqrt{5}}{2})^k(1-\sqrt{5}) \end{bmatrix}$$

$$VD^kV^{-1}x_1 = \frac{1}{4\sqrt{5}}\begin{bmatrix} 2 & -2 \\ \sqrt{5}-1 & \sqrt{5}+1 \end{bmatrix}\begin{bmatrix} (\frac{1+\sqrt{5}}{2})^k(\sqrt{5}+1) \\ (\frac{1-\sqrt{5}}{2})^k(1-\sqrt{5}) \end{bmatrix}$$

$$= \frac{1}{4\sqrt{5}}\begin{bmatrix} 2(\frac{1+\sqrt{5}}{2})^k(\sqrt{5}+1)-2(\frac{1-\sqrt{5}}{2})^k(1-\sqrt{5}) \\ 4(\frac{1+\sqrt{5}}{2})^k-4(\frac{1-\sqrt{5}}{2})^k \end{bmatrix}$$

Hence, we get for F_k:

$$F_k = \frac{1}{\sqrt{5}}\left[(\tfrac{1+\sqrt{5}}{2})^k-(\tfrac{1-\sqrt{5}}{2})^k\right] = \frac{1}{\sqrt{5}}\left[\lambda_1^k-\lambda_2^k\right].$$

f. $F_k \approx G_k = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^k$ for big values of k.

The rabbit population increases exponential.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_k	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
G_k	0.72	1.17	1.89	3.07	4.96	8.02	12.98	21.01	33.99	55.00	89.00	144.00	233.00	377.00	610.00