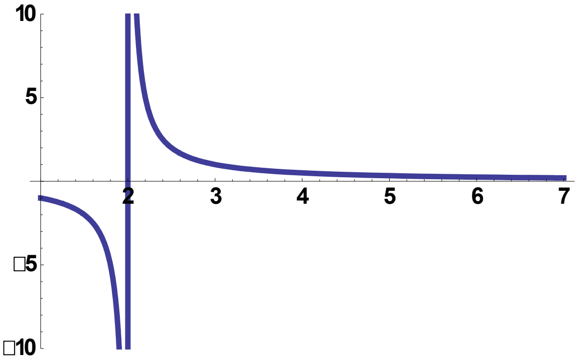
Exercises-Roots-Solutions

All figures shown in this document are created by NvG, UvA (2018)

1. Plot:

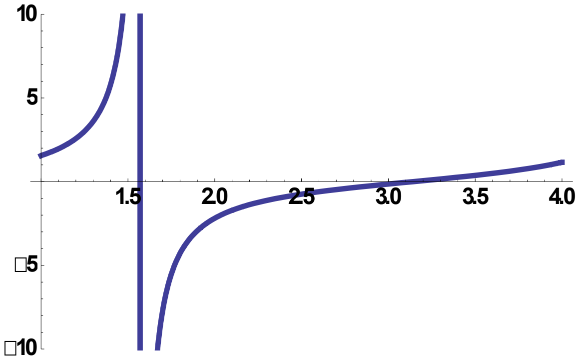


a. Bisection method terminates directly since there is no changing sign on the interval [3,7], i.e. $f(3) > 0$ and $f(7) > 0$.

b. Bisection method for het interval [1,7] converges to 2 (NO root!):

```
i= 1   a=     1.0000000000 b=     4.0000000000
i= 2   a=     1.0000000000 b=     2.5000000000
i= 3   a=     1.7500000000 b=     2.5000000000
i= 4   a=     1.7500000000 b=     2.1250000000
i= 5   a=     1.9375000000 b=     2.1250000000
i= 6   a=     1.9375000000 b=     2.0312500000
i= 7   a=     1.9843750000 b=     2.0312500000
i= 8   a=     1.9843750000 b=     2.0078125000
i= 9   a=     1.9960937500 b=     2.0078125000
i=10   a=     1.9960937500 b=     2.0019531250
```

Plot $f(x) = \tan(x)$:



c. Bisection method for the interval [3,4] converges to π:

```
i= 1   a=     3.0000000000 b=     3.5000000000
i= 2   a=     3.0000000000 b=     3.2500000000
i= 3   a=     3.1250000000 b=     3.2500000000
i= 4   a=     3.1250000000 b=     3.1875000000
i= 5   a=     3.1250000000 b=     3.1562500000
```

d. Bisection method for the interval [1,3] converges to $\pi / 2$ (NO root!):

```
i= 1   a=     1.0000000000 b=     2.0000000000
i= 2   a=     1.5000000000 b=     2.0000000000
i= 3   a=     1.5000000000 b=     1.7500000000
i= 4   a=     1.5000000000 b=     1.6250000000
i= 5   a=     1.5625000000 b=     1.6250000000
```

2a. Which function can be used to approximate $\sqrt{2}$ using the bisection method?

$f(x) = x^2 - 2$

b. Carry out 8 iterations of the bisection method for the interval [1.35, 1.45].

```
i= 1   a=     1.4000000000 b=     1.4500000000
i= 2   a=     1.4000000000 b=     1.4250000000
i= 3   a=     1.4125000000 b=     1.4250000000
i= 4   a=     1.4125000000 b=     1.4187500000
i= 5   a=     1.4125000000 b=     1.4156250000
i= 6   a=     1.4140625000 b=     1.4156250000
i= 7   a=     1.4140625000 b=     1.4148437500
i= 8   a=     1.4140625000 b=     1.4144531250
```

3. Determine algebraically whether the next functions have a unique fixed point for the following intervals

a. $g(x) = 1 - x^2/4$ on $[0,1]$

The derivative $g'(x) = -x/2$ is totally negative for the interval [0,1]. This means that $g(x)$ is monotonically decreasing, so $g(x) \in [g(1), g(0)] = [0,1]$ for $x \in [0,1]$.

Furthermore, $\max_{x\in[0,1]} |g'(x)| = 1/2 < 1$. There exists a unique fixed point.

b. $g(x) = 2^{-x}$ on [0, 1]

The derivative $g'(x) = -\ln(2)2^{-x}$ is totally negative for the interval [0,1]. Therefore, $g(x)$ is monotonically decreasing with $g(0) = 1$ and $g(1) = 1/2$, so $1/2 \le g(x) \le 1$ for $x \in [0,1]$. In addition, $\max_{x\in[0,1]} |g'(x)| = |g'(0)| = \ln(2) \approx 0.693147 < 1$. Hence, there exists a unique fixed point for the interval [0,1].

c. $g(x) = 1/x$ on [0.5, 2]

The derivative $g'(x) = -1/x^2$ is totally negative for the interval [0,2], so $g(x)$ is monotonically decreasing with $g(1/2) = 2$ and $g(2) = 1/2$. However, $\max_{x\in[1/2,2]} |g'(x)| = |g'(1/2)| = 4 > 1$, so there does not exist a unique fixed point for the interval.

Hint: suppose $g \in C[a,b]$, then there exists a unique fixed point if

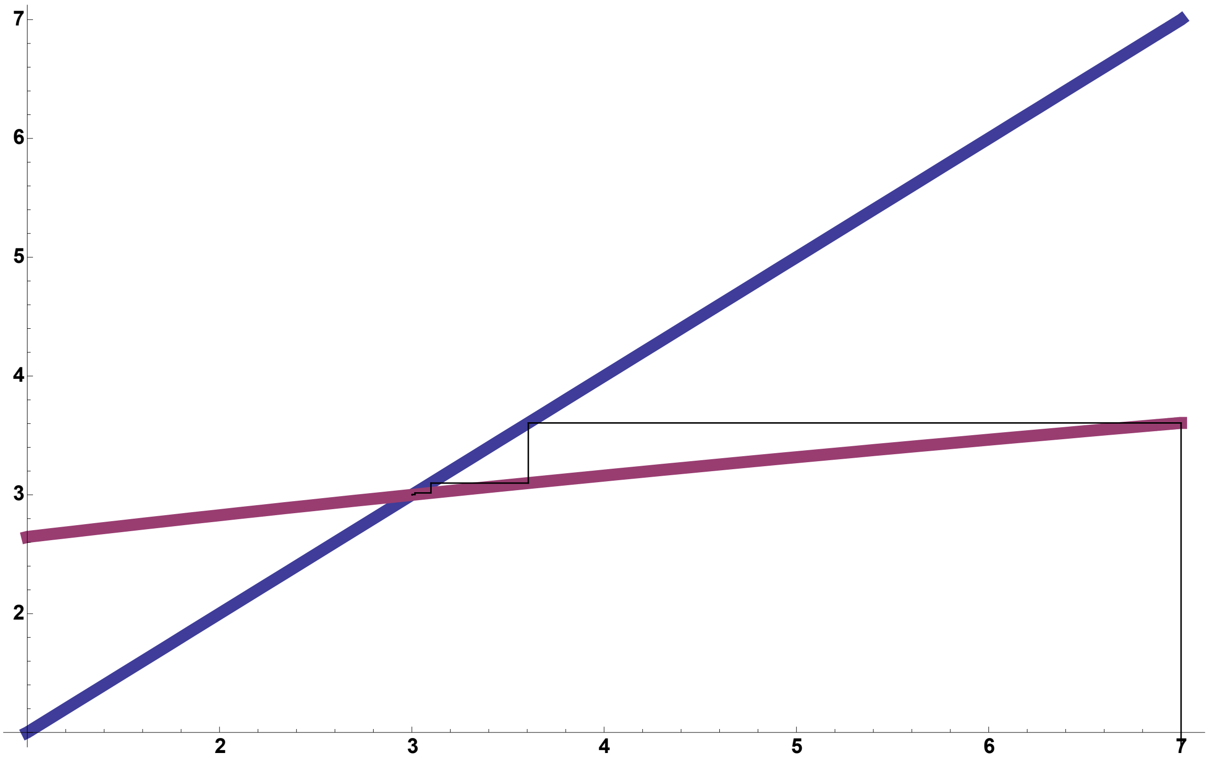
(i) $y = g(x) \in [a,b]$ for all $x \in [a,b]$

(ii) $|g'(x)| < 1$ for all $x \in [a,b]$

4. Determine graphically if the fixed point method converges for

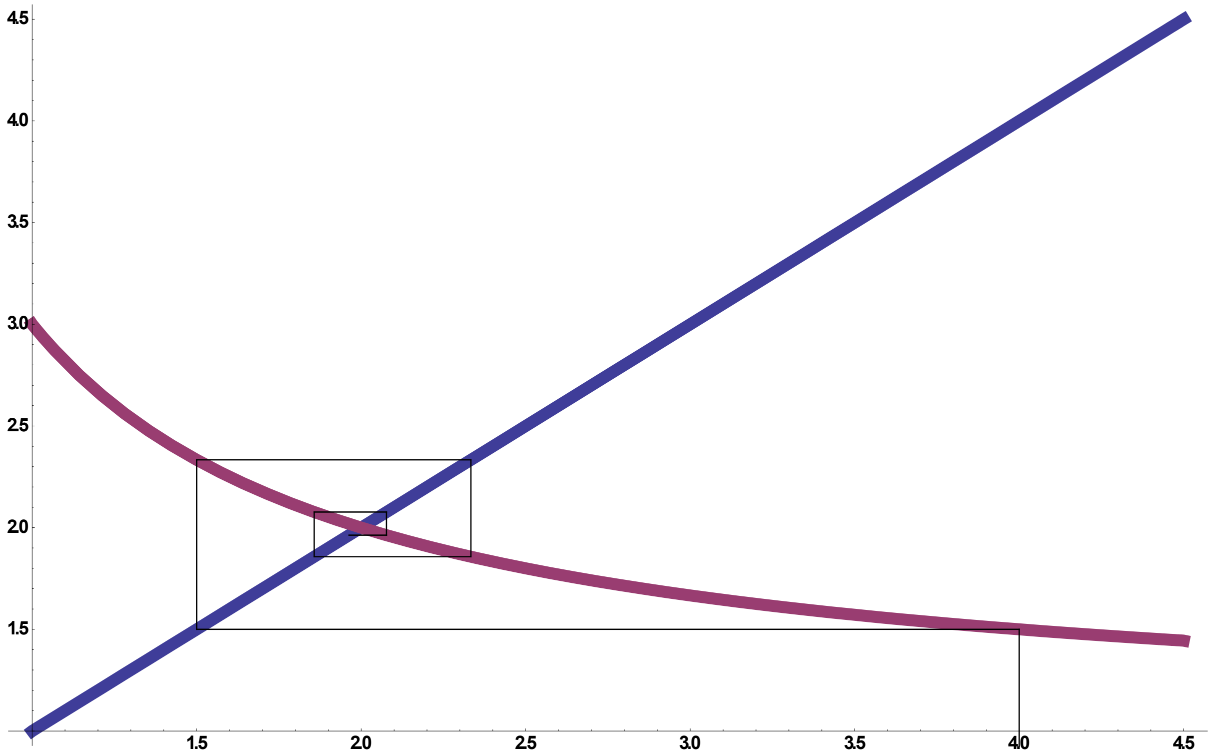
a. $f(x) = \sqrt{6+x}$ for $x_0 = 7$.

Note: $x = \sqrt{6+x} \leftrightarrow x^2 = 6 + x \leftrightarrow x^2 - x - 6 = 0 \leftrightarrow (x-3)(x+2) = 0 \rightarrow x = 3 \vee x = -2$



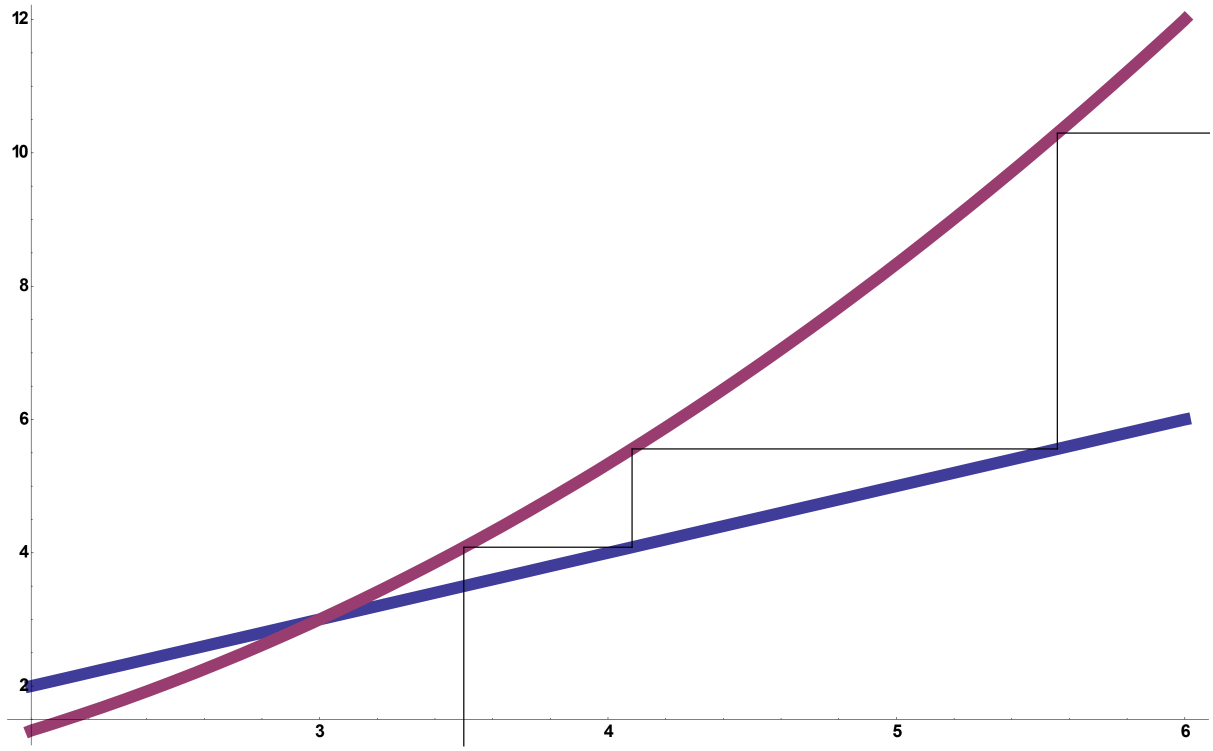
b. $g(x) = 1 + 2/x$ for $x_0 = 4$.

Note: $x = 1 + 2/x \leftrightarrow x^2 = x + 2 \leftrightarrow x^2 - x - 2 = 0 \leftrightarrow (x-2)(x+1) = 0 \rightarrow x = 2 \vee x = -1$



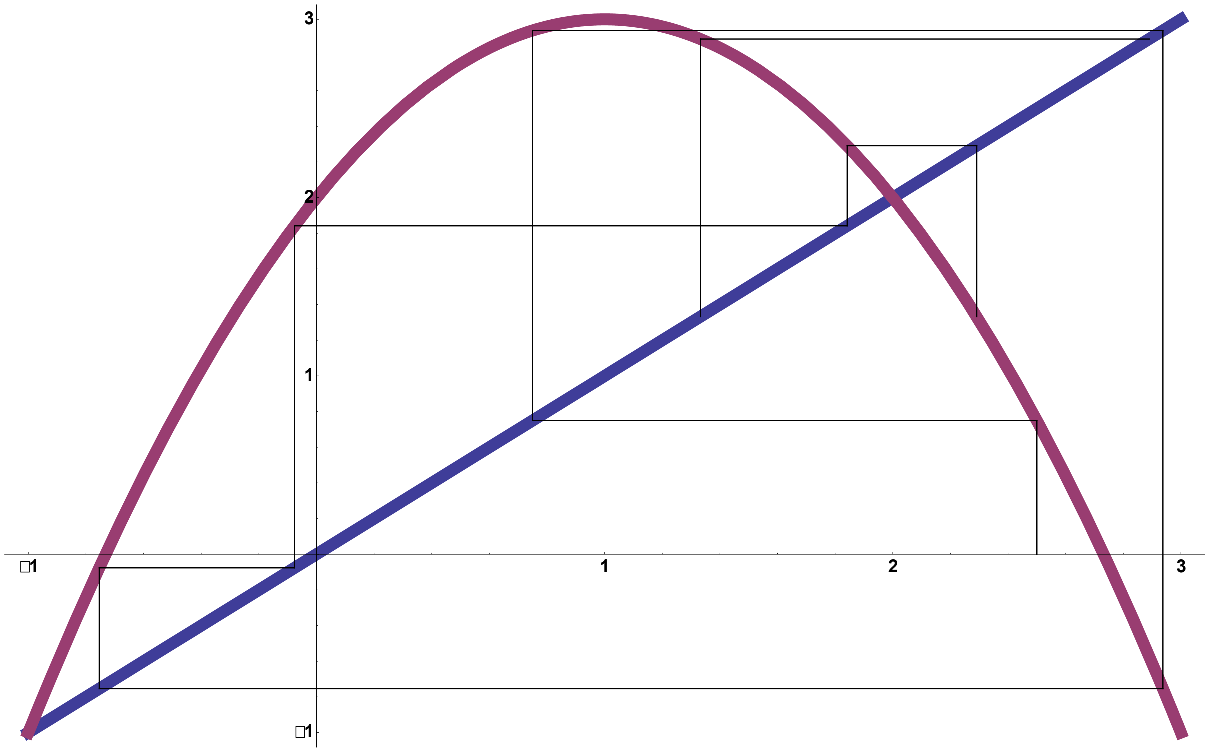
c. $h(x) = x^2/3$ for $x_0 = 3.5$.

Note: $x = x^2/3 \leftrightarrow x^2 - 3x = 0 \leftrightarrow x(x-3) = 0 \rightarrow x = 0 \vee x = 3$



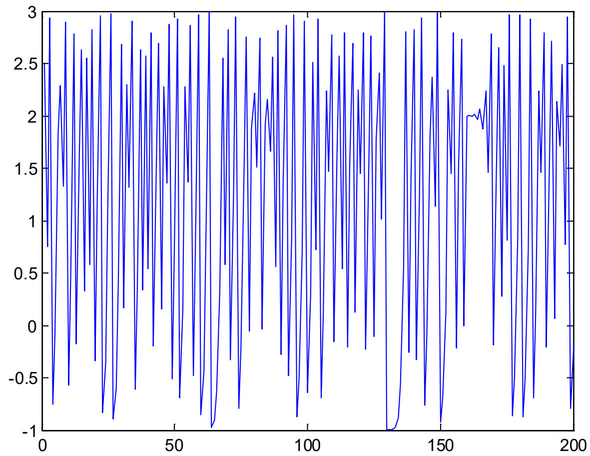
d. $y(x) = -x^2 + 2x + 2$ for $x_0 = 2.5$.

Note: $x = -x^2 + 2x + 2 \leftrightarrow x^2 - x - 2 = 0 \leftrightarrow (x+1)(x-2) = 0 \rightarrow x = -1 \vee x = 2$



```
i= 1   f(x)= 3.6056   g(x)=   1.500000   h(x)=4.083e+00   y(x)=   0.750000
i= 2   f(x)= 3.0993   g(x)=   2.333333   h(x)=5.558e+00   y(x)=   2.937500
i= 3   f(x)= 3.0165   g(x)=   1.857143   h(x)=1.030e+01   y(x)= -0.753906
i= 4   f(x)= 3.0027   g(x)=   2.076923   h(x)=3.534e+01   y(x)= -0.076187
i= 5   f(x)= 3.0005   g(x)=   1.962963   h(x)=4.163e+02   y(x)=   1.841821
i= 6   f(x)= 3.0001   g(x)=   2.018868   h(x)=5.777e+04   y(x)=   2.291337
i= 7   f(x)= 3.0000   g(x)=   1.990654   h(x)=1.113e+09   y(x)=   1.332449
i= 8   f(x)= 3.0000   g(x)=   2.004695   h(x)=4.126e+17   y(x)=   2.889478
i= 9   f(x)= 3.0000   g(x)=   1.997658   h(x)=5.674e+34   y(x)= -0.570126
i=10   f(x)= 3.0000   g(x)=   2.001172   h(x)=1.073e+69   y(x)=   0.534703
i=11   f(x)= 3.0000   g(x)=   1.999414   h(x)=3.838e+137   y(x)=   2.783499
i=12   f(x)= 3.0000   g(x)=   2.000293   h(x)=4.911e+274   y(x)= -0.180869
```

This last function does not lead to convergence.
Below, you can find a graph of $y(x)$ for the first 200 iterations.



5. Suppose $f(x) = x^2 - 2$.

a. Derive the iteration equation for the Newton-Raphson algorithm.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2 - 2}{2x_i} = \tfrac{1}{2} x_i + \frac{1}{x_i}$$

b. Carry out 3 iterations to approximate $\sqrt{2}$ using the starting value $x_0 = 1.4$.

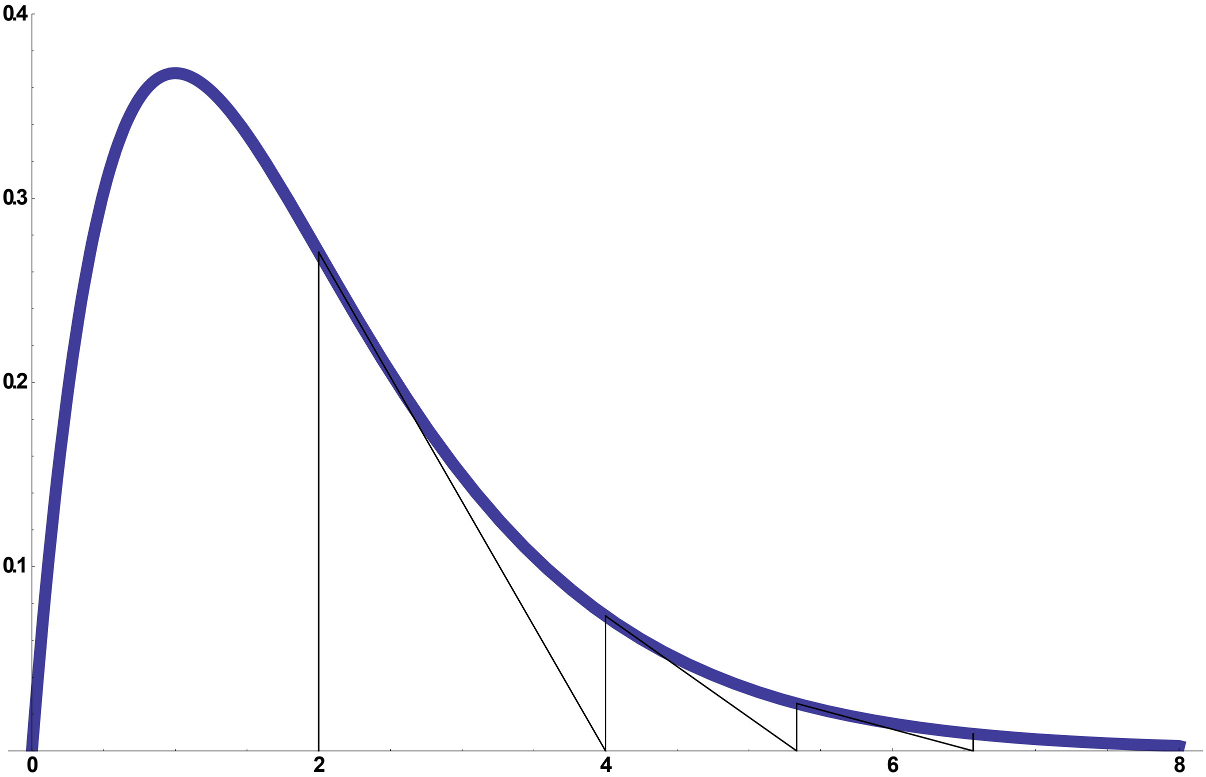
$$x_1 = \tfrac{1}{2}1.4 + \frac{1}{1.4} \approx 1.4142857$$

$$x_2 = \tfrac{1}{2}1.4142857 + \frac{1}{1.4142857} \approx 1.4142136$$

$$x_3 = \tfrac{1}{2}1.4142136 + \frac{1}{1.4142136} \approx 1.4142136$$

Hence, we observe that the convergence is much faster than the bisection method (see 2*b*).

6. Determine graphically what happens if the Newton-Raphson algorithm is applied to $f(x) = xe^{-x}$ for $x_0 = 2$.



$x_0 = 2, x_1 = 4, x_2 = 5\frac{1}{3}, \ldots$; Note: $x_i - f(x_i)/f'(x_i) = x_i^2/(x_i - 1)$.

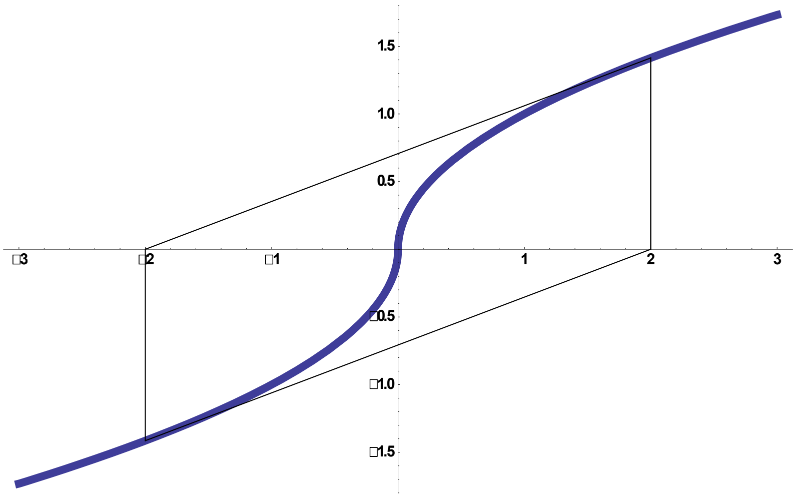
7. Let $f(x) = \begin{cases} \sqrt{x} & x \ge 0 \\ -\sqrt{-x} & \text{otherwise.} \end{cases}$

a. Derive the iteration equation for the Newton-Raphson algorithm.

For $x \ge 0$, we get: $x_{i+1} = x_i - \dfrac{\sqrt{x_i}}{\frac{1}{2\sqrt{x_i}}} = x_i - 2x_i = -x_i$

For $x < 0$, we get: $x_{i+1} = x_i - \dfrac{-\sqrt{-x_i}}{\frac{1}{2\sqrt{-x_i}}} = x_i - 2x_i = -x_i$

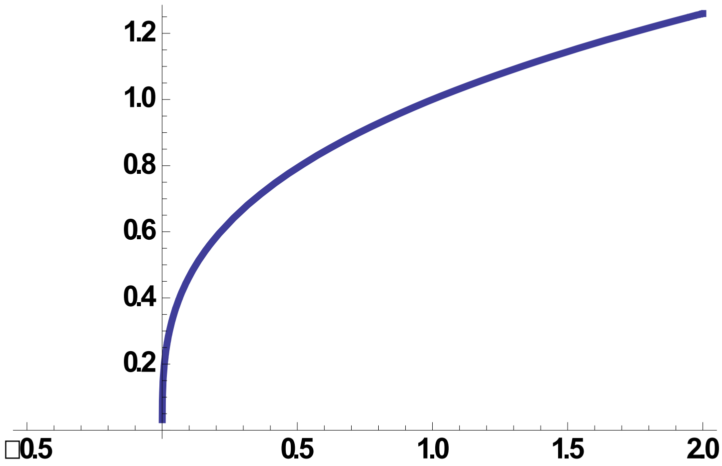
b. What will happen for a random stating value different from zero?
Infinite loop (see figure below for $x_0 = 2$).



8. Can Newton-Raphson be used to solve $f(x)=0$ for $f(x)=x^{1/3}$? Motivate your answer.

Newton-Raphson equation: $x_{i+1}=x_i-\dfrac{x_i^{1/3}}{\frac{1}{3}x_i^{-2/3}}=x_i-3x_i=-2x_i$

When you start with a positive starting value $(x_0>0)$, then the next value with be negative $(x_1<0)$. But $f(x_1)$ is not uniquely defined! For instance, $\sqrt[3]{-8}$ represents the roots of the polynomial x^3+8. One is real, namely –2, but the other two are complex, namely $1+i\sqrt{3}$ and $1-i\sqrt{3}$.



9. Use the Secant method with $x_0=-2.6$ and $x_1=-2.4$ to approximate the root $x=-2$ for the function $f(x)=x^3-3x+2$.

$$x_{i+1}=x_i-f(x_i)\frac{x_i-x_{i-1}}{f(x_i)-f(x_{i-1})}$$

$$=x_i-(x_i^3-3x_i+2)\frac{x_i-x_{i-1}}{(x_i^3-3x_i+2)-(x_{i-1}^3-3x_{i-1}+2)}$$

$$x_2=-2.4-((-2.4)^3-3\cdot(-2.4)+2)\frac{-2.4-(-2.6)}{((-2.4)^3-3\cdot(-2.4)+2)-((-2.6)^3-3\cdot(-2.6)+2)}$$

$$=-2.4-(-4.624)\frac{0.2}{-4.624-(-7.776)}=-2.1066$$

Using the secant function of the PC tutorial *PC5b - Roots*, we get:

```
> f<-function(x){x^3-3*x+2}
> secant(f,-2.6,-2.4)
At iteration 1 approximation is: -2.106599
At iteration 2 approximation is: -2.022641
At iteration 3 approximation is: -2.001511
At iteration 4 approximation is: -2.000023
At iteration 5 approximation is: -2
At iteration 6 approximation is: -2
```