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On a test statistic for the Kuramoto order parameter of synchronization: An illustration for group synchronization during rocking chairs

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ABSTRACT

We discuss a quantitative approach to detect phase synchronization in noisy experimental multivariate data. To this end, we derive a test statistic based on the Kuramoto order parameter. We derive criteria for rejecting the null hypothesis of zero phase synchronization at given significance levels. We apply the test statistic to a recently conducted study on group behavior that used a rocking chair paradigm. In this context, we quantify the emergence of phase synchronization within small groups of participants.

1. Introduction

Synchronization of oscillatory systems is a benchmark phenomena of nonlinear sciences that has been extensively studied. Typically, two oscillatory processes exhibiting the same oscillation frequency are said to be synchronized when their phases are locked. In this context, we refer to synchronization as phase synchronization (more general notions of synchronization have been introduced recently including n:m phase locking). Synchronization of two signals has been studied in various fields of the life sciences (brain activity e.g. [1–6], cardiorespiration [7], Parkinson patients [8], mother–fetus heart beat synchronizations [9]) and in many applications in physics and engineering sciences (for a review see e.g. [10]). Phase synchronization of bivariate time series has been quantified using various measures such as averaged phase difference [3], entropy [8], mutual information [11], phase distributions [4], phase diffusion index and other measures [10,12–14]. Synchronization in many-body systems (e.g., a large set of oscillators) has been studied as a collective phenomenon [15]. These studies have also been motivated by several systems of the animated world that exhibit synchronization as pointed out in Winfree's seminal works [16]. Examples in this regard are synchronization of firefly flashings [17,18], synchronized applause [19,20], and most recently the Millennium bridge instability [21,22]. Phase synchronization in spike-like oscillatory voltage signals of neural network units has been quantified using the mean voltage signal [23]. Similarly, for chaotic many-body systems the macroscopic oscillation amplitude may be used [24]. Phase synchronization of coupled phase oscillators may be quantified using the so-called order function [25]. However, one of the measures that has most frequently been used to quantify phase synchronization in many-body systems is the order parameter or cluster amplitude introduced by Kuramoto [15].

The Kuramoto order parameter is usually defined in the thermodynamic limit (i.e., for systems composed of a large number N of oscillatory units such that the limit $N \to \infty$ can be considered as a good approximation). As opposed to this classical application of the Kuramoto order parameter, the goal of the present effort is to use the order parameter for systems composed of a relatively small number of oscillatory units. That is, we are interested in studying phase synchronization that can be observed in multivariate time series where the number N of time series or trajectories is relatively small and the thermodynamic limit does not apply (i.e., N is of the order of 10 or 100). An essential problem in the analysis of such multivariate signals is the construction of appropriate hypothesis tests to test for phase synchronization. In Section 2 we will formulate two types of null hypotheses of zero phase synchronization and will discuss statistical tests related to these hypotheses.

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In Section 3 we will apply the hypothesis tests proposed in Section 2 to study the emergence of group behavior in an experiment involving small participant groups rocking in rocking chairs. This social coordination paradigm was chosen to test the proposed statistic, given that previous research [26] has demonstrated that not only can the rocking chair movements of visually coupled individuals be intentionally synchronized, but that spontaneous and/or intermittent synchronization can also unintentionally occur between visually coupled individuals; that is, when movement synchronization is not part of an interaction goal. In short, this methodology provided a way to examine the test statistic proposed in Section 2 under strong (intentional) and weak (spontaneous) levels of group synchrony, and moreover, chance synchronization (see below). We achieved this by instructing groups of six participants, arranged in a circle, to rock at a self-selected (comfort mode) frequency under three conditions of visual coupling. The first condition provided a measure of chance synchronization by instructing participants to rock while having their eyes closed (i.e., participants were not visually coupled). The second condition provided a measure of spontaneous synchronization by instructing participants to ignore their co-participants and rock at their own personal frequency, while keeping their eyes open (i.e., participants were visually coupled). The third condition provided a measure of intentional group synchronization by instructing participants to synchronize their rocking chair movements as a group.

2. Test statistic

2.1. Order parameters and null hypotheses of zero synchronization

We consider N angular variables $u_k \in [0, 2\pi]$ that are stroboscopically observed during T time points $t_1, \ldots, t_T$. That is, we consider N trajectories described by the functions $u_k(t_p)$ with $p = 1, \ldots, T$ and $k = 1, \ldots, N$. At every time point t, the Kuramoto order parameter $r(t)$ of the phases $u_k(t)$ is defined by [15,27–29]

$$r(t) = \frac{1}{N}\left|\sum_{k=1}^{N} \exp\{\mathrm{i}\, u_k(t)\}\right| \tag{1}$$

with $t \in \{t_1, \ldots, t_T\}$, $\mathrm{i} = \sqrt{-1}$, and $r(t) \in [0, 1]$. The order parameter is usually studied in the thermodynamic limit $N \to \infty$. In contrast, in what follows, we assume that N is finite. That is, we focus on a finite multi-particle or multi-agent system. In addition to the instantaneous order parameter $r(t)$ defined by Eq. (1), we consider the time average

$$r_{av} = \frac{1}{T}\sum_{p=1}^{T} r(t_p). \tag{2}$$

In general, experimental data will yield an averaged order parameter $r_{av} > 0$. In order to test whether the observation that r_{av} is larger than zero reflects statistical significance, we consider the squared instantaneous order parameter

$$y(t) = r^2(t) \tag{3}$$

and its time average

$$y_{av} = \frac{1}{T}\sum_{p=1}^{T} y(t_p). \tag{4}$$

Note that while by definition (3) we have $y(t) = r^2(t)$, for the time-averaged parameters, in general, we have $y_{av} \neq r_{av}^2$. Furthermore, note that in the special case of perfect phase synchronization we have $u_1(t) = u_2(t) = \cdots = u_N(t)$ for all time points $t_1, \ldots, t_T$. In this case, we obtain $r(t) = y(t) = r_{av} = y_{av} = 1$.

In what follows, we consider two types of null hypotheses that both state that the trajectories u_k do not exhibit phase synchronization. The strong null hypothesis states the following: (a) the angular variables $u_k(t)$ are uniformly distributed in $[0, 2\pi]$ for any time point t and for all k; (b) fluctuations u_k are 'spatially' independent in the sense that $u_k(t)$ and $u_m(t)$ are statistically independent random variables for all time points t provided that $k \neq m$; and (c) all trajectories $u_k(t)$ are statistically independent in time which implies that the trajectories exhibit zero autocorrelations. Let us define the joint probability density

$$P(u_k, t; u_m, t') = \left\langle \delta(u_k - u_k(t))\delta(u_m - u_m(t'))\right\rangle, \tag{5}$$

where $\delta(\cdot)$ is the Dirac delta function. Note that in what follows the bracket $\langle\cdot\rangle$ denotes an ensemble average. Then, the strong null hypothesis states that the following relation holds:

$$P(u_k, t; u_m, t') = W(u_k)W(u_m) \tag{6}$$

for $t \neq t'$ and arbitrary $k, m \in [1, N]$ with

$$W(x) = \frac{1}{2\pi}. \tag{7}$$

The weak null hypothesis of zero phase synchronization states that conditions (a) and (b) are satisfied. However, we do not require that condition (c) holds. That is, we do without the statistical independence in time. Trajectories $u_k(t)$ may have finite autocorrelations. Under the weak null hypothesis, Eq. (6) is still satisfied for $k \neq m$. However, for $k = m$ we have in general

$$P(u_k, t; u_k, t') \neq W(u_k)W(u_k). \tag{8}$$

2.2. Construction of statistical tests

Our next objective is to determine the test statistic of y. More precisely, we want to determine the mean $\langle y\rangle$ of y, the standard deviation σ_y of y, and the shape of the probability density $F(y)$ of y assuming that one of the two aforementioned null hypotheses is true. As far as the mean is concerned, we have

$$\begin{aligned} y(t) &= \frac{1}{N^2}\sum_k \exp\{\mathrm{i}\, u_k(t)\}\sum_m \exp\{-\mathrm{i}\, u_m(t)\} \\ &= \frac{1}{N^2}\sum_{k,m} \exp\{\mathrm{i}\,[u_k(t) - u_m(t)]\}, \end{aligned} \tag{9}$$

which can be written as

$$y(t) = \frac{1}{N^2}\left\{N + 2\sum_{k,m>k} \cos\left(u_k(t) - u_m(t)\right)\right\}. \tag{10}$$

There are $(N^2 - N)/2$ cosine terms in Eq. (10). In the special case of perfect phase synchronization ($u_1 = u_2 = \cdots = u_N$) we have $\cos(u_k - u_m) = 1$ and re-obtain the result $y(t) = N^{-2}[N + 2(N^2 - N)/2] = 1$ mentioned earlier. The first moment $\langle y(t)\rangle$ of $y(t)$ can be computed from

$$\langle y(t)\rangle = \frac{1}{N^2}\left\{N + 2\sum_{k,m>k} \left\langle\cos\left(u_k(t) - u_m(t)\right)\right\rangle\right\}. \tag{11}$$

Using the trigonometric relation

$$\cos\left(u_k(t) - u_m(t)\right) = \cos(u_k)\cos(u_m) + \sin(u_k)\sin(u_m) \tag{12}$$

and exploiting property (a) of the strong or weak null hypothesis, we find

$$\begin{aligned} &\left\langle\cos\left(u_k(t) - u_m(t)\right)\right\rangle \\ &\quad = \langle\cos(u_k)\rangle\,\langle\cos(u_m)\rangle + \langle\sin(u_k)\rangle\,\langle\sin(u_m)\rangle = 0. \end{aligned} \tag{13}$$

Consequently, the first moment of the instantaneous squared order parameter is given by

$$\langle y(t)\rangle = \frac{1}{N}. \tag{14}$$

Next we compute the first moment $\langle y_{av}\rangle$ of the time-averaged parameter y_{av}:

$$\langle y_{av}\rangle = \frac{1}{T}\left\langle \sum_{p=1}^{T} y(t_p)\right\rangle. \tag{15}$$

Under the strong null hypothesis we have zero autocorrelations. This implies

$$\left\langle \sum_{p=1}^{T} y(t_p)\right\rangle = \sum_{p=1}^{T}\left\langle y(t_p)\right\rangle. \tag{16}$$

As a result, we see that

$$\langle y_{av}\rangle = \frac{1}{N}. \tag{17}$$

Let us turn to the weak null hypothesis. In this case, we obtain Eq. (17) again due to the fact that y_{av} is a linear function with respect to its arguments $y(t_1), \ldots, y(t_T)$. In detail, we have

$$\begin{aligned}
\langle y\rangle &= \frac{1}{TN^2}\left\{TN + 2\sum_{k,m>k}\left\langle \sum_{p=1}^{T}\cos\left(u_k(t_p)-u_m(t_p)\right)\right\rangle\right\} \\
&= \frac{1}{TN^2}\left\{TN + 2\sum_{k,m>k}\int \sum_{p=1}^{T}\cos\left(u_{k,p}-u_{m,p}\right)\right. \\
&\quad \left.\times\ \tilde{P}(\mathbf{u}_1, t_1; \mathbf{u}_2, t_2; \ldots; \mathbf{u}_N, t_T)\prod_{j=1}^{N}\prod_{w=1}^{T}\mathrm{d}u_{j,w}\right\} \\
&= \frac{1}{TN^2}\left\{TN + 2\sum_{k,m>k}\int \sum_{p=1}^{T}\cos\left(u_{k,p}-u_{m,p}\right)\right. \\
&\quad \left.\times\ P(u_{k,p}, t_p; u_{m,p}, t_p)\,\mathrm{d}u_{k,p}\,\mathrm{d}u_{m,p}\right\} \\
&= \frac{1}{TN^2}\left\{TN + 2\sum_{k,m>k}\int \sum_{p=1}^{T}\cos\left(u_{k,p}-u_{m,p}\right)\right. \\
&\quad \left.\times\left(\frac{1}{2\pi}\right)^2\mathrm{d}u_{k,p}\,\mathrm{d}u_{m,p}\right\} \\
&= \frac{1}{N}.
\end{aligned} \tag{18}$$

Note that in Eq. (18) the variables $u_{k,p}$ are the coordinates of the random variables $u_k(t_p)$ and $\mathbf{u}_p$ denotes the vector $\mathbf{u}_p = (u_{1,p}, \ldots, u_{N,p})$. Furthermore, in Eq. (18) we have used the joint probability density

$$\tilde{P}(\mathbf{u}_1, t_1; \ldots; \mathbf{u}_N, t_T) = \left\langle \prod_{k=1}^{N}\prod_{p=1}^{T}\delta\left(u_{k,p}-u_k(t_p)\right)\right\rangle. \tag{19}$$

Let us determine next the second moment $\langle y_{av}^2\rangle$ and eventually the variance σ_{av}^2 of y_{av}. We first recognize that

$$y^2(t) = \frac{1}{N^4}\left\{N^2 + 4N\sum_{k,m>k}\cos(u_k(t)-u_m(t)) + 4\left[\sum_{k,m>k}\cos(u_k(t)-u_m(t))\right]^2\right\}. \tag{20}$$

As discussed above, we have $\left\langle\sum_{k,m>k}\cos(u_k(t)-u_m(t))\right\rangle = 0$ under the strong and weak null hypotheses, see Eq. (13). Therefore, we conclude that

$$\left\langle y^2(t)\right\rangle = \frac{1}{N^4}\left\{N^2 + 4\left\langle\left[\sum_{k,m>k}\cos(u_k(t)-u_m(t))\right]^2\right\rangle\right\}. \tag{21}$$

A detailed calculation shows that in the curled bracket $\{\cdot\}$ only the terms $\cos^2(u_k(t)-u_m(t))$ have a non-vanishing expectation value under both types of null hypotheses. Consequently, Eq. (21) becomes

$$\left\langle y^2(t)\right\rangle = \frac{1}{N^4}\left\{N^2 + 4\sum_{k,m>k}\left\langle\cos^2(u_k(t)-u_m(t))\right\rangle\right\}. \tag{22}$$

Using the trigonometric relation (11), we find that

$$\begin{aligned}
&\left\langle\cos^2(u_k(t)-u_m(t))\right\rangle \\
&\quad = \left\langle\cos^2(u_k)\right\rangle\left\langle\cos^2(u_m)\right\rangle + \left\langle\sin^2(u_k)\right\rangle\left\langle\sin^2(u_m)\right\rangle \\
&\qquad + 2\underbrace{\left\langle\cos(u_k)\cos(u_m)\sin(u_k)\sin(u_m)\right\rangle}_{=0}.
\end{aligned} \tag{23}$$

Since $\left\langle\cos^2(u)\right\rangle = (2\pi)^{-1}\int_0^{2\pi}\cos^2(x)\,\mathrm{d}x = 0.5$ holds, we have $\left\langle\cos^2(u_k(t)-u_m(t))\right\rangle = 1/2$. There are $(N^2-N)/2$ of such terms. Therefore, Eq. (22) becomes

$$\left\langle y^2(t)\right\rangle = \frac{1}{N^4}\left\{N^2 + 4\frac{(N^2-N)}{4}\right\} = \frac{2}{N^2} - \frac{1}{N^3}. \tag{24}$$

We can use this result to compute $\left\langle y_{av}^2\right\rangle$:

$$\left\langle y_{av}^2\right\rangle = \frac{1}{T^2}\sum_{p,w}\left\langle y(t_p)y(t_w)\right\rangle. \tag{25}$$

The strong null hypothesis implies that correlations between $y(t)$ and $y(t')$ vanish for $t \neq t'$ such that $\sum_{p,w}\left\langle y(t_p)y(t_w)\right\rangle = \sum_p \left\langle y^2(t_p)\right\rangle + \sum_{p,w\neq p}\left\langle y(t_p)\right\rangle\langle y(t_w)\rangle$. From Eqs. (24) and (25) it follows that

$$\left\langle y_{av}^2\right\rangle = \frac{1}{T}\left(\frac{2}{N^2} - \frac{1}{N^3}\right) + \frac{T-1}{T}\frac{1}{N^2}. \tag{26}$$

From Eqs. (14) and (26) it is clear that the variance $\sigma_{av}^2 = \left\langle y_{av}^2\right\rangle - \langle y_{av}\rangle^2$ reads

$$\sigma_{av}^2 = \frac{1}{N^2 T}\left(1 - \frac{1}{N}\right) \tag{27}$$

under the strong null hypothesis. Accordingly, the standard deviation is given by

$$\sigma_{av} = \frac{1}{N}\sqrt{\left(1-\frac{1}{N}\right)\frac{1}{T}}. \tag{28}$$

The shape of the probability density F of y_{av} can be determined using the central limit theorem. Recall that under the strong null hypothesis we assume that the random variables $u_k(t)$ are statistically independent from each other both 'spatially' and in time. This implies that the values $y(t_1), \ldots, y(t_T)$ are statistically independent from each other. Therefore, for large enough T (rule of thumb $T > 30$) the distribution of the time average $y_{av} = T^{-1}\sum_p y(t_p)$ is approximately given by a normal distribution. The mean of the normal distribution equals $\langle y_{av}\rangle = 1/N$ and the standard deviation $\sigma_{y(av)}$ of the normal distribution equals the standard deviation σ_t of $y(t)$ divided by $\sqrt{T}$. Using Eqs. (14) and (24), we obtain

$$\sigma_t^2 = \left\langle y^2(t)\right\rangle - \langle y(t)\rangle^2 = \frac{1}{N^2}\left(1 - \frac{1}{N}\right) \tag{29}$$

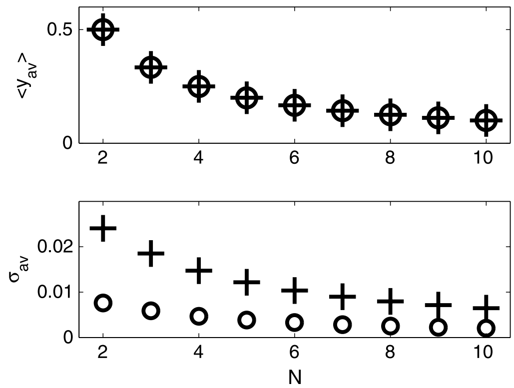

Fig. 1. Top panel: first moment $\langle y_{av} \rangle$ as a function of N computed from Eq. (14). Bottom panel: standard deviation σ_{av} as a function of N calculated from Eq. (28). Parameters: $T = 216$ (plus signs); $T = 2160$ (circles).

and

$$\sigma_t = \frac{1}{N}\sqrt{\left(1 - \frac{1}{N}\right)}. \tag{30}$$

Consequently, the derivation of $\sigma_{y(av)} = \sigma_t/\sqrt{T}$ by means of the central limit theorem yields Eq. (28) again: $\sigma_{y(av)} = \sigma_{av}$.

In short, we conclude that for sufficiently large T the test statistic shows good approximation to a normal distribution defined by

$$F(y_{av}) = \frac{1}{\sqrt{2\pi\sigma_{av}^2}} \exp\left\{-\frac{\left(y_{av} - N^{-1}\right)^2}{2\sigma_{av}^2}\right\}. \tag{31}$$

Since the test statistic is a normal distribution with known variance, the strong null hypothesis can be tested by means of a z test. Fig. 1 illustrates the dependencies of $\langle y_{av} \rangle$ (top panel) and σ_{av} (bottom panel) on the number of signals N and the number of data points T being considered. Fig. 2 illustrates the normal distribution (31) for multivariate time series involving $N = 4$, $N = 5$, and $N = 6$ signals. The exact result (31) as well as a probability density obtained from a simulation is shown. The distributions thus obtained are almost identical.

2.3. z tests for individual trials and samples using the strong null hypothesis

From our previous considerations it follows that for any given numbers of N and T, the so-called population mean μ and the variance σ^2 of the test statistic for the strong null hypothesis is known. We have $\mu = 1/N$ and σ^2 is defined by Eq. (27). With these results in hand, we can test the strong null hypothesis for single trials. Let us assume a particular trial yields a certain order parameter $r_{av,obs} > 0$ and a measure $y_{av,obs}$. In order to test the strong null hypothesis for this trial, we compute the z score

$$z = \frac{y_{av,obs} - \mu}{\sigma_{av}}. \tag{32}$$

Depending on the significance level α, we can read off the critical z value $z_{c,\alpha}$ from the normal unit table. By comparing z with $z_{c,\alpha}$ we arrive at a decision whether or not to reject the strong null hypothesis. If $z < z_{c,\alpha}$ then we do not reject the strong null hypothesis. That is, we accept the hypothesis that the trajectories do not exhibit synchronization. If $z > z_{c,\alpha}$ we reject the strong null hypothesis. Rejecting the strong null hypothesis supports the alternative hypothesis, namely, that the finiteness of the order parameter $r_{av,obs}$ is not due to a by-chance-effect. Rather, the trajectories exhibit a statistically significant degree of phase synchronization.

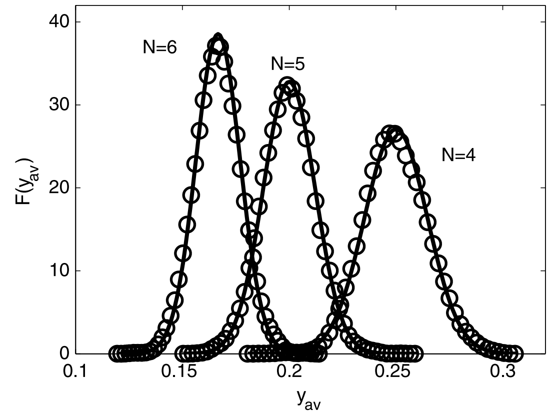

Fig. 2. Test statistic as obtained from Eq. (31) (solid lines) and as computed from numerical simulations (circles). Circles: for every N we computed numerically a set of 3000 parameters y_{av} and calculated from this set the probability density F (matlab kernel density estimator). Parameters: $N = 6$ (left), $N = 5$ (middle), $N = 4$ (right), $T = 216$ (note that the width σ_{av} of the distribution used in Section 3 is smaller by a factor 10 because the time series discussed in Section 3 will involve about $T = 21\,600$ data points).

Let us assume we repeat the experiment R times and in doing so obtain a sample of order parameters $r^{(1)}_{av,obs}, \ldots, r^{(R)}_{av,obs}$ and a sample of values $y^{(1)}_{av,obs}, \ldots, y^{(R)}_{av,obs}$. From these observations we compute the sample mean values $\bar{r} = R^{-1}\sum_j r^{(j)}_{av,obs}$ and $M = R^{-1}\sum_j y^{(j)}_{av,obs}$, respectively. In general, we will find $\bar{r} > 0$. In order to answer the question whether the observed finiteness of $\bar{r}$ is due to a by-chance-effect or reflects statistical significance, we compute the z score of the sample mean M like

$$z = \frac{M - \mu}{\sigma_M} \tag{33}$$

with $\mu = 1/N$ and $\sigma_M = \sigma_{av}/\sqrt{R}$. We proceed then as described above. For $z < z_{c,\alpha}$ we do not reject the strong null hypothesis. For $z > z_{c,\alpha}$ we reject the strong null hypothesis and conclude that the observation of $\bar{r} > 0$ bears statistical significance.

2.4. t test for the weak null hypothesis

However, we may have reason to believe that the weak null hypothesis is true. In particular, we may compute autocorrelation functions from the experimentally observed phase trajectories $u_k(t)$ and find that they exhibit autocorrelations above the significance level. In this case, the strong null hypothesis is violated. The weak null hypothesis is not necessarily violated. If the weak null hypothesis is true, trajectories do not exhibit synchronization but may exhibit autocorrelations. At this point it is important to note that the test statistic derived in Section 2.2 does not necessarily apply to the weak null hypothesis. Since autocorrelations might be larger than zero, we conclude that correlations of $y(t)$ and $y(t')$ for $t' \neq t$ do not necessarily vanish. This implies that the expression $\sum_{p,w\neq p}\left\langle y(t_p)y(t_w)\right\rangle$ is not necessarily equal to $\sum_{p,w\neq p}\left\langle y(t_p)\right\rangle\langle y(t_w)\rangle$. This argument can be shown explicitly using the joint probability $\tilde{P}$ occurring in Eq. (18). From our observation that the equivalence $\sum_{p,w\neq p}\left\langle y(t_p)y(t_w)\right\rangle = \sum_{p,w\neq p}\left\langle y(t_p)\right\rangle\langle y(t_w)\rangle$ does not necessarily hold, we conclude further that the second moment $\left\langle y_{av}^2\right\rangle$ under the weak null hypothesis does not necessarily correspond to Eq. (26). As a result, the variance of the test statistic is unknown (and in general will depend on the details of the autocorrelation functions). Nevertheless, we can test the weak null hypothesis because the mean of $\langle y_{av} \rangle$ under the weak null hypothesis is known to be equal to $1/N$ (see Section 2.2). In order to test the weak null hypothesis, we need to consider a sample of observations $r^{(1)}_{av,obs}, \ldots, r^{(R)}_{av,obs}$ yielding a sample mean $\bar{r} > 0$. In particular, we need to have the corresponding sample of values $y^{(1)}_{av,obs}, \ldots, y^{(R)}_{av,obs}$ at our disposal with sample mean M. We use the latter sample to conduct a t test. Let s denote the sample standard deviation computed from the observations $y^{(1)}_{av,obs}, \ldots, y^{(R)}_{av,obs}$. Then the t score is

$$t = \frac{M - \mu_y}{s_M} \tag{34}$$

with $s_M = s/\sqrt{R}$. The critical t value $t_{c,\alpha,df}$ can be determined from t tables and depends on the required significance level α and the degrees of freedom $df = R - 1$. For $t < t_{c,\alpha,df}$ we do not reject the weak null hypothesis. For $t > t_{c,\alpha,df}$ we reject the weak null hypothesis and support the alternative hypothesis. That is, in the latter case we conclude that the trajectories exhibit a statistically significant degree of synchronization. In other words, we conclude that the observed finiteness of the order parameter $\bar{r}$ is not due to a by-chance-effect.

3. Application to human group behavior: synchronization during rocking chairs

3.1. Experimental setup

Participants. Six groups ($R = 6$) of six participants (36 participants in total) were recruited for the study. All participants were Colby college undergraduate students who completed the experiment for partial course credit or monetary reward (USD 6.00). All participants were naive to the purpose of the study.

Materials. Participants sat and rocked in six identical wooden rocking chairs. The chairs were positioned evenly around a central 10×10 cm target that stood on a 5 cm wide by 1.2 m high stand and formed of a circle with a radius of 1.25 m (the radius was from the center of target to the front of the chairs. The Euclidean up-down (z) movements of each rocking chair's headrest was recorded at 120 Hz using a magnetic tracking system (Polhemus Liberty, Polhemus Corporation, Colchester, VT). A carpet was used to minimize the noise of chair motion. In addition, participants wore ear-plugs to eliminate possibility of an auditory coupling.

Procedure. Upon arrival, participants were randomly assigned to one of the six chairs. Participants were informed that the study was investigating the ergonomics or rocking chairs and that six participants were being run simultaneously to increase the speed of data collection. Participants were instructed to ignore each other, refrain from conversing with each other, and rock at their own self-selected (comfort mode) frequency. Prior to completing the experimental trials participants were given a 60 s eyes closed practice session to establish their comfort mode frequency. Note that the above cover story was employed to ensure that any synchronization observed in the control and spontaneous trials (see below) was not intentional (see [26]).

Each group completed five 3 min trials in the following order: one 'eyes closed' (control) trial; two 'eyes open' spontaneous trials, and two 'eyes open' intentional trials. For the 'eyes closed' trial, participants were instructed to rock at their own self-selected tempo with their eyes closed. This trial allowed for an experimental measure of chance level synchrony, as participants had no visual information about their co-participants' movements. For the 'spontaneous' trials, the participants were instructed to rock at their own self-selected frequency while looking (eyes open) directly at the central target. Instructing participants to look at the central target ensured participants had visual information about their co-participants rocking chair movements and at the same time prevented them from focusing solely on one individual in the group. For the 'intentional' trials, the participants were informed of the studies true purpose (investigating group synchrony) and were instructed to rock at a self-selected frequency while attempting to synchronize their rocking chair movements as a group. To control for looking direction, all participants were again instructed to look at the central target. No instructions as to the form or pattern of synchrony were provided.

3.2. Data analysis

We determined for each participant and condition the peak oscillation frequency. To this end, we computed the power spectra from the recorded movements in z direction. The peak oscillation frequency of a movement trajectory was defined as the frequency at which the spectrum exhibited its peak. We computed the standard deviation SDfreq from the individual peak oscillation frequencies in order to obtain a measure for the degree to which participants produced oscillations with the same frequency.

In order to determine the amount of phase synchronization exhibited by a group, we described the oscillatory movements of participants or chairs by means of running phases $\phi_1(t), \ldots, \phi_N(t)$. That is, the phases $\phi_1(t), \ldots, \phi_N(t)$ were monotonically increasing functions of time. (In particular, after completion of the first oscillation cycle we had $\phi = 2\pi$; after completion of the second oscillation cycle we had $\phi = 4\pi$ and so on.) A typical example of a deterministic model for such running phases is $\phi(t) = \omega t + \phi_0$, where ω is the oscillation frequency and ϕ_0 an initial phase angle. For each movement trajectory $z(t)$ we computed the running phase $\phi(t)$ using the projection of z into the re-scaled phase space $(\tilde{z}, \tilde{v}_z)$ spanned by the coordinates $\tilde{z} = z(t)/\max\{|z|\}$ and $\tilde{v}_z = v_z/\max\{|v|\}$, where v_z is defined as $v_z = \mathrm{d}z/\mathrm{d}t$. That is, $\phi(t)$ was defined by $\phi(t) = \arctan(\tilde{v}/\tilde{x})$. In order to determine whether or not the running phases $\phi_k(t)$ thus obtained exhibited phase synchronization, we examined the trajectories in a rotating frame defined by the motion of the cluster phase q. We were interested in studying synchronization based on relative phase relationships. Accordingly, when participants of a group in our experiment produced oscillations with phase lags that were more or less constant with respect to the cluster phase, then we considered this group behavior as exhibiting relative phase synchronization even if the phase lags were different from zero. A possible deterministic model for a group of trajectories exhibiting relative phase synchronization is $\phi_k(t) = \omega t + \phi_{k,0}$, where the variables $\phi_{k,0}$ represent different offset phases. A possible stochastic model would be the Kuramoto model defined by $\mathrm{d}\phi_k(t)/\mathrm{d}t = \omega - \kappa N^{-1}\sum_{m=1}^{N}\sin(\phi_k - \phi_m) + \xi_k(t)$, where $\kappa > 0$ is a sufficiently large coupling constant and $\xi_k(t)$ is a Langevin force [15].

In detail, we evaluated the running phases $\phi_k(t_p)$ as follows. First, we determined the instantaneous cluster phase $q(t)$ by calculating

$$v(t)\exp\{\mathrm{i}q(t)\} = \frac{1}{N}\sum_{k=1}^{N}\exp\{\mathrm{i}\phi_k(t)\}. \tag{35}$$

Consequently, the phases in the rotating frame were given by $\phi_k(t) - q(t)$. Second, we computed the mean offset phases δ_k of participants using circular statistics:

$$d_k\exp\{\mathrm{i}\delta_k\} = \frac{1}{T}\sum_{p=1}^{T}\exp\left\{\mathrm{i}\left[\phi_k(t_p) - q(t_p)\right]\right\}. \tag{36}$$

Consequently, the offset free phases in the rotating frame were given by the expressions $\phi_k(t) - q(t) - \delta_k$. We denoted these phases by $u_k(t)$. If we observed phase synchronization of the trajectories $u_k(t)$, then we said we observed that the original running phases

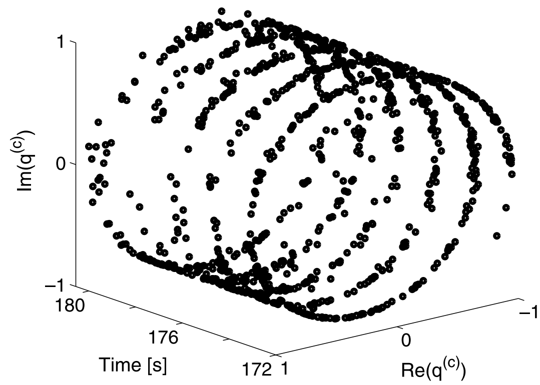

Fig. 3. Evolution of the cluster phase $q(t)$ computed from Eq. (35). 5 cycles are shown (oscillation frequency of about 0.6 Hz). Real and imaginary parts of $q^{(c)} = \exp\{iq(t)\}$ are depicted versus time.

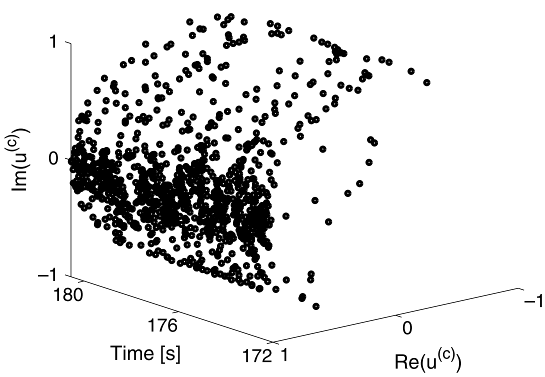

Fig. 4. Evolution of the phase angle u describing the behavior of an individual participant relative to the cluster phase q shown in Fig. 3. Real and imaginary parts of $u^{(c)} = \exp\{iu(t)\}$ are depicted versus time.

ϕ_k exhibited relative phase synchronization in the rotating frame. In short, in a third step, we computed the angular variables

$$u_k(t) = \phi_k(t) - q(t) - \delta_k \tag{37}$$

for every chair k and all time points $t \in \{t_1, \ldots, t_T\}$. Having constructed the phase trajectories $u_k(t)$, we computed the instantaneous Kuramoto order parameter $r(t)$ from Eq. (1) and the time-averaged order parameter r_{av} from Eq. (2). Moreover, for hypothesis testing purposes, we computed $y(t) = r^2(t)$ as defined by Eq. (3) and calculated y_{av} from Eq. (4). Fig. 3 illustrates the evolution of the cluster phase $q(t)$ over time as obtained from one of the groups (group 6, first intentional trial, see Section 3.3 below) for the final 1000 samples. Fig. 3 shows how $q(t)$ evolves on the unit cycle. To this end, we computed $q^{(c)} = \exp\{iq(t)\}$ and plotted the unit cycle in the y–z plane and time in the x direction. The random walk of the offset free relative phase u computed from the movement data of one participant (participant 1 of group 6, first intentional trial, see Section 3.3 below) by means of Eq. (37) is shown in Fig. 4. The variable u in Fig. 4 is again represented as an angle on a unit cycle using $u^{(c)} = \exp\{iu\}$. In line with the definition (37) the phase u is centered around a phase angle of zero which means that the real part of $u^{(c)}$ is close to 1 and the imaginary part of $u^{(c)}$ is close to zero.

3.3. Results

Oscillation frequencies. As mentioned above, in our study there were six participants rocking six chairs back and forth. The sensor of one chair was broken. Therefore, we evaluated data only from $N = 5$ chairs. The standard deviations of the participant frequencies for all groups and conditions are reported in Table 1. We averaged the two repetitions for the spontaneous and intentional condition and thus obtained for all three conditions one measure per group. We used these measures for further data analysis. The mean values of SDfreq are shown in Fig. 5. The largest standard deviations were found on average for the 'eyes closed' condition. The smallest standard deviations could be observed on average for the intentional condition. An F test did not yield conclusive results.[1] Therefore, we carried out three pairwise t-tests comparing the conditions 'eyes closed' with spontaneous, 'eyes closed' with intentional and spontaneous with intentional. We used a family-wise error of $\alpha_{FW} = 0.05$ (and a single test significance level of $\alpha = 0.01$; Bonferroni correction). Only the difference between the unintentional and the intentional conditions was significant ($t(5) = 4.14$, $p < 0.01$). Visual inspection of the power spectra confirmed that during the intentional condition participants exhibited almost the same oscillation frequency. Fig. 6 shows the power spectra obtained from group 6 (first intentional trial). In contrast, visual inspection of the power spectra obtained during the 'eyes closed' and spontaneous condition revealed considerable differences among the oscillation frequencies of participants.

Table 1
Experimental observations: SDfreq.

Group	Eyes closed	Spontaneous		Intentional	
		1	2	1	2
1	0.265	0.259	0.050	0.025	0.008
2	0.051	0.062	0.046	0.042	0.006
3	0.087	0.055	0.041	0.003	0.002
4	0.366	0.094	0.095	0.000	0.008
5	0.032	0.059	0.055	0.005	0.000
6	0.036	0.053	0.041	0.000	0.000

Fig. 5. Mean values of SDfreq reported in Table 1 (error bars represent standard deviations).

Phase synchronization. Table 2 summarizes the observed parameters $y_{av,obs}$. From the entries in Table 2 we see that in the 'eyes closed' condition and the spontaneous condition the observed values of $y_{av,obs}$ were relative close to the theoretical value $\mu = 1/N = 0.2$ derived in Section 2 assuming zero phase synchronization.

For the 'eyes closed' condition, we conducted z tests for individual trials. We used a significance level of $\alpha = 0.05$ which gave us $z_c = 1.64$. Each trial involved $T = 21\,600$ time points. From Eq. (28) we obtained $\sigma_{av} = 0.0012$. The z scores computed from Eq. (32) with $\mu = 1/N = 0.2$ are listed in Table 3. Significant

[1] Mauchly's test was significant and the result obtained under the assumption of sphericity was in contradiction with the Greenhouse–Geisser result.

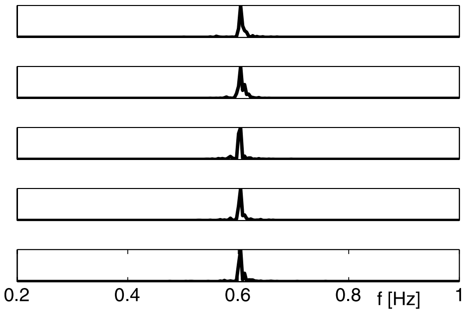

Fig. 6. Typical power spectra observed for 'intentional' group sessions. From top to bottom: power spectra computed for participants 1, 2, 3, 4, and 6. Power is expressed in arbitrary units (vertical axis). Data are taken from group 6 (first intentional session) and are related to Figs. 3 and 4.

Table 2
Experimental observations: $y_{av,obs}$.

Group	Eyes closed	Spontaneous		Intentional	
		1	2	1	2
1	0.2067	0.2349	0.2013	0.1652	0.5626
2	0.2421	0.2134	0.2277	0.4199	0.3000
3	0.1954	0.2010	0.1688	0.7253	0.6236
4	0.1975	0.2097	0.1956	0.5592	0.6480
5	0.2008	0.2132	0.1966	0.7484	0.7293
6	0.2024	0.2424	0.1881	0.7057	0.6608

Table 3
Eyes closed individual trials: z tests of the strong null hypothesis.

Group	$y_{av,obs}$	z score	Sig. [Y/N]
1	0.2067	5.58	Y
2	0.2421	35.08	Y
3	0.1954	−3.83	N
4	0.1975	−2.08	N
5	0.2008	0.67	N
6	0.2024	2.00	Y

Table 4
t tests of the weak null hypothesis.

Condition	$\bar{r}$	M	t score	Sig. [Y/N]
Eyes closed	0.4080	0.2075	1.05	N
Spontaneous	0.4089	0.2077	1.42	N
Intentional	0.7158	0.5707	5.47	Y

and not significant z scores are indicated in the last column. Observations from three groups (3, 4, 5) were consistent with the strong null hypothesis (not significant). The observations from the remaining three groups (1, 2, 6) were inconsistent with the strong null hypothesis (significant).

For each group we averaged the two repetitions under the unintentional and intentional condition. In doing so, we obtained three samples with 6 observations for all three conditions. The sample means $\bar{r}$ and M are reported in Table 4. We conducted for each condition a t test at a significance level of $\alpha = 0.05$. We used $R = 6$ and $df = 5$ which gave us $t_c(5) = 2.02$. The t scores computed from Eq. (34) with $\mu = 1/N = 0.2$ are listed in Table 4 as well. Significant and not significant results are indicated. The observations under the 'eyes closed' and spontaneous condition were consistent with the weak null hypothesis, whereas the degree of synchronization observed in the intentional condition was inconsistent with the weak null hypothesis.

4. Conclusions

We proposed a method to quantify phase synchronization. That is, the issue was to determine whether the observation of a finite degree of phase synchronization should be explained as the result of by-chance effects or as the result of a collective behavior of the multi-particle or multi-agent system under consideration. To this end, we derived a test statistic for the Kuramoto order parameter. The test statistic applies to the null hypothesis which states that there is zero phase synchronization. We distinguished between the strong and the weak null hypothesis, where under the strong null hypothesis it is assumed that phase trajectories do not exhibit correlations (and in fact exhibit statistical independence in time). The strong null hypothesis basically assumes that the experimentally observed phase dynamics of an individual subsystem or agent is given by a superposition of a trend and an uncorrelated noise source. After removing the trend, that is, if we consider the phase dynamics in a moving frame, only the uncorrelated noise source is left. The weak null hypothesis does not require that the phase processes are pure random processes. They might exhibit correlations. Time-continuous and time-discrete models of coupled phase oscillators for subthreshold coupling strength describe such a situation (i.e. the oscillators are coupled but the coupling forces are not strong enough relative to the fluctuating forces in order to establish a synchronized collective pattern).

We applied the test statistic to experimental data taken from a recently conducted (pilot) study on human group synchronization. In this study six participants were sitting in a circle in rocking chairs and were swinging back and forth at comfortable self-selected oscillation frequencies. In one condition they intentionally tried to synchronize their rocking patterns. In two other conditions they had their eyes closed or they had their eyes open but no explicit instruction was given to synchronize. In the latter condition synchronization might emerge spontaneously. In all conditions, we observed that the Kuramoto order parameter was about 0.4 or larger. That is, the order parameter assumed finite values that were definitely different from zero (recall that the range of the Kuramoto parameter is the interval [0,1]; so 0.4 on this scale can be considered by all means as a large value). In Section 3 we showed that the value of 0.4 obtained in the 'eyes closed' condition was due to by-chance effects – as expected. Therefore, the experiment nicely illustrated that the observation of a finite (and even relatively large) order parameter is not sufficient to conclude that there is phase synchronization. The rocking chair experiment clearly demonstrates that there is a need for statistical hypothesis testing.

We expected to see that the 'eyes closed' condition was consistent with the zero phase synchronization hypothesis. In fact, some of the sessions of the 'eyes closed' condition were consistent with the strong null hypothesis of zero phase synchronization. However, others were not. In the end, we obtained an inconclusive result. This does not come as a surprise given the fact that the phase trajectories were produced by adults swinging back and forth in rocking chairs. These adults in rocking chairs can be regarded as pendulum systems involving a considerable amount of inertia. Therefore, movement trajectories observed in such a paradigm are likely to show correlations in time — which violates one of the inbuilt assumptions of the strong null hypothesis.

Using the weak null hypothesis, we arrived at a simple set of conclusions. The observed phase synchronization of about 0.4 for the 'eyes closed' condition was not significant — as expected. The observed phase synchronization of about 0.4 for the spontaneous condition was not significant. The observed phase synchronization of about 0.7 for the intentional condition was significant — as expected.

Having arrived at these conclusions, we need to discuss the issue of frequency synchronization which is related to the issue of phase synchronization. As mentioned in Section 3.3, visual inspection of the power spectra obtained from 'intentional sessions' showed that participants were rocking back and forth with approximately the same oscillation frequencies (see also Fig. 6). The deviations between oscillation frequencies among participants were relatively small as is clear from the standard deviations reported in Table 1 (see columns 4 and 5). That is, we are inclined to say that during the 'intentional sessions' groups exhibited frequency synchronization. Under this assumption the aforementioned conclusion of having observed a significant phase synchronization of $\bar{r} = 0.7$ indeed means that we reject the null hypothesis of zero phase synchronization. In contrast, for the two remaining conditions the overall conclusions should be formulated more carefully.

The data obtained in our study provided us with the impression that frequency synchronization was not established or established only to a small degree during the 'eyes closed' and spontaneous conditions. This speculation was motivated on two grounds. First, we found that SDfreq for the spontaneous condition was significantly larger than for the intentional condition (Section 3.3; see also Fig. 5). Second, for the case of the 'eye closed' and spontaneous sessions we plotted (figures not shown) power spectra of individual participants. Comparing these spectra (by visual inspection) we observed that oscillation frequencies varied among participants to a relatively large extent — in particular relative to the power spectra as shown in Fig. 6. In view of these findings, we are inclined to say that the observed zero phase synchronization during these two conditions is probably due to a vanishing frequency synchronization. This line of argument is also supported by fundamental models of coupled phase oscillators with different eigenfrequencies. Roughly speaking, when differences between eigenfrequencies become large relative to the coupling strength then synchronization is lost (for more details see [15,30–32]). In closing these considerations, we should point out that the concepts of frequency and phase synchronization are model-based concepts. Using oscillator models we can clearly distinguish between systems exhibiting frequency synchronization but no phase synchronization and systems exhibiting both frequency and phase synchronization. When we are dealing with experimental data and observe overlapping and smeared-out power spectra but no significant phase synchronization as measured by the Kuramoto order parameter, then it is difficult to make a strong claim that there is significant frequency synchronization but not statistically significant phase synchronization. In such cases, one may simply state significant 'synchronization' was not observed. In any case, from an applied point of view we can reject that the multi-particle or multi-agent system under consideration exhibits collective behavior.

Movement coordination and synchronization between people is known to have many positive effects for social interaction, from increasing rapport [33–35] to facilitating the performance of a social task [36–38]. To date, however, nearly all of the research on the emergence of between-person movement synchronization, including the effects of such synchronization on social interaction, has been restricted to two person, so-called 'interpersonal', synchronization (see [39] for a review). Very few research studies have examined the synchronization that occurs between 3 or more individuals and of those studies that have examined group synchronization [40], no objective measures of group synchronization were obtained, meaning that the conclusions drawn from these studies were largely subjective. The significance of the test statistic proposed here is that it provides researchers interested in group synchrony an objective and easy to use method to measure the magnitude of such synchrony and, moreover, to examine whether and how the magnitude of such synchrony influences the 'social dynamics' of group interaction. More generally, the test statistic examined here will aid social scientists in investigating the dynamic time-dependent structure of group behavior, with respect to not only movement synchrony and coordination, but a broad spectrum of human perception and action phenomena.

As mentioned in the introduction, phase synchronization has been observed in a variety of disciplines both in the animate and inanimate world. Therefore, in addition to the example provided in Section 3, there is a wide applicability of the tests for phase synchronization discussed in Section 2. However, in particular in life sciences several synchronization phenomena have been observed in field experiments (synchronized firefly flashings, pedestrian walking, applause) rather than in laboratory experiments. The systems considered in such field experiments are typically composed of a relatively large number of units. Laboratory experiments supporting this kind of field work has typically been carried out by studying the properties of a single unit (e.g., the response of an individual firefly to a light stimulus) or by investigating a dyad (two units including their couplings). The tools developed in the present work support the design of laboratory experiments involving small groups of more than two units. Research on groups goes beyond research on single units and dyads. Such research devoted to study more than two interacting units will contribute to address the general question to what extent our knowledge about dyads and single units can be scaled-up and applied for understanding many-body systems composed of a large number of units.

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