Consistency of invariance-based randomization tests

Edgar Dobriban
University of Pennsylvania

July 2, 2022

Many thanks

- Conference organizing committe

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- And congratulations to all other awardees!

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- Look forward to more involvement

The paper I am presenting

CONSISTENCY OF INVARIANCE-BASED RANDOMIZATION TESTS

BY EDGAR DOBRIBAN[1]

[1]*Department of Statistics and Data Science, The Wharton School, University of Pennsylvania, dobriban@wharton.upenn.edu*

Invariance-based randomization tests—such as permutation tests, rotation tests, or sign changes—are an important and widely used class of statistical methods. They allow drawing inferences under weak assumptions on the data distribution. Most work focuses on their type I error control properties, while their consistency properties are much less understood.

We develop a general framework and a set of results on the consistency of invariance-based randomization tests in signal-plus-noise models. Our framework is grounded in the deep mathematical area of representation theory. We allow the transforms to be general compact topological groups, such as rotation groups, acting by general linear group representations. We study test statistics with a generalized sub-additivity property.

We apply our framework to a number of fundamental and highly important problems in statistics, including sparse vector detection, testing for low-rank matrices in noise, sparse detection in linear regression, and two-sample testing. Comparing with minimax lower bounds we develop, we find perhaps surprisingly that in some cases, randomization tests detect signals at the minimax optimal rate.

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- This work develops a general framework to study them under the alternative hypothesis (consistency, power)

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  - For instance, compare $T(X_1, \ldots, X_n)$ with $T(X_{\pi(1)}, \ldots, X_{\pi(n)})$ for permutations π, reject if $T(X)$ large enough—Permutation test

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  - For instance, $T(X) > \max(T(G_1X), \ldots, T(G_{19}X))$ happens on average at most 1 out of 20 times (each of the 20 equally likely to be the maximum), so $\alpha = 0.05$ ok.

Flowchart

$$
\begin{tikzcd}
& \text{Test statistic } T(X) \arrow[ddr] & \\
& \text{Test statistic after transform } T(G_1X) \arrow[dr] & \\
\text{Data } X \arrow[uur] \arrow[ur] \arrow[ddr] & & T(X) > q_{1-\alpha}\left(T(X), T(G_1X), \ldots, T(G_KX)\right) \\
& \vdots & \\
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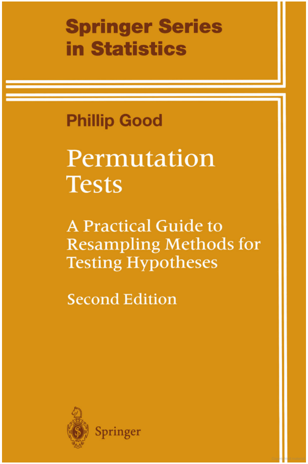
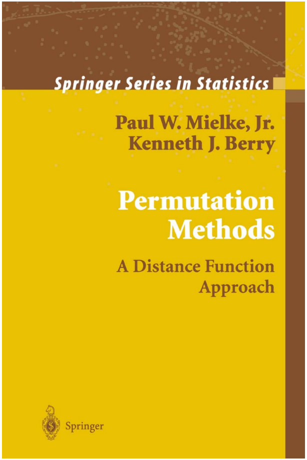
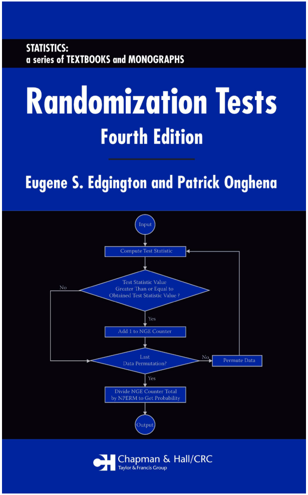
Books on permutation and randomization tests







Randomization tests are important: Neuroscience

Proceedings of the National Academy of Sciences of the United States of America

> Current Issue > vol. 113 no. 28 > Anders Eklund, 7900–7905, doi: 10.1073/pnas.1602413113

Cluster failure: Why fMRI inferences for spatial extent have inflated false-positive rates

Anders Eklund[a,b,c,1], Thomas E. Nichols[d,e], and Hans Knutsson[a,c]

Randomization tests are important: Genomics

Gene set enrichment analysis: A knowledge-based approach for interpreting genome-wide expression profiles

Aravind Subramanian[a,b], Pablo Tamayo[a,b], Vamsi K. Mootha[a,c], Sayan Mukherjee[d], Benjamin L. Ebert[a,e], Michael A. Gillette[a,f], Amanda Paulovich[g], Scott L. Pomeroy[h], Todd R. Golub[a,e], Eric S. Lander[a,c,i,j,k], and Jill P. Mesirov[a,k]

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  - Main surprise: Randomization tests work almost as well as parametric tests, with much fewer assumptions

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- Non-examples: functions of fast growth, e.g., $x \mapsto \exp(x)$

Main result

Theorem (informal). Consider sequence of signal-plus-noise problems with above conditions. Assume there is $t > 0$, such that the following hold:

1. **Strong signal.** For any signal s, there is $\tilde{t}(s) > 0$ such that

$$T(s) > \psi^{-2}\tilde{t}(s) + \psi^{-1}(\psi^{-1} + 1)t. \tag{1}$$

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2. **Noise control.** $P(T(N) \leqslant t) \to 1$ and $P(T(-N) \leqslant t) \to 1$.
3. **Bound on randomized statistic**.
$$P_G\left(T(Gs) \leqslant \tilde{t}(s)\right) \to 1.$$

The randomization test is consistent, i.e.,

$$P_{G_1,\ldots,G_K,X}\left(T(X) > q_{1-\alpha}\left[T(X), T(G_1X), \ldots, T(G_KX)\right]\right) \to 1.$$

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- Proof approach: deterministic separating sequence (between $T(X)$ and randomized $T(GX)$)
- Examples
  - Detecting sparse vectors
  - Detecting low-rank matrices
  - Sparse detection in linear regression

Follow-up work inspired by ours

FASTER EXACT PERMUTATION TESTING: USING A REPRESENTATIVE SUBGROUP

BY NICK W. KONING[1], JESSE HEMERIK[2]

[1]*Econometric Institute, Erasmus University Rotterdam, n.w.koning@ese.eur.nl*

[2]*Biometris, Wageningen University & Research, jesse.hemerik@wur.nl*

Abstract

Non-parametric tests based on permutation, rotation or sign-flipping are examples of so-called group-invariance tests. These tests rely on invariance of the null distribution under a set of transformations that has a group structure, in the algebraic sense. Such groups are often huge, which makes it computationally infeasible to use the entire group. Hence, it is standard practice to test using a randomly sampled set of transformations from the group. This random sample still needs to be substantial to obtain good power and replicability. We improve upon the standard practice by using a well-designed subgroup of transformations instead of a random sample. We show this can yield a more powerful and fully replicable test with the same number of transformations. For a normal location model and a particular design of the subgroup, we show that the power improvement is equivalent to the power difference between a Monte Carlo Z-test and Monte Carlo t-test. In our simulations, we find that we can obtain the same power as a test based on sampling with just half the number of transformations, or equivalently, more power for the same computation time. These benefits come entirely 'for free', as our methodology relies on an assumption of invariance under the subgroup, which is implied by invariance under the entire group.

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  - Valid under weak non-parametric assumption of symmetry; typically not true of tests with critical values chosen via parametric models
- Result: consistent when $\|s\|_\infty > 2(1+\varepsilon) \cdot \|n^{-1} \sum_{i=1}^n N_i\|_\infty$ with probability $\to 1$; for any $\varepsilon > 0$.

Numerical example

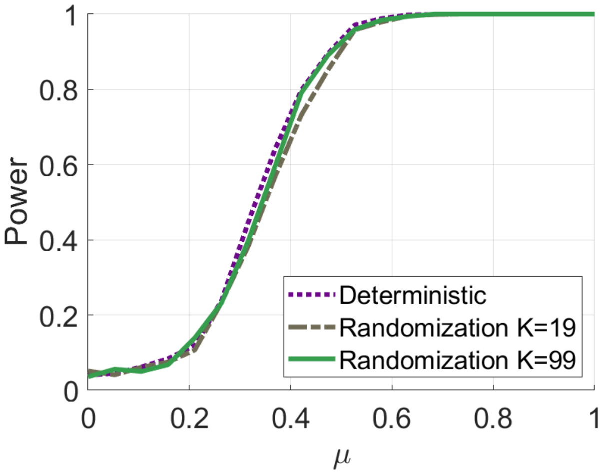

Figure: Evaluating the power of a randomization test in comparison with the deterministic test as a function of signal strength in sparse vector detection: standard normal noise, $n = p = 100$, $s = (\mu, 0, \ldots, 0)^\top$, $\alpha = 0.05$, 1000 repetitions.

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- Noise with finite Ψ-Orlicz norm $\Psi(x) = \exp(\ln[x+1]^\kappa) - 1$ (tail decay $\sim \exp(-\ln[x+1]^\kappa)$), $s \sim \exp[(\log p)^{1/\kappa}]/\sqrt{n}$.

Intriguing numerical example: two-sample t-test

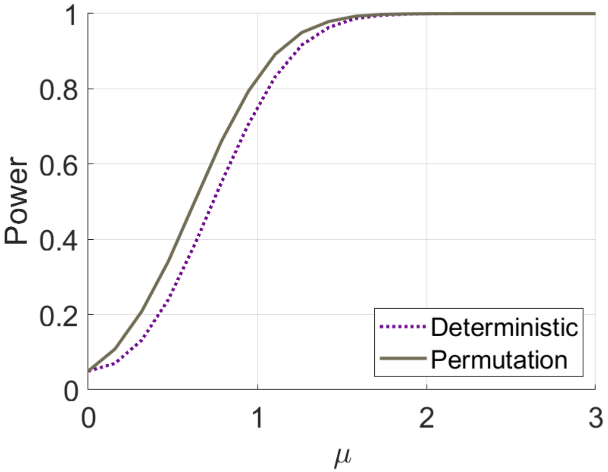

Figure: Evaluating the power of the permutation test in comparison with the deterministic test as a function of signal strength in two-sample testing, using t-test: standard normal noise, $n_1 = n_2 = 15$, $p = 1$, $s_1 - s_2 = (\mu, 0, \ldots, 0)^\top$, $\alpha = 0.05$, $K = 99$, 100,000 repetitions.

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- Thank you!