# New Digital Infinite Impulse Response Filters on Atomic Function $\mathrm{h}_a(x)$

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**Abstract**— Digital infinite impulse response low pass filters on atomic functions $\mathrm{h}_a(x)$ are first proposed. Due to its compactly support and the presence of interval where it is constant, atomic function $\mathrm{h}_a(x)$ can play the role of an ideal low pass filter magnitude response. Since $\mathrm{h}_a(x)$ is an infinitely differentiable function its spectrum decay is plenty fast. The synthesis of new infinite impulse response filters is performed by using analogue filter prototypes. Herewith the recently designed construction method for analogue low pass filter with magnitude response approximating function $\mathrm{h}_a(x)$ is applied. This method is based on representation of truncated entire function using Cauchy integral formula. To carry out the approximation the partial Fourier sums of $\mathrm{h}_a(x)$ function are used in the formula. Rational fractions which approximate atomic function $\mathrm{h}_a(x)$ can be found by replacement of integral with Riemann sum. Then, the problem of integration contour optimization need to be solved. The obtained rational functions have the convenient form of representation with sum of partial fractions. After receiving the nonnegative rational fraction which defines the frequency response of analogue filter the transfer function of digital infinite impulse response filter is obtained using standard transforms. New digital filters are stable since the poles of their transfer functions are situated inside the unit circle. The magnitude responses of new digital infinite impulse response filters approximates the infinitely smooth atomic function $\mathrm{h}_a(x)$. This property makes the new low pass filters substantially different from classical low pass infinite impulse response filters on Butterworth, Chebyshev and elliptical analogue prototypes which magnitude responses with order increase are approaching to discontinuous rectangular function. Due to the shape of new filters magnitude responses their impulse responses are fast decaying functions.

## 1. INTRODUCTION

Compactly supported and infinitely differentiable atomic functions [1] are widely applied in signal processing and other fields. The atomic function $\mathrm{h}_a(x)$ considered in this study due to its properties can be used for low-pass filters construction.

The Kravchenko-Kotelnikov [2] series form theoretical foundation of filtering methods using filters with frequency response based on the $\mathrm{h}_a(x)$. This series is a generalization of the Whittaker-Shannon-Kotelnikov series and compares favorably with the latter in a small value of the truncation error. For the purpose of practical implementation of the Kravchenko-Kotelnikov theorem, filters with magnitude response similar to the function $\mathrm{h}_a(x)$ were previously designed: digital filters with finite impulse response (FIR) [3], and then analogue filters [4, 5]. The presented here research on infinite impulse response (IIR) filters is a natural continuation of the works on analogue filters based on atomic functions.

## 2. FUNCTION $\mathrm{h}_a(x)$

Atomic function $\mathrm{h}_a(x)$ is compactly supported solution of functional-differential equation [1]

$$y(x) = \frac{a^2}{2}\left(y(ax+1) - y(ax-1)\right), \quad a > 1,$$

satisfying the normalization condition

$$\int\limits_{\mathbb{R}} y(x)\,dx = 1.$$

Graphics of $\mathrm{h}_a(x)$ for $a = 2.5$ and $a = 3$ are shown in Fig. 1.

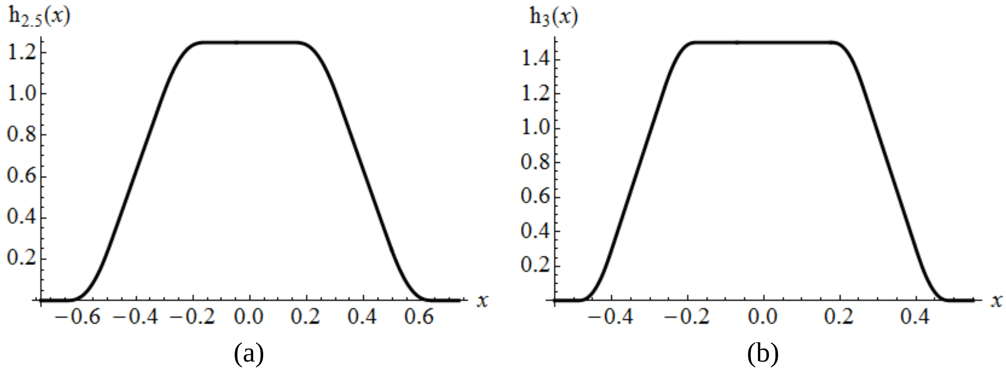


Figure 1: Graphics of $\mathrm{h}_a(x)$ function for (a) $a = 2.5$, (b) $a = 3$.

Support of $\mathrm{h}_a(x)$ is the segment $\left[-\frac{1}{a-1}, \frac{1}{a-1}\right]$. Spectrum $F_a(t)$ of $\mathrm{h}_a(x)$ is infinite product

$$F_a(t) = \prod_{k=1}^{\infty} \operatorname{sinc}\left(\frac{t}{a^k}\right).$$

Function $\mathrm{h}_a(x)$ may be presented in the form of Fourier integral

$$\mathrm{h}_a(x) = \frac{1}{2\pi} \int_{\mathbb{R}} F_a(t) e^{jtx}\, dt.$$

In practical applications for computation of atomic function $\mathrm{h}_a(x)$ on the segment $x \in \left[-\frac{1}{a-1}, \frac{1}{a-1}\right]$ the following formula is applied:

$$\mathrm{h}_a(x) \approx (a-1)\left(\frac{1}{2} + \sum_{l=1}^{L} P_a^{(K)}\left(\pi(a-1)l\right)\cos\left(\pi(a-1)lx\right)\right), \tag{1}$$

where

$$P_a^{(K)}(t) = \prod_{k=1}^{K} \operatorname{sinc}\left(\frac{t}{a^k}\right)$$

and $K$, $L$ are some given positive integers. Convergence rate of sum in the right-hand part of (1) decrease with increasing of parameter $a$.

If $a > 2$ then $\mathrm{h}_a(x)$ has the property

$$\mathrm{h}_a(x) \equiv \frac{a}{2}, \quad |x| \leq \frac{a-2}{a(a-1)}. \tag{2}$$

For each $a > 1$ holds

$$\mathrm{h}_a(x) \equiv 0, \quad |x| > \frac{1}{a-1}. \tag{3}$$

Properties (2) and (3) allow us to consider $\mathrm{h}_a(x)$ as magnitude response of some ideal low-pass filter. Another useful property of $\mathrm{h}_a(x)$ is the rapid decay of its spectrum for small values of the parameter $a$.

## 3. SYNTHESIS OF ANALOGUE PROTOTYPES

For construction of analogue filters with magnitude response based on $\mathrm{h}_a(x)$ method of approximation $\mathrm{h}_a^2(x)$ with rational functions is required at first. The applicable in practice method was first described in [4]. It is based on equality which is derived from Cauchy integral formula.

Let $C$ is a contour passing through points $z = \pm 1$ and bounding simply connected region $D$ on the complex plane $\mathbb{C}$, $D$ include the interval $(-1, 1)$, $\varphi(z)$ — some analytic in region $D$ function, $\omega$

— real variable. Then the Cauchy integral formula implies the equality

$$\frac{1}{2\pi j}\oint_C \frac{\varphi(z)}{z-\omega} = \begin{cases} \varphi(\omega), & |\omega|<1, \\ 0, & |\omega|>1. \end{cases} \tag{4}$$

Consider the function $\varphi_a(\omega) = \frac{4}{a^2}\mathrm{h}_a^2\left(\frac{\omega}{a-1}\right)$. Coefficients $a_m$, $m = 0, 1, 2, \dots$, of its Fourier expansion on the segment $-1 \le \omega \le 1$

$$\varphi_a(\omega) = \sum_{m=0}^{\infty} a_m \cos(\pi m \omega) \tag{5}$$

are determined by the formulae

$$\begin{aligned} a_0 &= \frac{c_0^2}{2} + \sum_{l=0}^{\infty} \frac{1}{2} c_l^2, \\ a_k &= \sum_{l=0}^{k} \frac{1}{2} c_{k-l} c_l + \sum_{l=k+1}^{\infty} c_{l-k} c_l, \quad k \ge 1, \end{aligned}$$

where $c_k$, $k = 0, 1, 2, \dots$, are coefficients of Fourier expansion of function $\frac{2}{a}\mathrm{h}_a\left(\frac{\omega}{a-1}\right)$:

$$\begin{aligned} c_0 &= \frac{a-1}{a}, \\ c_k &= \frac{2(a-1)}{a} F_a((a-1)\pi k), \quad k > 0. \end{aligned}$$

Partial sums $\varphi_a^{(M)}(\omega)$ of series (5)

$$\varphi_a^{(M)}(\omega) = \sum_{m=0}^{M-1} a_m \cos(\pi m \omega)$$

are entire functions then (4) holds for $\varphi(z) = \varphi_a^{(M)}(z)$ and its right-hand part should be approximation of $\varphi_a(\omega)$. Let contour $C$ be an ellipse centered at the origin and passing through $z = \pm 1$, $z = \pm jb$, where $b > 0$ is parameter,

$$C = \{(x, y) : x = \cos t, y = b \sin t\}.$$

If we select grid $t_l = \frac{\pi}{2n} + l\frac{\pi}{n}$, $l = 0, \dots, 2n-1$, on the segment $t \in [0, 2\pi]$ and replace integral from left-hand part of (4) with its Riemann sum approximation then we get the function

$$H_{b,M}(\omega) = \frac{1}{2nj} \sum_{l=0}^{2n-1} \frac{\varphi_a^{(M)}(\cos t_l + jb \sin t_l)(-\sin t_l + jb \cos t_l)}{\cos t_l + jb \sin t_l - \omega}. \tag{6}$$

Function $H_{b,M}(\omega)$ is a rational fraction. For selected integration grid its numerator and denominator are real polynomials containing only even powers of the variable $\omega$. Moreover, for fixed $b$ and sufficiently large $M$ and $n$

$$H_{b,M}(\omega) \approx \varphi_a(\omega).$$

Uniform error $\varepsilon$ of approximation of function $\varphi_a(\omega)$ by the fraction $H_{b,M}(\omega)$ is defined by formula

$$\varepsilon = \min_{\omega \in (-\infty, +\infty)} \left| \varphi_a(\omega) - H_{b,M}(\omega) \right|. \tag{7}$$

Value of (7) depends on parameter $b$ and number of terms $M$ of partial sum $\varphi_a^{(M)}(\omega)$, $\varepsilon \equiv \varepsilon(b, M)$. The problem of minimizing the value $\varepsilon$ is equivalent to the problem of minimizing the functional

$$\Phi(b_{opt}, M_{opt}) = \min_{b>0, M} \Phi(b, M), \tag{8}$$

where

$$\Phi(b, M) = \min_{\omega \in (-\infty, +\infty)} \left(\varphi_a(\omega) - H_{b,M}(\omega)\right)^2 .$$

For fixed $M$ the problem of the form

$$\Phi(b_{opt}, M) = \min_{b>0} \Phi(b, M), \tag{9}$$

could be solved numerically with gradient descent method. If a finite set of numbers $M$, $M = 1, \dots, M_1 < \infty$, is given, the numerical solution of (8) for this set can be found by enumerating solutions to problems of the form (9). However, in such formulation of the problem (it is only known that $b > 0$) nothing is known about the existence of a solution (9). In addition, obtained function $H_{b,M}(\omega)$ does not necessarily have the property of non-negativity required for synthesis of stable analogue filter.

In [5] are presented two theorems which make it possible to select a finite segment $b \in [b_0, b_1]$ of (9) solution search, outside of which the condition $\varepsilon > \varepsilon_1$ is satisfied, where $\varepsilon_1$ is some given value of error. A theorem on sampling of $[b_0, b_1]$ is also proved in such a way that the solution of the discrete problem differs little from the solution of the continuous problem (9). That all makes possible to find numerical solutions of (8) meeting non-negativity condition

$$H_{b,M}(\omega) \geq 0. \tag{10}$$

The problem (9) sampling method from [5], although it makes it possible to find for non-negative numerical solutions, is still quite cumbersome. Therefore, if desired, to facilitate the search process (without guaranteeing the quality of the obtained solution), one can set an arbitrary grid on $[b_0, b_1]$ and then obtain the solution for a set of points of this grid by brute-force search. It is even easier to apply the gradient descent method to solve (9) for $M = 1, \dots, M_1$ and check the non-negativity of the obtained functions $H_{b,M}(\omega)$ by determining their zeroes (fraction $H_{b,M}(\omega)$ does not have real poles for given numerical integration grid). Sometimes the solutions (9) obtained by the gradient descent method are non-negative and at the same time approximate $\varphi_a(\omega)$ with acceptable accuracy.

When non-negative function $H_{b,M}(\omega)$ is obtained then basing on it, one can build an analogue filter with magnitude response $|H(j\omega)|$ meeting the equality

$$|H(j\omega)| = \sqrt{H_{b,M}(\omega)} \tag{11}$$

or [6]

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = H_{b,M}(\omega).$$

To obtain frequency response function $H(j\omega)$ one have to find zeroes and poles of $H_{b,M}(\omega)$ and choose from them lying in the upper half-plane. The filter transfer function $H(s)$ is defined as

$$H(j\omega) \mid_{\omega=-js} = H(s).$$

## 4. DESIGN OF IIR DIGITAL FILTERS WITH MAGNITUDE RESPONSES ON ATOMIC FUNCTIONS $\mathbf{h}_a(x)$

If analogue filter with known transfer function is designed, then it's possible to transform it into digital IIR filter by several known methods. Here we consider two of these methods: impulse invariance method and bilinear transform.

Impulse invariance method allows to keep the shape of impulse response while turning from analogue filter to digital. This is an important advantage, since the impulse responses of continuous filters with magnitude response based on $\varphi_a(\omega)$ functions decay rather quickly for small $a$. Let designed analogue filter $h(t)$ of order $n$ has a transfer function $H(s)$ and expansion of $H(s)$ into elementary fractions has the form

$$H(s) = \sum_{l=0}^{n-1} \frac{\xi_l}{s - p_l}, \tag{12}$$

where $s_l$, $l = 0, \dots, n-1$, — poles of the function $H(s)$, $\xi_l$, $l = 0, \dots, n-1$, — coefficients of expansion (12). Then the transfer function $H_d(z)$ of digital IIR filter based on filter $h(t)$ is defined by [7]

$$H_d(z) = \sum_{l=0}^{n-1} \frac{\xi_l}{1 - z^{-1} e^{p_l T}}, \tag{13}$$

where $T$ is sampling interval. In this case, the coefficients $h_d(k)$ of the impulse response of the digital filter have the property

$$h_d(k) = h(kT).$$

If the value of error (7) is small than magnitude responses $|H(j\omega)|$ of analogue filters constructed according to the method discussed in previous section have the property

$$\begin{aligned} &|H(j\omega)| \approx 1, \ \ |\omega| < \frac{a-2}{a}, \\ &|H(j\omega)| \approx 0, \ \ |\omega| > 1. \end{aligned} \tag{14}$$

Then relation between stopband edge frequency and passband edge frequency of digital filter $\{h_d(k)\}$ is given by selection of parameter $a$ of function $\varphi_a(\omega)$, which is approximated by $|H(j\omega)|^2$. According to (14) interval $T$ in (13) meets inequality $T < \pi$ [7]. If construction $\{h_d(k)\}$ with given stopband edge frequency $0 < \theta_1 < \pi$ is required, then sampling interval is selected $T = \theta_1$.

Note that, in the general case, the number of zeroes of the function $H(j\omega)$ satisfying the relation (11) is only one less than the number of poles, therefore $|H(j\omega)|$ for $\omega > 1$ decays rather slowly. This negatively affects the passband deviation of digital filters with transfer function (13). Magnitude responses of digital IIR filters on the function $\varphi_a(\omega)$ obtained with impulse invariance method are shown in in Figs. 2(a), (b).

The transition from the transfer function $H(s)$ of the analogue low-pass filter $h(t)$ to the transfer function $H_d(z)$ of the IIR filter $\{h_d(k)\}$ using a bilinear transform is given by the relation formula between the variables $s$ and $z$ [6]

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}. \tag{15}$$

In this case, the relation between the analogue frequency $\omega$ and the digital frequency $\theta$ is not linear [6]

$$\omega = \frac{2}{T}\tan\left(\frac{\theta}{2}\right). \tag{16}$$

Then magnitude response $|H_d(e^{j\theta})|$ and impulse response $\{h_d(k)\}$ differ in shape from the characteristics of the original analogue filter. The method of constructing analogue filters described in Section 3 allows one to obtain digital filters without such disadvantage, that is, with the desired not distorted in frequency shape of magnitude response.

Let it be necessary to approximate some finite function $\psi(\omega)$ (we assume that $\psi(\omega)$ is even) with support $[-1, 1]$ and given Fourier coefficients $a_0, a_1, \ldots, a_k, \ldots$,

$$\psi(\omega) = \sum_{m=0}^{\infty} a_m \cos(\pi m \omega), \ \ \omega \in [-1, 1]. \tag{17}$$

Partial sums of (17) $\psi^{(M)}(\omega)$ are

$$\psi^{(M)}(\omega) = \sum_{m=0}^{M-1} a_m \cos(\pi m \omega).$$

Substituting $\psi^{(M)}(\omega)$ instead of $\varphi_a^{(M)}(\omega)$ in formula (6) we get the fraction $H_{b,M}(\omega)$ approximating $\psi(\omega)$ with some accuracy. Then, the transfer function $H(s)$ of the analogue filter is constructed in according to the method described in Section 3.

Let

$$\psi(\omega) = \varphi_a\left(\frac{2}{T}\text{arctan}\left(\omega\tan\frac{T}{2}\right)\right).$$

It is clear that if $|H(j\omega)| \approx \sqrt{\psi(\omega)}$, than corresponding to (16), for $H_d(z)$ obtained by formula

$$H_d(z) = H\left(\frac{T}{2}\frac{1-z^{-1}}{1+z^{-1}}\cot\frac{T}{2}\right) \tag{18}$$

holds

$$\left|H_d\left(e^{j\theta}\right)\right| \approx \sqrt{\varphi_a\left(\frac{\theta}{T}\right)}, \quad \theta \in [0, \pi].$$

Stopband edge frequency $\theta_1$ of digital filter with transfer function (18) is given by $\theta_1 = T$. Note that the shape of magnitude response $\left|H_d\left(e^{j\theta}\right)\right|$ of digital filter also badly affected by the slow decay of the frequency responses of analogue prototypes. In the case of bilinear transform, the resulting digital filter can have a noticeable stopband deviation. Examples of magnitude responses of IIR filters based on $\varphi_a(\omega)$ obtained with bilinear transform are presented in Figs. 2(c), (d).

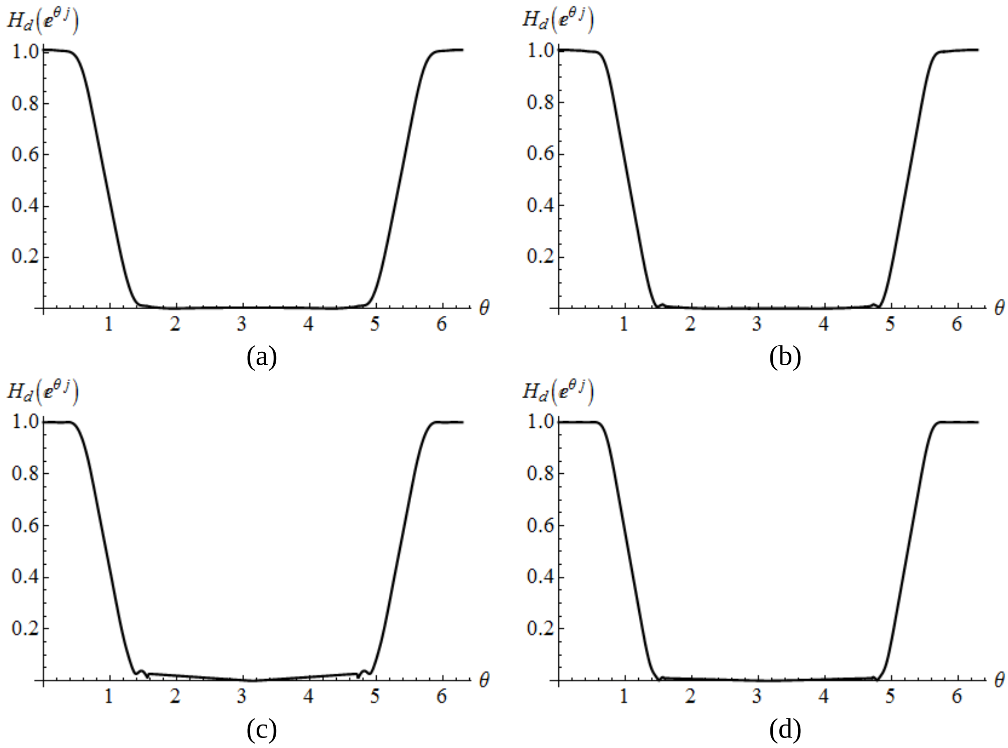


Figure 2: Magnitude responses $|H_d(e^{j\theta})|$ of digital IIR filters on function $\mathrm{h}_a(\omega)$ obtained with impulse invariance and bilinear transform methods for $n = 20, 30$; (a) $a = 2.5$, impulse invariance method, $n = 20$, (b) $a = 3$, impulse invariance method, $n = 30$, (c) $a = 2.5$, bilinear transform method, $n = 20$, (d) $a = 3$, bilinear transform method, $n = 30$.

## 5. NUMERICAL EXPERIMENT

New filters can be applied to increase the sampling rate [8, 9] of digital signals. Discrete signal

$$y(k) = \begin{cases} f(k\Delta), & k \geq 0, \\ 0, & k < 0, \end{cases}$$

where step $\Delta = \frac{\pi}{12}$ and

$$f(t) = \operatorname{sinc}^3((t-13)/2) + \frac{1}{2}\operatorname{sinc}^2(t-20) + \frac{1}{6}\operatorname{sinc}^2((t-33)/2),$$

is physically realizable and has the frequency response $Y(e^{j\theta}) = Z[\{y(k)\}](e^{j\theta})$, satisfying the relation $|Y(e^{j\theta})| \approx 0$ if $\theta \in \left(\frac{\pi}{6}, \frac{11\pi}{6}\right)$. Discarding all elements of the sequence $\{y(k)\}$ with numbers $k$ not divisible by 3, we obtain the discrete signal $\{y_0(m)\}$, defined by the expression

$$y_0(m) = y(3m), \quad m \in \mathbb{Z}.$$

Let us state the problem of recovering the sequence $\{y(k)\}$ from the sequence $\{y_0(m)\}$, or, which is equivalent, the problem of increasing the $\{y_0(m)\}$ sampling rate into 3 times. The upsampling is performed by computing the discrete convolution $\{\hat{y}(k)\}$ of the signal $\{y_1(k)\}$, which is equal to

$$y_1(k) = \begin{cases} y_0(m), & k = 3m, \\ 0, & k \neq 3m, \end{cases}$$

with impulse response $\{h(k)\}$ of digital low-pass filter

$$\hat{y}(k) = \sum_{l=-\infty}^{\infty} y_1(l)h(k-l). \tag{19}$$

If IIR filter is applied then elements of the sequence $\{\hat{y}(k)\}$ are defined by formula [7]

$$\hat{y}(k) = \sum_{l=0}^{P} b_l y_1(k-l) - \sum_{l=1}^{Q} a_l \hat{y}(k-l). \tag{20}$$

In (20) $a_k$, $k = 1, \ldots, P$, and $b_k$, $k = 0, \ldots, Q$, are coefficients of filter $\{h(k)\}$ transfer function $H(z)$

$$H(z) = \frac{\sum\limits_{k=0}^{P} b_k z^{-k}}{1 + \sum\limits_{k=1}^{Q} a_k z^{-k}}. \tag{21}$$

Magnitude response $|Y_1(e^{j\theta})|$ of the signal $\{y_1(k)\}$ is presented in Fig. 3. It consists of three copies of magnitude response $|Y(e^{j\theta})|$ of signal $y(m)$.

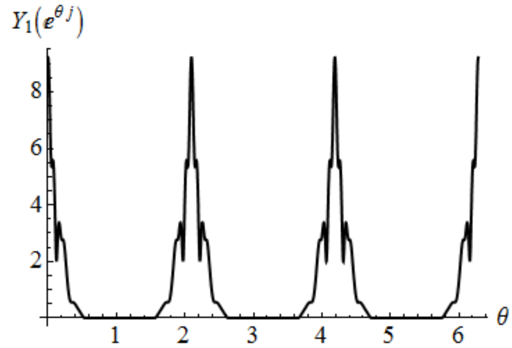


Figure 3: Magnitude response $|Y_1(e^{j\theta})|$ of the signal $\{y_1(k)\}$.

Signal $\{y(k)\}$ may be recovered from $\{y_0(m)\}$ by suppression of two redundant copies situated in $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ band. Then low-pass filter $\{h(k)\}$ in formula (19) must have passband edge frequency $\theta_0 \geq \frac{\pi}{6}$ and stopband edge frequency $\theta_1 \leq \frac{\pi}{2}$.

In the process of the signal $\{y(k)\}$ recovering, we will use four different filters of order $n = 30$: the atomic filter $\{h_1(k)\}$ and the Butterworth filter $\{h_2(k)\}$ obtained by the impulse invariance method; the atomic filter $\{h_3(k)\}$ and the Butterworth filter $\{h_4(k)\}$ obtained by the bilinear transform method. Let's describe analogue filter prototypes of $\{h_1(k), \ldots, h_4(k)\}$.

Prototype of $\{h_1(k)\}$ is analogue filter based on $\varphi_3(\omega)$ with magnitude response $|H_1(j\omega)| = \sqrt{H_{b,M}(\omega)}$, where $b = 0.29471$, $M = 13$. With given values of parameters $b$ and $M$ function $H_{b,M}(\omega)$ is non-negative and approximation error (7) is equal to $1.1 \cdot 10^{-3}$.

Prototype of $\{h_2(k)\}$ has magnitude response $|H_2(j\omega)| = \sqrt{B(1.25 \cdot \omega)}$, where $B(\omega)$ is Butterworth function [6]

$$B(\omega) = \frac{1}{1 + \omega^{2n}}.$$

The magnitude response $|H_3(j\omega)|$ of $\{h_3(k)\}$ filter prototype meets equality $|H_3(j\omega)| = \sqrt{H_{b,M}(\omega)}$ where $H_{b,M}(\omega)$ with values of parameters $b = 0.14883$, $M = 16$ approximates function $\varphi_3\left(\frac{2}{T}\arctan\left(\omega \tan\frac{T}{2}\right)\right)$ with error $\varepsilon = 1.7 \cdot 10^{-3}$.

For the construction of $\{h_4(k)\}$ analogue filter with magnitude response $|H_4(j\omega)| = \sqrt{B\left(1.25\omega\frac{T}{2}\tan^{-1}\frac{T}{2}\right)}$ is used.

Then each of described analogue filters with magnitude responses $|H_l(j\omega)|$, $l = 1,\dots,4$, is converted into digital one $\{h_l(k)\}$. For $l = 1,2$ formula (13) is used. For $l = 3$ and $l = 4$ the formulae (18) and (15) are applied, respectively. The sampling interval for all 4 filters is $T = \frac{\pi}{2}$. Coefficients of IIR filters are determined by converting their transfer functions to the form (21). The sequences $\{h_l(k)\}$, $l = 1,\dots,4$, are shown in Fig. 4 (hereinafter, discrete sequences of the form $\{x(k)\}$ are graphically presented as broken lines with nodes at the points $(k; x(k))$).

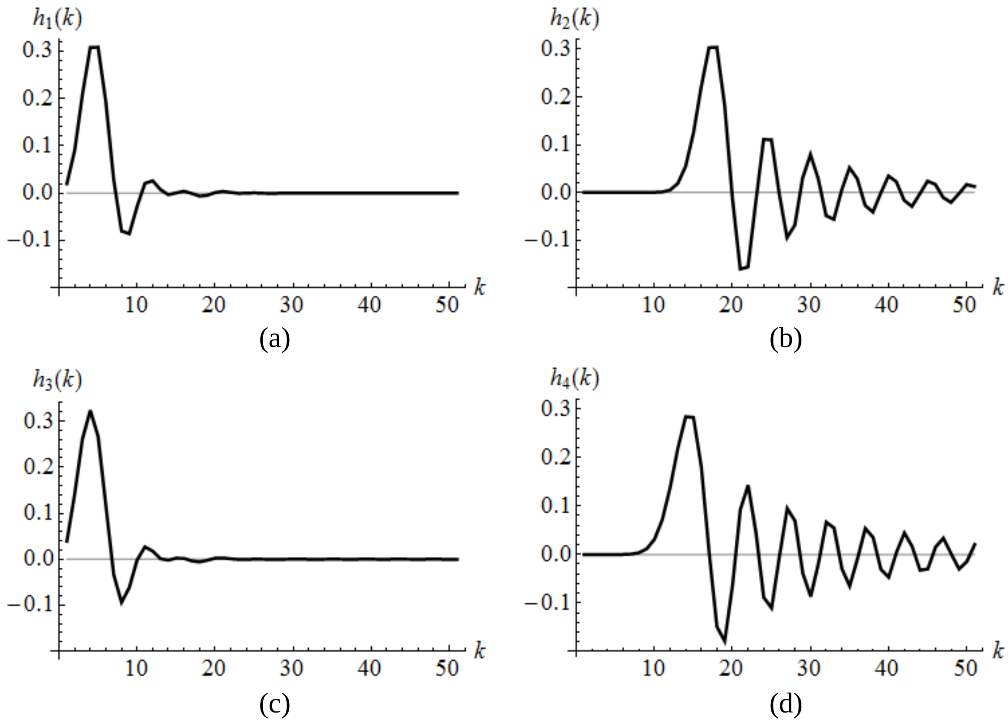


Figure 4: Sequences of impulse response coefficients of used filters (a) $\{h_1(k)\}$, (b) $\{h_2(k)\}$, (c) $\{h_3(k)\}$, (d) $\{h_4(k)\}$.

The plots in Fig. 4 well demonstrate the difference between the used atomic and classical IIR filters: the impulse responses of the former decay much faster.

Using the found coefficients of the transfer functions of the filters $\{h_l(k)\}$, $l = 1,\dots,4$, we apply the formula (20) to increase the sampling rate of the signal $\{y_0(k)\}$. Note that for this upsampling problem output sequence obtained by formula (20) should be equal to $\{\hat{y}(k)/3\}$ [8, 9], then it must be multiplied by gain factor which is equal to 3. The sequences $\{\hat{y}(k)\}$ obtained for each of the 4 filters (without taking into account the delay) are shown in Fig. 5.

The plots of $\{\hat{y}(k)\}$ shown in Fig. 5 differ little. A small recovery error occurred when atomic filter $\{h_3(k)\}$ obtained by the bilinear transform method was applied.

Now let's discard 15 samples $y_0(0),\dots,y_0(14)$ of the original signal $\{y_0(k)\}$ replacing them with zeroes and form corresponding sequence $\{y_1(k)\}$ according to the rule

$$y_1(k) = \begin{cases} y_0(m), & k = 3m \text{ and } m > 14, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Results of signal $\{y(k)\}$ recovery from the sequence (22) are presented in Fig. 6.

Discarding the samples of the signal $\{y_0(m)\}$ practically did not affect the quality of $\{y(k)\}$ recovery using filters $\{h_1(k)\}$, $\{h_3(k)\}$ based on the atomic function $\mathrm{h}_3(\omega)$. A significant error occurred when applying the filters $\{h_2(k)\}$ and $\{h_4(k)\}$. The worst result is obtained while the filter $\{h_4(k)\}$ constructed from the analogue Butterworth filter by bilinear transform was used to recovery.

Similar results were obtained in previous experiments on continuous signal recovery with analogue filters (see [4, 5]).

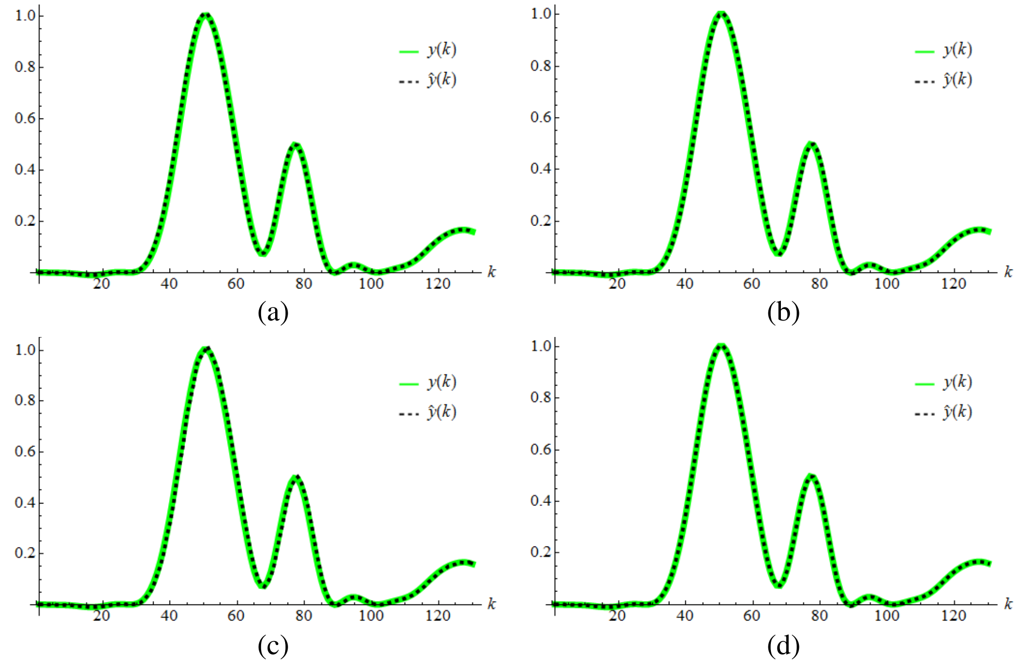


Figure 5: Original signal $\{y(k)\}$ (green line) and recovered signal $\{\hat{y}(k)\}$ (black dotted line), obtained by application of filters, (a) $\{h_1(k)\}$, (b) $\{h_2(k)\}$, (c) $\{h_3(k)\}$, (d) $\{h_4(k)\}$.

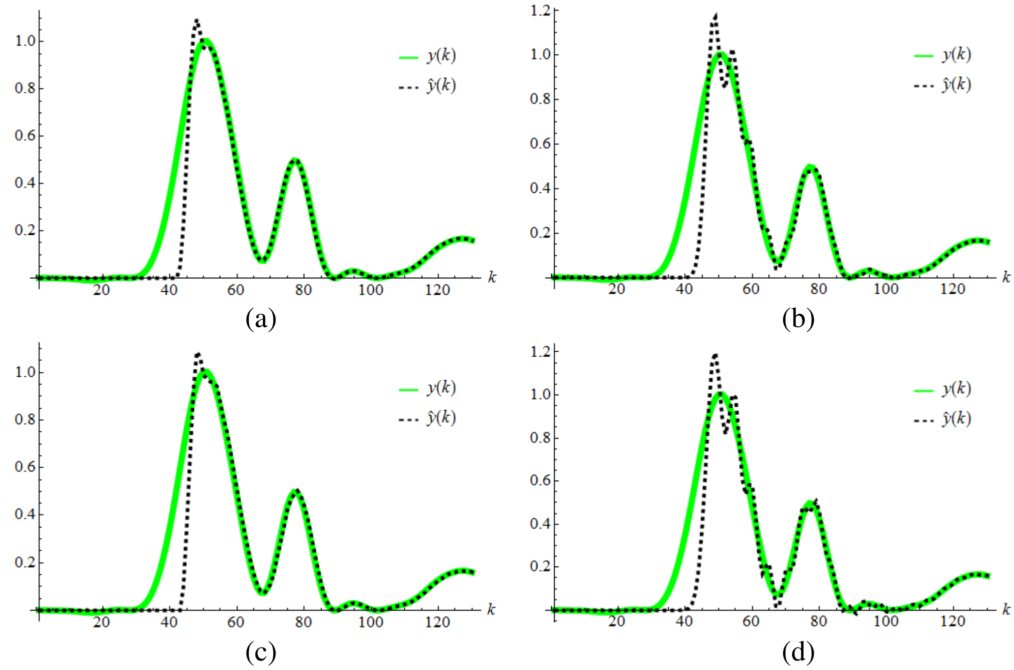


Figure 6: Original signal $\{y(k)\}$ (green line) and recovered $\{\hat{y}(k)\}$ (black dotted line) from the sequence with 15 samples discarded $\{y_0(m)\}$ by application of filters, (a) $\{h_1(k)\}$, (b) $\{h_2(k)\}$, (c) $\{h_3(k)\}$, (d) $\{h_4(k)\}$.

## 6. CONCLUSION

Digital IIR filters with frequency response based on atomic functions $\mathrm{h}_a(x)$ are first presented. A method for constructing their analogue prototypes based on the technique of approximating the squares of atomic functions by rational fractions is described.

Two methods of an analogue prototype to a digital filter conversion are considered: the impulse invariance method and the bilinear transform. A method of constructing of digital filters based on atomic functions with the desired not distorted in digital frequency magnitude response shape using bilinear transform is described.

A numerical experiment on a digital signal sampling rate increase was performed. In the experiment, filters obtained both by the impulse invariance method and by means of bilinear transform were used: two IIR filters with frequency response based on the atomic function $\mathrm{h}_3(x)$ and two classic filters based on Butterworth functions. The new filters in contrast to the classical ones turned out to be insensitive to discarding samples of the reconstructed signal.

This paper is Dedicated to the 95th anniversary of the birth of the outstanding scientist of our time, the creator of atomic, R-functions theories and non-Archimedean calculus, Academician of the National Academy of Sciences of Ukraine Vladimir Logvinovich Rvachev.

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