On the statistical foundations of adversarially robust learning

Edgar Dobriban
Wharton, UPenn
joint work with Hamed Hassani, David Hong, and Alex Robey
slides at: github.com/dobriban/talks

October 27, 2020

Overview

Overview

Statistics in the present era

- Our age is characterized by a fast-growing interest in leveraging data
- Data Science, Analytics, AI, Machine Learning, Big Data ... Statistics?
- Dean Erika James:

  "Analytics and AI will be central to business going forward. Need to make sure we are world leaders. Business Analytics is the fastest growing concentation."

- Wharton initiatives: Analytics at Wharton ($5M), AI at Wharton ($5M)
- Penn initiatives: SEAS Data Science ($25M), AI Major
- National initiatives:
  1. National AI R&D Plan: "AI as an Administration Priority"
  2. Nat'l AI Institutes ($250M); NSF-Simons Collab ($20M); ...
- Statistics is key to all this

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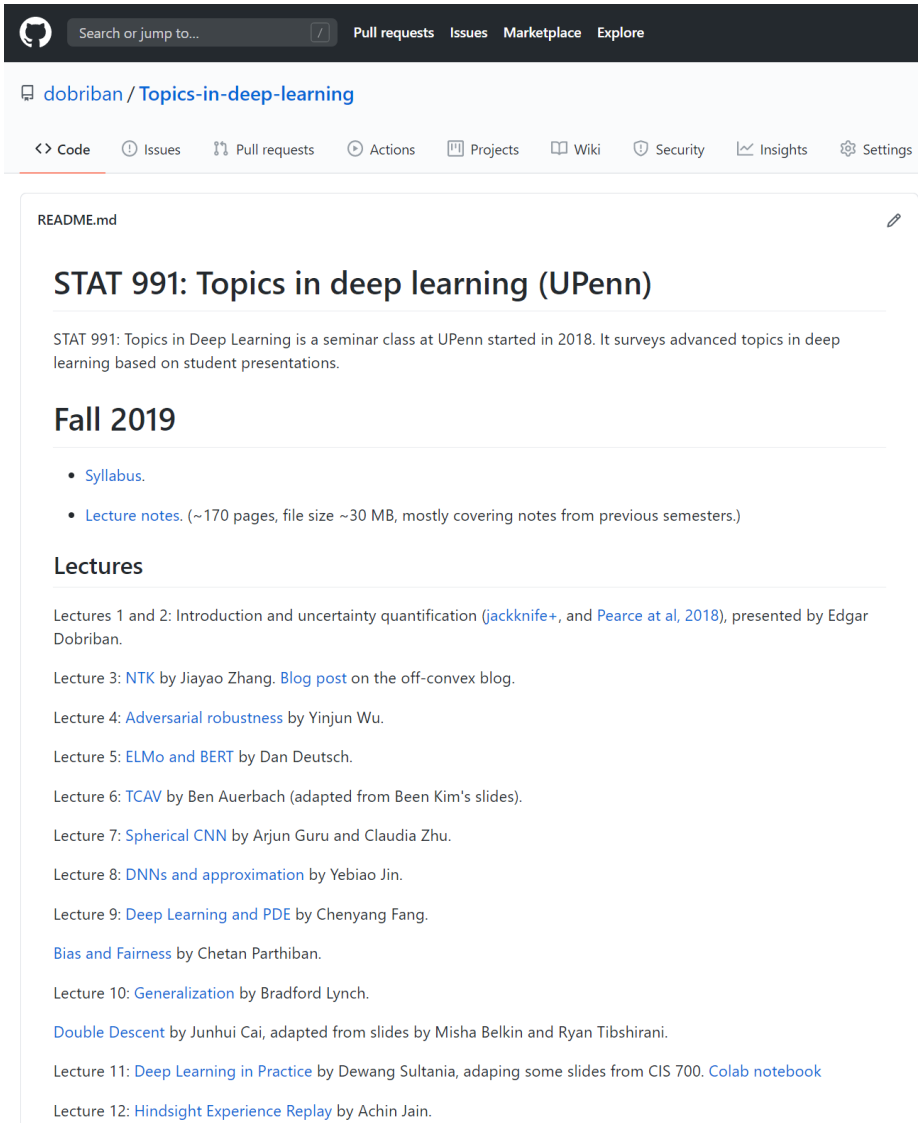
Some activities



Notes for Stat 991: Topics in Deep Learning

Edgar Dobriban*

January 28, 2020

Abstract

Deep learning has achieved many empirical successes, and has attracted considerable attention. However, it continues to be poorly understood. This advanced seminar course will explore several topics in deep learning. We will discuss both theory and applications.

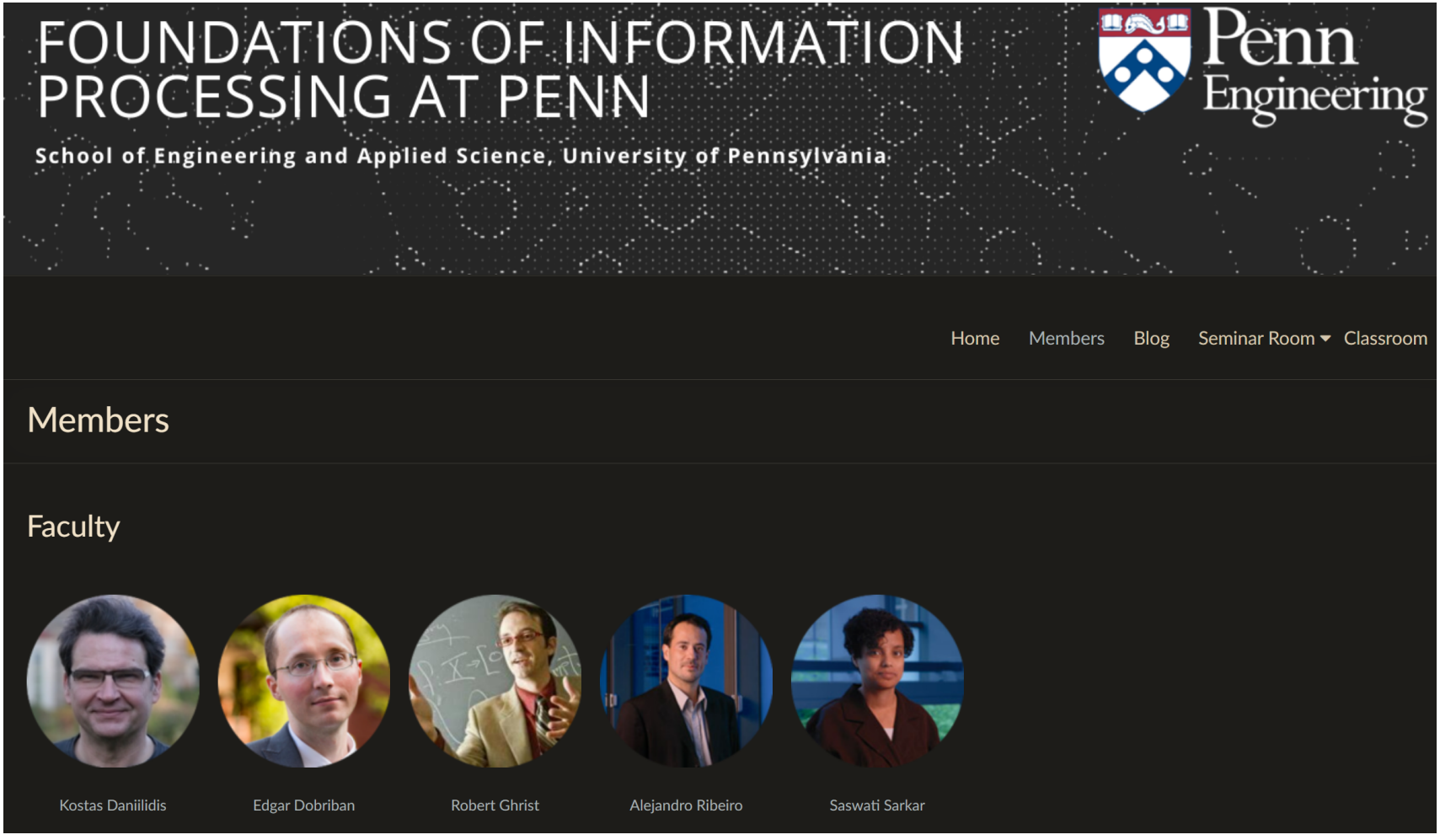
Contents

*Wharton Statistics Department, University of Pennsylvania. dobriban@wharton.upenn.edu. These notes draw inspiration from many sources, including David Donoho's course Stat 385 at Stanford, Andrew Ng's Deep Learning course on deeplearning.ai, David Silver's RL course, Tony Cai's reading group at Wharton. Disclaimer: the notes may contain factual and typographical errors. Thanks to several people who have provided parts of the notes, including Zongyu Dai, Georgios Kissas, Jane Lee, Barry Plunkett, Matteo Sordello, Yibo Yang, Bo Zhang, Yi Zhang, Carolina Zheng. The images included in this note are subject to copyright by their rightful owners, and are included here for educational purposes.

Figure: 170pg. notes

Some activities: GPU Machines

Some activities



Some activities

Friday, September 4, 2020

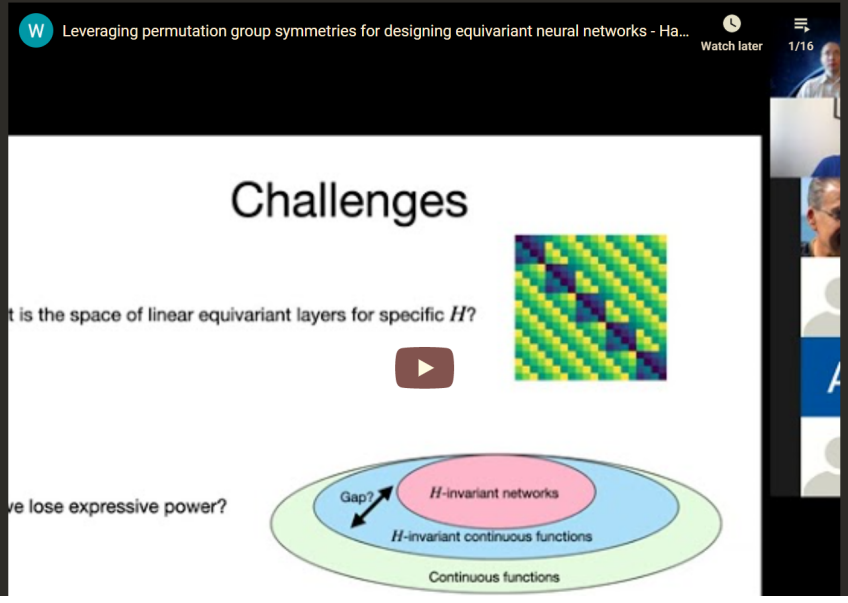
Workshop on Equivariance and Data Augmentation

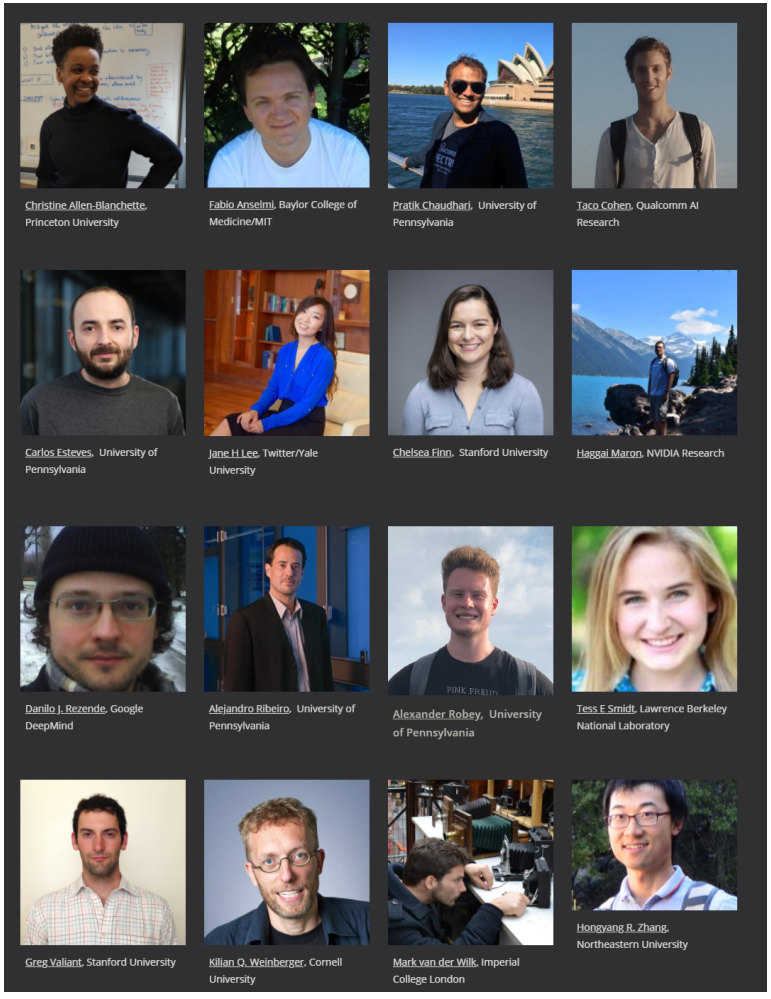
online, hosted by the University of Pennsylvania

Exploiting symmetry in structured data is a powerful way to improve the learning and generalization ability of AI systems, and extract higher quality information, in applications ranging from vision, imaging, and NLP to robotics. This is exemplified by convolutional neural nets, which are an ubiquitous architecture. Recently, there has been a great deal of progress to develop improved equivariant and invariant learning architectures, as well as improved data augmentation methods. There has also been progress on the theoretical foundations of the area, from the perspectives of statistics and optimization. The notion of adding data via data augmentation also arises in problems such as adversarial robustness. This workshop will bring together leading researchers in the area to discuss the state of the art of the field. The activity is part of the Center for Foundations of Information Processing at Penn, supported by NSF TRIPODS.

Due to Covid-19, the workshop will be held online as a Zoom webinar. While this means less in-person interaction, it also means that the talks are accessible to everyone for free, without the costs of travel.

Recorded talks are available as a Youtube playlist. Individual videos are also linked below. The link below takes you to the playlist in the "play all" mode.





Some activities

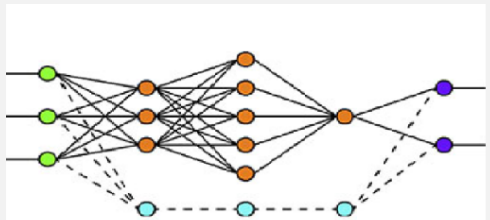
National Science Foundation
WHERE DISCOVERIES BEGIN

Search

Announcement

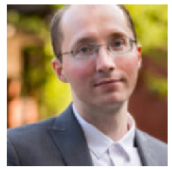
NSF and the Simons Foundation partner to uncover foundations of deep learning

Mathematicians, statisticians, engineers and theoretical computer scientists come together to work on deep learning and AI



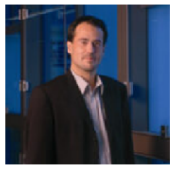

Emmanuel Candès (Stanford)


Ingrid Daubechies (Duke)


Edgar Dobriban (UPenn)


Rong Ge (Duke)


George Pappas (UPenn)


Alejandro Ribeiro (UPenn)


Guillermo Sapiro (Duke)


Shankar Sastry (UC Berkeley)

Some activities

C3.ai DTI

OVERVIEW ABOUT RESEARCH EVENTS NEWS CONTACT

Workshops

The Analytical Foundations of Deep Learning: Interpretability and Performance Guarantees

October 19-21 & 23, 2020

9 am to 2 pm PT (Noon to 5 pm ET) Daily

Attend: Zoom Meeting (You may need to authenticate through your institution's Zoom account before joining)

Watch: YouTube Channel (From our YouTube Channel select the "Live Now" stream)

Day 4 (Friday, Oct. 23): Brainstorm and Discussion

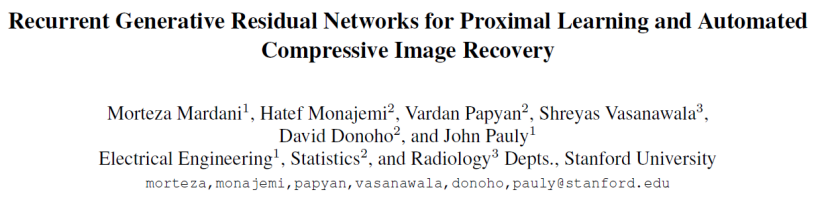
9 am – 10:30 am: Robustness

Lead: Edgar Dobriban (University of Pennsylvania)

Participants: Sebastien Bubeck (Microsoft Research), Jinghui Chen (University of California, Los Angeles), Soheil Feizi (University of Maryland), Micah Goldblum (University of Maryland), Zico Kolter (Carnegie Mellon University), Omar Montasser (Toyota Technological Institute at Chicago), Cyrus Rashtchian (University of California, San Diego), Aditi Raghunathan (Stanford University), Alex Robey (University of Pennsylvania), Chong You (University of California, Berkeley), Hongyang Zhang (Toyota Technological Institute at Chicago)

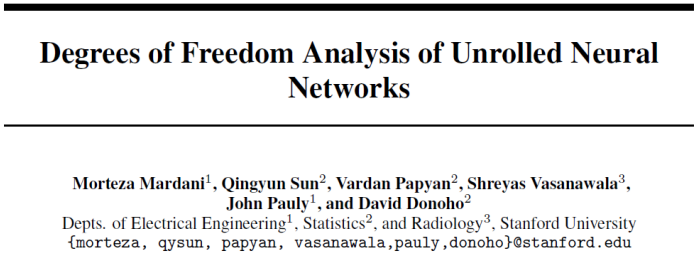
Some considerations

- Criticism of deep learning (resource inefficiency etc) by prominent statisticians (e.g., Donoho)... followed by work from the same people



Recurrent Generative Residual Networks for Proximal Learning and Automated Compressive Image Recovery

Morteza Mardani[1], Hatef Monajemi[2], Vardan Papyan[2], Shreyas Vasanawala[3], David Donoho[2], and John Pauly[1]
Electrical Engineering[1], Statistics[2], and Radiology[3] Depts., Stanford University
morteza,monajemi,papyan,vasanawala,donoho,pauly@stanford.edu



Degrees of Freedom Analysis of Unrolled Neural Networks

Morteza Mardani[1], Qingyun Sun[2], Vardan Papyan[2], Shreyas Vasanawala[3], John Pauly[1], and David Donoho[2]
Depts. of Electrical Engineering[1], Statistics[2], and Radiology[3], Stanford University
{morteza, qysun, papyan, vasanawala,pauly,donoho}@stanford.edu

Neural Proximal Gradient Descent for Compressive Imaging

Morteza Mardani[1], Qingyun Sun[4], Shreyas Vasawanala[2], Vardan Papyan[3], Hatef Monajemi[3], John Pauly[1], and David Donoho[3]
Depts. of [1]Electrical Eng., [2]Radiology, [3]Statistics, and [4]Mathematics; Stanford University
morteza,qysun,vasanawala,papyan,monajemi,pauly,donoho@stanford.edu

Prevalence of neural collapse during the terminal phase of deep learning training

Vardan Papyan[a,1], X. Y. Han[b,1], and David L. Donoho[a,2]

[a]Department of Statistics, Stanford University, Stanford, CA 94305-4065; and [b]School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14850

Some takeaways

- To ensure that our voice is heard, and that proper methods are used, statisticians need to be involved
- Our contributions are encouraged and valued
- There is a learning curve, and not everyone is welcoming (but worth it)

Overview

This talk

- Emerging area of adversarial robustness in AI/Machine Learning
- Answers to some fundamental statistical questions

Robustness

- Want our results to not be affected by accidental perturbations of the data
- Long history: median (middle ages), L_1 regression (Boscovich, Laplace), M-estimation (Huber, Rousseeuw, ...), ...
- This talk: test-time (adversarial) robustness of prediction methods (?2004–2013–...)

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Adversarial robustness

- Let $f(x, \hat{\theta})$ be a predictor (classifier, regression function) estimated/learned from data
- For a new test datapoint x, we predict $\hat{y}(x) = f(x, \hat{\theta})$
- Suppose an adversary perturbs $x \to x' = x + \delta$ for some "small" adversarially chosen δ
- Want prediction $\hat{y}(x')$ to be robust/stable (not change much)

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Modern methods are not robust

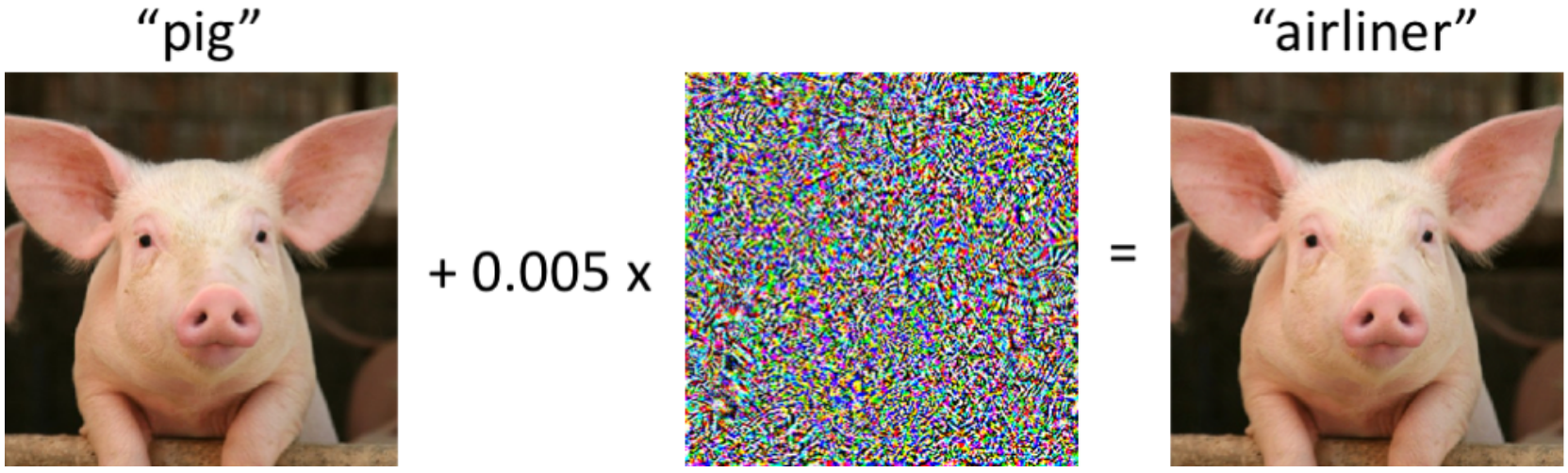

Figure: `gradientscience.org/intro_adversarial/`, by Madry & Schmidt.

- Key challenge in deploying learning algorithms for real problems

Modern methods are not robust: Adversarial patch

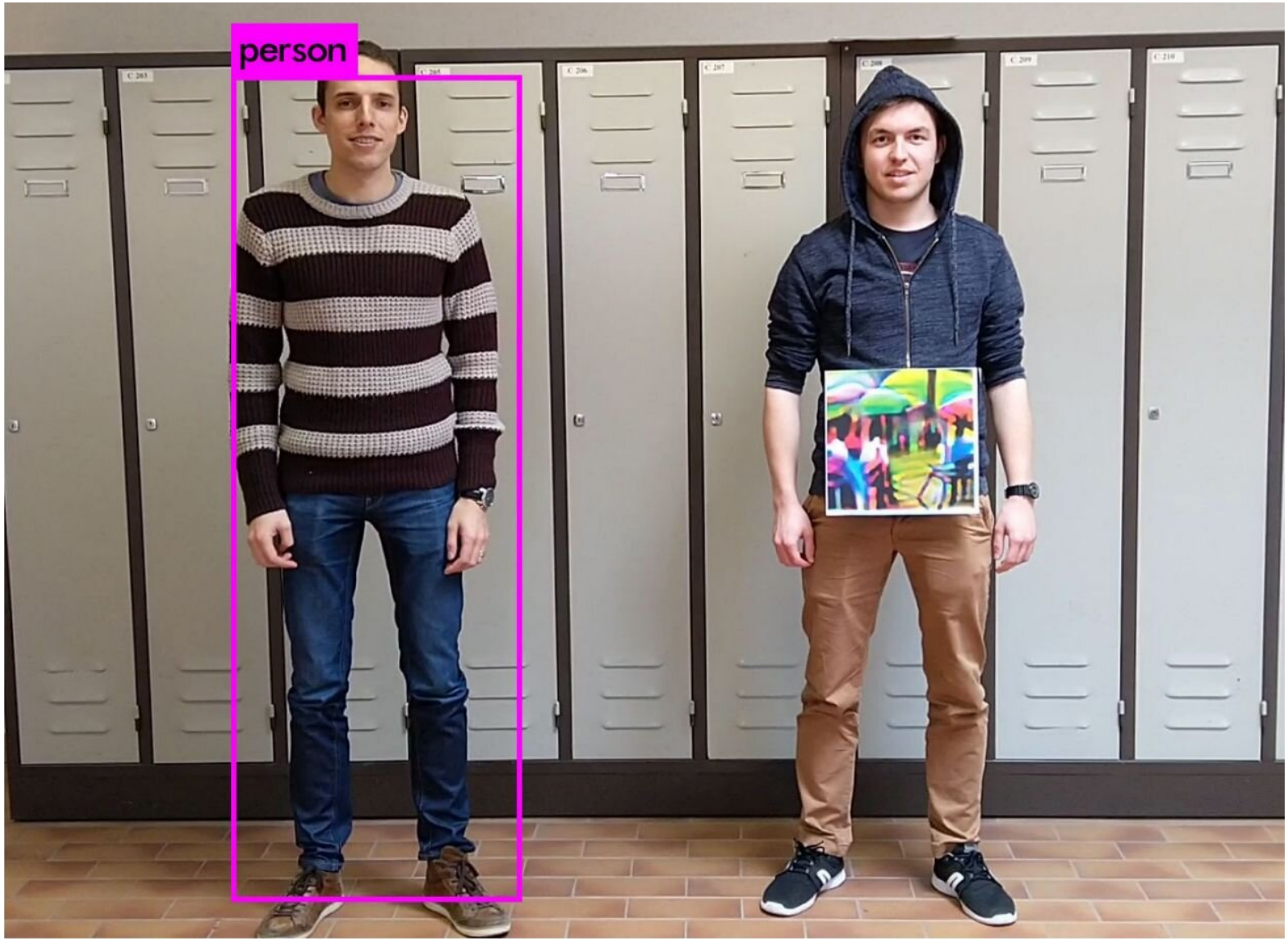

Figure: Thys et al. Fooling automated surveillance cameras: adversarial patches to attack person detection, arXiv:1904.08653

Unprecedented interest

Towards deep learning models resistant to adversarial attacks

A Madry, A Makelov, L Schmidt, D Tsipras… - arXiv preprint arXiv …, 2017 - arxiv.org

Recent work has demonstrated that deep neural networks are vulnerable to adversarial examples---inputs that are almost indistinguishable from natural data and yet classified incorrectly by the network. In fact, some of the latest findings suggest that the existence of …

☆ Cited by 2234 Related articles All 7 versions »

Key questions

- Is there a fundamental (unavoidable) tradeoff between robustness and accuracy?
- What are the optimally robust statistical learning methods?
- How does this depend on the data distribution?

Overview

Model

- Classification problem: features $x \in \mathbb{R}^p$, label $y \in \{\pm 1\}$.
- Classification error:

$$\Pr_{x,y}\{\hat{y}(x) \neq y\} = \mathbb{E}_x \Pr_{y|x}\{\hat{y}(x) \neq y\}$$

- At test time adversary perturbs input by δ, where $\|\delta\| \leq \varepsilon$, ($\|\cdot\|$ is norm).
- Goal: construct classifier $\hat{y} : \mathbb{R}^p \to \{\pm 1\}$ minimizing robust risk: probability of mistake after adversarial perturbation

$$R(\hat{y}, \varepsilon, \|\cdot\|) = \Pr_{x,y}\{\exists_{\delta:\|\delta\|\leq\varepsilon}\ \hat{y}(x+\delta) \neq y\}$$

- Standard risk: $\varepsilon = 0$: $R_s(\hat{y}) = R(\hat{y}, 0, \|\cdot\|) = \Pr_{x,y}\{\hat{y}(x) \neq y\}$
- Bayes optimal classifier: optimal choice for each x

$$\hat{y}(x) \in \operatorname{argmax}_{c \in C} \Pr_{y|x}(y = c)$$

- *Cannot* make separate optimal choices for each x when $\varepsilon > 0$

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Geometry of robust classifiers

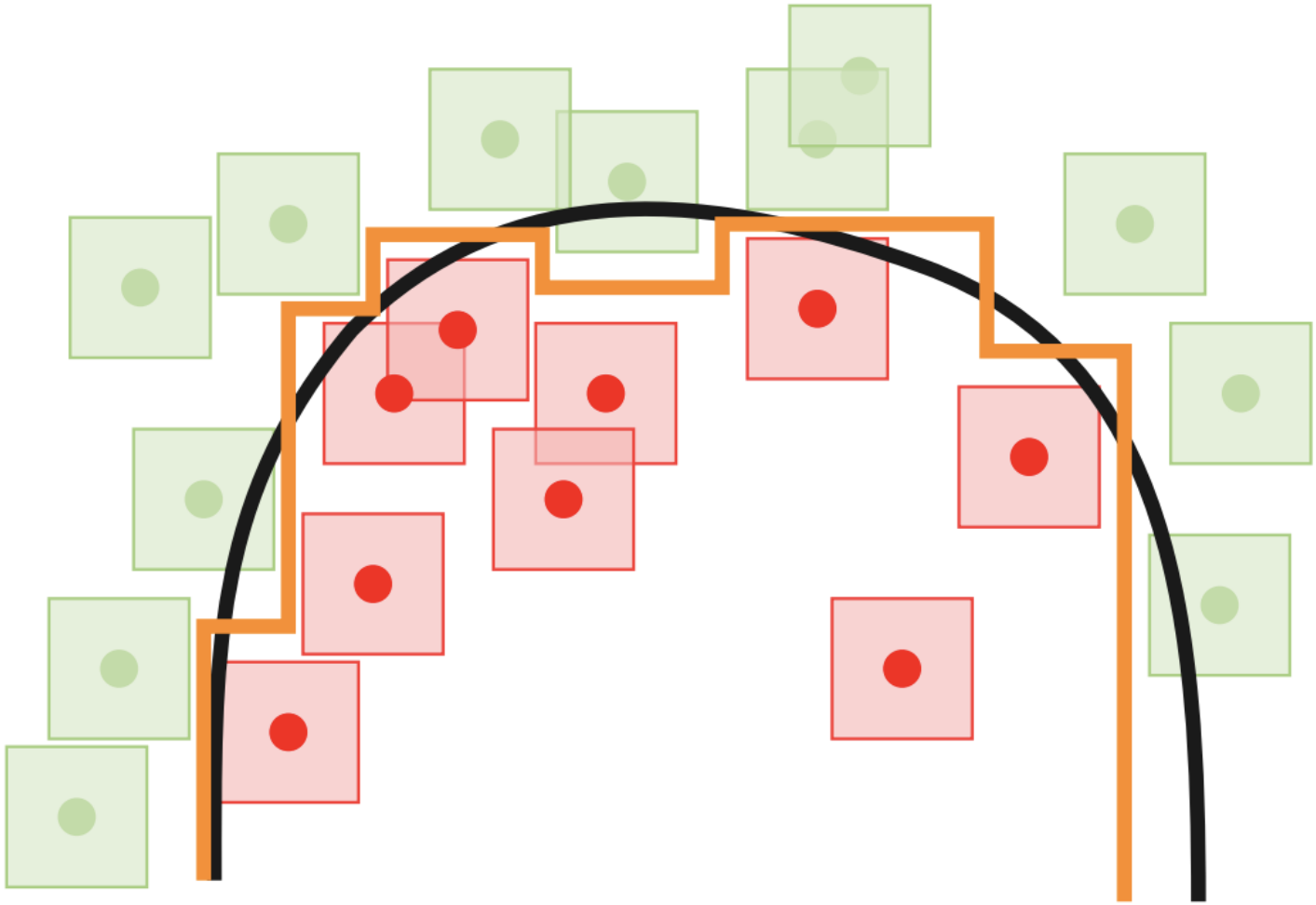


Figure: Figure from Yang el al., 2020

Two-class Gaussian Model

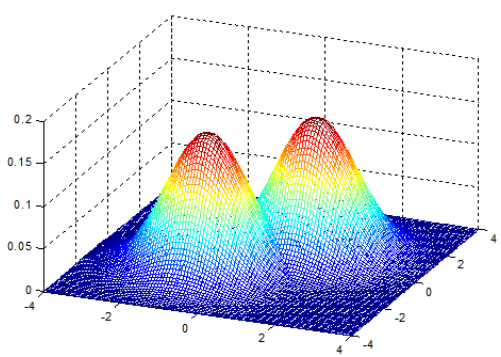
- $y \in \{\pm 1\}$, $P(y = 1) = \pi$
- Isotropic Gaussian class-conditional distribution (wlog $\sigma^2 = 1$):

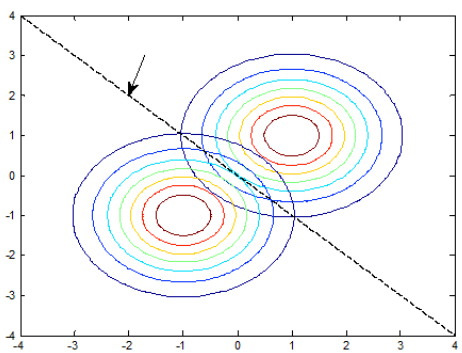
$$x|y \sim \mathcal{N}(y\mu, \sigma^2 I_p),$$

- Bayes optimal (non-robust) classifier is linear (special case of Fisher's LDA):

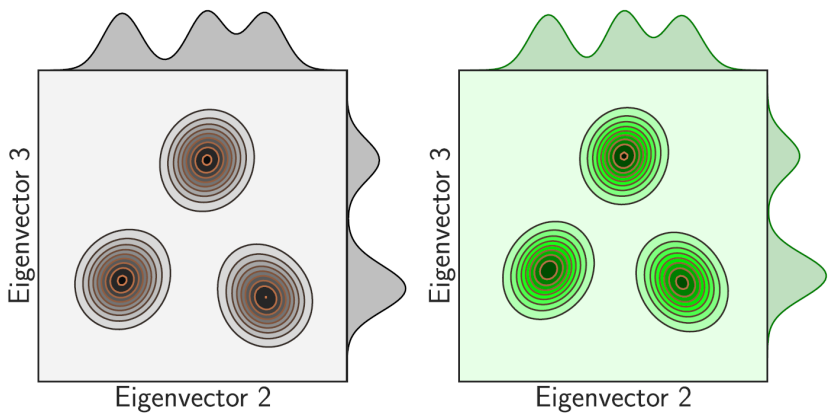
$$\hat{y}_b(x) = \text{argmax}_{c \in C} \Pr_{y|x}(y = c) = \text{sign}(x^\top \mu - q/2),$$

$q := \ln\{(1 - \pi)/\pi\}$ is the log-odds-ratio





(a) Puente et al., 2010



(b) Seddik et al. 2020: GAN $\approx$ GMM

Two-class Gaussian Model

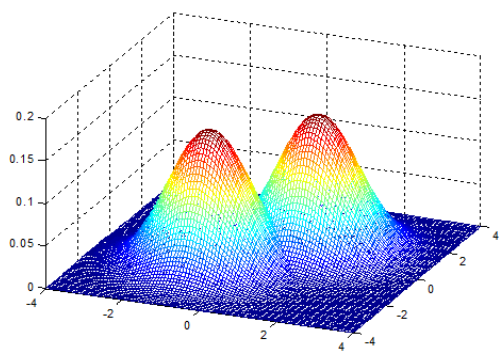
- $y \in \{\pm 1\}$, $P(y = 1) = \pi$
- Isotropic Gaussian class-conditional distribution (wlog $\sigma^2 = 1$):

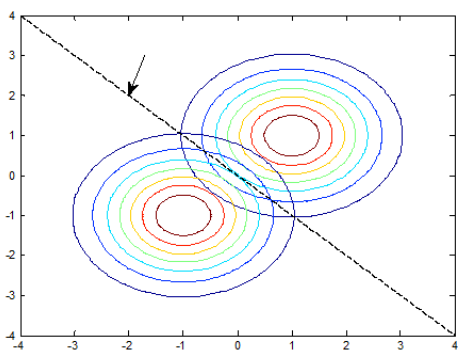
$$x|y \sim \mathcal{N}(y\mu, \sigma^2 I_p),$$

- Bayes optimal (non-robust) classifier is linear (special case of Fisher's LDA):

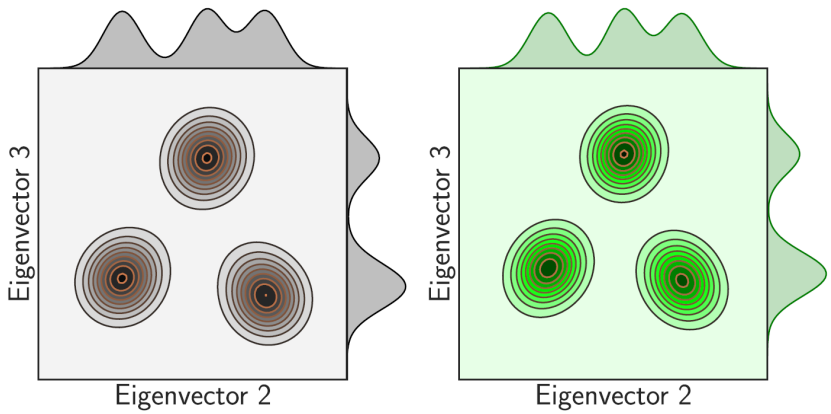
$$\hat{y}_b(x) = \text{argmax}_{c \in C} \Pr_{y|x}(y = c) = \text{sign}(x^\top \mu - q/2),$$

$q := \ln\{(1 - \pi)/\pi\}$ is the log-odds-ratio





(a) Puente et al., 2010



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Two-class Gaussian Model

- Bayes risk

$$R_{\mathrm{Bay}}(\mu, \pi) = \pi \cdot \Phi\left(\frac{q}{2\|\mu\|_2} - \|\mu\|_2\right) + (1-\pi) \cdot \bar{\Phi}\left(\frac{q}{2\|\mu\|_2} + \|\mu\|_2\right),$$

  Φ normal c.d.f, $\bar{\Phi} := 1 - \Phi$, normal tail d.f.

- Two-class Gaussian model fundamental and has rich history (Pearson, Fisher, Wald, ...). Serves as basis for many modern high-dim developments

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Questions

- Can we find the optimal robust classifiers?
- Challenge: decisions made at each point influence neighborhood, does not decouple
- Tradeoffs? Dependence on data distribution?

Optimal ℓ_2-robust two-class classifiers

Take $\|\cdot\| = \|\cdot\|_2$ to be the ℓ_2 norm.

Theorem

In the two-class Gaussian model, the optimal ℓ_2 robust classifier is the linear classifier:

$$\hat{y}^*(x) := \text{sign}\left\{x^\top \mu \left(1 - \frac{\varepsilon}{\|\mu\|_2}\right)_+ - q/2\right\},$$

where $q = \ln\{(1-\pi)/\pi\}$ and $(x)_+ = \max(x, 0)$. Moreover, the corresponding optimal robust risk is

$$R^*_{\text{rob}}(\mu, \pi; \varepsilon) := R_{\text{Bay}}\left\{\mu\left(1 - \frac{\varepsilon}{\|\mu\|_2}\right)_+, \pi\right\}.$$

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Optimal ℓ_2-robust two-class classifiers: Discussion

- ℓ_2 robust problem equivalent to standard prob. with reduced effective mean $\mu \mapsto \mu \left(1 - \frac{\varepsilon}{\|\mu\|_2}\right)_+$
  - $\|\mu\|_2 \geq \varepsilon \implies$ nontrivial classification impossible
  - consistent with "adv. rob. generalization requires more data" (Schmidt et al., 2018)
- also equivalent to standard prob. with amplified class imbalance $q \mapsto q / \left(1 - \frac{\varepsilon}{\|\mu\|_2}\right)_+$

$$\hat{y}^*(x) = \operatorname{sign}\left\{ x^\top \mu - \frac{q}{2\left(1 - \frac{\varepsilon}{\|\mu\|_2}\right)_+} \right\},$$

  1. robustness requires predicting majority $\hat{y}$ more often
- subtle tradeoff depends on class balance:
  - classes balanced ($q = \ln\{(1-\pi)/\pi\} = 0$): same classifier optimal for all $\varepsilon \geq 0$; no tradeoff
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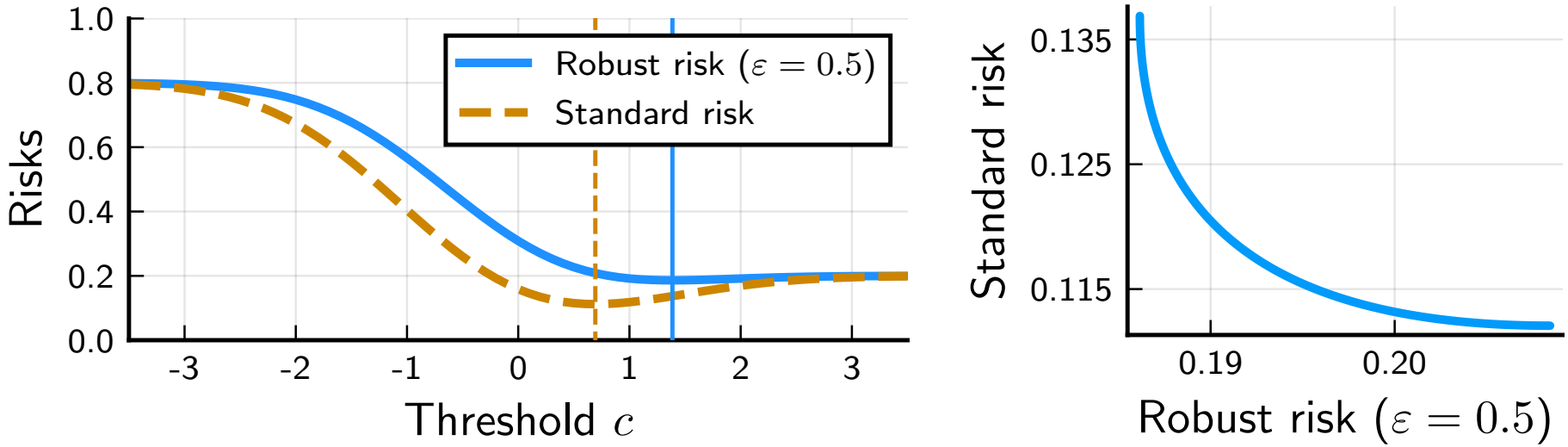
Tradeoffs between robustness and accuracy



(a) Risks as functions of threshold c; vertical lines at optimal thresholds.

(b) Standard v.s. robust risk for thresholds between minima.

Figure: Tradeoffs between standard and robust risks. $\|\mu\|_2 = 1$, $\pi = 0.2$, $\varepsilon = 0.5$. Risks of the linear classifier $\hat{y}(x) = \text{sign}(x^\top \mu - c)$ as a function of the threshold c.

Optimal ℓ_2-robust two-class classifiers: Discussion

- Robust problem equivalent to adding noise

$$\mathcal{N}\left(0, \left[\frac{1}{\left(1-\frac{\varepsilon}{\|\mu\|_2}\right)_+} - 1\right] \cdot I_p\right)$$

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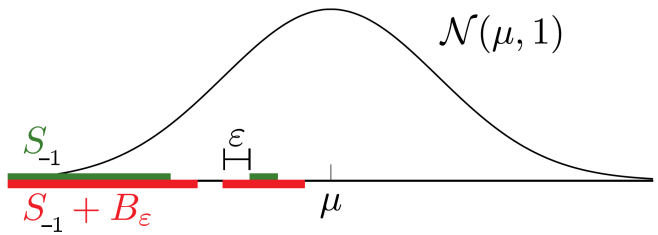
Optimal ℓ_2-robust two-class classifiers: Proof (wlog $\mu > 0$)

$$R(\hat{y}) = \pi \cdot P_{x|y=+1}(S_{-1} + B_\varepsilon) + (1-\pi) \cdot P_{x|y=-1}(S_{+1} + B_\varepsilon)$$
$$S_{\pm 1} = \{x : \hat{y}(x) = \pm 1\}$$



- Step 1: linear classifiers are admissible
  - can project to 1-D
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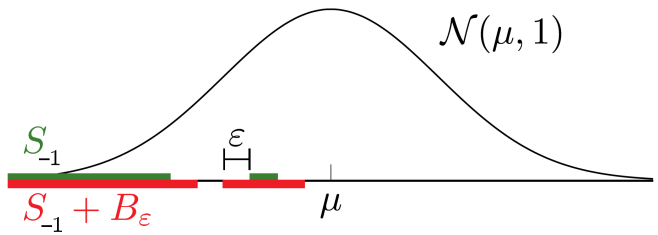

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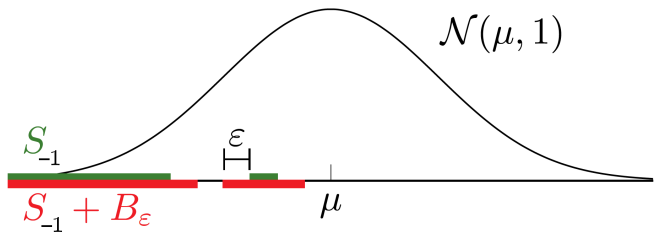

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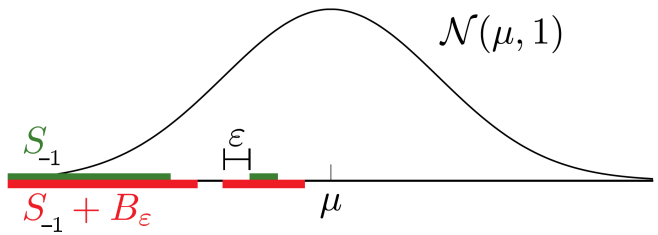

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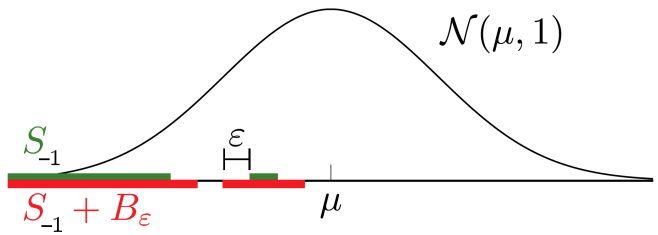

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Optimal ℓ_2-robust two-class classifiers: Extensions

- Weighted combinations: $R(\hat{y}, Q) = \mathbb{E}_{\varepsilon \sim Q} R(\hat{y}, \varepsilon)$. e.g., $R(\hat{y}, 0) + \lambda \cdot R(\hat{y}, \varepsilon)$ for $\lambda > 0$.
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Approximately optimal ℓ_2-robust two-class classifiers

- Important to know what the approximately optimal classifiers are. But: seems very hard
- For now, results in 1-D, and special classifiers
- γ: Gaussian measure
- $\gamma^*(S)$ is Gaussian deficit of S; approximability by half-lines

$$\gamma^*(S) := \inf_{\text{half-lines } H} \gamma(S \Delta H)$$

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Approximately optimal robust classifiers

1. Two-class Gaussian problem in 1-D
2. Classifiers whose classification regions $S_\pm$ are unions of intervals with endpoints in $[-M, M]$
3. ε is less than the half-width of all intervals

Theorem (All robust classifiers are close to linear)

Define $\tau = \tau(\varepsilon, M, \mu) = \varepsilon \exp\{-[(M+\mu)\varepsilon + \varepsilon^2/2]\}$. *Then, for some universal constant* $c > 0$,

$$R_{\text{rob}}(\hat{y}, \varepsilon) \geq R_{\text{Bay}} + \tau \cdot c \cdot [\pi \cdot \gamma^*(S_- - \mu)^2 + (1-\pi) \cdot \gamma^*(S_+ + \mu)^2].$$

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Approximately optimal robust classifiers: Proof

- Hinges on recent results from robust isoperimetry (Cianchi et al., 2011; Mossel and Neeman, 2015).
- "If a set is approximately isoperimetric (its boundary measure is close to minimal for its volume), then it is close to a hyperplane"
- Have not been used in statistics/machine learning/data science before?

Approximately optimal robust classifiers: Tools

The Annals of Probability
2015, Vol. 43, No. 3, 971–991
DOI: 10.1214/13-AOP860

ROBUST DIMENSION FREE ISOPERIMETRY IN GAUSSIAN SPACE[1]

By Elchanan Mossel and Joe Neeman

University of California, Berkeley

We prove the first robust dimension free isoperimetric result for the standard Gaussian measure γ_n and the corresponding boundary measure γ_n^+ in $\mathbb{R}^n$. The main result in the theory of Gaussian isoperimetry (proven in the 1970s by Sudakov and Tsirelson, and independently by Borell) states that if $\gamma_n(A) = 1/2$ then the surface area of A is bounded by the surface area of a half-space with the same measure, $\gamma_n^+(A) \leq (2\pi)^{-1/2}$. Our results imply in particular that if $A \subset \mathbb{R}^n$ satisfies $\gamma_n(A) = 1/2$ and $\gamma_n^+(A) \leq (2\pi)^{-1/2} + \delta$ then there exists a half-space $B \subset \mathbb{R}^n$ such that $\gamma_n(A \Delta B) \leq C \log^{-1/2}(1/\delta)$ for an absolute constant C. Since the Gaussian isoperimetric result was established, only recently a robust version of the Gaussian isoperimetric result was obtained by Cianchi et al., who showed that $\gamma_n(A \Delta B) \leq C(n)\sqrt{\delta}$ for some function $C(n)$ with no effective bounds. Compared to the results of Cianchi et al., our results have optimal (i.e., no) dependence on the dimension, but worse dependence on δ.

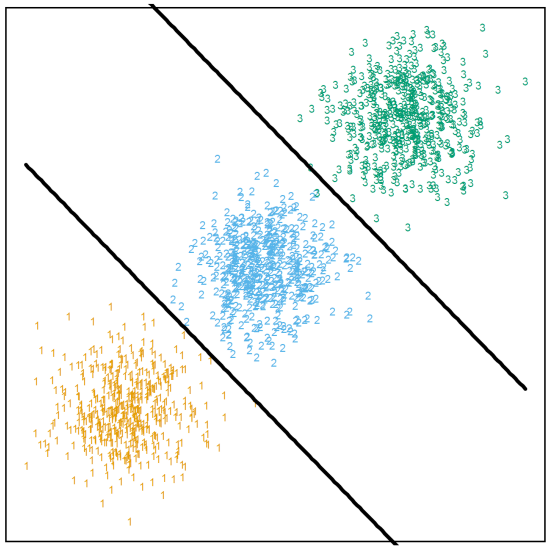
ON THE ISOPERIMETRIC DEFICIT IN GAUSS SPACE

By A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli

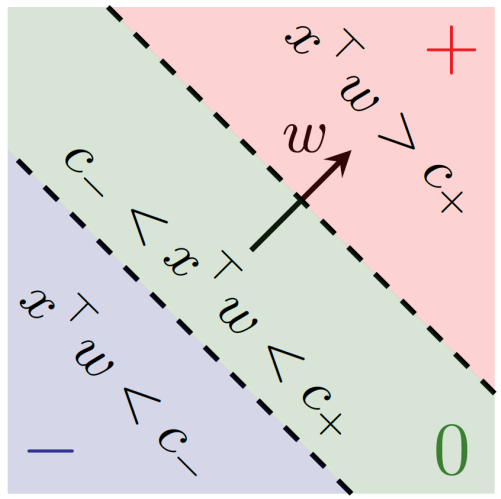
Optimal ℓ_2-robust three-class classifiers

- More general setting:
  1. three classes $C = \{-1, 0, 1\}$: $y = \pm 1$ w.p. $\pi_\pm$ and $y = 0$ w.p. π_0, where $\pi_+, \pi_0, \pi_- \in [0, 1]$ sum to unity and are the class proportions.
  2. $x|y \sim \mathcal{N}(y\mu, I_p)$
- Bayes optimal classif. is a linear-interval classif. ($w = \mu$, $c_+ \geq c_-$):

$$\hat{y}_{int}(x; w, c_+, c_-) := \begin{cases} +1 & \text{if } x^\top w \geq c_+, \\ 0 & \text{if } c_- \leq x^\top w < c_+, \\ -1 & \text{if } x^\top w \leq c_-. \end{cases}$$



(a) Figure: Hastie et al., 2009



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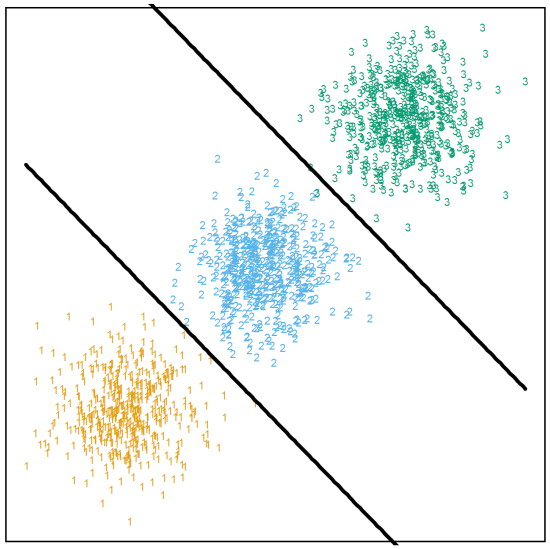

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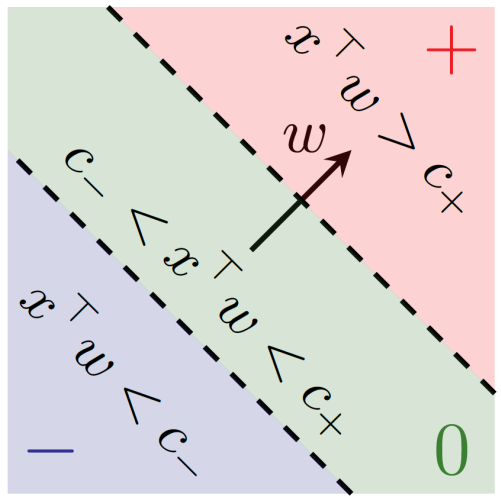

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Optimal ℓ_2-robust three-class classifiers

Theorem (ctd)

Case 2 (large zero class). *Otherwise,* $c_+^* - c_-^* > 2\varepsilon\|\mu\|_2$, *with*

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The cutoff α^* *is the unique solution to* ($\gamma := \sqrt{\pi_+/\pi_-}$):

$$(\gamma + \gamma^{-1}) R^*_{\text{rob}}\{\mu, \frac{\gamma}{\gamma + \gamma^{-1}}; \varepsilon\} \tag{1}$$

$$= (\gamma + \alpha) R^*_{\text{rob}}\{\frac{\mu}{2}, \frac{\gamma}{\gamma + \alpha}; \varepsilon\} + (\gamma^{-1} + \alpha) R^*_{\text{rob}}\{\frac{\mu}{2}, \frac{\gamma^{-1}}{\gamma^{-1} + \alpha}; \varepsilon\} - \alpha,$$

for $\alpha \geq \exp\{-(\|\mu\|_2 - 2\varepsilon)^2/2\}$.

Optimal ℓ_2-robust three-class classifiers

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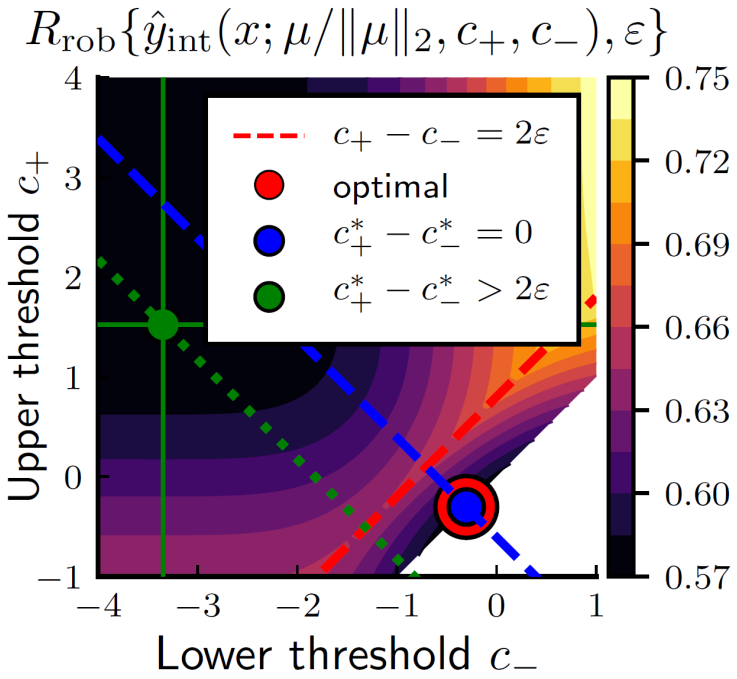
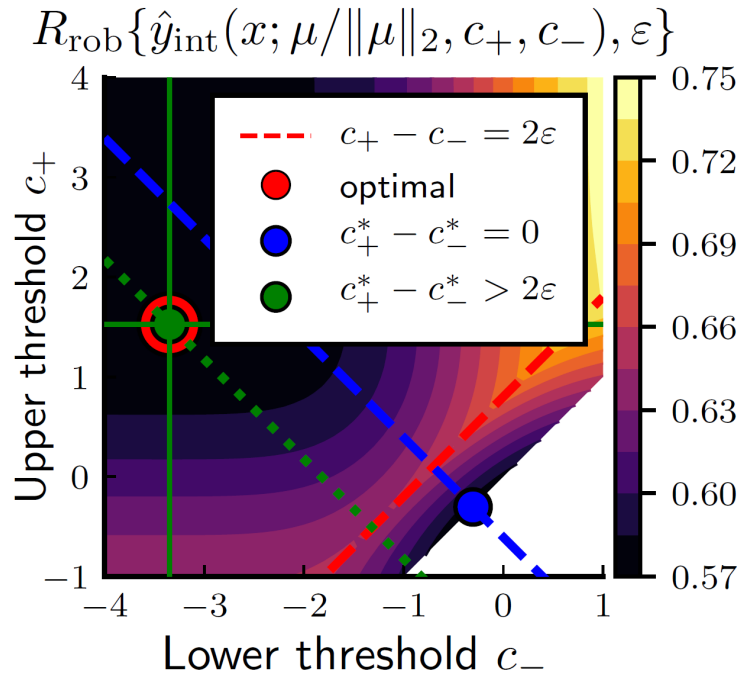
Optimal ℓ_2-robust three-class classifiers: Discussion

- Class (im)balance crucial. Zero class:
  1. Small: thresholds coincide with two-class robust classif (-1 vs 1)
  2. Large: thresholds coincide with two-class robust classif (-1 vs 0), (0 vs 1)
- Discontinuity: optimal thresholds can be discontinuous in problem parameters



(b) $\pi_0 = 42.00\%$



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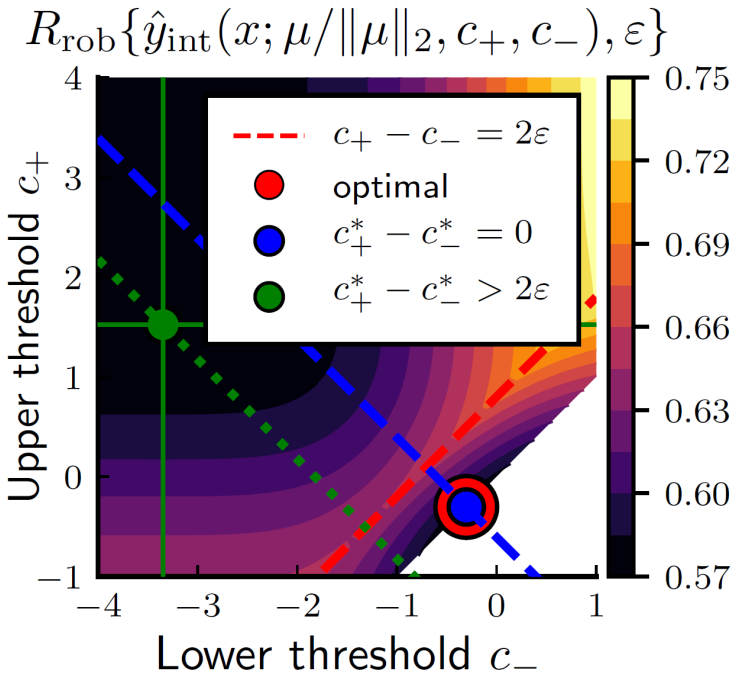

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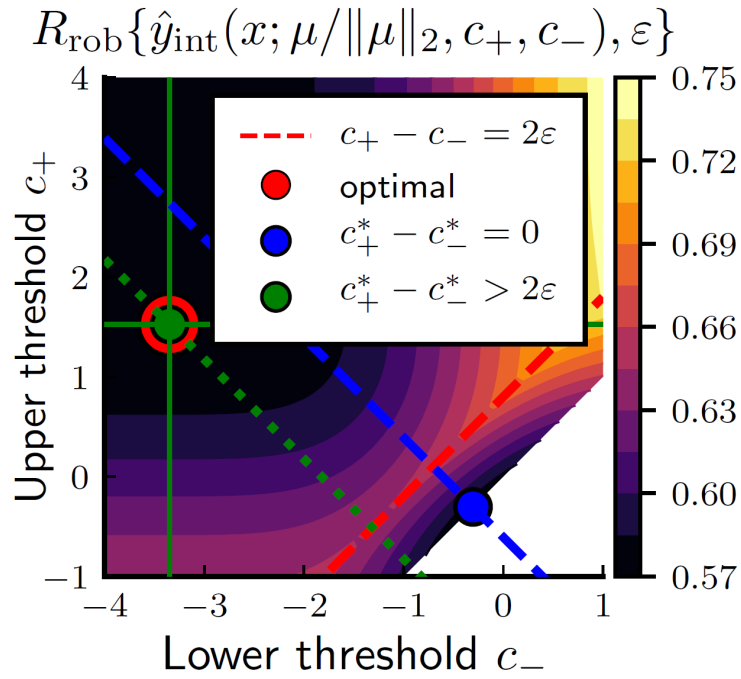

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Other results

- Thm: Opt. robust linear classifiers for ℓ_∞ perturbations are $\hat{y}^*(x) = \text{sign}\{x^\top \eta_\varepsilon(\mu) - q/2\}$, where η is soft-thresholding

$$\eta_\varepsilon(x) := \begin{cases} x - \varepsilon, & \text{if } x \geq \varepsilon, \\ 0, & \text{if } x \in (-\varepsilon, \varepsilon), \\ x + \varepsilon, & \text{if } x \leq -\varepsilon, \end{cases}$$

- Thm: For any norm-bounded perturbations, classification calibration for linear classif. and decreasing surrogate loss ℓ holds if
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Related works: Can only mention a few. See paper

- Adv. ex. are unavoidable
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Summary

- Adversarial robustness: active, important area
- Multi-class Gaussian models:
  1. Bayes-optimal classifiers (new proof techniques)
  2. Implications: tradeoffs, imbalance
  3. Classification calibration, Conv. rates
- Many problems remain
- slides at: `github.com/dobriban/talks`
- Thanks!

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