Chapter 10: Root Finding

- Starting values
- Closed interval methods (roots are searched within an interval)
  - Bisection
- Open methods (no interval)
  - Fixed Point
  - Newton-Raphson
  - Secant Method

10.1 Introduction

Roots are solutions for solving $f(x) = 0$. We can distinguish:

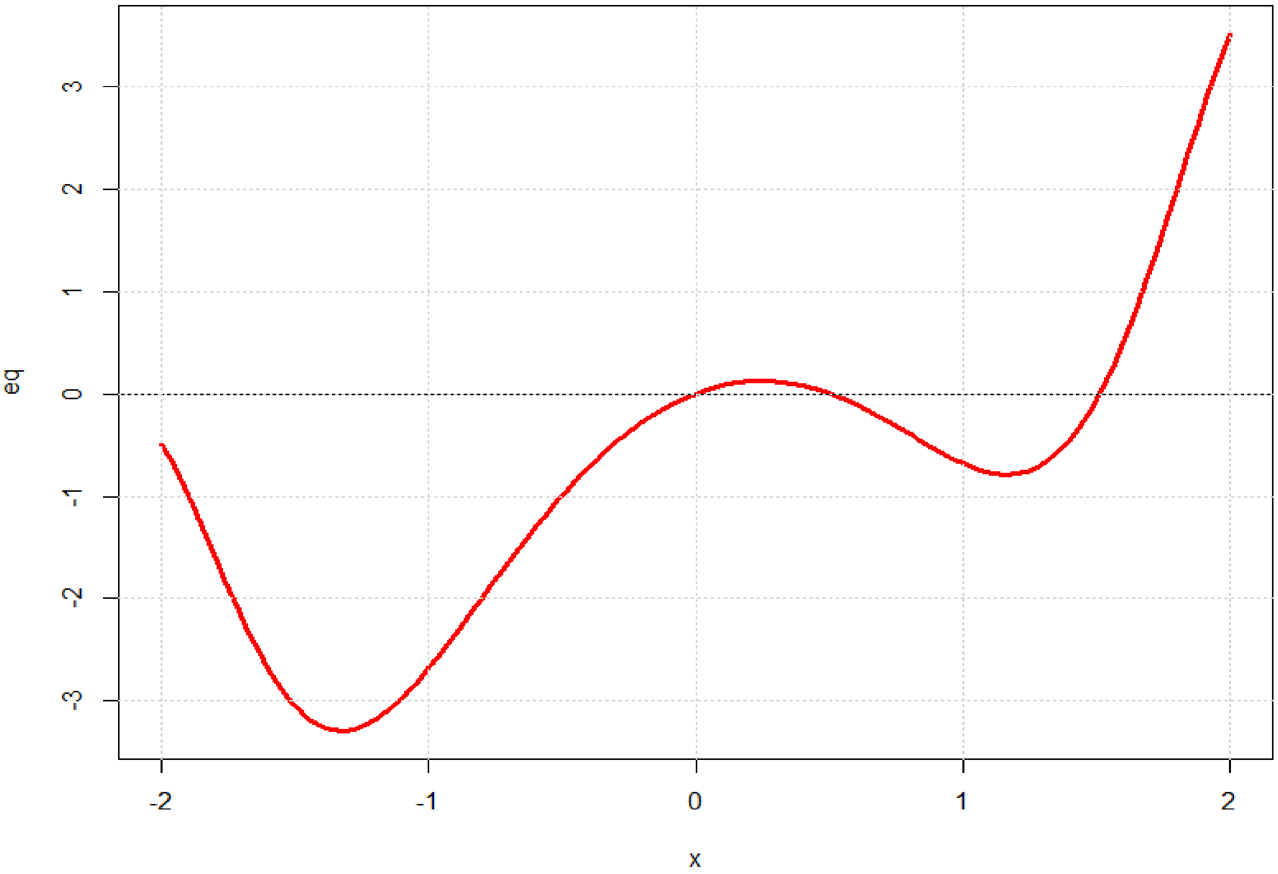
- Closed methods
  - convergence guaranteed
- Open methods
  - Divergence may occur
  - if convergence occurs, it is usually faster (rate is higher)

To approximate a root, the starting value is important: if possible, first draw a graph

Starting Values

Code

```
f <- function(x){x-2*sin(x^2)}
plot(f,from=-2,to=2)
grid()
```



10.5 Bisection

Assumption: interval $[a, b]$ contains one root

In this method, the interval is split in two:

- if the function changes sign (+ to – or – to +), then the function is evaluated in the middle
- the location of the root should lie in the subinterval in which the product changes sign

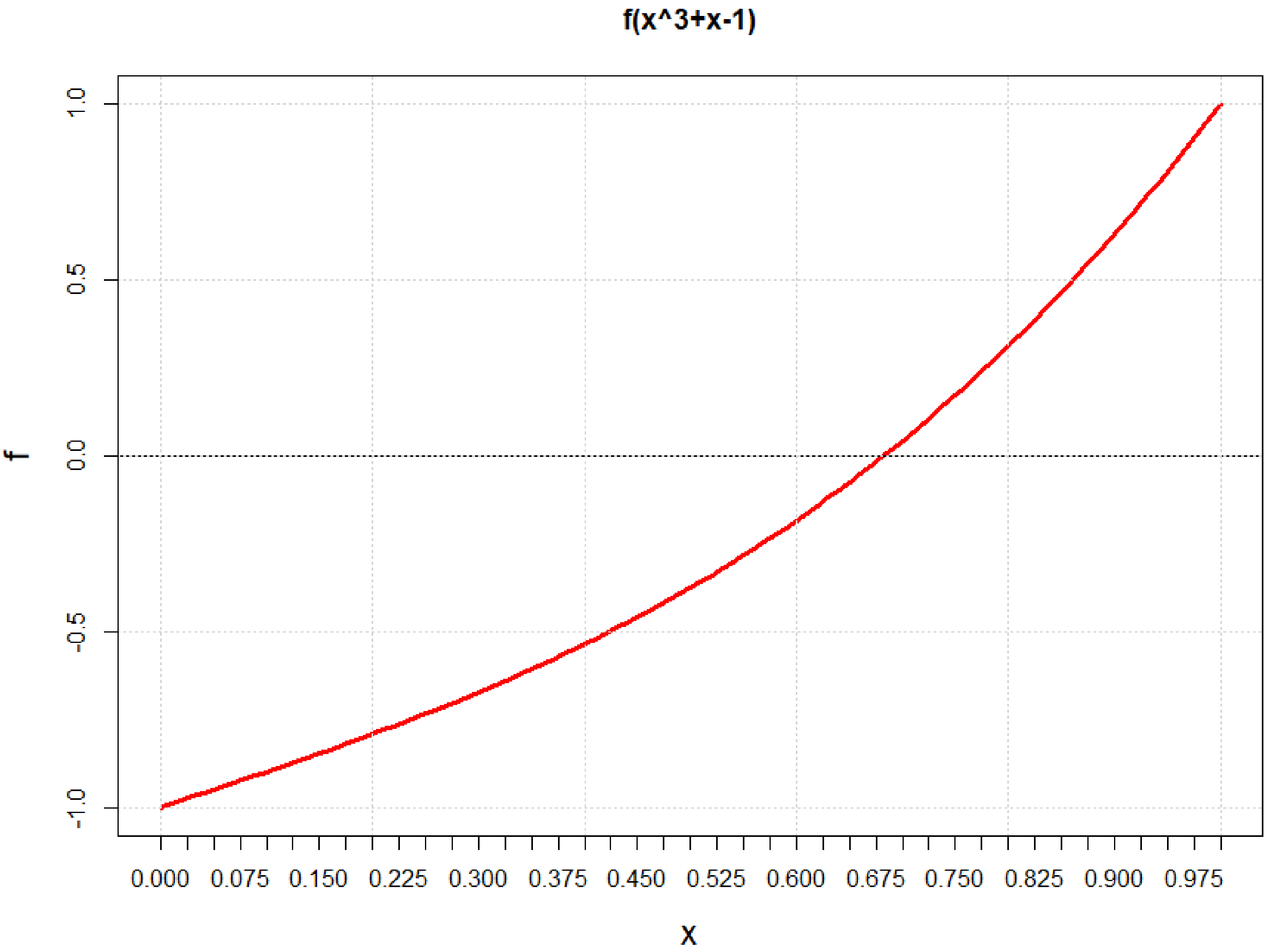
Algorithm: start with $x_l < x_r$ such that $f(x_l)f(x_r) < 0$

1. put $x_m = (x_l + x_r)/2$; if $f(x_m) = 0$, then stop
2. if $f(x_l)f(x_m) < 0$, then put $x_r = x_m$; otherwise put $x_l = x_m$
3. if $x_r - x_l \leq \varepsilon$ then stop, otherwise go back to 1

The absolute error is halved in each iteration:

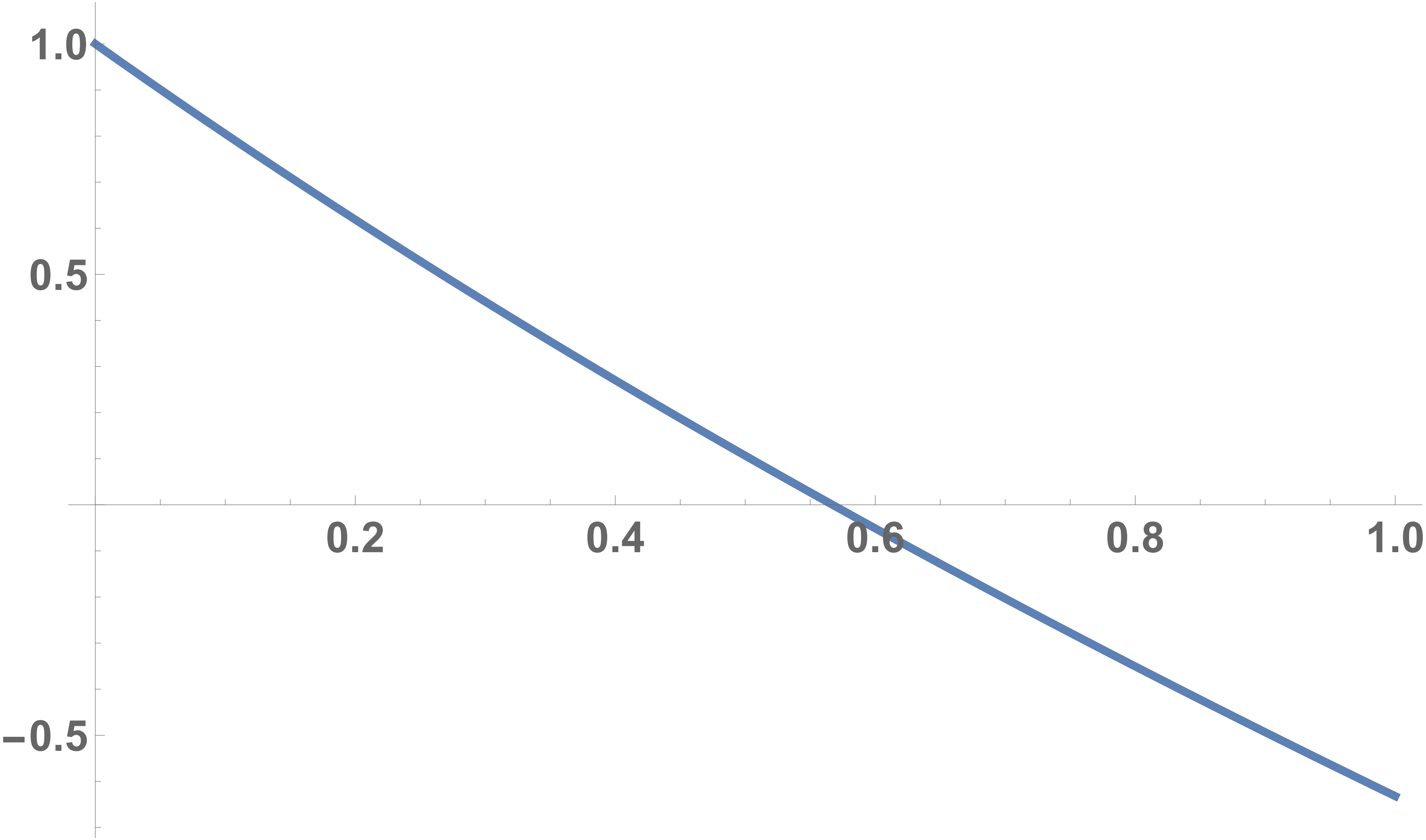
- length interval after n iterations: $\frac{b-a}{2^n}$
- if ε is a given tolerance, then $n > \frac{log\left(\frac{b-a}{\varepsilon}\right)}{log\, 2}$

10.5 Bisection - Example



10.5 Bisection – Another Example

$f(x) = \exp(-x) - x$



10.2 Fixed Point

Rewrite equation $f(x) = 0$ in $x = g(x)$

This can usually be done is several ways: for instance $f(x) = x - 2\,sin(x^2)$

(i) $x = 2\,sin(x^2)$ → `g <- 2*sin(x^2)`

(ii) $x = \sqrt{sin^{-1}(x/2)}$ → `g <- sqrt(asin(x/2)))`

Scheme: $x_{n+1} = g(x_n) \quad n = 0,1,\ldots$

- Starting value: x_0 (how to choose?)
- When to stop?
    - $f(x_n) \approx 0$ if $x_{n+1} \approx g(x_n)$: $|x_{n+1} - x_n| < \varepsilon$ (ε small)
    - maximum number of iterations (to prevent infinite loop)

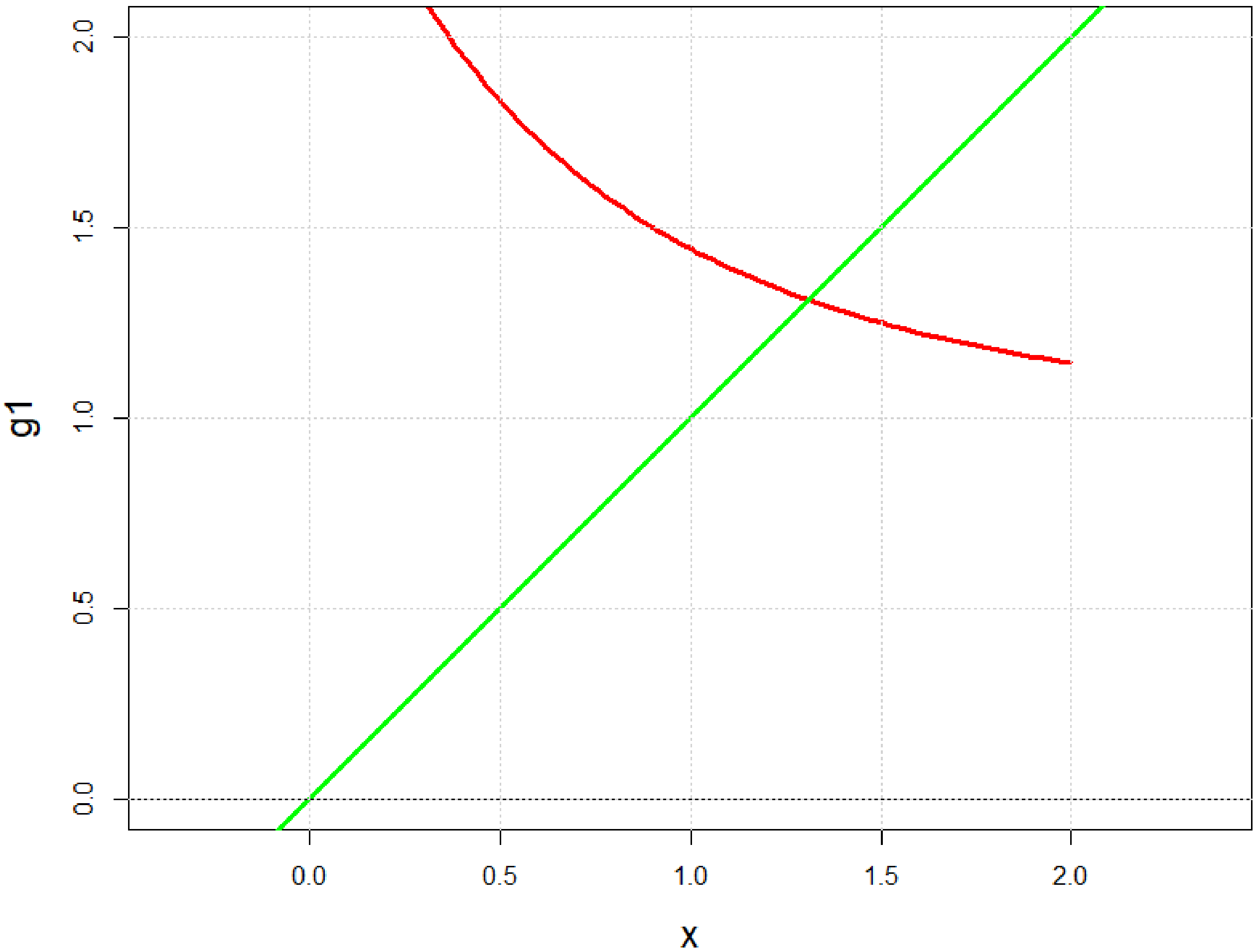
Method only works if $|g'(x)| < 1$

`g <- 2*sin(x^2)` → finds root 0

`g <- sqrt(asin(x/2)))` → finds root close to 0.5

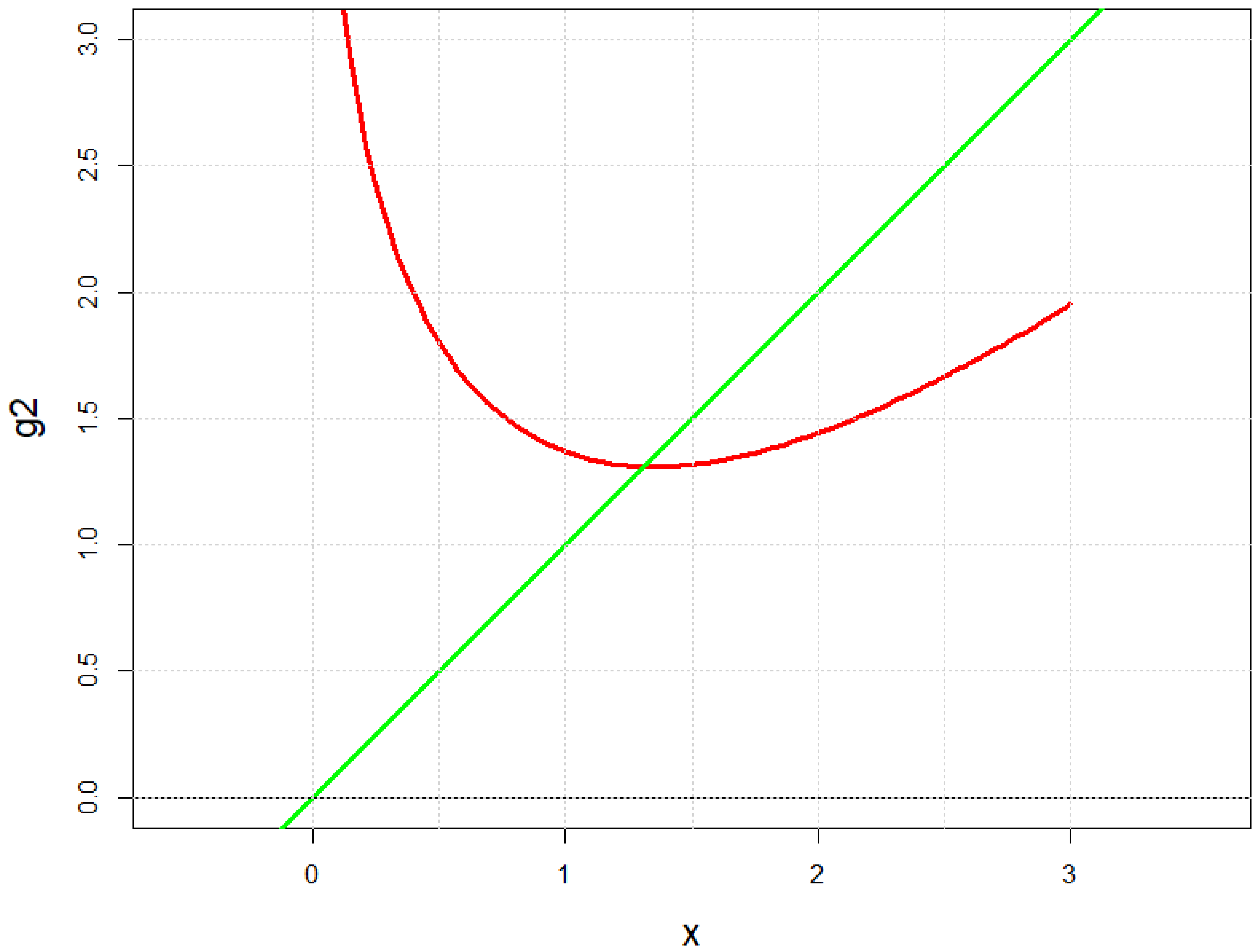
10.2 Fixed Point: example

$f(x) = \log(x) - \exp(-x)\,;\, g_1(x) = \exp\left(\exp(-x)\right)$



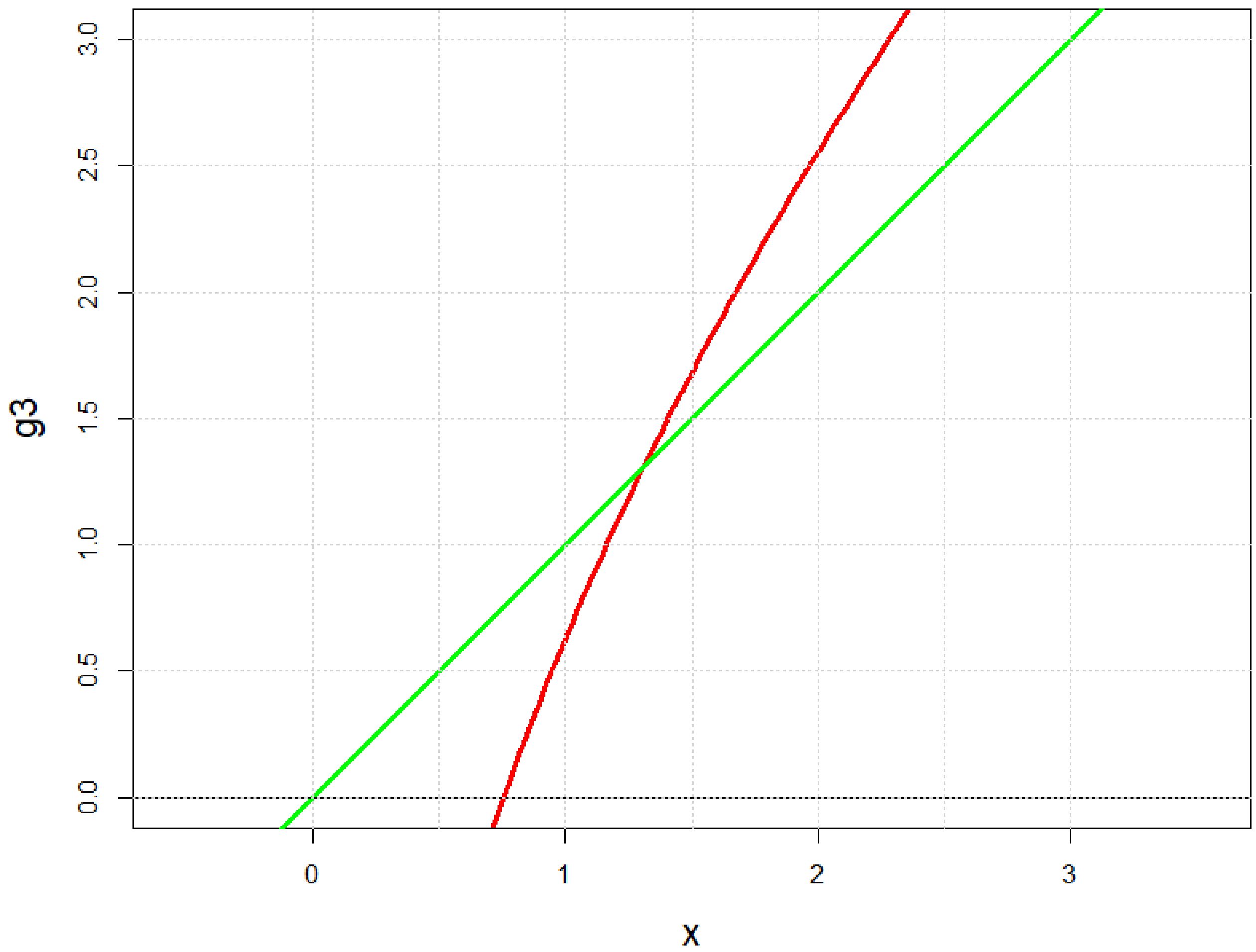
10.2 Fixed Point: example

$f(x) = \log(x) - \exp(-x)\,;\, g_2(x) = x - \log(x) + \exp\,(-x)$



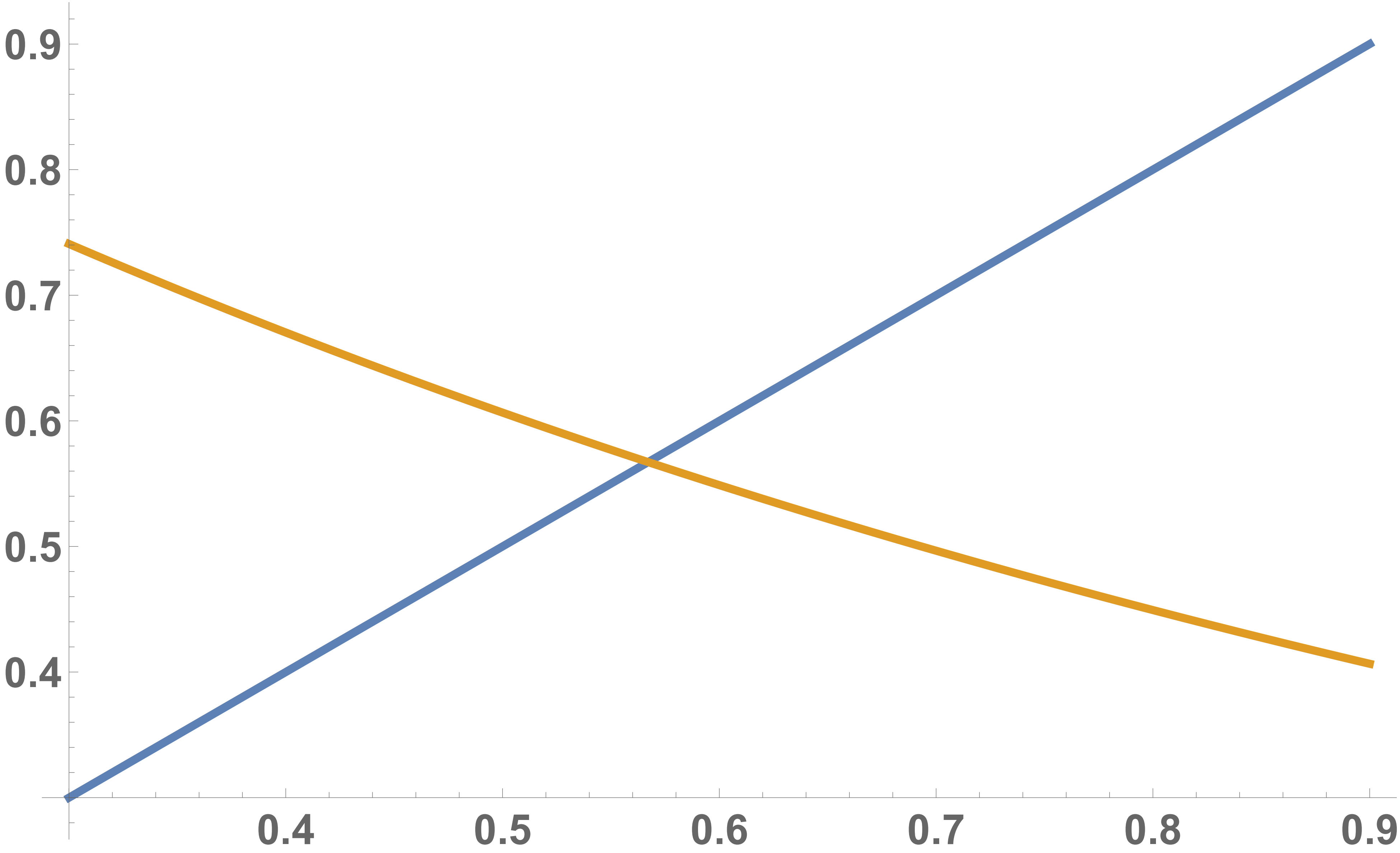
10.2 Fixed Point: example

$f(x) = \log(x) - \exp(-x)\,;\, g_3(x) = x + \log(x) - \exp\,(-x)$



10.2 Fixed Point: another example

$f(x) = \exp(-x) - x; \quad g(x) = \exp(-x)$



10.2 Fixed Point: convergence

a) Convergent: $0 \leq g' < 1$
b) Convergent: $-1 < g' \leq 0$
c) Divergent: $g' > 1$
d) Divergent: $g' < -1$

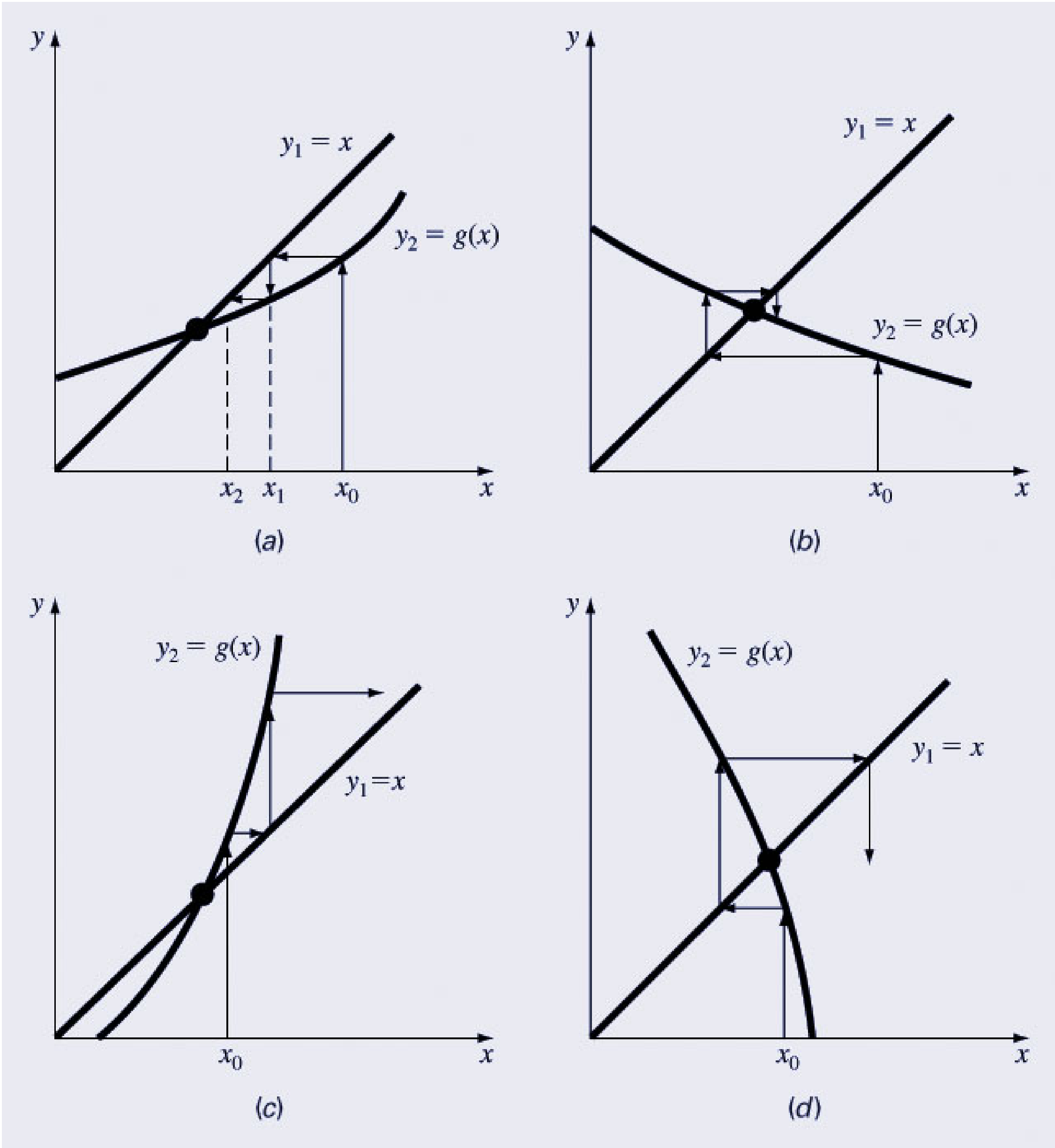


Figure 6.3 from Chapra, S. C. (2007) Applied Numerical Methods With Matlab: For Engineers And Scientists. McGraw Hill

10.2 Fixed Point: convergence

Definition fixed point: $g(x) = x$

If (i) *g* is a continuous function on the closed interval [*a*,*b*]
(ii) $g: [a, b] \to [a, b]$
(iii) *g* is differentiable with $|g'(x)| \leq k < 1$ for all $x \in (a, b)$,
then the sequence $\{x_{n+1} = g(x_n)\}$ converges to the fixed point *x* for every $x_0 \in [a, b]$.

Proof:

$$
\begin{aligned}
|x_{n+1} - x| &= |g(x_n) - g(x)| && \text{by definition of } x_{n+1} \text{ and } x \\
&= |g'(c)||x_n - x| && \text{Mean value theorem} \\
&\leq k|x_n - x| && g' \text{ bounded by } k \\
&\leq k^2|x_{n-1} - x| && \text{repeat previous 3 steps} \\
&\cdots \\
&\leq k^{n+1}|x_0 - x|
\end{aligned}
$$

Because $0 < k < 1$, we get $\lim_{n\to\infty} |x_{n+1} - x| \leq \lim_{n\to\infty} k^{n+1}|x_0 - x| = 0$.

Mean value theorem: if $f \in C[a, b]$ and differentiable, then there exists a point $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a) \to f(b) - f(a) = f'(c)(b - a)$

10.3 Newton-Raphson

Taylor approximation:

$$f(x_{i+1}) = f(x_i) + f^{'}(x_i)h + O(h^2)$$

Suppose $f(x_{i+1}) = 0$ & $O(h^2) = 0$,
then

$$0 = f(x_i) + f^{'}(x_i)h, \; h = (x_{i+1} - x_i)$$

Solving for h: $(x_{i+1} - x_i) = -\frac{f(x_i)}{f'(x_i)}$

Iteration scheme Newton-Raphson:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

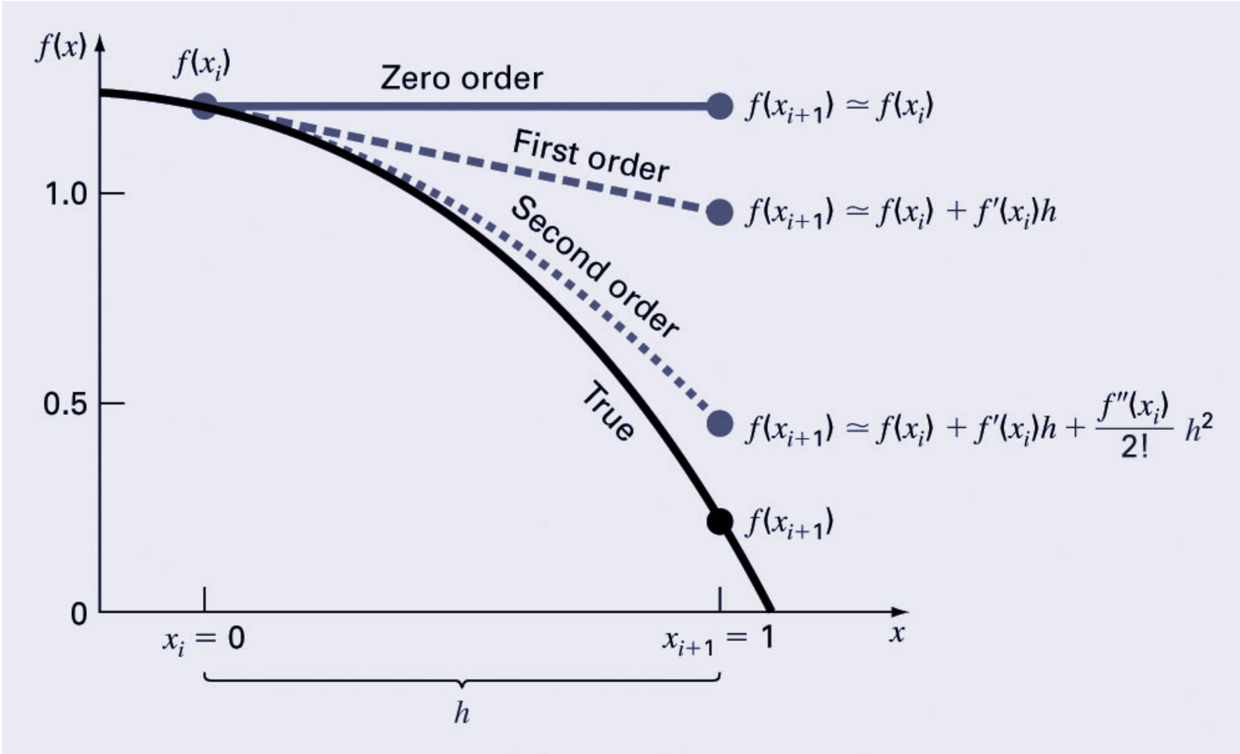


Figure 4.7 from Chapra, S. C. (2007) Applied Numerical Methods With Matlab: For Engineers And Scientists. McGraw Hill

10.3 Newton-Raphson - Graphical

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

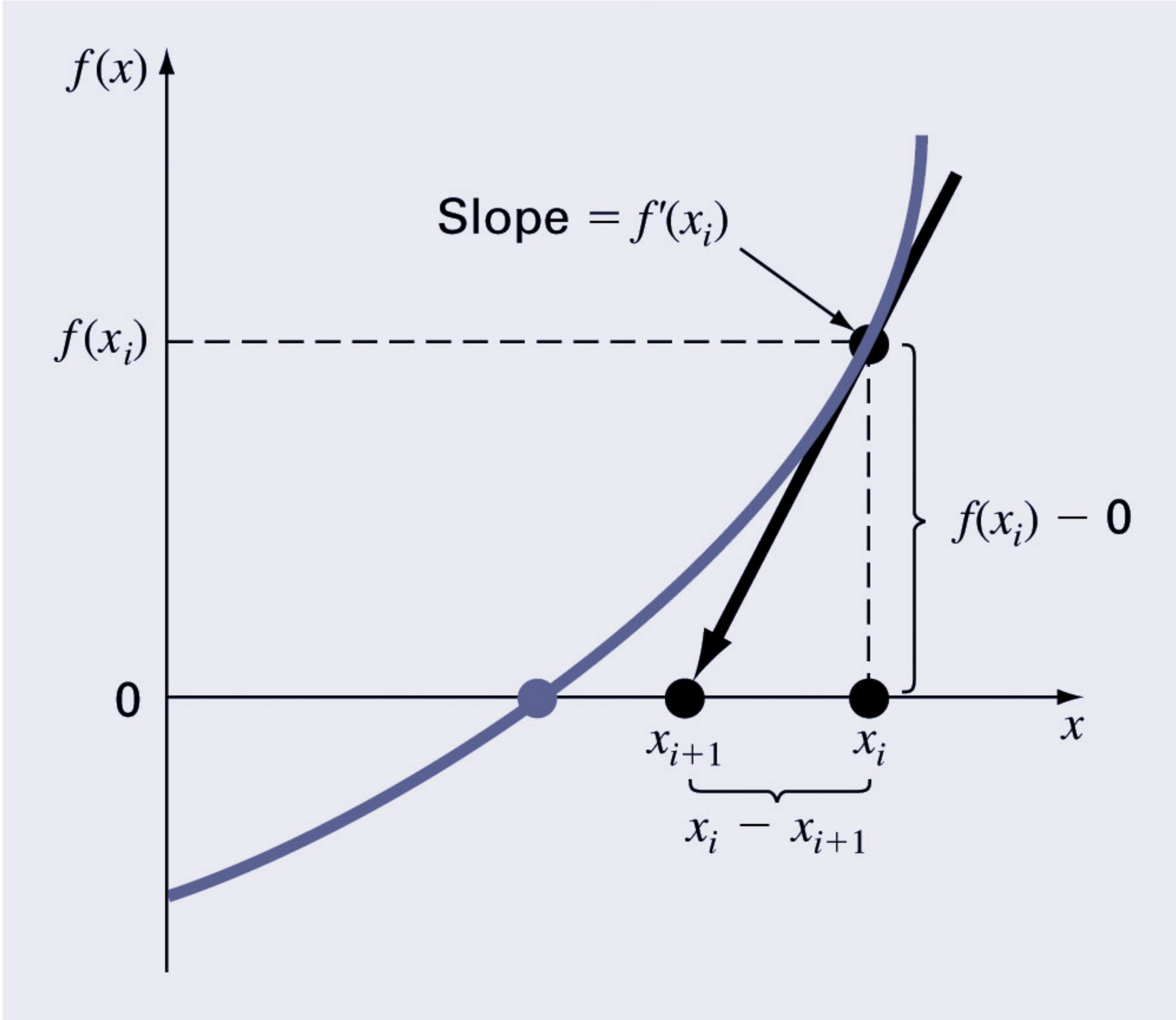
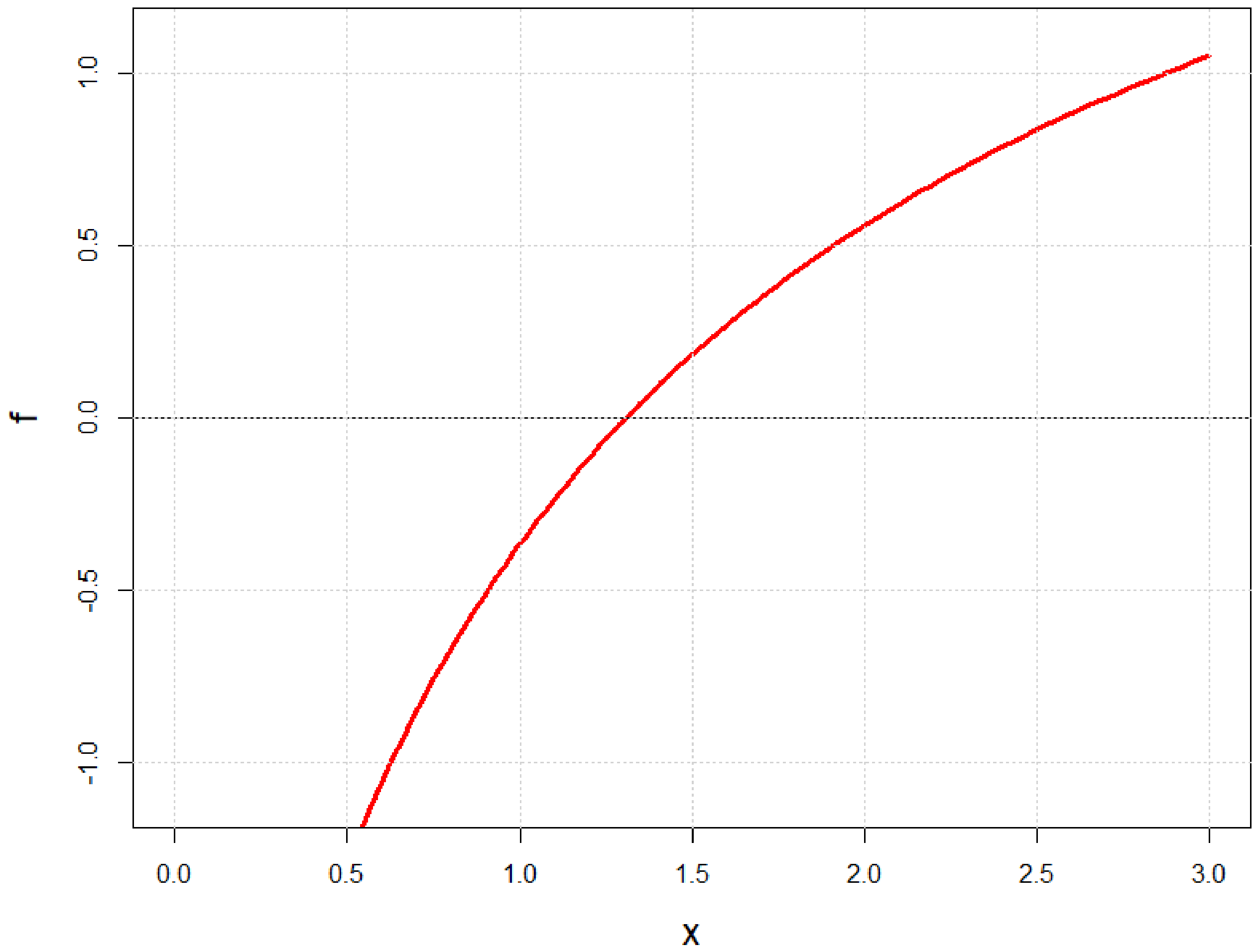


Figure 6.3 from Chapra, S. C. (2007) Applied Numerical Methods With Matlab: For Engineers And Scientists. McGraw Hill

10.3 Newton-Raphson - Example

$$f(x) = \log(x) - \exp(-x)\,; f'(x) = 1/x + \exp\,(-x)$$



10.4 Secant Method

Approximate the derivative $f'(x)$ numerical:

- def. $f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta)-f(x)}{\delta}$
- finite difference: $f'(x_i) \approx \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$ [forward difference]

Newton Raphson: $x_{i+1} = x_i - f(x_i)\frac{1}{f'(x_i)}$

- forward difference: $\frac{1}{f'(x_i)} \approx \frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})}$

Secant Method: $x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})}$ now: $i = 1,2,\ldots$

- no derivative needed
- now, two starting values: x_0, x_1
- variations:
  - backwards/central differences