Estimating Functionals of the Joint Distribution of Potential Outcomes with Optimal Transport

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Introduction

The fundamental problem of causal inference

It is impossible to observe the [treated outcome] and [untreated outcome] on the same unit and, therefore, it is impossible to observe the effect...

(Holland, 1986)

- Parameters of the joint distribution of potential outcomes are not point identified.
- **This paper**
  - shows optimal transport characterizes sharp bounds,
  - accomodates noncompliance through a standard IV model, and
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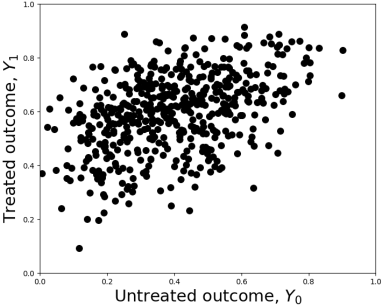

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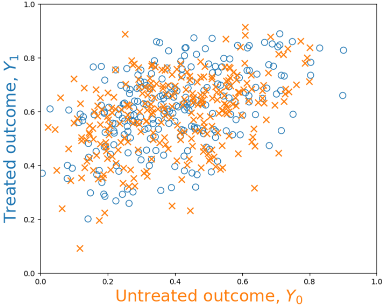

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$$\text{Observed outcome } Y = D\,Y_1 + (1 - D)\,Y_0$$

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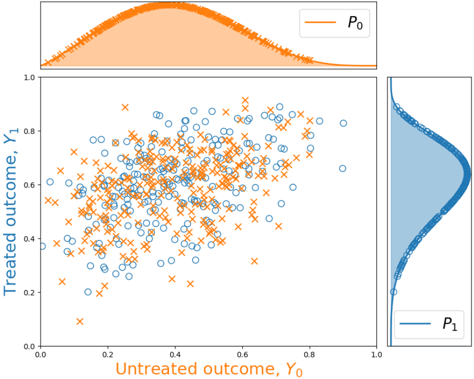

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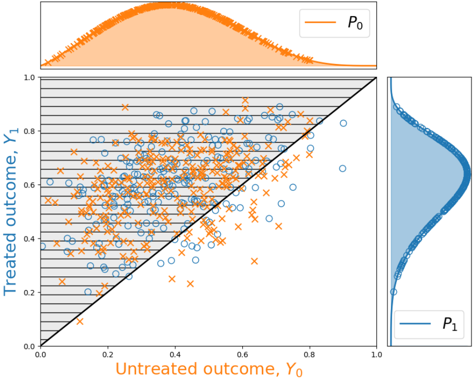

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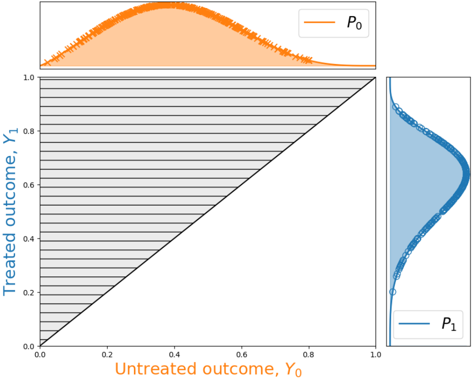
- For example, what share of units benefit from treatment?

Example 1: the share benefiting from treatment



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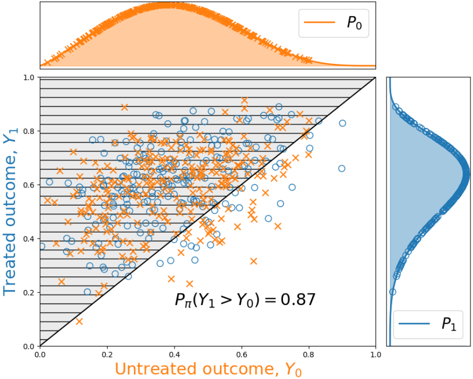

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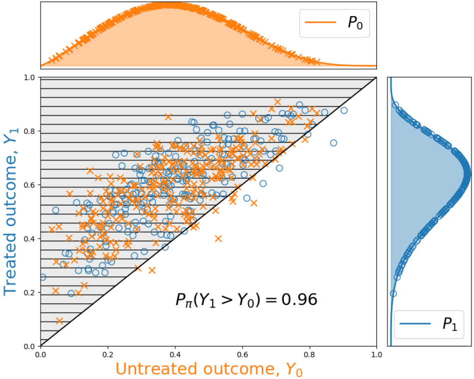

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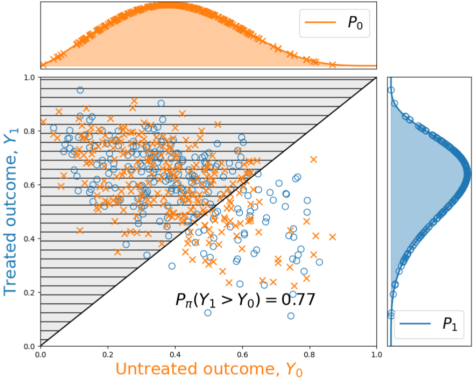

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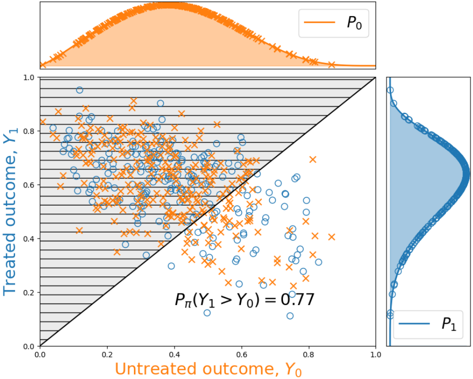

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- Optimizing $P(Y_1 > Y_0)$ over $\Pi(P_1, P_0)$ implies bounds:

$$\min_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0) \qquad \max_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0)$$

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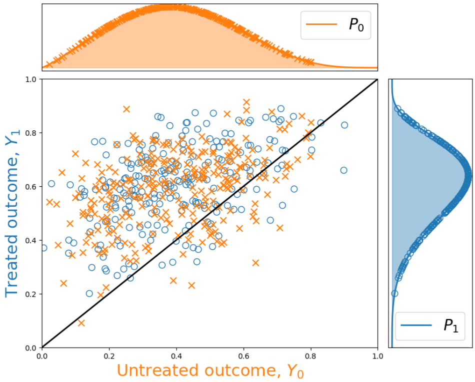

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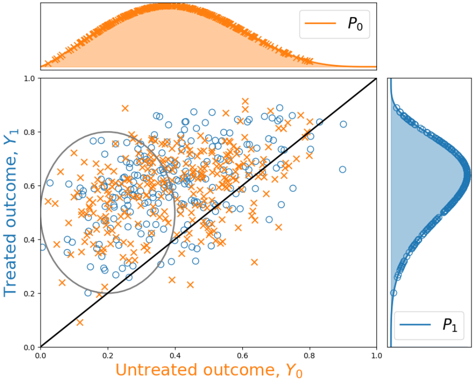

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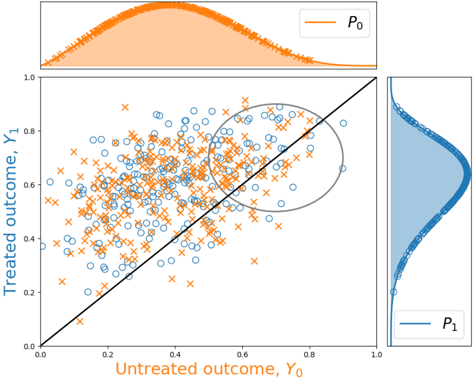

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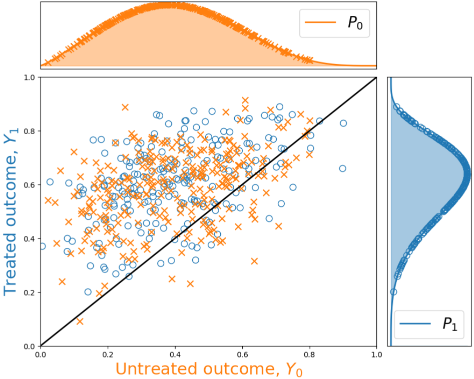

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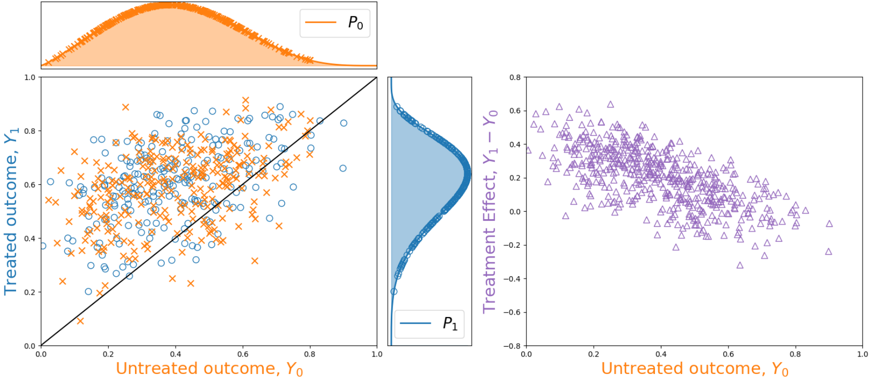

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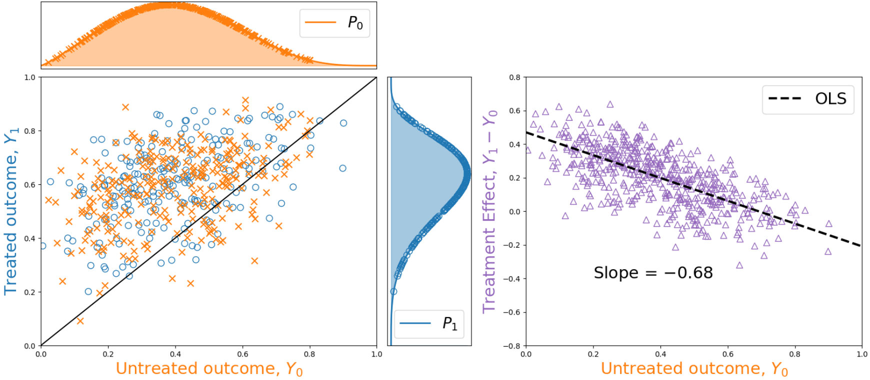

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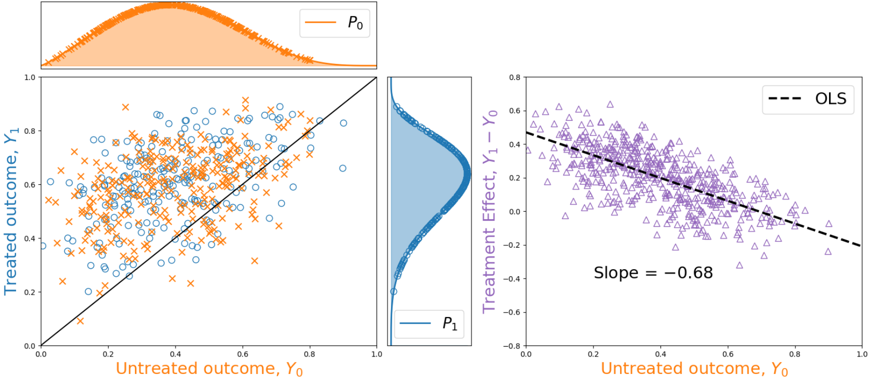

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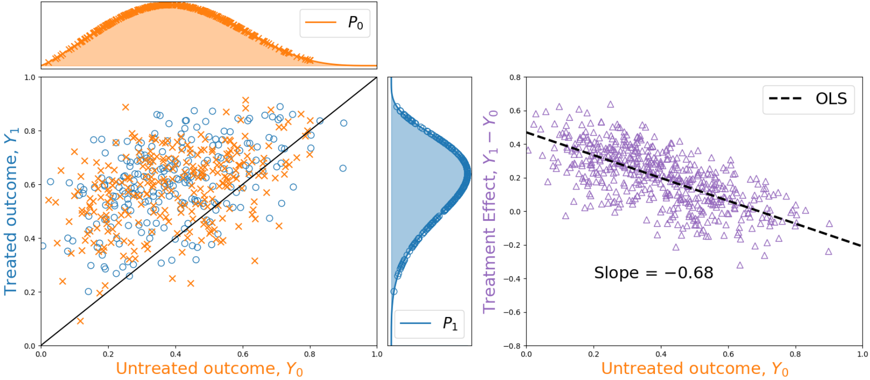

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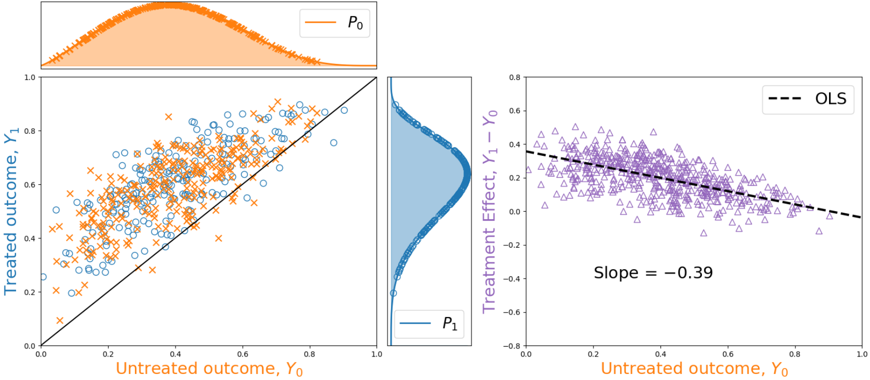

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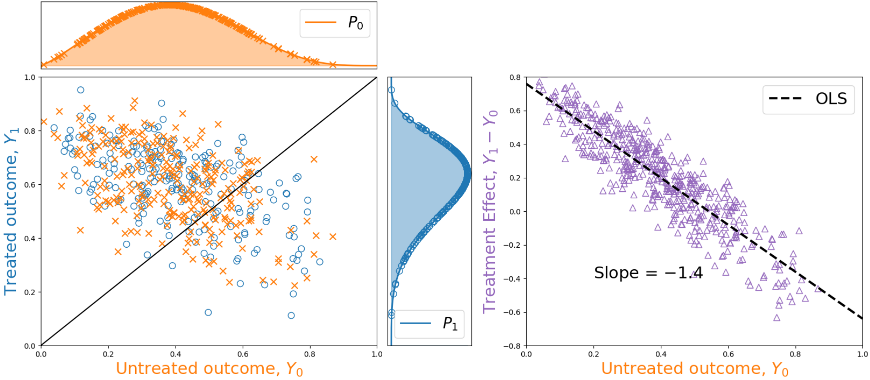

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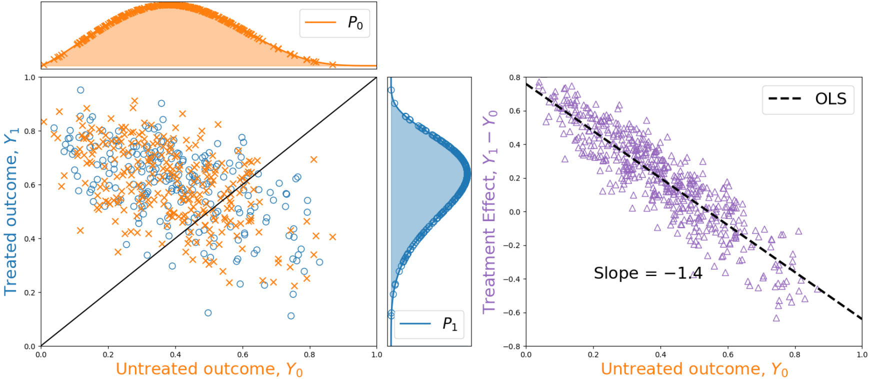

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- Empirical application: who sees larger benefits from the NSW job training?

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  - CDF or quantiles of $Y_1 - Y_0$: Manski (1997), Heckman et al. (1997), Firpo (2007), Fan and Park (2010), Fan and Park (2012), Firpo and Ridder (2019), Callaway (2021), Frandsen and Lefgren (2021).
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Overview

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1. Setting and parameter class
2. Identification
3. Estimators
4. Simulations
5. Application

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Assumption 1 (Setting, simplified) $\{Y_i, D_i, X_i\}_{i=1}^n$ is an i.i.d. sample with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \qquad D \in \{0, 1\}, \qquad X \in \mathcal{X} = \{x_1, \ldots, x_M\}$$

generated from a distribution satisfying

(i) Potential outcomes: $Y = DY_1 + (1 - D)Y_0$

(ii) Unconfoundedness: $(Y_1, Y_0) \perp D \mid X$

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- In the paper, binary IV satisfying monotonicity condition (Imbens and Angrist, 1994).

Setting w/IV

Parameter class

- Parameter of interest:
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where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$

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Assumption 2 (Cost function) Either

(i) $c(y_1, y_0)$ is Lipschitz continuous and $\mathcal{Y}$ is compact, or

(ii) $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ and the CDFs $F_{d|x}(y) = P(Y_d \leq y \mid X = x)$ are continuous.

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Remark: If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ but $F_{d|x}(\cdot)$ are not continuous, inference remains valid for an outer identified set.

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Remark: Assumption 3 (iii) is relaxed in the paper.

Full assumption 3

Parameter class: motivating examples

- Share benefiting: $P(Y_1 > Y_0)$
  - Allcott et al. (2020): deactivating Facebook affects subjective well-being.

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Overview

Optimal transport

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Choose a joint distribution with given marginals to minimize costs.
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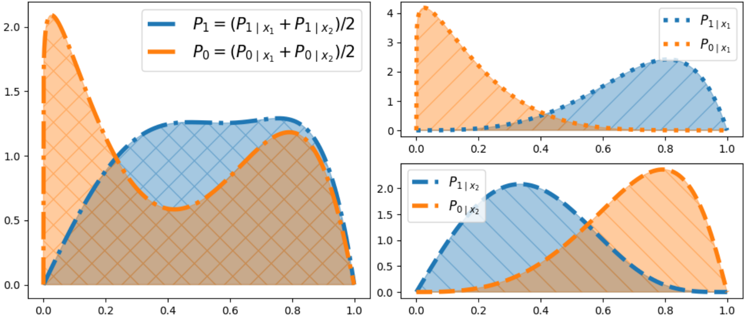

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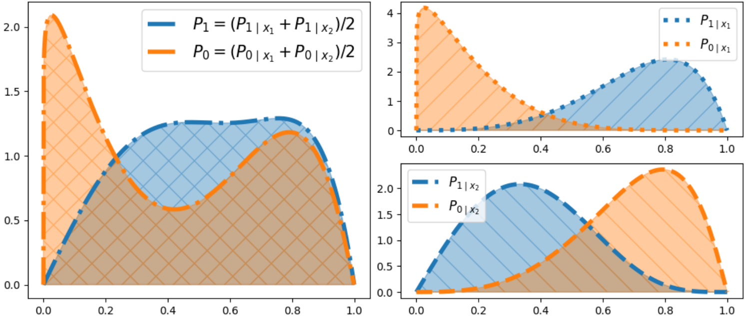

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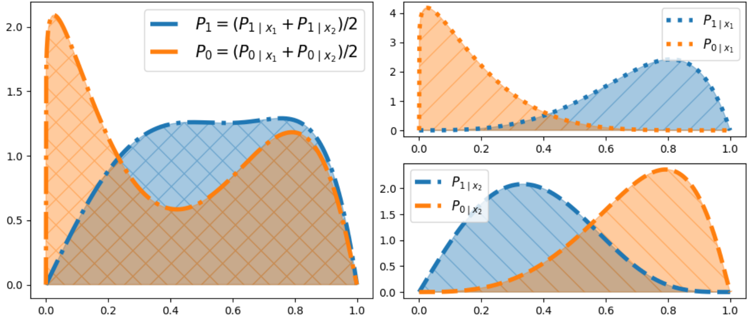

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Theorem (identification)

Suppose assumptions 1, 2, and 3 are satisfied. Then the sharp identified set for $\gamma = g(\theta, \eta)$ *is* $[\gamma^L, \gamma^H]$.

CDF? IV Aside Quantile details

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- Strong duality holds under the cost function assumptions. Each problem is attained, too.

Estimators: recall identification

- Distributions of $Y_d \mid X = x \sim P_{d|x}$:

$$E_{P_{d|x}}[f(Y_d)] = \frac{E[f(Y)\mathbb{1}\{D = d, X = x\}]}{P(D = d, X = x)}$$

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CDF?

Estimators: sample analogues

- Estimate $P_{d|x}$ with sample analogues $\hat{P}_{d|x}$:

$$E_{\hat{P}_{d|x}}[f(Y_d)] = \frac{\frac{1}{n}\sum_{i=1}^{n} f(Y_i)\mathbb{1}\{D_i = d, X_i = x\}}{\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{D_i = d, X_i = x\}}$$

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- Estimate the endpoints of $[\gamma^L, \gamma^H]$ with plug-in estimators. For c continuous,

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CDF?

Estimators: computing $OT_c(\hat{P}_{1|x}, \hat{P}_{0|x})$

$$OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{(\varphi,\psi)\in\Phi_c} E_{\hat{P}_{1|x}}[\varphi(Y_1)] + E_{\hat{P}_{0|x}}[\psi(Y_0)].$$

- To evaluate $E_{\hat{P}_{d|x}}[f(Y_d)]$ for any function f,

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- Dimension is reduced by ignoring φ_i, ψ_i, and constraints where $\omega_{d,x,i} = 0$.

Convergence in distribution: theorem

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Theorem (Weak convergence)

Suppose assumptions 1, 2, and 3 hold. Then

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) \xrightarrow{L} T'_P(\mathbb{G})$$

where $\sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ *and* $T'_P(\cdot)$ *is the Hadamard directional derivative of* $T(\cdot)$ *at* P*.*

$T(\cdot)$ details | Proof sketch

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- Compute $T(\mathbb{P}_n^*)$ the same way as $T(\mathbb{P}_n)$: let $\omega_{d,x,i}^* = \frac{\mathbb{1}\{D_i^*=d, X_i^*=x\}/n}{\frac{1}{n}\sum_{j=1}^n \mathbb{1}\{D_j^*=d, X_j^*=x\}}$,

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Precise assumption 4

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Theorem (Bootstrap consistency)

Suppose assumptions 1, 2, 3, and 4 hold. Then $T'_P(\mathbb{G})$ is bivariate normal, and conditional on $\{Y_i, D_i, X_i\}_{i=1}^n$,

$$\sqrt{n}(T(\mathbb{P}_n^*) - T(\mathbb{P}_n)) \xrightarrow{L} T'_P(\mathbb{G})$$

in outer probability.

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- Assumption 4 may hold without this lemma's conditions.

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- The paper shows a consistent alternative.
  - Follows Fang and Santos (2019): estimating the derivative $T'_P(\cdot)$.
  - Implementation is more involved, but still computationally tractable.

Overview

Simulations: parameter and DGP

- Parameter $\gamma = \theta = P(Y_1 - Y_0 \leq \delta)$ has simple bounds:

$$\gamma^L = \sup_y \{F_1(y) - F_0(y-\delta)\}, \qquad \gamma^H = 1 + \inf_y \{F_1(y) - F_0(y-\delta)\}$$

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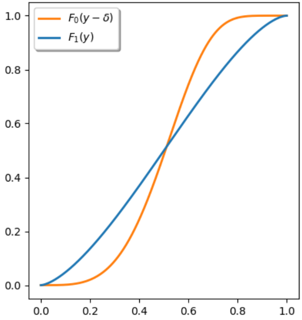
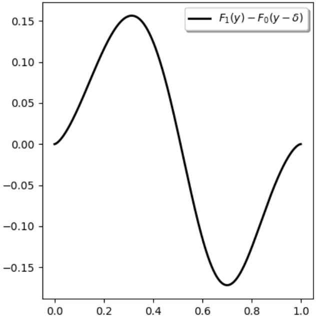
- For simplicity: no X, $P(D = 1) = 1/2$, distributions of Y_1, Y_0:





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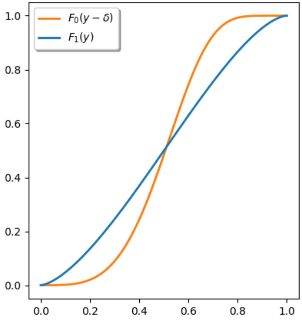

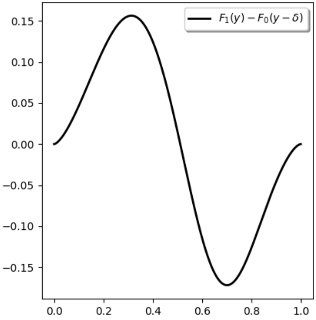

- Unique solutions $\implies$ bootstrap is valid.

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  (ii) For each $b = 1, \ldots, B$, draw $\{Y^*_{i,b}, D^*_{i,b}, X^*_{i,b}\}_{i=1}^n$ to define $\mathbb{P}^*_{n,b}$ and compute:

  $$(\hat{\gamma}^{L*}_b, \hat{\gamma}^{H*}_b) = T(\mathbb{P}^*_{n,b})$$

  (iii) Let $\hat{c}_{1-\alpha}$ be the $1-\alpha$ quantile of $\{\max\{\sqrt{n}(\hat{\gamma}^{L*}_b - \hat{\gamma}), -\sqrt{n}(\hat{\gamma}^{H*}_b - \hat{\gamma}^H)\}\}_{b=1}^B$, and

  $$CI = [\hat{\gamma}^L - \hat{c}_{1-\alpha}/\sqrt{n}, \hat{\gamma}^H + \hat{c}_{1-\alpha}/\sqrt{n}]$$

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- Bootstrap bias corrected confidence interval:

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n	Bias		St. Dev.		Emp. Coverage
	$\hat{\gamma}^L$	$\hat{\gamma}^H$	$\hat{\gamma}^L$	$\hat{\gamma}^H$	CI
100	0.047	-0.051	0.065	0.066	0.900
200	0.031	-0.031	0.049	0.049	0.917
300	0.030	-0.021	0.040	0.040	0.893

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n	Bias $\hat{\gamma}^L_{BC}$	Bias $\hat{\gamma}^H_{BC}$	St. Dev. $\hat{\gamma}^L_{BC}$	St. Dev. $\hat{\gamma}^H_{BC}$	Emp. Coverage CI_{BC}
100	0.021	-0.026	0.071	0.071	0.927
200	0.013	-0.015	0.052	0.051	0.953
300	0.015	-0.007	0.042	0.042	0.957

Overview

A randomized job training experiment

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Table: Balance table

	base inc.	age	yrs. educ.	HS dropout	black	hispanic	married
control	3672.49	24.45	10.19	0.81	0.80	0.11	0.16
	(6521.53)	(6.59)	(1.62)	(0.39)	(0.40)	(0.32)	(0.36)
treated	3571.00	24.63	10.38	0.73	0.80	0.09	0.17
	(5773.13)	(6.69)	(1.82)	(0.44)	(0.40)	(0.29)	(0.37)

Note: Standard deviations in parentheses.

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$$\gamma = \frac{\mathrm{Cov}(Y_1 - Y_0, Y_0)}{\mathrm{Var}(Y_0)} = \frac{\overbrace{E[(Y_1 - Y_0)Y_0]}^{\theta} - (E[Y_1] - E[Y_0])E[Y_0]}{E[Y_0^2] - (E[Y_0])^2}$$

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- *Interpretation:* $\gamma < 0$ implies workers with below average Y_0 tend to see above average $Y_1 - Y_0$

NSW results

- Discretized age and baseline income are informative covariates.
  - age bins: $[16, 23]$, $(23, \infty)$
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Table: Estimates of bounds for γ, the OLS Slope

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NSW results: conditional on covariate values

Table: Estimates conditional on covariate values

age	base inc.	$\hat{\gamma}^L_{BC}$	$\hat{\gamma}^H_{BC}$	95% CI_{BC}	n
(16, 23]	0	-1.97	0.28	[-2.26, 0.56]	140
	(0, 4000]	-1.74	-0.15	[-1.9, 0.01]	141
	(4000, ∞)	-1.45	-0.44	[-1.63, -0.27]	90
(23, ∞)	0	-2.13	0.81	[-2.65, 1.33]	187
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- This subset's vulnerable individuals see larger benefits from treatment.

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- Summary:
  - Parameters of the joint distribution of potential outcomes are not point identified.
  - Sharp bounds are characterized with optimal transport.
  - Sample analogue estimators are computationally and analytically attractive.

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Appendix: full setting

Assumption 1 (Setting). $\{Y_i, D_i, Z_i, X_i\}_{i=1}^n$ is an i.i.d. sample, with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \qquad D \in \{0,1\}, \qquad Z \in \{0,1\}, \qquad X \in \mathcal{X} = \{x_1, \dots, x_M\}$$

generated from a distribution satisfying

(i) Potential outcomes: $Y = DY_1 + (1-D)Y_0$,

(ii) Potential treatment statuses: $D = ZD_1 + (1-Z)D_0$, with $D_z \in \{0,1\}$,

(iii) Instrument exogeneity: $(Y_1, Y_0, D_1, D_0) \perp Z \mid X$,

(iv) Monotonicity: $D_1 \geq D_0$ almost surely,

(v) Existence of compliers: $P(D_1 > D_0, X = x) > 0$ for each x, and

(vi) $P(X = x, Z = z) > 0$ for each (x, z)

- Terminology: always-taker, complier, defier, never-taker.

	$D_0 = 1$	$D_0 = 0$
$D_1 = 1$	Always-takers	Compliers
$D_1 = 0$	~~Defiers~~	Never-takers

- Monotonicity rules out defiers. Focus on distribution of compliers.

Setting Aside, IV

Appendix: identification of $P(Y_1 - Y_0 \leq \delta)$

- $OT_c(P_1, P_0)$ is well behaved (attained, strong duality holds, etc) when $c(y_1, y_0)$ is bounded and lower semicontinuous
- If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, let

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$$

$$\theta_x^L = OT_{c_L}(P_{1|x}, P_{0|x}), \qquad \theta_x^H = 1 - OT_{c_H}(P_{1|x}, P_{0|x})$$

- The form of the bounds remains the same:

$$\theta^L = E[\theta_X^L], \qquad \theta^H = E[\theta_X^H]$$

$$\gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta), \qquad \gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

- Identified sets are still sharp when CDFs are continuous:

$$F_{d|x}(y) = P(Y_d \leq y \mid X = x)$$

Identification | Identification Thm | Estimators

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Bounds on $\theta = P(Y_1 - Y_0 \leq \delta)$ are found with

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\},$$
$$\theta^L = OT_{c_L}(P_1, P_0), \qquad \theta^H = 1 - OT_{c_H}(P_1, P_0)$$

Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

Appendix: aside, CDF results are conservative when continuity fails

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  Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

- As a byproduct, recover the famed **Makarov bounds** studied by Fan and Park (2010)

$$\theta^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \qquad \theta^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Bounds on $\theta = P(Y_1 - Y_0 \leq \delta)$ are found with

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\},$$
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  Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

- As a byproduct, recover the famed **Makarov bounds** studied by Fan and Park (2010)

$$\theta^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \qquad \theta^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

- **Furthermore,** $\mathbb{1}\{y_1 - y_0 < \delta\} \leq \mathbb{1}\{y_1 - y_0 \leq \delta\}$ implies the bounds are conservative: $\Theta_{ID} \subseteq [\theta^L, \theta^H]$ whether or not F_d are continuous.

Identification | Identification Thm | Estimators

Appendix: full assumption 3

- Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R}$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$.

Assumption 3 (Function of moments)

(i) $E[\|\eta_d(Y)\|^2] < \infty$ for $d = 1, 0$,

(ii) $g(\cdot, \eta)$ is continuous, and

(iii) the functions

$$g^L(t^L, t^H, e) = \min_{t \in [t^L, t^H]} g(t, e), \qquad g^H(t^L, t^H, e) = \max_{t \in [t^L, t^H]} g(t, e)$$

are continuously differentiable at $(t^L, t^H, e) = (\theta^L, \theta^H, \eta)$.

Remark: A3 (ii), (iii) implied by g continuously differentiable and $g(\cdot, \eta)$ monotonic

Back

Appendix: quantiles

- Suppose the parameter of interest is q_τ solving
$$P(Y_1 - Y_0 \leq q_\tau) = \tau$$

- View CDF bounds as a function: $\theta(\delta) = P(Y_1 - Y_0 \leq \delta)$
$$\begin{aligned} c_{L,\delta}(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 < \delta\}, & c_{H,\delta}(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 > \delta\}, \\ \theta_x^L(\delta) &= OT_{c_L}(P_{1|x}, P_{0|x}), & \theta_x^H(\delta) &= 1 - OT_{c_H}(P_{1|x}, P_{0|x}) \\ \theta^L(\delta) &= E[\theta_X^L(\delta)] & \theta^H(\delta) &= E[\theta_X^H(\delta)] \end{aligned}$$
and let $Q_{I,\tau}$ be the sharp identified set for q_τ.

Lemma (Identification of q_τ). Suppose assumptions 1 and 2(ii) hold. Then $q \in Q_{I,\tau}$ if and only if $\theta^L(q) \leq \tau \leq \theta^H(q)$.

Examples

Appendix: aside, IV

- Identification extends easily to IV.

Ident. Thm.

Appendix: aside, IV

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- Consider the binary IV potential outcomes framework of Abadie (2003): $Z \in \{0, 1\}$,

Ident. Thm. IV Setting

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Ident. Thm. IV Setting

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- Consider the binary IV potential outcomes framework of Abadie (2003): $Z \in \{0, 1\}$,

$$D = ZD_1 + (1 - Z)D_0, \qquad (Y_1, Y_0, D_1, D_0) \perp Z \mid X,$$

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- Consider the binary IV potential outcomes framework of Abadie (2003): $Z \in \{0, 1\}$,

$$D = ZD_1 + (1 - Z)D_0, \qquad (Y_1, Y_0, D_1, D_0) \perp Z \mid X, \qquad D_1 \geq D_0$$

Ident. Thm. IV Setting

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units with $D_1 > D_0$ are known as *compliers*.

Ident. Thm. IV Setting

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  units with $D_1 > D_0$ are known as *compliers*.
- This model identifies marginal distributions of potential outcomes of compliers:

$$Y_d \mid D_1 > D_0, X = x$$

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  units with $D_1 > D_0$ are known as *compliers*.
- This model identifies marginal distributions of potential outcomes of compliers:

$$Y_d \mid D_1 > D_0, X = x$$

- Same identification applies to parameters conditional on compliance. E.g.,

$$P(Y_1 > Y_0 \mid D_1 > D_0)$$

Ident. Thm. IV Setting

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

Weak convergence theorem

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

- View $\mathbb{P}_n$, P as bounded functions on $\mathcal{F}$:

$$\ell^\infty(\mathcal{F}) = \left\{ g : \mathcal{F} \to \mathbb{R} \; ; \; \|g\|_\infty = \sup_{f \in \mathcal{F}} |g(f)| < \infty \right\}$$

Weak convergence theorem

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

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$$\ell^\infty(\mathcal{F}) = \left\{ g : \mathcal{F} \to \mathbb{R} \; ; \; \|g\|_\infty = \sup_{f \in \mathcal{F}} |g(f)| < \infty \right\}$$

- The map $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$ is described by $P \mapsto (P_{1|x}, P_{0|x}, \eta)$ and

$$\begin{aligned} \theta_x^L &= OT_c(P_{1|x}, P_{0|x}), & \theta_x^H &= -OT_{-c}(P_{1|x}, P_{0|x}) \\ \theta^L &= E[\theta_X^L], & \theta^H &= E[\theta_X^H] \\ \gamma^L &= \min_{t \in [\theta^L, \theta^H]} g(t, \eta), & \gamma^H &= \max_{t \in [\theta^L, \theta^H]} g(t, \eta) \end{aligned}$$

Weak convergence theorem

Appendix: proof sketch (1/3)

1. Will view P, $\mathbb{P}$ as maps in $\ell^\infty(\mathcal{F})$ for Donsker set $\mathcal{F}$ (defined later), and $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$.

Weak convergence theorem

Appendix: proof sketch (1/3)

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2. To show $T(\cdot)$ is (Hadamard) directionally differentiable, suffices to show OT_c is directionally differentiable.

Weak convergence theorem

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1. Will view P, $\mathbb{P}$ as maps in $\ell^\infty(\mathcal{F})$ for Donsker set $\mathcal{F}$ (defined later), and $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$.

2. To show $T(\cdot)$ is (Hadamard) directionally differentiable, suffices to show OT_c is directionally differentiable.

3. By strong duality,

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi,\psi)\in\Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$

$$\Phi_c = \{(\varphi, \psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

Weak convergence theorem

Appendix: proof sketch (2/3)

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi,\psi)\in\Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$

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4. Φ_c is a large set, but much of it can be ignored:

Weak convergence theorem c-concavity

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4. Φ_c is a large set, but much of it can be ignored:
   - If $\varphi(y_1) \leq \tilde{\varphi}(y_1)$, then $E_{P_{1|x}}[\varphi(Y_1)] \leq E_{P_{1|x}}[\tilde{\varphi}(Y_1)]$

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5. This observation leads to

$$\sup_{(\varphi,\psi)\in\Phi_c} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] = \sup_{(\varphi,\psi)\in\Phi_c\cap(\mathcal{F}_c\times\mathcal{F}_c^c)} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] \quad (1)$$

   (i) if $c(y_1, y_0)$ is L-Lip. and $\mathcal{Y}$ is compact, $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are L-Lip. and universally bounded.

   (ii) if $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, $\mathcal{F}_c$ is the set of intervals, $\mathcal{F}_c^c$ the complements of intervals.

Weak convergence theorem | c-concavity

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6. Finally, $\Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)$ is compact and $E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$ is continuous

Weak convergence theorem | c-concavity

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   (i) if $c(y_1, y_0)$ is L-Lip. and $\mathcal{Y}$ is compact, $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are L-Lip. and universally bounded.

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$\implies$ OT_c, and therefore $T(\cdot)$, are Hadamard directionally differentiable.

Weak convergence theorem | c-concavity

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

Weak convergence theorem

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

8. $\mathcal{F}$ is Donsker $\implies \sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

Weak convergence theorem

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

8. $\mathcal{F}$ is Donsker $\implies \sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

9. Functional delta method implies the result,

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) = \sqrt{n}(T(\mathbb{P}_n) - T(P)) \xrightarrow{L} T'_P(\mathbb{G}).$$

Weak convergence theorem

Appendix: c-concavity

$$OT_c(P_1, P_0) = \sup_{(\varphi,\psi)\in\Phi_c} \underbrace{E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)]}_{J(\varphi,\psi)},$$

- Define the c-transforms:

$$\varphi^c(y_0) = \inf_{y_1}\{c(y_1, y_0) - \varphi(y_1)\}, \qquad \psi^c(y_1) = \inf_{y_0}\{c(y_1, y_0) - \psi(y_0)\}$$

call φ^c (and ψ^c) c-concave functions.

Proof sketch | Weak convergence theorem

Appendix: c-concavity

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- For any $(\varphi, \psi) \in \Phi_c = \{(\varphi, \psi)\ ;\ \varphi(y_1) + \psi(y_0) \le c(y_1, y_0)\}$,

Proof sketch | Weak convergence theorem

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Proof sketch Weak convergence theorem

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  (ii) If $(\varphi, \psi) \in \Phi_c$, then $\psi(y_0) \leq \varphi^c(y_0)$ for all y_0, so

Proof sketch Weak convergence theorem

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Proof sketch Weak convergence theorem

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  $\implies$ The dual problem can be restricted to c-concave functions.

Proof sketch Weak convergence theorem

Appendix: c-concavity

$$OT_c(P_1, P_0) = \sup_{(\varphi,\psi)\in\Phi_c} \underbrace{E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)]}_{J(\varphi,\psi)},$$

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  $\Longrightarrow$ The dual problem can be restricted to c-concave functions.

- c-concave functions often inherit properties of c:

Proof sketch   Weak convergence theorem

Appendix: c-concavity

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- c-concave functions often inherit properties of c:
  - Lipschitz continuity, boundedness, etc.

Proof sketch | Weak convergence theorem

Appendix: c-concavity

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  $\Longrightarrow$ The dual problem can be restricted to c-concave functions.

- c-concave functions often inherit properties of c:
  - Lipschitz continuity, boundedness, etc.
  - These properties are used to define $\mathcal{F}_c$ and $\mathcal{F}_c^c$

Proof sketch | Weak convergence theorem

Appendix: formal assumption 4

- Let P be the distribution of an observation: $(Y, D, Z, X) \sim P$.
- Let $\mathcal{Y}_{d,x}$ be the support of $Y \mid D = d, X = x$, and $\mathbb{1}_{\mathcal{Y}_{d,x}}(y) = \mathbb{1}\{y \in \mathcal{Y}_{d,x}\}$
- Define c_L, c_H:
  (i) If assumption 2 (i) holds, let $c_L = c(y_1, y_0)$ and $c_H(y_1, y_0) = -c(y_1, y_0)$.
  (ii) If assumption 2 (ii) holds, let $c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}$ and $c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$.

Assumption 4 (Unique solutions) For each $x \in \mathcal{X}$, each $c \in \{c_L, c_H\}$, and any

$$(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \underset{(\varphi,\psi)\in\Phi_c\cap(\mathcal{F}_c\times\mathcal{F}_c^c)}{\arg\max} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)],$$

there exists $s \in \mathbb{R}$ such that

$$\mathbb{1}_{\mathcal{Y}_{1,x}} \times \varphi_1 = \mathbb{1}_{\mathcal{Y}_{1,x}} \times (\varphi_2 + s),\ P - a.s., \quad \mathbb{1}_{\mathcal{Y}_{0,x}} \times \psi_1 = \mathbb{1}_{\mathcal{Y}_{0,x}} \times (\psi_2 - s),\ P - a.s.$$

Assumption 4 Why c_L, c_H?