Comparing Classes of Estimators: When does Gradient Descent Beat Ridge Regression in Linear Models?

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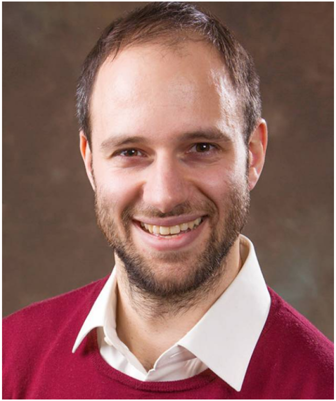

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Overview

Overview

Modern Machine Learning

Modern Machine Learning

- Big Datasets: ImageNet, Common Crawl, MS-Coco, ...
- Big Models: GPT-3, Megatron-Turing NLG, WideResNet, ...

Backbone of many algorithms is Stochastic Gradient Descent

$$\beta_0 \sim P_0 \qquad \beta_{t+1} = \beta_t - \eta \widehat{g}_t \qquad \mathbb{E}[\widehat{g}_t] = \nabla f(\beta_t) \quad t = 0, 1, \ldots$$

Motivations for (Stochastic) Gradient Descent

Can *generally* be split in two directions:

Computation/Approximation

- Cheap gradient estimation through mini-batch sub-sampling
- Backpropagation + auto. diff allow for expressive models
- Accelerated computations with GPUs

Statistics/Regularization (this work)

- Implicit Regularization: GD induces a *good* inductive bias
- Canonical example: on over-parametrized least squares, GD from $\beta_0 = 0$ converges to *Least Norm* interpolating solution

Shrinkage methods for Linear Models

Vast amount of work focusing on (non-sparse) shrinkage methods ...

- **Early works on shrinkage methods** - Tikhonov [1943], Stein [1956], James and Stein [1961], Hoerl and Kennard [1970], Stein [1981], ...
- **Inverse Problems (GD = Landweber Damping)** - Landweber [1951], Engl et al. [1996], Bissantz et al. [2007], Caponnetto and De Vito [2007], Yao et al. [2007], Bauer et al. [2007], Raskutti et al. [2014], Rosasco and Villa [2015], Blanchard and Mücke [2018], Pagliana and Rosasco [2019], Lin et al. [2020], ...

Classical View / Takeaway

Explicit Regularization (e.g., Ridge Regression) is Gold Standard & GD is an Approximation

Gradient Descent “Magic”

More recently, a focus on understanding (S)GD *Magic*

- (Early Stopped Grad Flow + Ridge) $\gg$ Ridge [Skouras et al., 1994]
- Bias towards certain principal components - textbook result, see also Belkin et al, Wu et al. [2020]
- Mild sample inflation: Single-pass SGD vs Ridge - [Zou et al., 2021]
- ...

Our perspective

- Modern methods depend on many tuning parameters (learning rate, batch size, regularization strength). Empirically, performance can depend strongly on them.
- Can we understand theoretically how performance is affected? Especially, what is the *sensitivity to suboptimal hyperparameters?*
- Our work that aims to to answer this (in linear models) by comparing *classes of estimators* (e.g., GD & Ridge)

Overview

Setup

Standard Linear Model: observe features $X \in \mathbb{R}^{n \times d}$, outcome $Y \in \mathbb{R}^n$

$$Y = X\beta_\star + \epsilon$$

Assumptions:

$$\mathbb{E}[\epsilon] = 0, \qquad \mathbb{E}[\epsilon\epsilon^\top] = \sigma^2 I_n, \qquad \mathbb{E}[\beta_\star\beta_\star^\top] = \frac{\psi}{d} I_d, \qquad \epsilon \perp\!\!\!\perp \beta_\star$$

Study average-case behavior $\mathbb{E}_{\beta_\star}[L_{\beta_\star}(\hat{\beta})]$ where $L_{\beta_\star}(\hat{\beta}) := \|\hat{\beta} - \beta_\star\|_2^2$

Random-effects model for $\beta_\star$: average-case analysis over problems where each feature has small effect.

Classes of Estimators

Focusing on two classes of estimators:

- **Ridge Regression**: for *regularization parameters* $\lambda > 0$, minimize $\frac{1}{2n}\|X\beta - Y\|_2^2 + \frac{\lambda}{2}\|\beta\|_2^2$, leading to

$$\widehat{\beta}_\lambda := \left(\frac{X^\top X}{n} + \lambda I_d\right)^{-1} \frac{X^\top Y}{n}$$

- **Gradient Descent**: for $\eta > 0$, $\widehat{\beta}_{\eta,0} = 0$, and for *iterates* $t \geq 0$

$$\begin{aligned}
\widehat{\beta}_{\eta,t+1} &= \widehat{\beta}_{\eta,t} - \eta \nabla_{\widehat{\beta}_{\eta,t}} \left(\frac{1}{2n}\|X\widehat{\beta}_{\eta,t} - Y\|_2^2\right) \\
&= \widehat{\beta}_{\eta,t} - \frac{\eta}{n} X^\top (X\widehat{\beta}_{\eta,t} - Y) \\
&= \sum_{\ell=0}^{t} \eta \left(I_d - \eta \frac{X^\top X}{n}\right)^{\ell} \frac{X^\top Y}{n}
\end{aligned}$$

A key classical Lemma

Optimal amount of ridge regularization $\lambda_\star := \frac{\sigma^2 p}{\psi n} = \frac{1}{SNR}\frac{p}{n}$.
GD (and other estimators $\Phi(\frac{X^\top X}{n})\frac{X^\top Y}{n}$ where $\Phi : \mathbb{R} \to \mathbb{R}$) <u>cannot</u> beat opt. tuned ridge regression in expectation

Lemma

Under the random-effects model, for any $\eta > 0$, $t > 0$

$$\underbrace{\mathbb{E}_{\epsilon,\beta_\star}[L_{\beta_\star}(\widehat{\beta}_{\eta,t})]}_{\text{Error of GD}} \geq \underbrace{\mathbb{E}_{\epsilon,\beta_\star}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]}_{\text{Opt. Tuned Ridge Regression}}$$

But $\lambda_\star$ depends upon the unknown $SNR = \psi/\sigma^2$
In practice *classes of models* are considered, e.g.,

$$\Gamma = \{\widehat{\beta}_\lambda : \lambda \in \{0, \delta, 2\delta, \ldots, 1\}\} \qquad \delta > 0$$

then picking $\operatorname{argmin}_{\widehat{\beta}\in\Gamma} \hat{\mathbb{E}}_{\epsilon,\beta_\star}[L_{\beta_\star}(\widehat{\beta})]$ e.g., via cross-validation

Overview

Minimax Optimality for a *Class of Models*

- Evaluate a class of models $\Gamma = \{\Phi_1, \ldots, \Phi_k\}$, where $\Phi_j : \mathbb{R}_+ \to \mathbb{R}_+$ is a *spectral shrinker*, so $\hat{\beta}_{\Phi_j} = \Phi_j(\frac{X^\top X}{n})\frac{X^\top Y}{n}$, via the smallest excess risk

$$\min_{\hat{\beta}\in\Gamma}\Big(\mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\hat{\beta})] - \mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]\Big)$$

- Consider a parameter space Θ of $\theta = (\psi, \sigma, X)$ (recall $\mathbb{E}[\epsilon\epsilon^\top] = \sigma^2 I_n$, $\mathbb{E}[\|\beta_\star\|^2] = \psi$)
- What is the minimax optimal class of k models? Find

$$\mathcal{M}_k = \inf_{\Gamma=\{\Phi_1,\ldots,\Phi_k\}} \sup_{(\psi,\sigma,X)\in\Theta} \min_{\hat{\beta}\in\Gamma}\Big(\mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\hat{\beta})] - \mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]\Big)$$

Ridge? GD? something else?

Minimax Optimality for Orthogonal Designs

Theorem

Wlog fix $\sigma = 1$. Suppose $\psi \in [\psi_-, \psi_+]$ and $X^\top X/n = I_p$. Then

$$\mathcal{M}_k = \frac{1}{k^2}\left(\frac{1}{\sqrt{1+\psi_-}} - \frac{1}{\sqrt{1+\psi_+}}\right)^2.$$

Optimal class $\Gamma = \{\Phi_1, \ldots, \Phi_k\}$ where $\Phi_j = 1 - \phi_j$ for $j = 1, \ldots, k$ is

$$\phi_j = \left[\frac{1}{x_+} + \left(j - \frac{1}{2}\right) c\right]^2 - \frac{c^2}{4}, \quad c = \frac{1}{k}\left(\frac{1}{x_-} - \frac{1}{x_+}\right), \quad x_\pm = \sqrt{1+\psi_\pm}$$

Excess risks of specific classes

Comparing shrinkage performance with k models

- **Minimax Grid**: We have

$$\text{Minimax excess risk} \sim \frac{1}{k^2}$$

- **Ridge Regression**: with uniform discretization

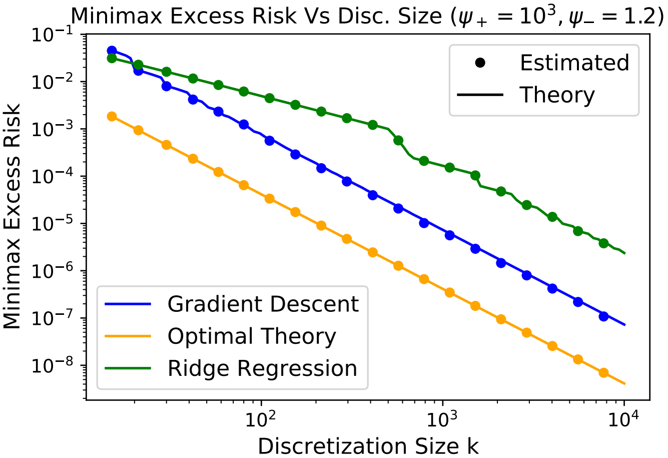
$$\text{Excess risk decays as} \begin{cases} \sim 1/k & \text{for } k \text{ small enough} \\ \sim 1/k^2 & \text{for } k \text{ large enough} \end{cases}$$

  (also have "switching" behavior)

- **Gradient Descent**: With $\eta \sim 1/k$,

$$\text{Excess risk} \sim \frac{1}{k^2}$$

Minimax Excess Risk



- Excess risk of Ridge Regression initially decays at the slow $O(1/k)$ rate
- Excess risks of GD and optimal method both decay at fast $O(1/k^2)$ rate.

Minimax Excess Risk: Proof sketch

- After some algebra, equivalent to

$$\inf_{\phi_1,\ldots,\phi_k\in\mathbb{R}} \sup_{x\in[x_-,x_+]} \min_{j\in[k]} \left| x\phi_j - \frac{1}{x} \right|.$$

- By the monotonicity properties of $g : (0,\infty)\times(0,\infty)\mapsto[0,\infty)$, $g(x,l) = \left|xl - \frac{1}{x}\right|$, for fixed $x_+^{-2} \le \phi_1 \le \ldots \le \phi_k \le x_-^{-2}$, the sup is

$$\max\left(\frac{1}{x_-} - x_-\phi_k, \max_{j=1,\ldots,k-1} \frac{\phi_{j+1}-\phi_j}{\sqrt{2(\phi_{j+1}+\phi_j)}}, x_+\phi_1 - \frac{1}{x_+} \right).$$

- If any two terms are not equal, can decrease the inf by changing some ϕ_j. By the compactness of $x_+^{-2} \le \phi_1 \le \ldots \le \phi_k \le x_-^{-2}$, the min is achieved; and all terms must be equal (say, to γ).
- Solve recursion $\phi_{j+1} - \phi_j = \gamma\cdot\sqrt{2(\phi_{j+1}+\phi_j)}$, with boundary conditions.

Overview

General Design Matrices - Setup

Design Matrix: eigenvalues of $X^\top X/n$ are $s_1 \geq s_2 \geq \cdots \geq s_r > 0$

Assumptions: Suppose $\psi = 1$ and that

$$1 \geq \frac{d}{n}\sigma^2 = \lambda_\star \geq \lambda_{\min} > 0$$

where $\lambda_{\min} > 0$ is known. Equivalently, $\sigma \geq \sigma_{\min} > 0$ where $\sigma_{\min}$ known.[1]

Model Classes: size $k > 1$

$$\mathcal{C}^{\mathrm{GD}}(\eta) := \{\widehat{\beta}_{\eta,j} : 1 \leq j \leq k\},$$
$$\mathcal{C}^{\mathrm{Ridge}}(\lambda_{\min}) := \{\widehat{\beta}_\lambda : \lambda \in \{\lambda_{\min}, \lambda_{\min} + \delta, \lambda_{\min} + 2\delta, \ldots, 1\},\}$$

where $\delta = (1 - \lambda_{\min})/(k-1)$

[1] Differs from previous parametrization with fixed σ, but equivalent to it.

Comparing Classes of Models

Definition

For any two finite classes of estimators $\mathcal{C}_1 = \{\hat{\beta}^u\}_{u \in U_1}$ and $\mathcal{C}_2 = \{\hat{\beta}^u\}_{u \in U_2}$, define the *relative sub-optimality ratio*[a]

$$\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2) := \frac{\min_{\hat{\beta} \in \mathcal{C}_1} \mathbb{E}_{\beta_\star, \epsilon}[L_{\beta_\star}(\hat{\beta})] - \mathbb{E}_{\beta_\star, \epsilon}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]}{\min_{\hat{\beta} \in \mathcal{C}_2} \mathbb{E}_{\beta_\star, \epsilon}[L_{\beta_\star}(\hat{\beta})] - \mathbb{E}_{\beta_\star, \epsilon}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]}.$$

[a] for $x > 0$, $x/0 = \infty$, and $0/0$ is undefined

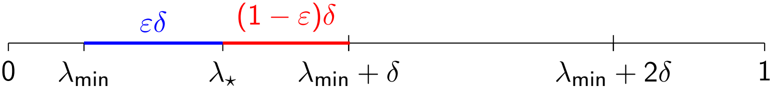
- If $\mathcal{S}(\mathcal{C}_1, \mathcal{C}_2) < 1$, best model in $\mathcal{C}_1$ outperforms best model in $\mathcal{C}_2$.
- Study $\mathcal{S}(\mathcal{C}^{\text{GD}}(\eta), \mathcal{C}^{\text{Ridge}}(\lambda_{\min}))$

General Design Matrices - Setup

- Provide general results based on eigenvalue decay
- Specialize them to slow decay (power law $i^{-\alpha}$, $\alpha < 1$); and fast decay (power law $i^{-\alpha}$, $\alpha > 1$; exponential). Focus presentation on this

General Design Matrices - Slow Decay Result

Let $\Gamma = \{\lambda_{\min}, \lambda_{\min} + \delta, \lambda_{\min} + 2\delta, \ldots, 1\}$ & $\mathrm{Dist}_\delta(\lambda_\star, \Gamma) = \frac{1}{\delta} \min_{\lambda \in \Gamma} |\lambda - \lambda_\star|$



$\mathrm{Dist}_\delta(\lambda_\star, \Gamma) = \min\{\varepsilon, 1-\varepsilon\}$ if $\lambda_\star = \lambda_{\min} + (j+\varepsilon)\delta$ for $j \leq k-2, \varepsilon \in (0,1)$.

Theorem (Slow Decay)

Suppose $s_i = i^{-\alpha}$ for $\alpha \in (0,1)$. There exists $r_{\alpha,\lambda_\star,k,\lambda_{\min}} > 1$ such that if

$$k \gtrsim \frac{1}{\lambda_\star \lambda_{\min}} \log\left(1 + \frac{1}{\lambda_\star}\right) \qquad r \geq r_{\alpha,\lambda_\star,k,\lambda_{\min}}$$

then with $\eta = 1/(k\lambda_{\min})$

$$\mathcal{S}(\mathcal{C}^{\mathrm{GD}}(\eta), \mathcal{C}^{\mathrm{Ridge}}(\lambda_{\min})) \simeq \frac{1}{\mathrm{Dist}_\delta(\lambda_\star, \Gamma)^2} \left(\frac{d}{n}\right)^2 \left(\frac{\sigma^2}{\sigma_{\min}}\right)^4.$$

General Design Matrices w/ Slow Decay - Result

If eigenvalues decay slowly, r is large and $\text{Dist}_\delta(\lambda_\star, \Gamma) = \Omega(1)$ then

$$\mathcal{S}(\mathcal{C}^{\text{GD}}(\eta), \mathcal{C}^{\text{Ridge}}(\lambda_{\min})) \simeq \left(\frac{d}{n}\right)^2 \left(\frac{\sigma^2}{\sigma_{\min}}\right)^4.$$

- If $d \simeq n$ and $\sigma_{\min} = \Omega(\sigma)$ then $\mathcal{S}(\mathcal{C}^{\text{GD}}(\eta), \mathcal{C}^{\text{Ridge}}(\lambda_{\min})) \simeq \sigma^4$
- (roughly) GD outperforms ridge regression in High-Dim. + Low Noise!

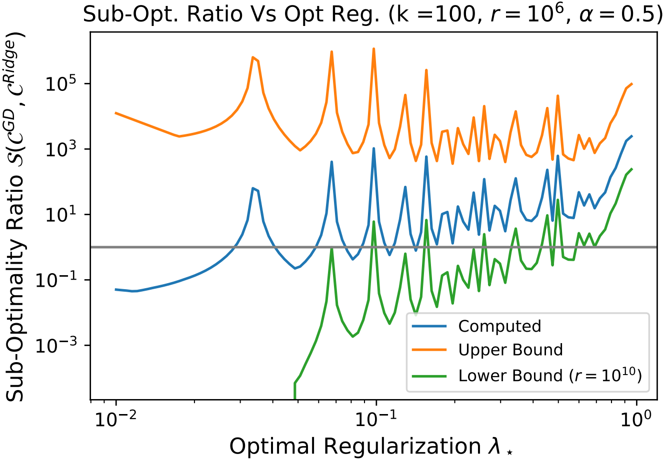
- For instance, if $r = d = n^q$ for $q \in (g(\alpha), 1)$, then[2]

$$\mathcal{S}(\mathcal{C}^{\text{GD}}(\eta), \mathcal{C}^{\text{Ridge}}(\lambda_{\min})) \simeq \frac{(\sigma^2/\sigma_{\min})^4}{n^{2(1-q)}}$$

Tends to zero as $n, d \to \infty$

[2] $g(\alpha) = (\alpha + 1)/(1 + \alpha(2 - \alpha))$

General Design Matrices w/ Slow Decay - Experiment



- Numerically compute $\mathcal{S}(\mathcal{C}^{\text{GD}}(\eta), \mathcal{C}^{\text{Ridge}}(\lambda_{\text{min}}))$ for fixed r.
- Evaluate our upper and lower bounds (constants $= 1$)
- Bounds capture $\nearrow$ trend and spiking due to discretization as a function of $\lambda_\star = p\sigma^2/n$.

General Design Matrices w/ Fast Decay - Result

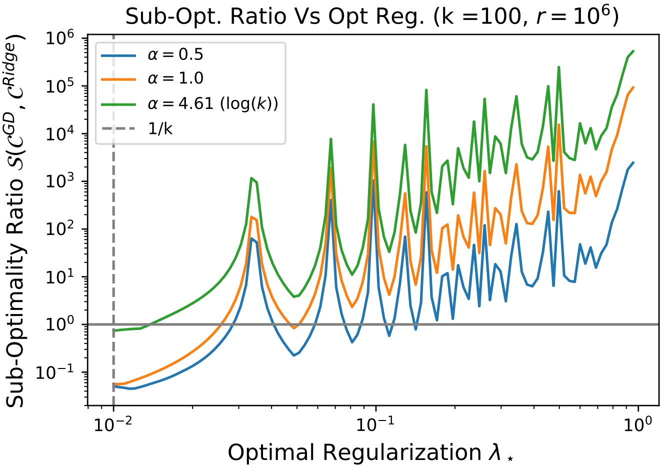
Theorem (Fast Decay)

Suppose $s_i = i^{-\alpha}$ for $\alpha > 1$ and $s_r \leq \lambda_\star$, and $k \gtrsim \frac{1}{\lambda_\star \lambda_{\min}} \log(1 + \frac{1}{\lambda_\star})$. Then with $\eta = 1/(k\lambda_{\min})$

$$\mathcal{S}(\mathcal{C}^{\mathrm{GD}}(\eta), \mathcal{C}^{\mathrm{Ridge}}(\lambda_{\min})) \gtrsim \frac{\exp(-O(\alpha))}{\mathrm{Dist}_\delta(\lambda_\star, \Gamma)^2} \left(\frac{d}{n}\right)^2 k^2 \sigma^4.$$

- Intuition: Ridge Reg. outperforms GD, so $\mathcal{S}(\mathcal{C}^{\mathrm{GD}}(\eta), \mathcal{C}^{\mathrm{Ridge}}(\lambda_{\min})) \gg 1$, when eigenvalues decay quickly.
- When $d \sim n$ and σ is a constant, nontrivial when $1 < \alpha \lesssim \log(k)$
- Also holds for exponentially decaying eigenvalues $s_i = e^{-\rho i}$ for $\rho > 0$.

General Design Matrices - Experiment w/ α



- Numerically compute $\mathcal{S}(\mathcal{C}^{\mathrm{GD}}(\eta), \mathcal{C}^{\mathrm{Ridge}}(\lambda_{\min}))$ for fixed r.
- $\log(k)$ seems to be roughly the threshold where $RR \gg GD$ uniformly for $\lambda_\star \geq 1/k$ (where our theory holds).

Overview

Proof Sketch - Setup

For a class of models $\mathcal{C}$, optimal excess risk is

$$\mathcal{E}(\mathcal{C}) := \min_{\hat{\beta}\in\mathcal{C}} \left\{ \mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\hat{\beta})] - \mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\hat{\beta}_{\lambda_\star})] \right\}.$$

Consider one estimator $\widehat{\beta}_\Phi = \Phi(\frac{X^\top X}{n})\frac{X^\top Y}{n}$. With $M_\Phi(s) = 1 - \Phi(s)s$

$$\mathcal{E}(\widehat{\beta}_\Phi) = \frac{\lambda_\star}{d}\sum_{i=1}^{r}\left(\frac{1}{\lambda_\star} + \frac{1}{s_i}\right)\left(M_\Phi(s_i) - \frac{\lambda_\star}{\lambda_\star + s_i}\right)^2.$$

Consider the classes $\mathcal{C} = (\Phi_1, \ldots, \Phi_k)$ and $\delta = (1 - \lambda_{\min})/(k-1)$

- **Ridge Regression**: $M_{\Phi_j}(s) = \lambda_j/(s + \lambda_j)$ where $\lambda_j = \lambda_{\min} + (j-1)\delta$
- **GD**: $M_{\Phi_j}(s) = (1 - \eta s)^j$ for $j = 1, \ldots, k$.

Proof Sketch - Ridge Regression

Recall that $\lambda_\star = \lambda_{\min} + (j + \varepsilon)\delta$ for some $j \in \{0, 1, \ldots, k-2\}, \varepsilon \in (0,1)$

$$\begin{aligned}
\mathcal{E}(\mathcal{C}^{\text{Ridge}}(\lambda_{\min}, k)) &= \lambda_\star \min_{t=1,\ldots,k} \frac{1}{d} \sum_{i=1}^{r} \left(\frac{1}{\lambda_\star} + \frac{1}{s_i}\right) \left(\frac{\lambda_t}{\lambda_t + s_i} - \frac{\lambda_\star}{\lambda_\star + s_i}\right)^2 \\
&= \lambda_\star \min_{\kappa \in \{-\varepsilon, 1-\varepsilon\}} \frac{1}{d} \sum_{i=1}^{r} \left(\frac{1}{\lambda_\star} + \frac{1}{s_i}\right) \frac{s_i^2 \kappa^2 \delta^2}{(\lambda_\star + \kappa\delta + s_i)^2 (\lambda_\star + s_i)^2}
\end{aligned}$$

Suppose $\varepsilon = 1/2$ and $\delta \leq \lambda_\star$, so the summands are $\frac{s_i}{\lambda_\star} \frac{\delta^2}{(s_i + \lambda_\star)^3}$; then

$$\mathcal{E}(\mathcal{C}^{\text{Ridge}}(\lambda_{\min}, k)) \simeq \frac{\lambda_\star \delta^2}{d} \Bigg(\sum_{s_i > \lambda_\star} \frac{\lambda_\star}{s_i^2} + \underbrace{\sum_{s_i \leq \lambda_\star} \frac{s_i}{\lambda_\star^3}}_{\textbf{Tail of Eigenvalue Distribution}} \Bigg). \tag{1}$$

Proof Sketch - Gradient Descent

Plugging in the class of models

$$\mathcal{E}(\mathcal{C}^{\mathrm{GD}}(\eta, k)) = \lambda_\star \min_{t=1,\dots,k} \frac{1}{d} \sum_{i=1}^{r} \left(\frac{1}{\lambda_\star} + \frac{1}{s_i}\right) \left((1-\eta s_i)^t - \frac{\lambda_\star}{\lambda_\star + s_i}\right)^2 .$$

Switch perspective: define per-eigenvalue optimal number of iterations $t^\star(s) = \log(1 + s/\lambda_\star)/\log(1/(1-\eta s))$ so that

$$(1-\eta s)^{t^\star(s)} = \frac{\lambda_\star}{\lambda_\star + s} \qquad \lim_{s\to 0} t^\star(s) = \frac{1}{\eta\lambda_\star}$$

Choosing global number of iterations $t = \lceil (\eta\lambda_\star)^{-1} \rceil$, can bound

$$\mathcal{E}(\mathcal{C}^{\mathrm{GD}}(\eta, k)) \lesssim \frac{\lambda_\star}{d} \left(\sum_{i:s_i>\lambda_\star} \frac{\lambda_\star}{s_i^2} + \sum_{i:s_i\le\lambda_\star} \frac{1}{s_i} \left(1 - (1-\eta s_i)^{\lceil (\eta\lambda_\star)^{-1}\rceil - t^\star(s_i)}\right)^2 \right)$$

Key calculation: bounding $\lceil (\eta\lambda_\star)^{-1} \rceil - t^\star(s_i)$

Proof Sketch - Gradient Descent

We have

$$\frac{1}{\eta\lambda_\star} - t^\star(s) = t^\star(0) - t^\star(s) \leq \frac{3}{2}\frac{s}{\eta\lambda_\star^2}$$

yielding $1 - (1-\eta s_i)^{\lceil(\eta\lambda_\star)^{-1}\rceil - t^\star(s_i)} \lesssim \eta s(1 + \frac{s}{\eta\lambda_\star^2})$ and thus

$$\mathcal{E}(\mathcal{C}^{\mathrm{GD}}(\eta, k)) \lesssim \frac{\lambda_\star}{d}\left(\sum_{i:s_i>\lambda_\star}\frac{\lambda_\star}{s_i^2} + \sum_{i:s_i\leq\lambda_\star}\max\left\{\eta^2 s_i, \frac{s_i^3}{\lambda_\star^4}\right\}\right). \tag{2}$$

Compare Tail Dependence $s_i < \eta\lambda_\star^2$

- **Ridge Regression**: $\frac{\delta^2 s}{\lambda_\star^3}$
- **Gradient Descent**: $\lambda_\star\eta^2 s$

Eigenvalue decay slow: Dominant error is in the tails, so

$$\frac{\lambda_\star\eta^2 s_i}{(\delta^2 s_i)/(\lambda_\star^3)} \approx \frac{\lambda_\star^4}{k^2\lambda_{\min}^2}\frac{1}{(1/k^2)} = \frac{\lambda_\star^4}{\lambda_{\min}^2} = \left(\frac{d}{n}\right)^2\left(\frac{\sigma^2}{\sigma_{\min}}\right)^2$$

where $\eta = 1/(k\lambda_{\min})$, $\delta = O(1/k)$.

Overview

Conclusion

Aim: compare classes of estimators accounting for (discrete) hyperparameters. Some intriguing results in linear models.

Orthogonal Designs:

- Min-Max-Min excess risk

$$\underbrace{\inf_{\Gamma=\{\Phi_1,\dots,\Phi_k\}}}_{\textbf{Model Class}} \underbrace{\sup_{\psi,\sigma,X}}_{\textbf{Nature}} \underbrace{\min_{\hat\beta\in\Gamma}}_{\textbf{Model Selection}} \underbrace{\left(\mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\hat\beta)] - \mathbb{E}_{\beta_\star,\epsilon}[L_{\beta_\star}(\widehat{\beta}_{\lambda_\star})]\right)}_{\textbf{Excess risk}}$$

- For k models, optimal class and GD: $O(1/k^2)$
- Ridge: “switching” between $O(1/k)$ & $O(1/k^2)$ $\implies$ GD $\gg$ Ridge

General Designs: compare GD & Ridge via relative suboptimality

- Fast eigenvalue decay $1 < \alpha \lesssim \log(k)$: Ridge $\gg$ GD
- Slow eigenvalue decay $0 < \alpha < 1$, small noise, high dim: GD $\gg$ Ridge

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