Estimating Functionals of the Joint Distribution of Potential Outcomes with Optimal Transport

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Introduction

The fundamental problem of causal inference

It is impossible to observe the [treated outcome] and [untreated outcome] on the same unit and, therefore, it is impossible to observe the effect...

(Holland, 1986)

- Parameters of the joint distribution of potential outcomes are not point identified.
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  - shows optimal transport characterizes sharp bounds,
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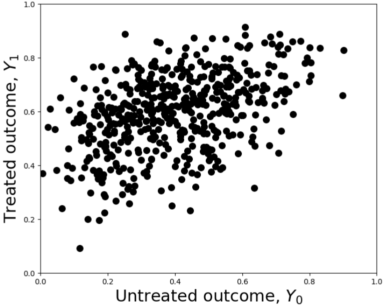

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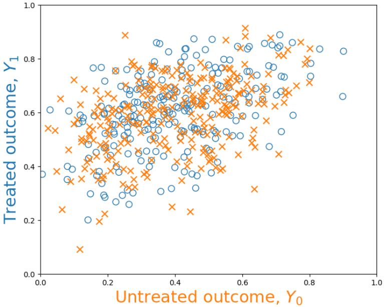

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$$\text{Observed outcome } Y = D Y_1 + (1 - D) Y_0$$

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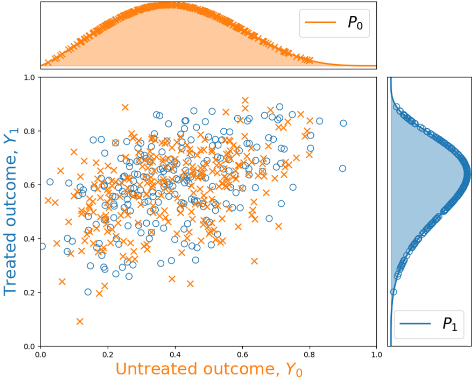

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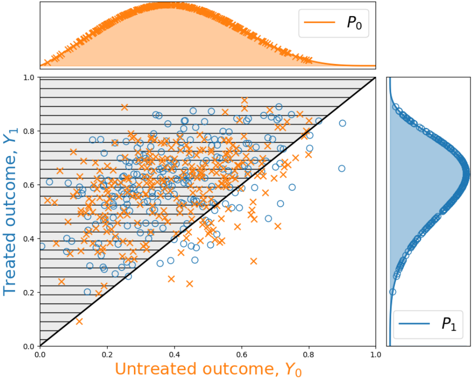

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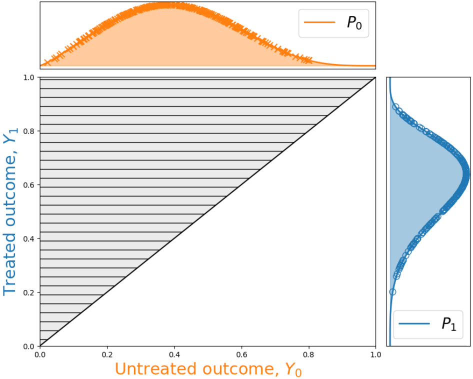
- For example, what share of units benefit from treatment?

Example 1: the share benefiting from treatment



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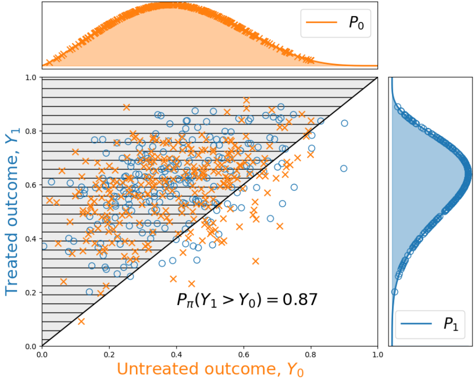

- Many joint distributions π share marginal distributions P_1, P_0:

$$\Pi(P_1, P_0) = \{\pi \ : \ \pi_1 = P_1, \ \pi_0 = P_0\}$$

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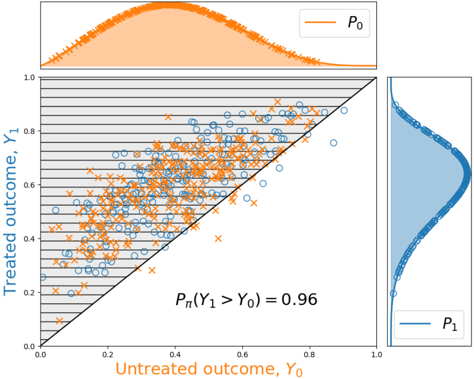

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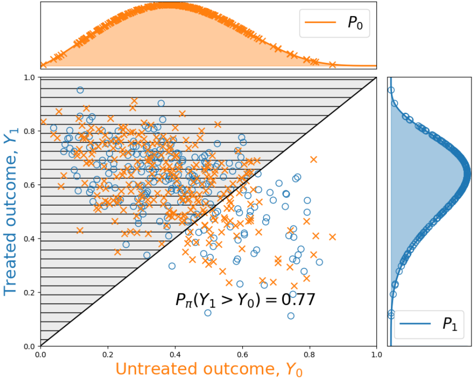

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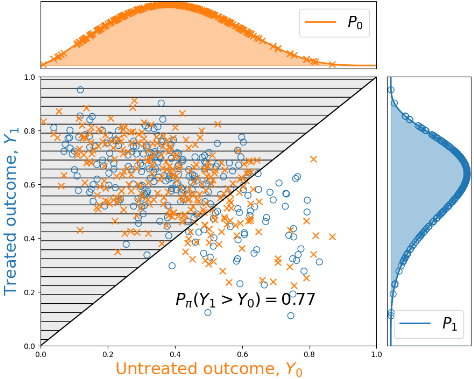

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- Optimizing $P(Y_1 > Y_0)$ over $\Pi(P_1, P_0)$ implies bounds:

$$\min_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0) \qquad\qquad \max_{\pi \in \Pi(P_1, P_0)} P_\pi(Y_1 > Y_0)$$

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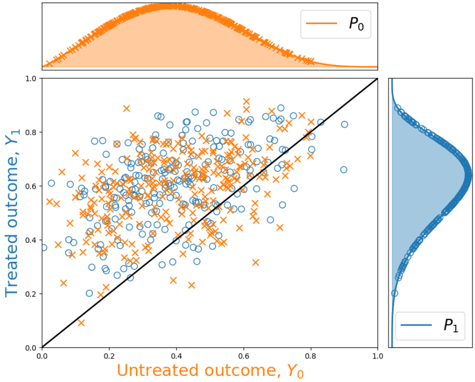

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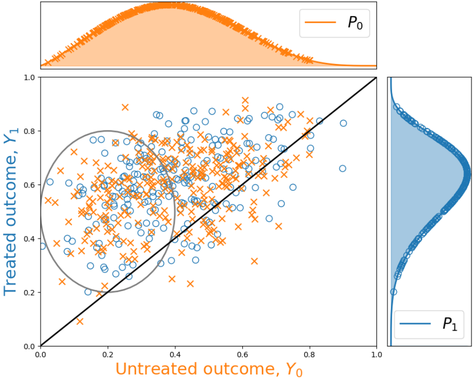

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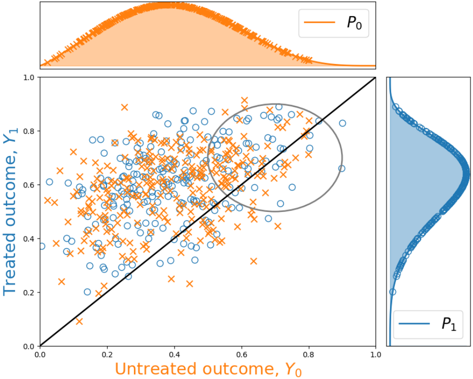

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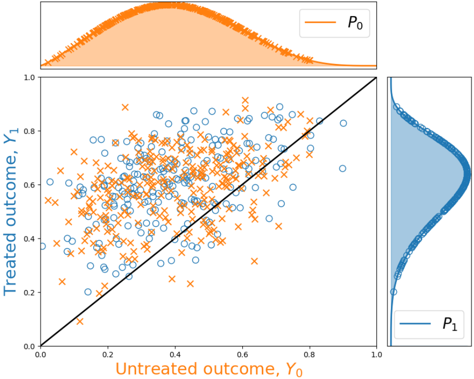

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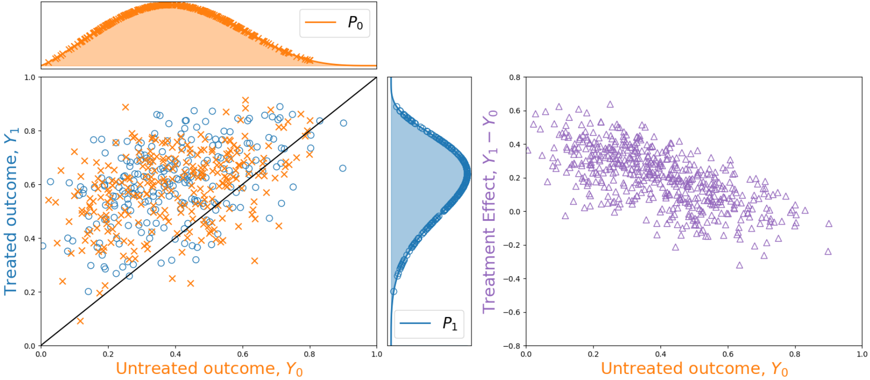

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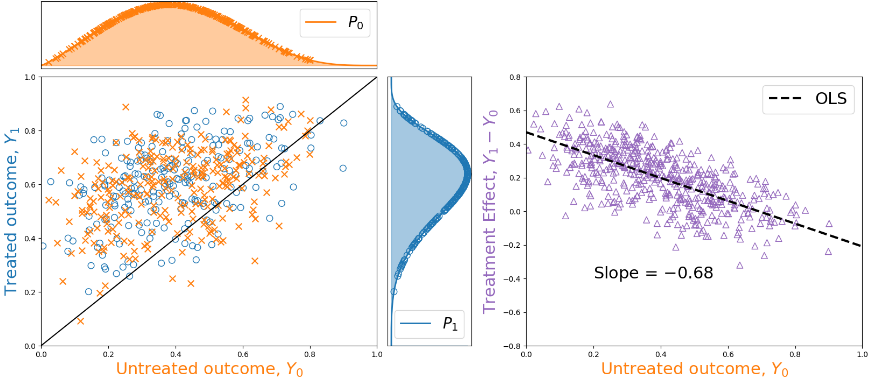

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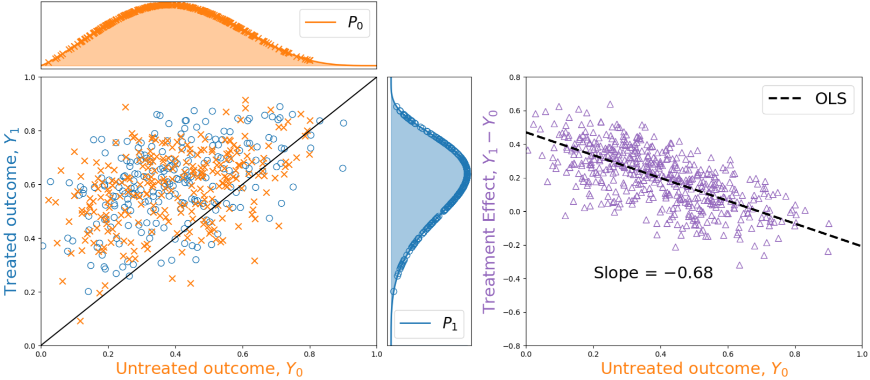

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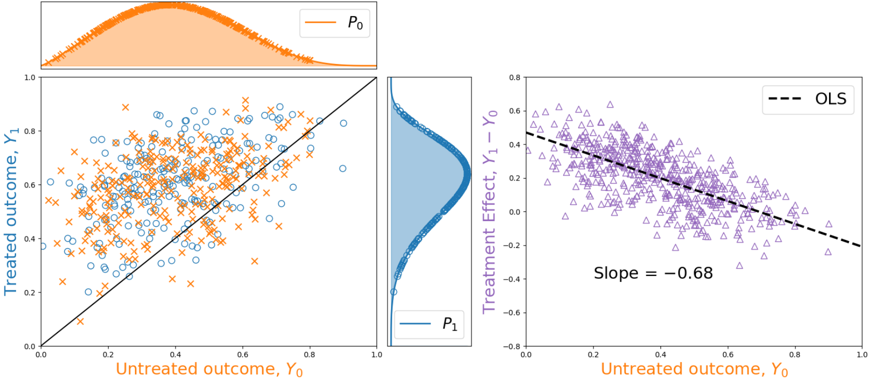

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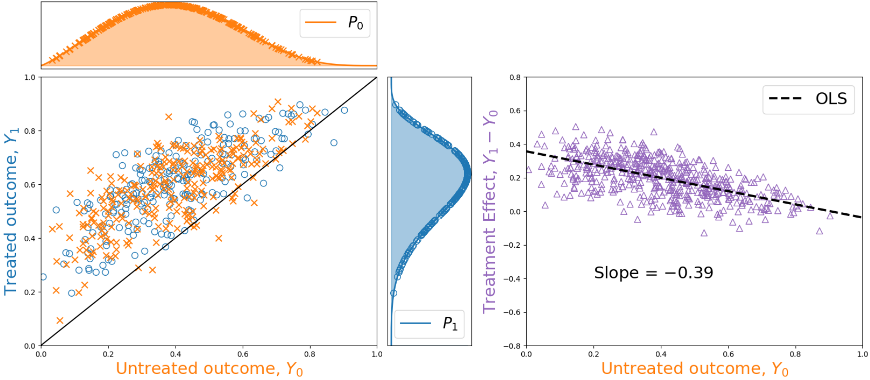

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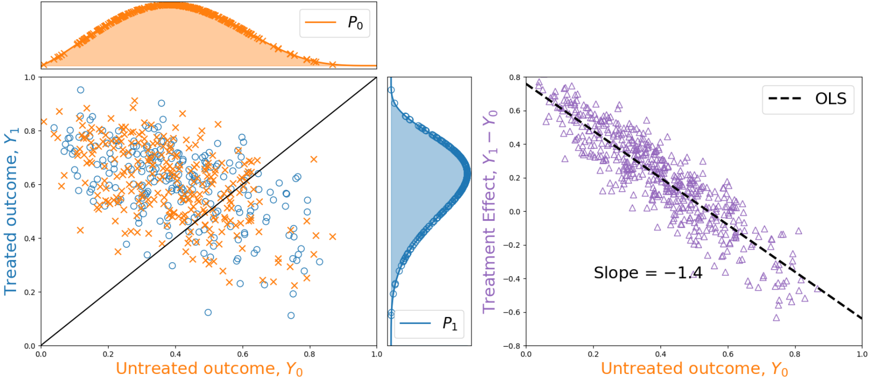

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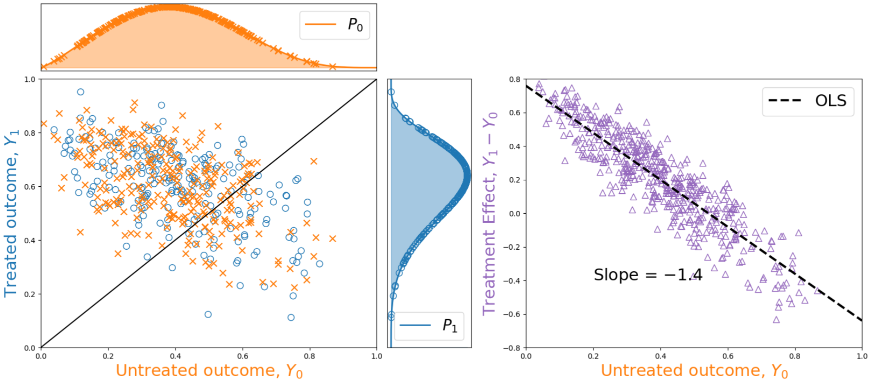

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- Empirical application: who sees larger benefits from the NSW job training?

Related literature

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  - CDF or quantiles of $Y_1 - Y_0$: Manski (1997), Heckman et al. (1997), Firpo (2007), Fan and Park (2010), Fan and Park (2012), Firpo and Ridder (2019), Callaway (2021), Frandsen and Lefgren (2021).
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Overview

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1. Setting and parameter class
2. Identification
3. Estimators
4. Simulations
5. Application

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Assumption 1 (Setting, simplified) $\{Y_i, D_i, X_i\}_{i=1}^n$ is an i.i.d. sample with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \qquad D \in \{0,1\}, \qquad X \in \mathcal{X} = \{x_1, \ldots, x_M\}$$

generated from a distribution satisfying

(i) Potential outcomes: $Y = DY_1 + (1-D)Y_0$

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- In the paper, binary IV satisfying monotonicity condition (Imbens and Angrist, 1994).

Setting w/IV

Parameter class

- Parameter of interest:
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Assumption 2 (Cost function) Either

(i) $c(y_1, y_0)$ is Lipschitz continuous and $\mathcal{Y}$ is compact, or

(ii) $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ and the CDFs $F_{d|x}(y) = P(Y_d \leq y \mid X = x)$ are continuous.

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Remark: If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$ but $F_{d|x}(\cdot)$ are not continuous, inference remains valid for outer identified set.

CDF Details

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Remark: Assumption 3 (iii) is relaxed in the paper.

Full assumption 3

Parameter class: motivating examples

- Share benefiting: $P(Y_1 > Y_0)$
  - Allcott et al. (2020): deactivating Facebook affects subjective well-being.

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Overview

Optimal transport

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Choose a joint distribution with given marginals to minimize costs.
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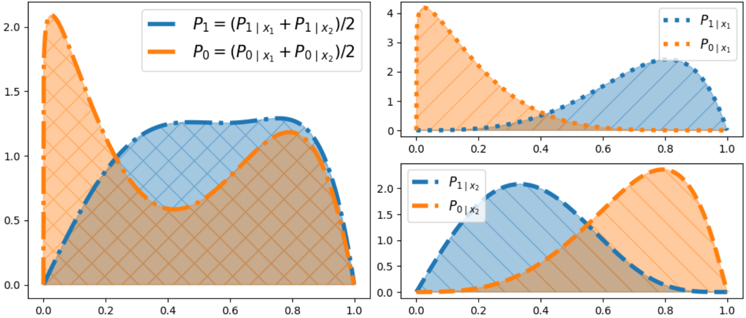

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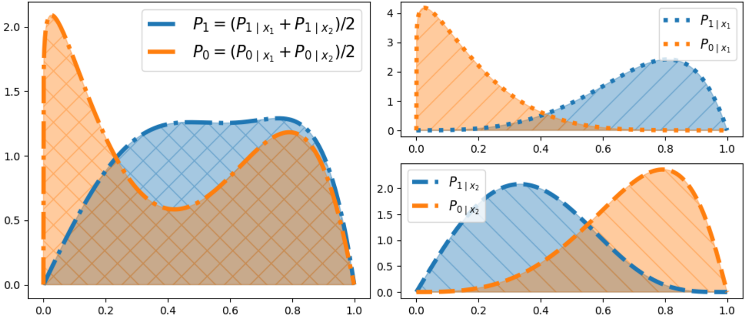

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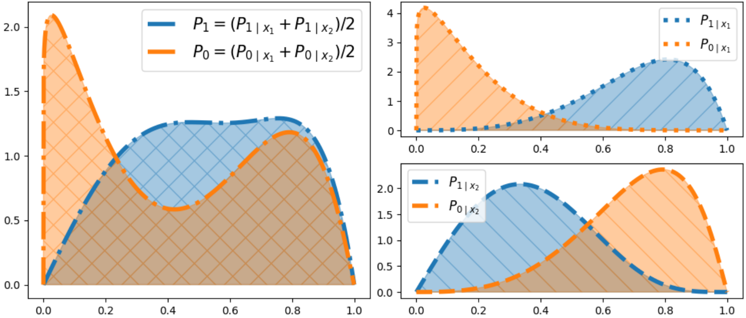

Theorem: identification

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Theorem (identification)

Suppose assumptions 1, 2, and 3 are satisfied. Then the sharp identified set for $\gamma = g(\theta, \eta)$ is $[\gamma^L, \gamma^H]$.

CDF? IV Aside Quantile details

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- The dual problem is used for estimation.
- Strong duality holds under the cost function assumptions. Each problem is attained, too.

Estimators: recall identification

- Distributions of $Y_d \mid X = x \sim P_{d|x}$:

$$E_{P_{d|x}}[f(Y_d)] = \frac{E[f(Y)\mathbb{1}\{D = d, X = x\}]}{P(D = d, X = x)}$$

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CDF?

Estimators: sample analogues

- Estimate $P_{d|x}$ with sample analogues $\hat{P}_{d|x}$:

$$E_{\hat{P}_{d|x}}[f(Y_d)] = \frac{\frac{1}{n}\sum_{i=1}^{n} f(Y_i)\mathbb{1}\{D_i = d, X_i = x\}}{\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{D_i = d, X_i = x\}}$$

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- Estimate the endpoints of $[\gamma^L, \gamma^H]$ with plug-in estimators. For c continuous,

$$\hat{\theta}^L_x = OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}), \qquad \hat{\theta}^H_x = -OT_{-c}(\hat{P}_{1|x}, \hat{P}_{0|x})$$

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CDF?

Estimators: computing $OT_c(\hat{P}_{1|x}, \hat{P}_{0|x})$

$$OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{(\varphi,\psi)\in\Phi_c} E_{\hat{P}_{1|x}}[\varphi(Y_1)] + E_{\hat{P}_{0|x}}[\psi(Y_0)].$$

- To evaluate $E_{\hat{P}_{d|x}}[f(Y_d)]$ for any function f,

$$E_{\hat{P}_{d|x}}[f(Y_d)] = \sum_{i=1}^{n} \omega_{d,x,i} \times f(Y_i), \qquad \omega_{d,x,i} = \frac{\mathbb{1}\{D_i = d, X_i = x\}/n}{\frac{1}{n}\sum_{j=1}^{n} \mathbb{1}\{D_j = d, X_j = x\}}.$$

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$$\begin{aligned} OT_c(\hat{P}_{1|x}, \hat{P}_{0|x}) = \max_{\{\varphi_i,\psi_i\}_{i=1}^{n}} & \sum_{i=1}^{n} \omega_{1,x,i} \times \varphi_i + \sum_{i=1}^{n} \omega_{0,x,i} \times \psi_i \\ \text{s.t. } & \varphi_i + \psi_j \leq c(Y_i, Y_j) \text{ for all } 1 \leq i,j \leq n, \end{aligned}$$

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$$\text{s.t. } \varphi_i + \psi_j \le c(Y_i, Y_j) \text{ for all } 1 \le i,j \le n,$$

- Dimension is reduced by ignoring φ_i, ψ_i, and constraints where $\omega_{d,x,i} = 0$.

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Theorem (Weak convergence)

Suppose assumptions 1, 2, and 3 hold. Then

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) \xrightarrow{L} T'_P(\mathbb{G})$$

where $\sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ *and* $T'_P(\cdot)$ *is the Hadamard directional derivative of* $T(\cdot)$ *at* P.

$T(\cdot)$ details Proof sketch

Inference: bootstrap

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- Compute $T(\mathbb{P}_n^*)$ the same way as $T(\mathbb{P}_n)$: let $\omega_{d,x,i}^* = \frac{\mathbb{1}\{D_i^*=d, X_i^*=x\}/n}{\frac{1}{n}\sum_{j=1}^n \mathbb{1}\{D_j^*=d, X_j^*=x\}}$,

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Precise assumption 4

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Theorem (Bootstrap consistency)

Suppose assumptions 1, 2, 3, and 4 hold. Then $T_P'(\mathbb{G})$ *is bivariate normal, and conditional on* $\{Y_i, D_i, X_i\}_{i=1}^n$,

$$\sqrt{n}(T(\mathbb{P}_n^*) - T(\mathbb{P}_n)) \xrightarrow{L} T_P'(\mathbb{G})$$

in outer probability.

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- Assumption 4 may hold without this lemma's conditions.

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- The paper shows a consistent alternative.
  - Follows Fang and Santos (2019): estimating the derivative $T'_P(\cdot)$.
  - Implementation is more involved, but still computationally tractable.

Overview

Simulations: parameter and DGP

- Parameter $\gamma = \theta = P(Y_1 - Y_0 \leq \delta)$ has simple bounds:

$$\gamma^L = \sup_y \{F_1(y) - F_0(y-\delta)\}, \qquad \gamma^H = 1 + \inf_y \{F_1(y) - F_0(y-\delta)\}$$

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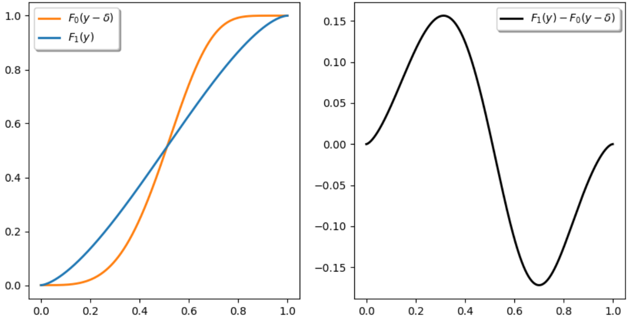

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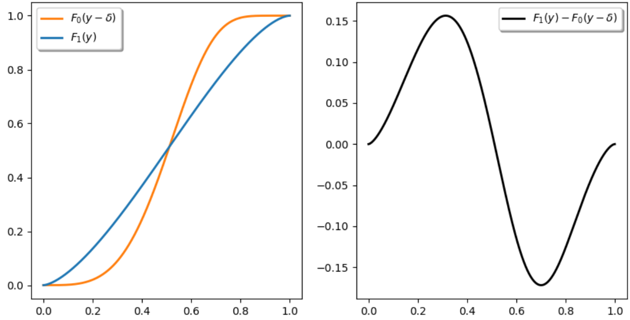

- Unique solutions $\implies$ bootstrap is valid.

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  (iii) Let $\hat{c}_{1-\alpha}$ be the $1-\alpha$ quantile of $\{\max\{\sqrt{n}(\hat{\gamma}_b^{L*} - \hat{\gamma}), -\sqrt{n}(\hat{\gamma}_b^{H*} - \hat{\gamma}^H)\}\}_{b=1}^B$, and

$$CI = [\hat{\gamma}^L - \hat{c}_{1-\alpha}/\sqrt{n}, \hat{\gamma}^H + \hat{c}_{1-\alpha}/\sqrt{n}]$$

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$$(\widehat{bias}^L, \widehat{bias}^H) = \frac{1}{B}\sum_{b=1}^{B}(\hat{\gamma}^{L*}, \hat{\gamma}^{H*}) - (\hat{\gamma}^L, \hat{\gamma}^H),$$

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- Bootstrap bias corrected confidence interval:

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n	Bias		St. Dev.		Emp. Coverage
	$\hat{\gamma}^L$	$\hat{\gamma}^H$	$\hat{\gamma}^L$	$\hat{\gamma}^H$	CI
100	0.047	-0.051	0.065	0.066	0.900
200	0.031	-0.031	0.049	0.049	0.917
300	0.030	-0.021	0.040	0.040	0.893

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n	Bias		St. Dev.		Emp. Coverage
	$\hat{\gamma}^L_{BC}$	$\hat{\gamma}^H_{BC}$	$\hat{\gamma}^L_{BC}$	$\hat{\gamma}^H_{BC}$	CI_{BC}
100	0.021	-0.026	0.071	0.071	0.927
200	0.013	-0.015	0.052	0.051	0.953
300	0.015	-0.007	0.042	0.042	0.957

Overview

1. Setting and parameter class
2. Identification
3. Estimators
4. Simulations
5. Application

A randomized job training experiment

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Table: Balance table

	base inc.	age	yrs. educ.	HS dropout	black	hispanic	married
control	3672.49	24.45	10.19	0.81	0.80	0.11	0.16
	(6521.53)	(6.59)	(1.62)	(0.39)	(0.40)	(0.32)	(0.36)
treated	3571.00	24.63	10.38	0.73	0.80	0.09	0.17
	(5773.13)	(6.69)	(1.82)	(0.44)	(0.40)	(0.29)	(0.37)

Note: Standard deviations in parentheses.

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$$\gamma = \frac{\mathrm{Cov}(Y_1 - Y_0, Y_0)}{\mathrm{Var}(Y_0)} = \frac{\overbrace{E[(Y_1 - Y_0)Y_0]}^{\theta} - (E[Y_1] - E[Y_0])E[Y_0]}{E[Y_0^2] - (E[Y_0])^2}$$

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- *Interpretation:* $\gamma < 0$ implies workers with below average Y_0 tend to see above average $Y_1 - Y_0$

NSW results

- Discretized age and baseline income are informative covariates.
  - age bins: $[16, 23]$, $(23, \infty)$
  - baseline income bins: $[0, 0]$, $(0, 4000]$, $(4000, \infty)$

Table: Estimates of bounds for γ, the OLS Slope

	Lower Bound	Upper Bound	95% *CI*
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NSW results: conditional on covariate values

Table: Estimates conditional on covariate values

age	base inc.	$\hat{\gamma}^L_{BC}$	$\hat{\gamma}^H_{BC}$	95% CI_{BC}	n
(16, 23]	0	-1.97	0.28	[-2.26, 0.56]	140
	(0, 4000]	-1.74	-0.15	[-1.9, 0.01]	141
	$(4000, \infty)$	-1.45	-0.44	[-1.63, -0.27]	90
$(23, \infty)$	0	-2.13	0.81	[-2.65, 1.33]	187
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- This subset's vulnerable individuals see larger benefits from treatment.

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- Ongoing and future work:
  - Accomodate plausible support restrictions, such as $Y_1 \geq Y_0$.

Conclusion

- Summary:
  - Parameters of the joint distribution of potential outcomes are not point identified.
  - Sharp bounds are characterized with optimal transport.
  - Sample analogue estimators are computationally and analytically attractive.
- Ongoing and future work:
  - Accomodate plausible support restrictions, such as $Y_1 \geq Y_0$.
  - Support function approach to consider parameters depending on more than one joint moment.

References I

Abadie, Alberto (2003). "Semiparametric instrumental variable estimation of treatment response models". In: *Journal of econometrics* 113(2), pp. 231–263.

Allcott, Hunt et al. (2020). "The welfare effects of social media". In: *American Economic Review* 110(3), pp. 629–676.

Beaman, Lori et al. (2013). "Profitability of fertilizer: Experimental evidence from female rice farmers in Mali". In: *American Economic Review* 103(3), pp. 381–386.

Callaway, Brantly (2021). "Bounds on distributional treatment effect parameters using panel data with an application on job displacement". In: *Journal of Econometrics* 222(2), pp. 861–881.

Diamond, Alexis and Jasjeet S Sekhon (2013). "Genetic matching for estimating causal effects: A general multivariate matching method for achieving balance in observational studies". In: *Review of Economics and Statistics* 95(3), pp. 932–945.

Dunipace, Eric (2021). "Optimal transport weights for causal inference". In: *arXiv preprint arXiv:2109.01991*.

Efron, Bradley and Robert J Tibshirani (1994). *An introduction to the bootstrap*. CRC press.

Ekeland, Ivar, Alfred Galichon, and Marc Henry (2010). "Optimal transportation and the falsifiability of incompletely specified economic models". In: *Economic Theory* 42, pp. 355–374.

Fan, Yanqin and Sang Soo Park (2010). "Sharp bounds on the distribution of treatment effects and their statistical inference". In: *Econometric Theory* 26(3), pp. 931–951.

Fan, Yanqin and Sang Soo Park (2012). "Confidence intervals for the quantile of treatment effects in randomized experiments". In: *Journal of Econometrics* 167(2), pp. 330–344.

Fan, Yanqin, Xuetao Shi, and Jing Tao (2023). "Partial identification and inference in moment models with incomplete data". In: *Journal of Econometrics* 235(2), pp. 418–443.

Fang, Zheng and Andres Santos (2019). "Inference on directionally differentiable functions". In: *The Review of Economic Studies* 86(1), pp. 377–412.

Firpo, Sergio (2007). "Efficient semiparametric estimation of quantile treatment effects". In: *Econometrica* 75(1), pp. 259–276.

Firpo, Sergio and Geert Ridder (2019). "Partial identification of the treatment effect distribution and its functionals". In: *Journal of Econometrics* 213(1), pp. 210–234.

References II

Frandsen, Brigham R and Lars J Lefgren (2021). "Partial identification of the distribution of treatment effects with an application to the Knowledge is Power Program (KIPP)". In: *Quantitative Economics* 12(1), pp. 143–171.

Friebel, Guido et al. (2023). "What do employee referral programs do? Measuring the direct and overall effects of a management practice". In: *Journal of Political Economy* 131(3), pp. 633–686.

Galichon, Alfred and Marc Henry (2011). "Set identification in models with multiple equilibria". In: *The Review of Economic Studies* 78(4), pp. 1264–1298.

Gunsilius, Florian and Yuliang Xu (2021). "Matching for causal effects via multimarginal unbalanced optimal transport". In: *arXiv preprint arXiv:2112.04398*.

Haile, Philip A and Elie Tamer (2003). "Inference with an incomplete model of English auctions". In: *Journal of Political Economy* 111(1), pp. 1–51.

Heckman, James J, Jeffrey Smith, and Nancy Clements (1997). "Making the most out of programme evaluations and social experiments: Accounting for heterogeneity in programme impacts". In: *The Review of Economic Studies* 64(4), pp. 487–535.

Holland, Paul W (1986). "Statistics and causal inference". In: *Journal of the American statistical Association* 81(396), pp. 945–960.

Horowitz, Joel L (2001). "The bootstrap". In: *Handbook of econometrics*. Vol. 5. Elsevier, pp. 3159–3228.

Imbens, Guido W. and Joshua D. Angrist (1994). "Identification and Estimation of Local Average Treatment Effects". In: *Econometrica* 62(2), pp. 467–475.

Ji, Wenlong, Lihua Lei, and Asher Spector (2023). "Model-Agnostic Covariate-Assisted Inference on Partially Identified Causal Effects". In: *arXiv preprint arXiv:2310.08115*.

Manski, Charles F (1997). "Monotone treatment response". In: *Econometrica: Journal of the Econometric Society*, pp. 1311–1334.

Russell, Thomas M (2021). "Sharp bounds on functionals of the joint distribution in the analysis of treatment effects". In: *Journal of Business & Economic Statistics* 39(2), pp. 532–546.

Torous, William, Florian Gunsilius, and Philippe Rigollet (2021). "An optimal transport approach to causal inference". In: *arXiv preprint arXiv:2108.05858*.

Appendix: full setting

Assumption 1 (Setting). $\{Y_i, D_i, Z_i, X_i\}_{i=1}^n$ is an i.i.d. sample, with

$$Y \in \mathcal{Y} \subseteq \mathbb{R}, \qquad D \in \{0,1\}, \qquad Z \in \{0,1\}, \qquad X \in \mathcal{X} = \{x_1, \ldots, x_M\}$$

generated from a distribution satisfying

(i) Potential outcomes: $Y = DY_1 + (1-D)Y_0$,

(ii) Potential treatment statuses: $D = ZD_1 + (1-Z)D_0$, with $D_z \in \{0,1\}$,

(iii) Instrument exogeneity: $(Y_1, Y_0, D_1, D_0) \perp Z \mid X$,

(iv) Monotonicity: $D_1 \geq D_0$ almost surely,

(v) Existence of compliers: $P(D_1 > D_0, X = x) > 0$ for each x, and

(vi) $P(X = x, Z = z) > 0$ for each (x, z)

- Terminology: always-taker, complier, defier, never-taker.

	$D_0 = 1$	$D_0 = 0$
$D_1 = 1$	Always-takers	Compliers
$D_1 = 0$	~~Defiers~~	Never-takers

- Monotonicity rules out defiers. Focus on distribution of compliers.

Setting Aside, IV

Appendix: identification of $P(Y_1 - Y_0 \leq \delta)$

- $OT_c(P_1, P_0)$ is well behaved (attained, strong duality holds, etc) when $c(y_1, y_0)$ is bounded and lower semicontinuous

- If $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, let

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$$
$$\theta_x^L = OT_{c_L}(P_{1|x}, P_{0|x}), \qquad \theta_x^H = 1 - OT_{c_H}(P_{1|x}, P_{0|x})$$

- The form of the bounds remains the same:

$$\theta^L = E[\theta_X^L], \qquad \theta^H = E[\theta_X^H]$$
$$\gamma^L = \min_{t \in [\theta^L, \theta^H]} g(t, \eta), \qquad \gamma^H = \max_{t \in [\theta^L, \theta^H]} g(t, \eta)$$

- Identified sets are still sharp when CDFs are continuous:

$$F_{d|x}(y) = P(Y_d \leq y \mid X = x)$$

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Bounds on $\theta = P(Y_1 - Y_0 \leq \delta)$ are found with

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\},$$
$$\theta^L = OT_{c_L}(P_1, P_0), \qquad \theta^H = 1 - OT_{c_H}(P_1, P_0)$$

  Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

Cost function assumption

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Bounds on $\theta = P(Y_1 - Y_0 \leq \delta)$ are found with

$$\begin{aligned} c_L(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 < \delta\}, & c_H(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 > \delta\}, \\ \theta^L &= OT_{c_L}(P_1, P_0), & \theta^H &= 1 - OT_{c_H}(P_1, P_0) \end{aligned}$$

  Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

- As a byproduct, recover the famed **Makarov bounds** studied by Fan and Park (2010)

$$\theta^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \qquad \theta^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

Cost function assumption

Appendix: aside, CDF results are conservative when continuity fails

$$OT_c(P_1, P_0) = \inf_{\pi \in \Pi(P_1, P_0)} E_\pi[c(Y_1, Y_0)]$$

- Bounds on $\theta = P(Y_1 - Y_0 \le \delta)$ are found with

$$c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}, \qquad c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\},$$
$$\theta^L = OT_{c_L}(P_1, P_0), \qquad \theta^H = 1 - OT_{c_H}(P_1, P_0)$$

  Using OT results, show that if marginal CDFs F_d are continuous then $\Theta_{ID} = [\theta^L, \theta^H]$.

- As a byproduct, recover the famed **Makarov bounds** studied by Fan and Park (2010)

$$\theta^L = \sup_y \{F_1(y) - F_0(y - \delta)\}, \qquad \theta^H = 1 + \inf_y \{F_1(y) - F_0(y - \delta)\}$$

- **Furthermore,** $\mathbb{1}\{y_1 - y_0 < \delta\} \le \mathbb{1}\{y_1 - y_0 \le \delta\}$ implies the bounds are conservative: $\Theta_{ID} \subseteq [\theta^L, \theta^H]$ whether or not F_d are continuous.

Cost function assumption

Appendix: full assumption 3

- Parameter of interest:

$$\gamma = g(\theta, \eta) \in \mathbb{R}$$

where $\theta = E[c(Y_1, Y_0)] \in \mathbb{R}$ and $\eta = (E[\eta_1(Y_1)], E[\eta_0(Y_0)]) \in \mathbb{R}^{K_1+K_0}$.

Assumption 3 (Function of moments)

(i) $E[\|\eta_d(Y)\|^2] < \infty$ for $d = 1, 0$,

(ii) $g(\cdot, \eta)$ is continuous, and

(iii) the functions

$$g^L(t^L, t^H, e) = \min_{t \in [t^L, t^H]} g(t, e), \qquad g^H(t^L, t^H, e) = \max_{t \in [t^L, t^H]} g(t, e)$$

are continuously differentiable at $(t^L, t^H, e) = (\theta^L, \theta^H, \eta)$.

Remark: A3 (ii), (iii) implied by g continuously differentiable and $g(\cdot, \eta)$ monotonic

Back

Appendix: quantiles

- Suppose the parameter of interest is q_τ solving
$$P(Y_1 - Y_0 \le q_\tau) = \tau$$

- View CDF bounds as a function: $\theta(\delta) = P(Y_1 - Y_0 \le \delta)$
$$\begin{aligned} c_{L,\delta}(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 < \delta\}, & c_{H,\delta}(y_1, y_0) &= \mathbb{1}\{y_1 - y_0 > \delta\}, \\ \theta^L_x(\delta) &= OT_{c_L}(P_{1|x}, P_{0|x}), & \theta^H_x(\delta) &= 1 - OT_{c_H}(P_{1|x}, P_{0|x}) \\ \theta^L(\delta) &= E[\theta^L_X(\delta)] & \theta^H(\delta) &= E[\theta^H_X(\delta)] \end{aligned}$$
and let $Q_{I,\tau}$ be the sharp identified set for q_τ.

Lemma (Identification of q_τ). Suppose assumptions 1 and 2(ii) hold. Then $q \in Q_{I,\tau}$ if and only if $\theta^L(q) \le \tau \le \theta^H(q)$.

Examples

Appendix: aside, IV

- Identification extends easily to IV.

Ident. Thm.

Appendix: aside, IV

- Identification extends easily to IV.
- Consider the binary IV potential outcomes framework of Abadie (2003):

$$D = ZD_1 + (1 - Z)D_0 \qquad (Y_1, Y_0, D_1, D_0) \perp Z \mid X, \qquad D_1 \geq D_0$$

units with $D_1 > D_0$ are known as *compliers*.

Ident. Thm. IV Setting

Appendix: aside, IV

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- Consider the binary IV potential outcomes framework of Abadie (2003):

$$D = ZD_1 + (1 - Z)D_0 \qquad (Y_1, Y_0, D_1, D_0) \perp Z \mid X, \qquad D_1 \geq D_0$$

  units with $D_1 > D_0$ are known as *compliers*.
- This model identifies marginal distributions of potential outcomes of compliers:

$$Y_d \mid D_1 > D_0, X = x$$

Ident. Thm. IV Setting Ident. Dist. Compliers

Appendix: aside, IV

- Identification extends easily to IV.
- Consider the binary IV potential outcomes framework of Abadie (2003):

$$D = ZD_1 + (1 - Z)D_0 \qquad (Y_1, Y_0, D_1, D_0) \perp Z \mid X, \qquad D_1 \geq D_0$$

  units with $D_1 > D_0$ are known as *compliers*.
- This model identifies marginal distributions of potential outcomes of compliers:

$$Y_d \mid D_1 > D_0, X = x$$

- Same identification applies to parameters conditional on compliance. E.g.,

$$P(Y_1 > Y_0 \mid D_1 > D_0)$$

Ident. Thm. IV Setting Ident. Dist. Compliers

Appendix: identification of $P_{d|x}$ with IV

- The marginal distribution of Y_d given $D_1 > D_0$ and $X = x$ is identified with

$$\begin{aligned} E_{P_{d|x}}[f(Y_d)] &= E[f(Y_d) \mid D_1 > D_0, X = x] \\ &= \frac{E[f(Y)\mathbb{1}\{D = d\} \mid Z = d, X = x] - E[f(Y)\mathbb{1}\{D = d\} \mid Z = 1-d, X = x]}{P(D = d \mid Z = d, X = x) - P(D = d \mid Z = 1-d, X = x)} \end{aligned}$$

- The marginal distribution of X given $D_1 > D_0$ is identified with

$$\begin{aligned} s_x &= P(X = x \mid D_1 > D_0) \\ &= \frac{[P(D = 1 \mid Z = 1, X = x) - P(D = 1 \mid Z = 0, X = x)]P(X = x)}{\sum_{x'}[P(D = 1 \mid Z = 1, X = x') - P(D = 1 \mid Z = 0, X = x')]P(X = x')} \end{aligned}$$

Aside, IV

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

Weak convergence theorem

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

- View $\mathbb{P}_n$, P as bounded functions on $\mathcal{F}$:

$$\ell^\infty(\mathcal{F}) = \left\{ g : \mathcal{F} \to \mathbb{R} \, ; \, \|g\|_\infty = \sup_{f \in \mathcal{F}} |g(f)| < \infty \right\}$$

Weak convergence theorem

Appendix: definition of T

- Proof defines a set of universally bounded functions

$$\mathcal{F} \subseteq \{f : \mathcal{Y} \times \{0,1\} \times \mathcal{X} \to \mathbb{R}\}$$

- View $\mathbb{P}_n$, P as bounded functions on $\mathcal{F}$:

$$\ell^\infty(\mathcal{F}) = \left\{ g : \mathcal{F} \to \mathbb{R} \, ; \, \|g\|_\infty = \sup_{f \in \mathcal{F}} |g(f)| < \infty \right\}$$

- The map $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$ is described by $P \mapsto (P_{1|x}, P_{0|x}, \eta)$ and

$$\begin{aligned} \theta_x^L &= OT_c(P_{1|x}, P_{0|x}), & \theta_x^H &= -OT_{-c}(P_{1|x}, P_{0|x}) \\ \theta^L &= E[\theta_X^L], & \theta^H &= E[\theta_X^H] \\ \gamma^L &= \min_{t \in [\theta^L, \theta^H]} g(t, \eta), & \gamma^H &= \max_{t \in [\theta^L, \theta^H]} g(t, \eta) \end{aligned}$$

Weak convergence theorem

Appendix: proof sketch (1/3)

1. Will view P, $\mathbb{P}$ as maps in $\ell^\infty(\mathcal{F})$ for Donsker set $\mathcal{F}$ (defined later), and $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$.

Weak convergence theorem

Appendix: proof sketch (1/3)

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2. To show $T(\cdot)$ is (Hadamard) directionally differentiable, suffices to show OT_c is directionally differentiable.

Weak convergence theorem

Appendix: proof sketch (1/3)

1. Will view P, $\mathbb{P}$ as maps in $\ell^\infty(\mathcal{F})$ for Donsker set $\mathcal{F}$ (defined later), and $T : \ell^\infty(\mathcal{F}) \to \mathbb{R}^2$.

2. To show $T(\cdot)$ is (Hadamard) directionally differentiable, suffices to show OT_c is directionally differentiable.

3. By strong duality,

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi,\psi)\in\Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$

$$\Phi_c = \{(\varphi,\psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

Weak convergence theorem

Appendix: proof sketch (2/3)

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi,\psi)\in\Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$
$$\Phi_c = \{(\varphi,\psi) : \varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)\}$$

4. Φ_c is a large set, but much of it can be ignored:

Weak convergence theorem | c-concavity

Appendix: proof sketch (2/3)

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4. Φ_c is a large set, but much of it can be ignored:
    - If $\varphi(y_1) \leq \tilde{\varphi}(y_1)$, then $E_{P_{1|x}}[\varphi(Y_1)] \leq E_{P_{1|x}}[\tilde{\varphi}(Y_1)]$

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   - Any pair (φ, ψ) where $\varphi(y_1) + \psi(y_0) \leq c(y_1, y_0)$ is "slack" can be ignored

Weak convergence theorem c-concavity

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5. This observation leads to

$$\sup_{(\varphi,\psi)\in\Phi_c} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] = \sup_{(\varphi,\psi)\in\Phi_c\cap(\mathcal{F}_c\times\mathcal{F}_c^c)} E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)] \quad (1)$$

   (i) if $c(y_1, y_0)$ is L-Lip. and $\mathcal{Y}$ is compact, $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are L-Lip. and universally bounded.

   (ii) if $c(y_1, y_0) = \mathbb{1}\{y_1 - y_0 \leq \delta\}$, $\mathcal{F}_c$ is the set of intervals, $\mathcal{F}_c^c$ the complements of intervals.

Weak convergence theorem | c-concavity

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6. Finally, $\Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)$ is compact and $E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$ is continuous

Appendix: proof sketch (2/3)

$$OT_c(P_{1|x}, P_{0|x}) = \sup_{(\varphi,\psi)\in\Phi_c} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)]$$

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   (i) if $c(y_1, y_0)$ is L-Lip. and $\mathcal{Y}$ is compact, $\mathcal{F}_c$ and $\mathcal{F}_c^c$ are L-Lip. and universally bounded.

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$\implies$ OT_c, and therefore $T(\cdot)$, are Hadamard directionally differentiable.

Weak convergence theorem | c-concavity

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

Weak convergence theorem

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

8. $\mathcal{F}$ is Donsker $\implies \sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

Weak convergence theorem

Appendix: proof sketch (3/3)

7. Define $\mathcal{F}$ to be union of $\mathcal{F}_c$ and $\mathcal{F}_c^c$ (and nuisance moments, all $\times$ indicators).

8. $\mathcal{F}$ is Donsker $\implies \sqrt{n}(\mathbb{P}_n - P) \xrightarrow{L} \mathbb{G}$ in $\ell^\infty(\mathcal{F})$.

9. Functional delta method implies the result,

$$\sqrt{n}((\hat{\gamma}^L, \hat{\gamma}^H) - (\gamma^L, \gamma^H)) = \sqrt{n}(T(\mathbb{P}_n) - T(P)) \xrightarrow{L} T_P'(\mathbb{G}).$$

Weak convergence theorem

Appendix: c-concavity

$$OT_c(P_1, P_0) = \sup_{(\varphi,\psi)\in\Phi_c} \underbrace{E_{P_1}[\varphi(Y_1)] + E_{P_0}[\psi(Y_0)]}_{J(\varphi,\psi)},$$

- Define the c-transforms:

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Proof sketch Weak convergence theorem

Appendix: c-concavity

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  $\Longrightarrow$ The dual problem can be restricted to c-concave functions.

Proof sketch | Weak convergence theorem

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Proof sketch | Weak convergence theorem

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  - Lipschitz continuity, boundedness, etc.

Proof sketch | Weak convergence theorem

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- c-concave functions often inherit properties of c:
  - Lipschitz continuity, boundedness, etc.
  - These properties are used to define $\mathcal{F}_c$ and $\mathcal{F}_c^c$

Proof sketch Weak convergence theorem

Appendix: formal assumption 4

- Let P be the distribution of an observation: $(Y, D, Z, X) \sim P$.
- Let $\mathcal{Y}_{d,x}$ be the support of $Y \mid D = d, X = x$, and $\mathbb{1}_{\mathcal{Y}_{d,x}}(y) = \mathbb{1}\{y \in \mathcal{Y}_{d,x}\}$
- Define c_L, c_H:
  1. (i) If assumption 2 (i) holds, let $c_L = c(y_1, y_0)$ and $c_H(y_1, y_0) = -c(y_1, y_0)$.
  2. (ii) If assumption 2 (ii) holds, let $c_L(y_1, y_0) = \mathbb{1}\{y_1 - y_0 < \delta\}$ and $c_H(y_1, y_0) = \mathbb{1}\{y_1 - y_0 > \delta\}$.

Assumption 4 (Unique solutions) For each $x \in \mathcal{X}$, each $c \in \{c_L, c_H\}$, and any

$$(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \underset{(\varphi,\psi) \in \Phi_c \cap (\mathcal{F}_c \times \mathcal{F}_c^c)}{\arg\max} E_{P_{1|x}}[\varphi(Y_1)] + E_{P_{0|x}}[\psi(Y_0)],$$

there exists $s \in \mathbb{R}$ such that

$$\mathbb{1}_{\mathcal{Y}_{1,x}} \times \varphi_1 = \mathbb{1}_{\mathcal{Y}_{1,x}} \times (\varphi_2 + s),\ P - a.s., \quad \mathbb{1}_{\mathcal{Y}_{0,x}} \times \psi_1 = \mathbb{1}_{\mathcal{Y}_{0,x}} \times (\psi_2 - s),\ P - a.s.$$

Assumption 4 | Why c_L, c_H?