Robustness to Missing Data: Breakdown Point Analysis

Daniel Ober-Reynolds

University of California, Los Angeles

doberreynolds@gmail.com

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Introduction

- Missing data is common, as are the selection concerns it raises
- Common solution: assume data are Missing (Completely) At Random
  - Impute or ignore incomplete observations, use standard methods
  - Convenient solution, often implausible justification
- This paper proposes an interpretable measure of selection, and estimates how much selection is needed to overturn a conclusion

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  - Only 50% of survey recipients responded.
  - When Lee (2009) sample selection bounds were applied, this conclusion could no longer be supported.

Related literature

- Missing data without MAR
  - Point identification: Heckman (1979), Das et al. (2003)
  - Partial identification: Manski (2005), Lee (2009)
  - Robustness/sensitivity analysis: Kline and Santos (2013)

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  - Missing data: Kline and Santos (2013)
  - Potential outcomes: Masten and Poirier (2020)
  - Omitted variable bias: Diegert et al. (2022)

$\Rightarrow$ This paper contributes a robustness exercise for missing data that

i. allows for any number of variables to be missing
ii. directly uses the researcher's GMM model
iii. requires no additional data or modeling (no exclusion restriction)
iv. gives results that are succinct and interpretable

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1. Introduction
2. Setting
3. Breakdown Point Analysis
4. Estimation
5. Simulations

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  - Example: first OLS coefficient is positive. $\mathbf{B}_0 = \{b \in \mathbf{B} \; ; \; b^{(1)} \leq 0\}$

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  - A quantitative measure of selection will allow meaningful discussion.

Quantifying selection: predictive power of (Y, X)

Sample is $\{D_i, D_i Y_i, X_i\}_{i=1}^n$, i.i.d.. $p_D = P(D = 1)$,

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$$f_1(y, x) = \frac{p_D(y, x)}{p_D} \qquad f_0(y, x) = \frac{1 - p_D(y, x)}{1 - p_D}$$

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  - Optimistic: D is independent of (Y, X).
    $\Longrightarrow$ $p_D(y, x) = p_D$, so $f_1 = f_0$ (data is MCAR)
  - Pessimistic: D is almost a function of (Y, X).
    $\Longrightarrow$ $p_D(y, x) \approx 1$ or 0; f_1 and f_0 look quite different

Quantifying selection with squared Hellinger

- Measure **selection** as the **squared Hellinger distance** between P_0 and P_1:

$$H^2(P_0, P_1) = \frac{1}{2} E_P \left[(\sqrt{f_0(Y,X)} - \sqrt{f_1(Y,X)})^2 \right]$$

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- $f_1(y,x) = p_D(y,x)/p_D$ and $f_0(y,x) = (1 - p_D(y,x))/p_D$ implies
$$H^2(P_0, P_1) = 1 - \frac{E_P \left[\sqrt{\text{Var}(D \mid Y, X)} \right]}{\sqrt{\text{Var}(D)}}$$

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  - Rules out selection mechanisms that "truncate" data; e.g. $D_i = \mathbb{1}\{Y_i \leq c\}$.

Other Selection Measures

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Nominally Identified Sets

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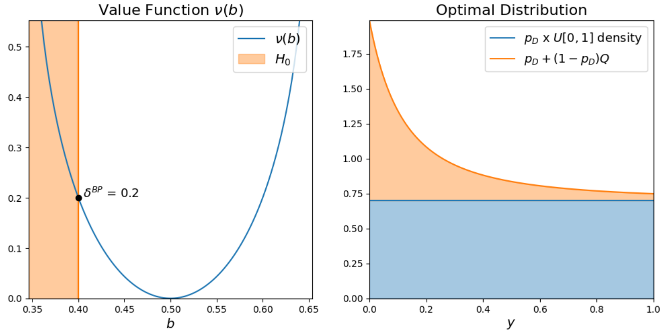

Simulations

Overview

Estimation overview

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Skip to Simulations

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Dual problem detail | Strong duality assumptions

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$$V(b) = \sup_{\lambda \in \mathbb{R}^{d_g + K}} E\left[\frac{\lambda^\intercal J(D) h(DY, X, b)}{1 - p_D} - \frac{Df^*(\lambda^\intercal h(DY, X, b))}{p_D}\right] \tag{2}$$

  a finite dimensional convex optimization problem.
  - f^*, J and h are known functions,
  - the expectation is wrt the distribution of (D, DY, X), and
  - K is the cardinality of $\mathrm{Supp}(X)$.

- Under regularity conditions, **strong duality** holds:

$$V(b) = \nu(b)$$

  - Assume this holds for all $b \in B \subseteq \mathbf{B}$, with $\inf_{b \in \mathbf{B}_0} \nu(b) = \inf_{b \in B \cap \mathbf{B}_0} \nu(b)$
  - $\implies$ we can focus on the dual problem.

Dual problem detail · Strong duality assumptions

Estimators

- With strong duality, the breakdown point is $\delta^{BP} = \inf_{b \in B \cap \mathbf{B}_0} \nu(b)$, where

$$\nu(b) = \sup_{\lambda \in \mathbb{R}^{d_g + K}} E\left[\underbrace{\frac{\lambda^\intercal J(D) h(DY, X, b)}{1 - p_D} - \frac{D f^*(\lambda^\intercal h(DY, X, b))}{p_D}}_{:=\varphi(D, DY, X, b, \lambda, p)} \right]$$

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- Straightforward sample analogue estimators: $\hat{\delta}_n^{BP} = \inf_{b \in \mathbf{B}_0} \hat{\nu}_n(b)$, where

$$\hat{\nu}_n(b) = \sup_{\lambda \in \mathbb{R}^{d_g + K}} \frac{1}{n} \sum_{i=1}^{n} \varphi(D_i, D_i Y_i, X_i, b, \lambda, \hat{p}_{D,n})$$

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- Under additional regularity conditions, estimators are consistent:

$$\hat{\nu}_n \xrightarrow{p} \nu \quad \text{in } \ell^\infty(B), \qquad \hat{\delta}_n^{BP} \xrightarrow{p} \delta^{BP}$$

Consistency Assumptions

Inference: asymptotic distributions

Theorem Under assumptions discussed in the paper,

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Theorem Suppose the same assumptions hold. Then $\mathbf{m}(\nu) = \arg\min_{b \in B \cap \mathbf{B}_0} \nu(b)$ is nonempty and

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- Follows from Hadamard directional differentiability of $\nu \mapsto \inf_{b \in B \cap \mathbf{B}_0} \nu(b)$ and the functional delta method (Fang and Santos (2019)).
- $\mathbf{m}(\nu)$ is plausibly a singleton: $\{b^i\}$. If so, $\sqrt{n}(\hat{\delta}_n^{BP} - \delta^{BP})$ is asymptotically normal.

Inference: lower confidence intervals

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- Reporting a **lower confidence interval** addresses this concern:

$$\lim_{n\to\infty} P\left(\underbrace{\hat{\delta}_n^{BP} - \frac{1}{\sqrt{n}}\hat{c}_{1-\alpha,n}}_{\widehat{CI}_{L,n}} \leq \delta^{BP}\right) = 1-\alpha$$

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- $\hat{c}_{1-\alpha,n}$ is estimated with the score bootstrap
  - Assuming $\mathbf{m}(\nu) = \arg\min_{b \in B \cap \mathbf{B}_0} \nu(b)$ is the singleton $\{b^i\}$, $\hat{c}_{1-\alpha,n}$ is computed with a computationally convenient procedure

Score Bootstrap

Overview

Simulations: uniform expectation

- Example: The sample is $\{D_i, D_i Y_i\}_{i=1}^n$, and $\beta = E[Y] \in \mathbb{R}$.

$$Y \mid D = 1 \sim \mathcal{U}[0,1], \qquad p_D = P(D = 1) = 0.7$$

The claim to be supported is $H_1 : \beta > 0.4$.

Illustration

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The claim to be supported is $H_1 : \beta > 0.4$.

- 250 simulations with $P(D = 1) = 0.7$, and $\delta^{BP} \approx 0.2$:

Table: Simulations, Squared Hellinger, Uniform, Mean

n	RMSE	Emp. Bias	Emp. CI Coverage	Ave. CI Length
1000	0.060	0.008	98.4	0.091
2000	0.040	0.005	97.6	0.063
3000	0.032	0.001	96.8	0.051
5000	0.024	0.003	96.4	0.040

Illustration

Simulations: OLS

- Consider a linear model
$$Y_1 = \beta_0 + \beta_1 X_1 + \beta_2 Y_2 + \beta_3 X_2 + \varepsilon = W^\intercal \beta + \varepsilon, \qquad E[W\varepsilon] = 0$$
where X_1, X_2 are discrete and Y_1, Y_2 are continuous.

- The conclusion to be investigated is $H_1 : \beta_1 > 0$. The observed data is $\{D_i, D_i Y_{i1}, D_i Y_{i2}, X_{i1}, X_{i2}\}_{i=1}^n$.

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- 250 simulations from a DGP with $P(D = 1) \approx 0.7$, and $\delta^{BP} \approx 0.2$:

Table: Simulations, Squared Hellinger, OLS

n	RMSE	Emp. Bias	Emp. CI Coverage	Ave. CI Length
1000	0.043	0.009	100.0	0.078
2000	0.033	0.005	98.0	0.052
3000	0.026	0.007	98.0	0.043
5000	0.017	0.002	98.0	0.032

- Empirical coverage suggests inference is conservative.

Conclusion

- Breakdown point analysis is a tractable approach to assessing how robust a conclusion is to relaxing common missing data assumptions.

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- For the conclusion $\beta \in \mathbf{B} \setminus \mathbf{B}_0$, the claim $\beta \in \mathbf{B}_0$ implies

$$\delta^{BP} \leq 1 - \frac{E_P[\sqrt{\mathrm{Var}(D \mid Y, X)}]}{\sqrt{\mathrm{Var}(D)}}$$

  If it is implausible (Y, X) predicts D this well, then $\beta \in \mathbf{B}_0$ is similarly implausible.

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- The breakdown point δ^{BP} is $\sqrt{n}$-estimable, and lower confidence intervals can be constructed with simple bootstrap procedures.

- Reporting $\hat{\delta}_n^{BP}$ and the lower confidence interval $\widehat{CI}_L$ is a succinct summary of a conclusion's robustness.

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Missing (completely) at random

- With i.i.d. sample $\{D_i, D_i Y_i, X_i\}_{i=1}^n$, where $D_i = \mathbb{1}\{Y_i \text{ is observed}\}$

$$(Y, X) \mid D = 1 \sim P_1, \qquad\qquad (Y, X) \mid D = 0 \sim P_0,$$
$$P = p_D P_1 + (1 - p_D) P_0$$

two common assumptions restore point identification of P

- **Missing completely at random** (MCAR) assumes $P_0 = P_1$
  - Testable: do distributions of X match? $P_{0X} = P_{1X}$?
  - Justifies dropping observations where $D_i = 0$

- **Missing at random** (MAR) assumes $Y \mid X = x, D = 0$ follows the same distribution as $Y \mid X = x, D = 1$
  - Not testable
  - Justifies imputating $Y \mid X = x, D = 0$ based on distribution of $Y \mid X = x, D = 1$

- Preliminary analysis may be based on either assumption.

Setting

Assumption: P_0 is dominated by P_1

- **Assumption:** P_0 is dominated by P_1, i.e. $P_0 \ll P_1$.
  - For any set A with $P_1((X, Y) \in A) = 0$, then $P_0((X, Y) \in A) = 0$.
  - Simplifies analysis considerably; set of possible P_0 characterized by densities wrt P_1
  - Allows squared Hellinger to be written as an f-divergence

- Some support assumption is typically necessary for an interesting exercise.
  - Example: $\beta = E[Y]$. P_1 and P_0 given by

$$\begin{array}{lll} P_1(Y=-1)=0.5 & P_1(Y=1)=0.5 & \\ P_0(Y=-1)=0.5 & P_0(Y=1)=0.5-\alpha & P_0(Y=y)=\alpha \end{array}$$

Then

$$H^2(P_0, P_1) = (\sqrt{0.5-\alpha} - \sqrt{0.5} + \sqrt{\alpha})^2$$

can be made **arbitrarily close to zero** by choice of $\alpha > 0$. For any $\alpha > 0$,

$$E_P[Y] = (1 - p_D)E_{P_0}[Y] = (1 - p_D)\alpha(y - 1)$$

can be made **any number** by choice of $y \in \mathbb{R}$.

f-divergence Squared Hellinger

Other selection measures: f-divergences

- Given a convex function $f : \mathbb{R} \to [0, \infty]$ satisfying $f(t) = \infty$ for $t < 0$ and taking a unique minimum of $f(1) = 0$, the corresponding f-**divergence** is the function given by

$$d_f(Q\|P) = \begin{cases} \int f\left(\frac{dQ}{dP}\right) dP & \text{if } Q \ll P \\ \infty & \text{otherwise} \end{cases} \tag{3}$$

- Many popular divergences can be written as f-divergences (when $Q \ll P$):

Name	Common formula	$f(t)$ when $t \geq 0$
Squared Hellinger	$H^2(Q,P) = \frac{1}{2}\int \left(\sqrt{\frac{dQ}{dP}(z)} - 1\right)^2 dP(z)$	$f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$
Kullback-Leibler (KL)	$KL(Q\|P) = \int \log\left(\frac{dQ}{dP}(z)\right) dQ(z)$	$f(t) = t\log(t) - t + 1$
"Reverse" KL	$KL(P\|Q) = \int \log\left(\frac{dP}{dQ}(z)\right) dP(z)$	$f(x) = -\log(t) + t - 1$
Cressie-Read	–	$f_\gamma(t) = \frac{t^\gamma - \gamma t + \gamma - 1}{\gamma(\gamma - 1)}$, $\gamma < 1$

Table: Common f-divergences

- Results in the paper allow any f-divergence (satisfying certain regularity conditions) to be used to measure selection

Squared Hellinger

Breakdown Point through Partial Identification

- Breakdown point analysis can be framed as an exercise in partial identification, as in Kline and Santos (2013), Masten and Poirier (2020), and Diegert et al. (2022).

- In this framing, consider assumptions of the form $H^2(P_0, P_1) \leq \delta$ for some $\delta > 0$.

- The *nominal* identified set $\mathbf{B}_{ID}(\delta)$ for β grows with δ. As long as $\mathbf{B}_{ID}(\delta) \subseteq \mathbf{B} \setminus \mathbf{B}_0$, it is clear the conclusion holds.

- The **breakdown point** δ^{BP} can then be defined as either:
  1. the largest δ for which $\mathbf{B}_{ID}(\delta) \subseteq \mathbf{B} \setminus \mathbf{B}_0$, or
  2. the smallest δ for which $\mathbf{B}_{ID}(\delta) \cap \mathbf{B}_0 \neq \varnothing$

Breakdown Point

Dual problem (detailed)

- The **dual problem** using squared Hellinger is

$$V(b) = \sup_{\lambda \in \mathbb{R}^{d_g+K}} E\left[\frac{\lambda^\intercal J(D)h(DY, X, b)}{1-p_D} - \frac{Df^*(\lambda^\intercal h(DY, X, b))}{p_D}\right]$$

where

$$J(D) = \begin{bmatrix} -DI_{d_g} & 0 \\ 0 & (1-D)I_K \end{bmatrix}, \qquad h(DY, X, b) = \begin{pmatrix} g(DY, X, b) \\ \mathbb{1}\{X = x_1\} \\ \vdots \\ \mathbb{1}\{X = x_K\} \end{pmatrix},$$

$$f^*(r) = \begin{cases} \frac{1}{2}\left(\frac{1}{1-2r} - 1\right) & \text{if } r < 1/2 \\ \infty & \text{o.w.} \end{cases}$$

and $\{x_1, \ldots, x_K\}$ is the support of X.

- $f^*(r) = \sup_{t\in\mathbb{R}}\{rt - f(t)\}$ is the **convex conjugate** of $f(t)$, the function defining the f-divergence used to measure selection.

Duality f-divergences

Formal assumptions: setting and strong duality

Assumption 1 (Setting) $\{D_i, D_iY_i, X_i\}_{i=1}^n$ is an i.i.d. sample from a distribution satisfying

(i) $p_D = P(D = 1) \in (0, 1)$

(ii) $X \mid D = 1$ and $X \mid D = 0$ have the same finite support $\{x_1, \ldots, x_K\}$

(iii) $E[\sup_{b \in \mathbf{B}} \|g(Y, X, b)\| \mid D = 1] < \infty$

Assumption 2 (Strong duality) $B \subseteq \mathbf{B}$ is such that $\inf_{b \in \mathbf{B}_0} \nu(b) = \inf_{b \in B \cap \mathbf{B}_0} \nu(b)$. Furthermore, for each $b \in B$,

(i) there exists $Q^b \in \mathbf{P}^b$ such that $0 < \frac{\partial Q^b}{\partial P_1}(y, x) < \infty$, almost-surely P_1.

(ii) $\lambda(b)$ solving the dual problem is in the interior of $\{\lambda \,;\, E[|f^*(\lambda^\intercal h(Y, X, b))| \mid D = 1] < \infty$.

Duality

Formal assumptions: consistency

Assumption 3 (Consistency)

(i) B is compact

(ii) $g(y, x, b)$ is continuous in b for all (y, x)

(iii) For each $b \in B$, $\{h_j(y, x, b)\}_{j=1}^{d_g+K}$ are linearly independent in the sense that for any $\lambda \neq 0 \in \mathbb{R}^{d_g+K}$,

$$P(\lambda^\intercal h(Y, X, b) \neq 0 \mid D = 1) > 0$$

(iv) For each $b \in B$, there exists a closed convex $\bar{\Lambda}^b$ with $\lambda(b) \in \text{int}(\bar{\Lambda}^b)$ such that $\bar{\Lambda}^B = \left\{(b, \lambda) \; ; \; b \in B, \; \lambda \in \bar{\Lambda}^b\right\}$ is copmact, and for some open $\mathcal{N} \subset \mathbb{R}$ containing p_D,

$$E\left[\sup_{p \in \mathcal{N}} \sup_{(b,\lambda) \in \bar{\Lambda}^B} |\varphi(D, DY, X, b, \lambda, p)|\right] < \infty,$$

$$E\left[\sup_{(b,\lambda) \in \bar{\Lambda}^B} \|\nabla_\lambda \varphi(D, DY, X, b, \lambda, p_D)\|\right] < \infty, \quad E\left[\sup_{(b,\lambda) \in \bar{\Lambda}^B} \|\nabla_\lambda^2 \varphi(D, DY, X, b, \lambda, p_D)\|\right] < \infty$$

If assumptions 1, 2, and 3 hold, then $\hat{\nu}_n \xrightarrow{p} \nu$ in $\ell^\infty(B)$ and $\hat{\delta}_n^{BP} \xrightarrow{p} \delta^{BP}$.

Estimators

Formal assumptions: inference

Let $\theta(b) = (\nu(b), \lambda(b), p_D)$, $\theta = (v, \lambda, p)$,

$$\phi(D, DY, X, b, \theta) = \phi(D, DY, X, b, v, \lambda, p) = \begin{pmatrix} \varphi(D, DY, X, b, \lambda, p) - v \\ \nabla_\lambda \varphi(D, DY, X, b, \lambda, p) \\ D - p \end{pmatrix},$$

$\Theta^b = \left\{\theta = (v, \lambda, p)\ ;\ v \in [0, \mathcal{V}],\ \lambda \in \bar{\Lambda}^b, p \in [\underline{p}, \overline{p}]\right\}$, and $\theta^B = \left\{(b, \theta)\ ;\ b \in B, \theta \in \Theta^b\right\}$.

Assumption 4 (Inference) Suppose that

(i) $\mathbf{B}_0$ is closed

(ii) B is convex

(iii) $g(z, b)$ is continuously differentiable with respect to b

(iv) $\hat{\theta}_n(b) = (\hat{\nu}_n(b), \hat{\lambda}_n(b), \hat{p}_{D,n}) \in \Theta^b$ for each b

(v) There exists $F(d, dy, x)$ such that

$$\sup_{b \in B} \sup_{\theta \in \Theta^b} \|\nabla_{(b,\theta)} \phi(d, dy, x, b, \theta)\| \le F(d, dy, x)$$

and $E[F(D, DY, X)^2] < \infty$.

If assumptions 1, 2, 3, and 4 hold, then

$$\sqrt{n}(\hat{\nu}_n - \nu) \xrightarrow{L} \mathbb{G}_\nu \text{ in } \ell^\infty(B), \qquad \text{and} \qquad \sqrt{n}(\hat{\delta}_n^{BP} - \delta^{BP}) \xrightarrow{L} \inf_{b \in \mathbf{m}(\nu)} \mathbb{G}_\nu(b) \text{ in } \mathbb{R}$$

Score bootstrap

- Let $\{W_i\}_{i=1}^n$ be i.i.d. scalars, independent of $\{D_i, D_i Y_i, X_i\}_{i=1}^n$, satisfying
  (i) $E[W] = 0$,
  (ii) $E[W^2] = 1$, and
  (iii) $E[|W|^{2+a}] < \infty$ for some $a > 0$.

- Let $\hat{\Phi}_n(b) = \frac{1}{n}\sum_{i=1}^n \nabla_\theta \phi(D_i, D_i Y_i, X_i, b, \hat{\theta}_n(b))$,

$$\hat{G}_n^*(b) = \hat{\Phi}_n(b)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \phi(D_i, D_i Y_i, X_i, b, \hat{\theta}_n(b))$$

and $\hat{G}_n^*(b, 1)$ be the first coordinate of the vector $\hat{G}_n^*(b)$.

Bootstrap procedure

1. Compute $\hat{b}_n^i = \arg\min_{b \in B \cap \mathbf{B}_0} \hat{\nu}_n(b)$,
2. Generate N bootstrap samples $\{W_i\}_{i=1}^n$ from a distribution with moments described above, and compute $\hat{G}_n^*(\hat{b}_n^i, 1)$ for each of the N bootstrap samples,
3. Let $\hat{c}_{1-\alpha,n}$ be the $1-\alpha$ quantile of $\{\hat{G}_{n,k}^*(\hat{b}_n^i, 1)\}_{k=1}^N$.

If assumptions 1, 2, 3, and 4 hold, and $\mathbf{m}(\nu) = \arg\min_{b \in B \cap \mathbf{B}_0} \nu(b)$ is the singleton $\{b^i\}$, then $\lim_{n\to\infty} P\left(\hat{\delta}_n^{BP} - \frac{1}{\sqrt{n}}\hat{c}_{1-\alpha,n} \le \delta^{BP}\right) = 1 - \alpha$.

Inference: lower confidence intervals