

Estimating Functionals of the Joint Distribution of Potential Outcomes with Optimal Transport

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Introduction

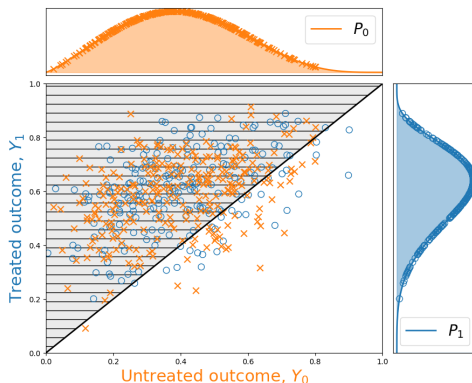
The fundamental problem of causal inference

It is impossible to observe the [treated outcome] and [untreated outcome] on the same unit and, therefore, it is impossible to observe the effect...

(Holland, 1986)

- ▶ Parameters of the **joint distribution of potential outcomes** are not point identified.
- ▶ **This paper**
 - shows **optimal transport** characterizes sharp bounds,
 - accomodates noncompliance through a standard IV model, and
 - provides simple, computationally convenient estimators.

The fundamental problem of causal inference

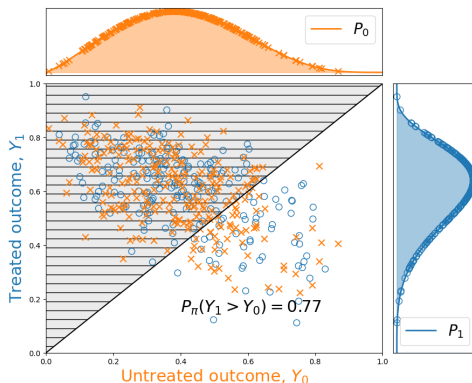


- ▶ Never observe (Y_1, Y_0) , because each unit is **treated** ($D = 1$) or **untreated** ($D = 0$):

$$\text{Observed outcome } Y = DY_1 + (1 - D)Y_0$$

- ▶ The marginal distributions P_1 and P_0 are identified - but have less information.
- ▶ For example, what share of units benefit from treatment?

Example 1: the share benefiting from treatment



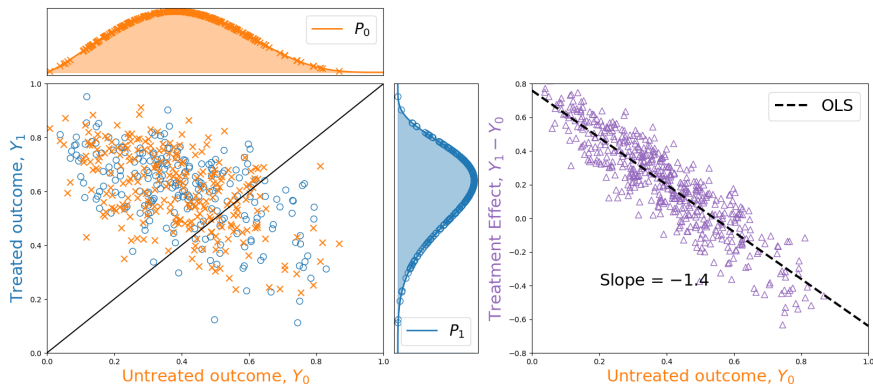
- ▶ Many joint distributions π share marginal distributions P_1 , P_0 :

$$\Pi(P_1, P_0) = \{\pi : \pi_1 = P_1, \pi_0 = P_0\}$$

- ▶ Optimizing $P(Y_1 > Y_0)$ over $\Pi(P_1, P_0)$ implies bounds:

$$\min_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0) \qquad \max_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0)$$

Example 2: who sees larger benefits from treatment?



- Do those with smaller Y_0 see larger $Y_1 - Y_0$?

$$\text{OLS slope} = \frac{\text{Cov}(Y_1 - Y_0, Y_0)}{\text{Var}(Y_0)} = \frac{E[(Y_1 - Y_0)Y_0] - (E[Y_1] - E[Y_0])E[Y_0]}{E[Y_0^2] - (E[Y_0])^2}$$

- Optimizing $E[(Y_1 - Y_0)Y_0]$ over $\Pi(P_1, P_0)$ implies bounds on OLS slope:

$$\min_{\pi \in \Pi(P_1, P_0)} E_{\pi}[(Y_1 - Y_0)Y_0]$$

$$\max_{\pi \in \Pi(P_1, P_0)} E_{\pi}[(Y_1 - Y_0)Y_0]$$

This paper

- ▶ Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R},$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$.

- Example 3: $\gamma = \text{Var}(Y_1 - Y_0) = E[(Y_1 - Y_0)^2] - (E[Y_1] - E[Y_0])^2$

- ▶ Characterize sharp identified set with optimal transport:

$$OT_c(P_1, P_0) = \min_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)]$$

- ▶ Propose and study sample analogue estimators of the bounds.
- ▶ Empirical application: who sees larger benefits from the NSW job training?

Related literature

► Joint distribution of potential outcomes

- CDF or quantiles of $Y_1 - Y_0$: Manski (1997), Heckman et al. (1997), Firpo (2007), Fan and Park (2010), Fan and Park (2012), Firpo and Ridder (2019), Callaway (2021), Frandsen and Lefgren (2021).
- General methods: Russell (2021) Fan et al. (2023), Ji et al. (2023), **this paper**.

► Optimal transport in econometrics

- Partial identification: Galichon and Henry (2011), Ekeland et al. (2010)
- Causal inference: Dunipace (2021), Günsilius and Xu (2021), Torous et al. (2021)
- Joint distribution of (Y_1, Y_0) : Ji et al. (2023), **this paper**.

⇒ **This paper contributes** identification and estimators that

- i. cover a large class of parameters while remaining tractable,
- ii. allow for simple bootstrap inference, and
- iii. accomodate noncompliance through a standard IV model.

Overview

- 1 Setting and parameter class
- 2 Identification
- 3 Estimators
- 4 Simulations
- 5 Application

Overview

1 Setting and parameter class

2 Identification

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Setting

- For this talk, focus on **unconfoundedness**.

Assumption 1 (Setting, simplified) $\{Y_i, D_i, X_i\}_{i=1}^n$ is an i.i.d. sample with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \quad D \in \{0, 1\}, \quad X \in \mathcal{X} = \{x_1, \dots, x_M\}$$

generated from a distribution satisfying

- (i) Potential outcomes: $Y = DY_1 + (1 - D)Y_0$
- (ii) Unconfoundedness: $(Y_1, Y_0) \perp D \mid X$
- (iii) $P(D = d, X = x) > 0$ for each (d, x)

- In the paper, **binary IV satisfying monotonicity condition** (Imbens and Angrist, 1994).

Setting w/IV

Parameter class

- Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R}$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$

Assumption 2 (Cost function) Either

- (i) $c(y_1, y_0)$ is Lipschitz continuous and \mathcal{Y} is compact, or
- (ii) $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ and the CDFs $F_{d|x}(y) = P(Y_d \leq y \mid X = x)$ are continuous.

Remark: If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ but $F_{d|x}(\cdot)$ are not continuous, inference remains valid for an outer identified set.

Parameter class

- Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R}$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$.

Assumption 3 (Function of moments, simplified)

- (i) $\eta_1(Y)$ and $\eta_0(Y)$ have finite second moments,
- (ii) $g(\cdot, \cdot)$ is continuously differentiable, and
- (iii) $g(\cdot, \eta)$ is monotonic.

Remark: Assumption 3 (iii) is relaxed in the paper.

Full assumption 3

Parameter class: motivating examples

- ▶ Share benefiting: $P(Y_1 > Y_0)$
 - Allcott et al. (2020): deactivating Facebook affects subjective well-being.
- ▶ Share benefiting above cost: $P(Y_1 - Y_0 > \text{cost})$
 - Friebe et al. (2023): employee referral programs increase grocery store profit.
- ▶ Who benefits more from treatment? $\text{Cov}(Y_1 - Y_0, Y_0)/\text{Var}(Y_0)$
 - **Application:** NSW job experience increases post-training annual income.
- ▶ Expected percent change: $E\left[\frac{Y_1 - Y_0}{Y_0}\right]$
 - This parameter is often approximated with $E[\log(Y_1) - \log(Y_0)]$.
- ▶ Quantiles of $Y_1 - Y_0$
 - Median is more representative than mean when distribution is skewed.

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Optimal transport

$$OT_c(P_1, P_0) = \min_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)]$$

- ▶ Choose a **joint distribution** with **given marginals** to minimize **costs**.
 - Feasible set: $\Pi(P_1, P_0) = \{\pi : \pi_1 = P_1, \pi_0 = P_0\}$
 - Cost function: $c(y_1, y_0)$
- ▶ Often interpreted in other contexts, but here intended literally.
- ▶ Attained under mild conditions.

Identification without covariates

- ▶ $\{Y_i, D_i\}_{i=1}^n$ identifies marginal distributions P_1 and P_0 .
- ▶ Identified set for $P_{1,0}$ is set of joint distributions with marginals P_1, P_0 :

$$\Pi(P_1, P_0) = \{\pi : \pi_1 = P_1, \pi_0 = P_0\}$$

- ▶ Bounds on $\theta = E_{P_{1,0}}[c(Y_1, Y_0)]$ for continuous c :

$$\begin{aligned}\theta^L &= \min_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)], & \theta^H &= \max_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)] \\ &= OT_c(P_1, P_0), & &= -OT_{-c}(P_1, P_0)\end{aligned}$$

- ▶ Bounds on $\gamma = g(\theta, \eta)$:

$$\gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta), \quad \gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

Identification with covariates

- ▶ $\{Y_i, D_i, X_i\}_{i=1}^n$ identifies marginal *conditional* distributions $P_{1|x}$ and $P_{0|x}$.

$$Y_d \mid X = x \sim P_{d|x}$$

- ▶ Identified set for $P_{1,0|x}$ is set of joint distributions with marginals $P_{1|x}$, $P_{0|x}$:

$$\Pi(P_{1|x}, P_{0|x}) = \{\pi_{1,0|x} : \pi_{1|x} = P_{1|x}, \pi_{0|x} = P_{0|x}\}$$

- ▶ Bounds on $\theta = E_{P_{1,0}}[c(Y_1, Y_0)] = E[\overbrace{E_{P_{1,0|x}}[c(Y_1, Y_0) \mid X]}^{:=\theta_X}]$ for continuous c :

$$\begin{aligned}\theta_X^L &= OT_c(P_{1|x}, P_{0|x}), & \theta_X^H &= -OT_{-c}(P_{1|x}, P_{0|x}) \\ \theta^L &= E[\theta_X^L], & \theta^H &= E[\theta_X^H]\end{aligned}$$

- ▶ Bounds on $\gamma = g(\theta, \eta)$ remain the same:

$$\gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta), \quad \gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

Covariates tighten identified bounds

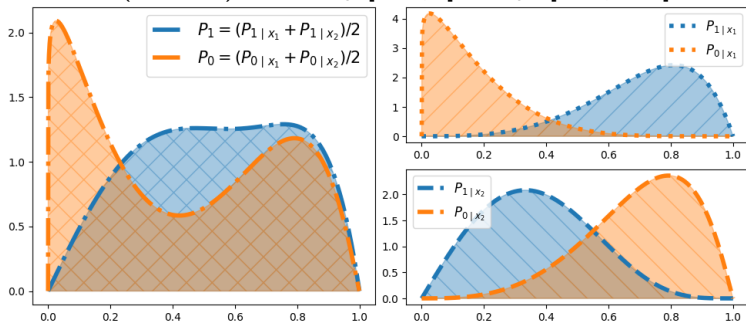
► Covariates tighten bounds,

$$OT_c(P_1, P_0) \leq \theta^L,$$

$$\theta^H \leq -OT_{-c}(P_1, P_0).$$

- Why? The optimization has **additional constraints**.
- $\theta^L = E[OT_c(P_{1|X}, P_{0|X})]$ looks for $\pi \in \Pi(P_1, P_0)$ **also matching** $(P_{1|X}, P_{0|X})$.

► Bounds on $P(Y_1 > Y_0)$: not sharp $[0.25, 1]$, sharp: $[0.44, 0.68]$.



Theorem: identification

► For continuous c ,

$$\text{Bounds on } \theta_x : \quad \theta_x^L = OT_c(P_{1|x}, P_{0|x}), \quad \theta_x^H = -OT_{-c}(P_{1|x}, P_{0|x})$$

$$\text{Bounds on } \theta : \quad \theta^L = E[\theta_X^L] \quad \theta^H = E[\theta_X^H]$$

$$\text{Bounds on } \gamma : \quad \gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta), \quad \gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

Theorem (identification)

Suppose assumptions 1, 2, and 3 are satisfied. Then the sharp identified set for $\gamma = g(\theta, \eta)$ is $[\gamma^L, \gamma^H]$.

CDF?

IV Aside

Quantile details

Overview

- 1 Setting and parameter class
- 2 Identification
- 3 Estimators**
- 4 Simulations
- 5 Application

Optimal transport

$$OT_c(P_1, P_0) = \underbrace{\min_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)]}_{\text{Primal Problem}} \overset{\text{Strong Duality}}{=} \underbrace{\max_{(\varphi, \psi) \in \Phi_c} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)]}_{\text{Dual Problem}}$$

$$\Pi(P_1, P_0) = \{\pi : \pi_1 = P_1, \pi_0 = P_0\} \quad \Phi_c = \{(\varphi, \psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

- ▶ The **primal problem** is used in identification.
- ▶ The **dual problem** is used for estimation.
- ▶ **Strong duality** holds under the cost function assumptions. Each problem is attained, too.

Estimators: recall identification

- Distributions of $Y_d \mid X = x \sim P_{d|x}$:

$$E_{P_{d|x}}[f(Y_d)] = \frac{E[f(Y)\mathbb{1}\{D = d, X = x\}]}{P(D = d, X = x)}$$

- Using strong duality,

$$OT_c(P_{1|x}, P_{0|x}) = \max_{(\varphi, \psi) \in \Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)].$$

- The identified set for γ is $[\gamma^L, \gamma^H]$, where for c continuous,

$$\begin{aligned}\theta_x^L &= OT_c(P_{1|x}, P_{0|x}), & \theta_x^H &= -OT_{-c}(P_{1|x}, P_{0|x}) \\ \theta^L &= E[\theta_X^L], & \theta^H &= E[\theta_X^H] \\ \gamma^L &= \min_{t \in [\theta^L, \theta^H]} g(t, \eta), & \gamma^H &= \max_{t \in [\theta^L, \theta^H]} g(t, \eta)\end{aligned}$$

Estimators: sample analogues

- Estimate $P_{d|x}$ with **sample analogues** $\hat{P}_{d|x}$:

$$E_{\hat{P}_{d|x}}[f(Y_d)] = \frac{\frac{1}{n} \sum_{i=1}^n f(Y_i) \mathbb{1}\{D_i = d, X_i = x\}}{\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{D_i = d, X_i = x\}}$$

- Using strong duality,

$$OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{(\varphi, \psi) \in \Phi_c} E_{\hat{P}_{1|x}}[\varphi(Y_1)] + E_{\hat{P}_{0|x}}[\psi(Y_0)].$$

- Estimate the endpoints of $[\gamma^L, \gamma^H]$ with plug-in estimators. For c continuous,

$$\hat{\theta}_x^L = OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}), \quad \hat{\theta}_x^H = -OT_{-c}(\hat{P}_{1|x}, \hat{P}_{0|x})$$

$$\hat{\theta}^L = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_{X_i}^L, \quad \hat{\theta}^H = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_{X_i}^H$$

$$\hat{\gamma}^L = \min_{t \in [\hat{\theta}^L, \hat{\theta}^H]} g(t, \hat{\eta}), \quad \hat{\gamma}^H = \max_{t \in [\hat{\theta}^L, \hat{\theta}^H]} g(t, \hat{\eta})$$

Estimators: computing $OT_c(\hat{P}_{1|x}, \hat{P}_{0|x})$

$$OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{(\varphi, \psi) \in \Phi_c} E_{\hat{P}_{1|x}}[\varphi(Y_1)] + E_{\hat{P}_{0|x}}[\psi(Y_0)].$$

- To evaluate $E_{\hat{P}_{d|x}}[f(Y_d)]$ for any function f , only the values $f_i = f(Y_i)$ matter.

$$E_{\hat{P}_{d|x}}[f(Y_d)] = \sum_{i=1}^n \omega_{d,x,i} \times f_i, \quad \omega_{d,x,i} = \frac{\mathbb{1}\{D_i = d, X_i = x\}/n}{\frac{1}{n} \sum_{j=1}^n \mathbb{1}\{D_j = d, X_j = x\}}.$$

- Computing $OT_c(\hat{P}_{1|x}, \hat{P}_{0|x})$ is straightforward **linear programming**:

$$\begin{aligned} OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{\{\varphi_i, \psi_i\}_{i=1}^n} & \sum_{i=1}^n \omega_{1,x,i} \times \varphi_i + \sum_{i=1}^n \omega_{0,x,i} \times \psi_i \\ \text{s.t. } & \varphi_i + \psi_j \leq c(Y_i, Y_j) \text{ for all } 1 \leq i, j \leq n, \end{aligned}$$

- Dimension is reduced by ignoring φ_i , ψ_i , and constraints where $\omega_{d,x,i} = 0$.

Convergence in distribution: theorem

- Let P be the distribution of an observation, and \mathbb{P}_n the empirical distribution.

$$(\hat{\gamma}^L, \hat{\gamma}^H) = T(\mathbb{P}_n), \quad (\gamma^L, \gamma^H) = T(P)$$

Theorem (Weak convergence)

Suppose assumptions 1, 2, and 3 hold. Then

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) \xrightarrow{L} T'_P(\mathbb{G})$$

where $\sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ and $T'_P(\cdot)$ is the Hadamard directional derivative of $T(\cdot)$ at P .

[T\(·\) details](#)

[Proof sketch](#)

Inference: bootstrap

- ▶ Estimating the asymptotic distribution is necessary for inference.
- ▶ The bootstrap provides an attractive procedure.
 - Bootstrap draw: $\{Y_i^*, D_i^*, X_i^*\}_{i=1}^n$
 - Bootstrap empirical distribution: \mathbb{P}_n^*
- ▶ Compute $T(\mathbb{P}_n^*)$ the same way as $T(\mathbb{P}_n)$: let $\omega_{d,x,i}^* = \frac{\mathbb{1}\{D_i^*=d, X_i^*=x\}/n}{\frac{1}{n} \sum_{j=1}^n \mathbb{1}\{D_j^*=d, X_j^*=x\}}$,

$$OT_c(\hat{P}_{1|x}^*, \hat{P}_{0|x}^*) = \max_{\{\varphi_i, \psi_i\}_{i=1}^n} \sum_{i=1}^n \omega_{1,x,i}^* \varphi_i + \sum_{i=1}^n \omega_{0,x,i}^* \psi_i$$

s.t. $\varphi_i + \psi_j \leq c(Y_i, Y_j)$ for all $1 \leq i, j \leq n$

Inference: bootstrap

Assumption 4 (Unique solutions, informal) For each instance of optimal transport in $T(P)$, the solution to the dual problem is suitably unique.

Theorem (Bootstrap consistency)

Suppose assumptions 1, 2, 3, and 4 hold. Then $T'_P(\mathbb{G})$ is bivariate normal, and conditional on $\{Y_i, D_i, X_i\}_{i=1}^n$,

$$\sqrt{n}(T(\mathbb{P}_n^*) - T(\mathbb{P}_n)) \xrightarrow{L} T'_P(\mathbb{G})$$

in outer probability.

Precise assumption 4

Inference: bootstrap

- ▶ Bootstrap works with assumption 4 (unique solutions)...when does that happen?

Lemma (Unique solutions) Suppose that

- (i) $c(y_1, y_0)$ is continuously differentiable, and
 - (ii) for each x , $\text{Supp}(Y_d \mid X = x) = [y_{d,x}^\ell, y_{d,x}^u]$ is bounded.
- then assumption 4 holds.

- ▶ Assumption 4 may hold without this lemma's conditions.

Inference: bootstrap alternative

- ▶ Only require assumptions 1, 2, and 3 to claim

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) \xrightarrow{L} T'_P(\mathbb{G}).$$

- ▶ But without assumption 4, $T'_P(\mathbb{G})$ may not be bivariate Normal,

⇒ The bootstrap is not consistent.

- ▶ The paper shows a consistent alternative.

- Follows Fang and Santos (2019): estimating the derivative $T'_P(\cdot)$.
- Implementation is more involved, but still computationally tractable.

Overview

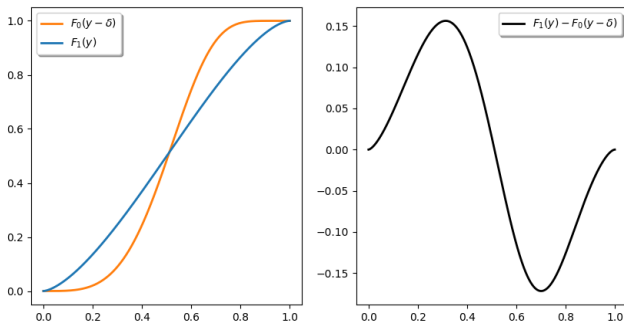
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Simulations: parameter and DGP

- ▶ Parameter $\gamma = \theta = P(Y_1 - Y_0 \leq \delta)$ has simple bounds:

$$\gamma^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \quad \gamma^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

- ▶ For simplicity: no X , $P(D = 1) = 1/2$, distributions of Y_1 , Y_0 :



- ▶ Unique solutions \implies bootstrap is valid.

Simulations: confidence set

► Asymptotic $1 - \alpha$ confidence set for $[\gamma^L, \gamma^H]$:

(i) Using $\{Y_i, D_i, X_i\}_{i=1}^n$, compute estimators:

$$(\hat{\gamma}^L, \hat{\gamma}^H) = T(\mathbb{P}_n)$$

(ii) For each $b = 1, \dots, B$, draw $\{Y_{i,b}^*, D_{i,b}^*, X_{i,b}^*\}_{i=1}^n$ to define $\mathbb{P}_{n,b}^*$ and compute:

$$(\hat{\gamma}_b^{L*}, \hat{\gamma}_b^{H*}) = T(\mathbb{P}_{n,b}^*)$$

(iii) Let $\hat{c}_{1-\alpha}$ be the $1 - \alpha$ quantile of $\{\max\{\sqrt{n}(\hat{\gamma}_b^{L*} - \hat{\gamma}), -\sqrt{n}(\hat{\gamma}_b^{H*} - \hat{\gamma}^H)\}\}_{b=1}^B$, and

$$CI = [\hat{\gamma}^L - \hat{c}_{1-\alpha}/\sqrt{n}, \hat{\gamma}^H + \hat{c}_{1-\alpha}/\sqrt{n}]$$

Simulations: finite sample bias and correction

- ▶ CI has exact *asymptotic* coverage. What about small samples?
 - max over sample averages is biased upward (Haile and Tamer, 2003).
 - Leads to $[\hat{\gamma}^L, \hat{\gamma}^H]$ that tend to be “too narrow” in small samples.
- ▶ Bootstrap bias correction (Efron and Tibshirani, 1994; Horowitz, 2001):

$$(\widehat{bias}^L, \widehat{bias}^H) = \frac{1}{B} \sum_{b=1}^B (\hat{\gamma}^{L*}, \hat{\gamma}^{H*}) - (\hat{\gamma}^L, \hat{\gamma}^H),$$

$$\hat{\gamma}_{BC}^L = \hat{\gamma}^L - \widehat{bias}^L, \quad \hat{\gamma}_{BC}^H = \hat{\gamma}^H - \widehat{bias}^H$$

- ▶ Bootstrap bias corrected confidence interval:

$$CI_{BC} = [\hat{\gamma}_{BC}^L - \hat{c}_{1-\alpha}/\sqrt{n}, \hat{\gamma}_{BC}^H + \hat{c}_{1-\alpha}/\sqrt{n}]$$

Simulations: results

- ▶ 300 simulations, 3,000 bootstrap draws, targeting 95% coverage.

Table: Simulations, $P(Y_1 - Y_0 \leq \delta)$

n	Bias		St. Dev.		Emp. Coverage CI
	$\hat{\gamma}^L$	$\hat{\gamma}^H$	$\hat{\gamma}^L$	$\hat{\gamma}^H$	
100	0.047	-0.051	0.065	0.066	0.900
200	0.031	-0.031	0.049	0.049	0.917
300	0.030	-0.021	0.040	0.040	0.893

Table: Simulations, $P(Y_1 - Y_0 \leq \delta)$, w/Bias Correction

n	Bias		St. Dev.		Emp. Coverage CI_{BC}
	$\hat{\gamma}_{BC}^L$	$\hat{\gamma}_{BC}^H$	$\hat{\gamma}_{BC}^L$	$\hat{\gamma}_{BC}^H$	
100	0.021	-0.026	0.071	0.071	0.927
200	0.013	-0.015	0.052	0.051	0.953
300	0.015	-0.007	0.042	0.042	0.957

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A randomized job training experiment

- ▶ The National Supported Work Demonstration Program (NSW)
 - Disadvantaged workers randomized to treatment (guaranteed job, meeting w/counselor) or control.
 - Diamond and Sekhon (2013) subsample: men, 297 treated and 425 control
 - Outcome Y is 1978 real earnings, one year after treatment ended.

Table: Balance table

	base inc.	age	yrs. educ.	HS dropout	black	hispanic	married
control	3672.49 (6521.53)	24.45 (6.59)	10.19 (1.62)	0.81 (0.39)	0.80 (0.40)	0.11 (0.32)	0.16 (0.36)
treated	3571.00 (5773.13)	24.63 (6.69)	10.38 (1.82)	0.73 (0.44)	0.80 (0.40)	0.09 (0.29)	0.17 (0.37)

Note: Standard deviations in parentheses.

Who saw larger benefits from treatment?

► *Question:* Who saw larger benefits from the NSW treatment?

► *Parameter:* The OLS slope coefficient $Y_1 - Y_0 = \alpha + \gamma Y_0 + \varepsilon$

$$\gamma = \frac{\text{Cov}(Y_1 - Y_0, Y_0)}{\text{Var}(Y_0)} = \frac{\overbrace{E[(Y_1 - Y_0)Y_0]}^{\theta} - (E[Y_1] - E[Y_0])E[Y_0]}{E[Y_0^2] - (E[Y_0])^2}$$

► *Interpretation:* $\gamma < 0$ implies workers with below average Y_0 tend to see above average $Y_1 - Y_0$

NSW results

► Discretized age and baseline income are informative covariates.

- age bins: $[16, 23]$, $(23, \infty)$
- baseline income bins: $[0, 0]$, $(0, 4000]$, $(4000, \infty)$

Table: Estimates of bounds for γ , the OLS Slope

	Lower Bound	Upper Bound	95% <i>CI</i>
No Covariates	-1.78	0.19	$[-2.01, 0.42]$
Disc. Age and Inc.	-1.72	0.00	$[-1.95, 0.22]$
With Bias Corr.	-1.73	0.04	$[-1.96, 0.27]$

NSW results: conditional on covariate values

Table: Estimates conditional on covariate values

age	base inc.	$\hat{\gamma}_{BC}^L$	$\hat{\gamma}_{BC}^H$	95% CI_{BC}	n
	0	-1.97	0.28	[-2.26, 0.56]	140
(16, 23]	(0, 4000]	-1.74	-0.15	[-1.9, 0.01]	141
	(4000, ∞)	-1.45	-0.44	[-1.63, -0.27]	90
	0	-2.13	0.81	[-2.65, 1.33]	187
(23, ∞)	(0, 4000]	-1.39	-0.16	[-1.93, 0.38]	56
	(4000, ∞)	-1.66	0.03	[-2.08, 0.45]	108

- ▶ Among young men with + base income, low Y_0 is associated with high $Y_1 - Y_0$.
- ▶ This subset's vulnerable individuals see larger benefits from treatment.

Conclusion

► Summary:

- Parameters of the **joint distribution of potential outcomes** are not point identified.
- Sharp bounds are characterized with **optimal transport**.
- Sample analogue estimators are **computationally and analytically attractive**.

► Ongoing and future work:

- Accomodate plausible **support restrictions**, such as $Y_1 \geq Y_0$.
- Support function approach to consider parameters depending on **more than one joint moment**.

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Appendix: full setting

Assumption 1 (Setting). $\{Y_i, D_i, Z_i, X_i\}_{i=1}^n$ is an i.i.d. sample, with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \quad D \in \{0, 1\}, \quad Z \in \{0, 1\}, \quad X \in \mathcal{X} = \{x_1, \dots, x_M\}$$

generated from a distribution satisfying

- (i) Potential outcomes: $Y = DY_1 + (1 - D)Y_0$,
- (ii) Potential treatment statuses: $D = ZD_1 + (1 - Z)D_0$, with $D_z \in \{0, 1\}$,
- (iii) Instrument exogeneity: $(Y_1, Y_0, D_1, D_0) \perp Z \mid X$,
- (iv) Monotonicity: $D_1 \geq D_0$ almost surely,
- (v) Existence of compliers: $P(D_1 > D_0, X = x) > 0$ for each x , and
- (vi) $P(X = x, Z = z) > 0$ for each (x, z)

► Terminology: always-taker, complier, defier, never-taker.

	$D_0 = 1$	$D_0 = 0$
$D_1 = 1$	Always-takers	Compliers
$D_1 = 0$	Defiers	Never-takers

► Monotonicity rules out defiers. Focus on distribution of compliers.

Appendix: identification of $P(Y_1 - Y_0 \leq \delta)$

- ▶ $OT_c(P_1, P_0)$ is well behaved (attained, strong duality holds, etc) when $c(y_1, y_0)$ is bounded and lower semicontinuous
- ▶ If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, let

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\},$$

$$\theta_x^L = OT_{c_L}(P_{1|x}, P_{0|x}),$$

$$c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$$

$$\theta_x^H = 1 - OT_{c_H}(P_{1|x}, P_{0|x})$$

- ▶ The form of the bounds remains the same:

$$\theta^L = E[\theta_X^L],$$

$$\gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta),$$

$$\theta^H = E[\theta_X^H]$$

$$\gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

- ▶ Identified sets are still sharp when CDFs are continuous:

$$F_{d|x}(y) = P(Y_d \leq y \mid X = x)$$

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_{\pi}[c(Y_1, Y_0)]$$

- ▶ Bounds on $\theta = P(Y_1 - Y_0 \leq \delta)$ are found with

$$\begin{aligned} c_L(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 < \delta\}, & c_H(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 > \delta\}, \\ \theta^L &= OT_{c_L}(P_1, P_0), & \theta^H &= 1 - OT_{c_H}(P_1, P_0) \end{aligned}$$

Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

- ▶ As a byproduct, recover the famed **Makarov bounds** studied by Fan and Park (2010)

$$\theta^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \quad \theta^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

- ▶ **Furthermore**, $\mathbb{1}\{y_1 - y_0 < \delta\} \leq \mathbb{1}\{y_1 - y_0 \leq \delta\}$ implies **the bounds are conservative**: $\Theta_{ID} \subseteq [\theta^L, \theta^H]$ **whether or not F_d are continuous**.

Appendix: full assumption 3

- Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R}$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$.

Assumption 3 (Function of moments)

- (i) $E[\|\eta_d(Y)\|^2] < \infty$ for $d = 1, 0$,
- (ii) $g(\cdot, \eta)$ is continuous, and
- (iii) the functions

$$g^L(t^L, t^H, e) = \min_{t \in [t^L, t^H]} g(t, e), \quad g^H(t^L, t^H, e) = \max_{t \in [t^L, t^H]} g(t, e)$$

are continuously differentiable at $(t^L, t^H, e) = (\theta^L, \theta^H, \eta)$.

Remark: A3 (ii), (iii) implied by g continuously differentiable and $g(\cdot, \eta)$ monotonic

Appendix: quantiles

- Suppose the parameter of interest is q_τ solving

$$P(Y_1 - Y_0 \leq q_\tau) = \tau$$

- View CDF bounds as a function: $\theta(\delta) = P(Y_1 - Y_0 \leq \delta)$

$$c_{L,\delta}(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\},$$

$$c_{H,\delta}(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\},$$

$$\theta_x^L(\delta) = OT_{c_L}(P_{1|x}, P_{0|x}),$$

$$\theta_x^H(\delta) = 1 - OT_{c_H}(P_{1|x}, P_{0|x})$$

$$\theta^L(\delta) = E[\theta_X^L(\delta)]$$

$$\theta^H(\delta) = E[\theta_X^H(\delta)]$$

and let $Q_{I,\tau}$ be the sharp identified set for q_τ .

Lemma (Identification of q_τ). Suppose assumptions 1 and 2(ii) hold. Then $q \in Q_{I,\tau}$ if and only if $\theta^L(q) \leq \tau \leq \theta^H(q)$.

Appendix: aside, IV

- Identification extends easily to IV.
- Consider the binary IV potential outcomes framework of Abadie (2003):

$$D = ZD_1 + (1 - Z)D_0 \quad (Y_1, Y_0, D_1, D_0) \perp Z \mid X, \quad D_1 \geq D_0$$

units with $D_1 > D_0$ are known as *compliers*.

- This model identifies marginal distributions of potential outcomes of compliers:

$$Y_d \mid D_1 > D_0, X = x$$

- Same identification applies to parameters conditional on compliance. E.g.,

$$P(Y_1 > Y_0 \mid D_1 > D_0)$$

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0, 1\} \times \mathcal{X} \rightarrow \mathbb{R}\}$$

- View \mathbb{P}_n, P as bounded functions on \mathcal{F} :

$$\ell^\infty(\mathcal{F}) = \left\{ g : \mathcal{F} \rightarrow \mathbb{R} ; \|g\|_\infty = \sup_{f \in \mathcal{F}} |g(f)| < \infty \right\}$$

- The map $T : \ell^\infty(\mathcal{F}) \rightarrow \mathbb{R}^2$ is described by $P \mapsto (P_{1|x}, P_{0|x}, \eta)$ and

$$\begin{aligned} \theta_x^L &= OT_c(P_{1|x}, P_{0|x}), & \theta_x^H &= -OT_{-c}(P_{1|x}, P_{0|x}) \\ \theta^L &= E[\theta_X^L], & \theta^H &= E[\theta_X^H] \\ \gamma^L &= \min_{t \in [\theta^L, \theta^H]} g(t, \eta), & \gamma^H &= \max_{t \in [\theta^L, \theta^H]} g(t, \eta) \end{aligned}$$

Weak convergence theorem

Appendix: proof sketch (1/3)

1. Will view P, \mathbb{P} as maps in $\ell^\infty(\mathcal{F})$ for Donsker set \mathcal{F} (defined later), and $T : \ell^\infty(\mathcal{F}) \rightarrow \mathbb{R}^2$.
2. To show $T(\cdot)$ is (Hadamard) directionally differentiable, suffices to show OT_c is directionally differentiable.
3. By strong duality,

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi, \psi) \in \Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$
$$\Phi_c = \{(\varphi, \psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

Weak convergence theorem

Appendix: proof sketch (2/3)

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi, \psi) \in \Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$

$$\Phi_c = \{(\varphi, \psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

4. Φ_c is a **large set**, but much of it can be **ignored**:

- If $\varphi(y_1) \leq \tilde{\varphi}(y_1)$, then $E_{P_{1|x}}[\varphi(Y_1)] \leq E_{P_{1|x}}[\tilde{\varphi}(Y_1)]$
- Any pair (φ, ψ) where $\varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)$ is “slack” can be ignored

5. This observation leads to

$$\sup_{(\varphi, \psi) \in \Phi_c} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] = \sup_{(\varphi, \psi) \in \Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] \quad (1)$$

- (i) if $c(y_1, y_0)$ is L -Lip. and \mathcal{Y} is compact, \mathcal{F}_c and \mathcal{F}_c^c are L -Lip. and universally bounded.
- (ii) if $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, \mathcal{F}_c is the set of intervals, \mathcal{F}_c^c the complements of intervals.

6. Finally, $\Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)$ is compact and $E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$ is continuous

$\implies OT_c$, and therefore $T(\cdot)$, are Hadamard directionally differentiable.

Appendix: proof sketch (3/3)

7. Define \mathcal{F} to be union of \mathcal{F}_c and \mathcal{F}_c^c (and nuisance moments, all \times indicators).
8. \mathcal{F} is Donsker $\implies \sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.
9. Functional delta method implies the result,

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) = \sqrt{n}(T(\mathbb{P}_n) - T(P)) \xrightarrow{L} T'_P(\mathbb{G}).$$

Weak convergence theorem

Appendix: c -concavity

$$OT_c(P_1, P_0) = \sup_{(\varphi, \psi) \in \Phi_c} \underbrace{E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)]}_{J(\varphi, \psi)},$$

- Define the c -transforms:

$$\varphi^c(y_0) = \inf_{y_1} \{c(y_1, y_0) - \varphi(y_1)\}, \quad \psi^c(y_1) = \inf_{y_0} \{c(y_1, y_0) - \psi(y_0)\}$$

call φ^c (and ψ^c) **c -concave** functions.

- For any $(\varphi, \psi) \in \Phi_c = \{(\varphi, \psi) ; \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$,

- (i) $(\varphi, \varphi^c) \in \Phi_c$
- (ii) If $(\varphi, \psi) \in \Phi_c$, then $\psi(y_0) \leq \varphi^c(y_0)$ for all y_0 , so
- (iii) $J(\varphi, \psi) \leq J(\varphi, \varphi^c)$ by monotonicity of $E_{P_d}[\cdot]$.

⇒ The dual problem can be restricted to c -concave functions.

- c -concave functions often **inherit properties of c** :

- Lipschitz continuity, boundedness, etc.
- These properties are used to define \mathcal{F}_c and \mathcal{F}_c^c

Appendix: formal assumption 4

- ▶ Let P be the distribution of an observation: $(Y, D, Z, X) \sim P$.
- ▶ Let $\mathcal{Y}_{d,x}$ be the support of $Y \mid D = d, X = x$, and $\mathbb{1}_{\mathcal{Y}_{d,x}}(y) = \mathbb{1}\{y \in \mathcal{Y}_{d,x}\}$
- ▶ Define c_L, c_H :
 - (i) If assumption 2 (i) holds, let $c_L = c(y_1, y_0)$ and $c_H(y_1, y_0) = -c(y_1, y_0)$.
 - (ii) If assumption 2 (ii) holds, let $c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}$ and $c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$.

Assumption 4 (Unique solutions) For each $x \in \mathcal{X}$, each $c \in \{c_L, c_H\}$, and any

$$(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \arg \max_{(\varphi, \psi) \in \Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)],$$

there exists $s \in \mathbb{R}$ such that

$$\mathbb{1}_{\mathcal{Y}_{1,x}} \times \varphi_1 = \mathbb{1}_{\mathcal{Y}_{1,x}} \times (\varphi_2 + s), \quad P - a.s., \quad \mathbb{1}_{\mathcal{Y}_{0,x}} \times \psi_1 = \mathbb{1}_{\mathcal{Y}_{0,x}} \times (\psi_2 - s), \quad P - a.s.$$

Assumption 4

Why c_L, c_H ?