# R-, Atomic and WA-system Functions Theory in Physical Problems

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**Abstract**— Review is concerned on results based on the works of outstanding Soviet mathematician academician of the National Academy of Sciences of Ukraine V. L. Rvachev. In 1960–70 in Kharkiv he worked on numerical solution of problems in mechanics. During this work Rvachev laid foundation of R-function theory. It presents mathematical apparatus for solution of inverse problem of analytical geometry by means of Boolean algebra. In particular it allows construction of equations which describe complex geometrical figures. It is widely applied in boundary value problems solution, numerical modeling of electromagnetic and thermal fields and in image processing. Another great invention of V. L. Rvachev is atomic function theory. Note 50-anniversary from publication of the first paper describing atomic function in 1971. Atomic functions are solutions of functional-differential equations of special form and may be represented as infinite convolutions of rectangular (in simplest case) pulses of decreasing length. They are both infinitely smooth and compactly supported. This unique property makes them widely applicable as basis and weight functions. We note some specific applications of atomic functions. Generalization of sampling theorem based on atomic functions was presented by V. F. Kravchenko. Generalized Kravchenko-Kotelnikov series have fast convergence and small truncation error. Family of wavelets based on scaled sums of shifts of atomic functions was presented by D. V. Churikov and A. V. Yurin. These wavelets are also called WA-systems. Recent work performed by K. A. Budunova is concerned on construction of FIR and IIR digital and analogue filters with magnitude response approximating atomic functions. In the review fundamental and applied works which develop generalize and apply V. L. Rvachev's ideas of $R$- and atomic function theory are discussed.

## 1. INTRODUCTION

This paper is Dedicated to the 95th anniversary of the birth of the outstanding scientist of our time, the creator of theories atomic, $R$-functions and non-Archimedean calculus, Academician of the National Academy of Sciences of Ukraine Vladimir Logvinovich Rvachev. Overview of some of the most basic ideas presented by V. L. Rvachev and recent results based on them is presented.

## 2. R-FUNCTIONS AND AUTOMATIC PROGRAMMING

The most known invention of V. L. Rvachev is $R$-function theory. The concept originally appeared in appendix to Rvachev's dissertation concerned on boundary value problems. If the problem contains boundary condition of the form $u|_{\partial\Omega} = \varphi$ then solution presents in the form $u = \varphi + \omega\psi$, where $\omega > 0$ inside the domain $\Omega$ and vanishes on the boundary $\partial\Omega$, $\psi$ is unknown function. Practically $\omega$ presents analytical description of geometrical object $\Omega$. Note that if concept of coordinates proposed by Rene Descartes presented solution of direct problem of analytical geometry allowing to construct domain $\Omega$ according to given equation then $R$-function theory presents solution of inverse problem providing method of construction equation from given object $\Omega$. The methodology is based on Boolean algebra providing representation between Boolean and ordinary functions. If the sign of function value fully determined by the signs of arguments then it is $R$-function. Then Boolean operations of conjunction and disjunction may be represented by corresponding $R$-functions. One of known options is $R_\alpha$ system presented by functions of $R$-conjunction $x_1 \wedge_\alpha x_2 = F(x_1, x_2, \alpha, -1)$ and $R$-disjunction $x_1 \vee_\alpha x_2 = F(x_1, x_2, \alpha, 1)$ (the last argument determines sign of square root)

$$R_\alpha: \quad F(x_1, x_2, \alpha, k) = \frac{1}{1+\alpha}\left(x_1 + x_2 + k\sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}\right). \tag{1}$$

Original object $\Omega$ is presented or approximated with a certain precision by union of intersections of primitive geometric objects. To each primitive object $\Omega_i$ corresponds function $\omega_i > 0$ inside $\Omega_i$, $\omega_i < 0$ outside $\Omega_i$ and vanishing on $\partial\Omega_i$. Operation of union of intersections corresponds to Boolean function ($\cup \sim \vee$, $\cap \sim \wedge$), which is represented by disjunctive canonical form. Replacing conjunctions and disjunctions by $R$-conjunctions and $R$-disjunctions (1) and substituting $\omega_i$ instead of corresponding arguments one can obtain function $w$ describing domain $\Omega$.

$R$-functions is now well-known mathematical apparatus. Large amount of papers, monographs [1–6], and reviews [7] are concerned on them. Scientific school in Kharkiv founded by Rvachev continues investigation of $R$-functions under the leadership of Rvachev's student and follower L. V. Kurpa.

Concept of $R$-functions could not be successful without computer aid. Then next idea of V. L. Rvachev was creation of automatic programming complex. Boundary value problem was formulated in standard form including all boundary conditions and form of domain. Then program code was generated automatically. It allows user to obtain numerical solution of his problems without immersion into mathematical background and programming tricks. Modern applied computation packages like MATLAB or Wolfram Mathematica are constructed in same paradigm.

## 3. ATOMIC FUNCTIONS

Let's consider not so famous but very basic and interesting concept of atomic functions. They provide effective local and smooth basis suitable for solution of boundary value problems and present useful addition to $R$-functions apparatus. Foundations of theory was laid by V. L. Rvachev, then the work was continued by V. A. Rvachev due to V. L. Rvachev was concerned on $R$-functions.

### 3.1. The Simplest Atomic Function up (x)

Atomic function up $(x)$ is defined as compactly supported solution of equation

$$y'(x) = 2y(2x+1) - 2y(2x-1) \tag{2}$$

normalized to unit square. It is solution of the problem stated by V. L. Rvachev in 1967. If we consider bell-shaped function then its derivative consists of hump corresponding to segment of function increasing and pit corresponding to segment of decreasing. Could it be that hump and pit of derivative are similar to the hump of the function? The original article [8] considering up $(x)$ was published in 1971.

Function up $(x)$ has several interesting properties making it native solution of different problems. Then it was independently found many times by different scientists. Its Fourier transform has form

$$\widehat{\mathrm{up}}(t) = \prod_{k=1}^{\infty} \mathrm{sinc}\left(\frac{t}{2^k}\right) \tag{3}$$

then it is infinite convolution of rectangular pulses with decreasing length and unit square

$$\mathrm{up}(x) = \phi(x) * 2\phi(2x) * \cdots * 2^n\phi(2^n x) * \ldots, \tag{4}$$

where $\phi(x)$ is standard rectangular pulse

$$\phi(x) = \begin{cases} 1 & |x| < 0.5, \\ 0.5 & |x| = 0.5, \\ 0 & |x| > 0.5. \end{cases} \tag{5}$$

In form (4) up $(x)$ was constructed in [9] by J. Fabius as a simple example of an infinitely differentiable non-analytic function which is the density function of weighted sum of uniformly distributed random variable.

If $\mathrm{up}(x) = \phi(x) * 2\mathrm{up}(2x)$ is a convolution of rectangular pulse and compactly supported function of unit square then it presents partition of unity

$$\sum_{k\in\mathbb{Z}} \mathrm{up}(x-k) \equiv 1. \tag{6}$$

If we combine $n+1$ first terms of convolution in (4) into B-spline of order $n$ we will have partition of polynomials

$$\sum_{k\in\mathbb{Z}} c_k \mathrm{up}(x - 2^{-n}k) = x^n. \tag{7}$$

Then according to the Strang-Fix condition [10] $\{\mathrm{up}(x-2^{-n}k)\}$ presents basis with good approximative properties. Ability of representation given function with sum of shifts of $\mathrm{up}(x)$ like everything in universe consists of atoms led V. L Rvachev to the idea to call new functions atomic.

Integration of (2) leads to

$$y(x)=\int_{-1}^{x}2y(2t+1)-2y(2t-1)dt. \tag{8}$$

Then $\mathrm{up}(x)$ may be considered as a fixed point of integral operator. Iterative process tending to $\mathrm{up}(x)$ may be performed in several ways. Analytical integration of (8) started from $y_0=0.5\phi(0.5x)$ leads to the sequence of perfect splines presented by V. A. Rvachev in [11]. Ordinary numerical integration leads to their numerical approximations. Wolfgang Hilberg in works [12, 13] got $\mathrm{up}(x)$ (he called it $\mathrm{hut}(x)$) as a signal obtained by loop-like electronic circuit. He also found that $\mathrm{hut}(x)$ satisfies Equation (2). In [14–16] another form of algorithm is presented. If we consider $y_k$ as a sequence of length $l_k$ then right hand part may be presented as a sequence of length $2l_k$ consisting of previous sequence and the same one with altered signs of elements. Integration is replaced with summation (each element of new sequence is sum of current and all previous elements of previous sequence). Normalized elements of sequence obtained on each iteration may be considered as approximations of values of atomic function $\{\mathrm{up}(x-2^{-n}k)\}$. If initial sequence consists of only one element 1 then $l_k=2^k$. If we change the order of operations performing $n$ times duplicating of sequence length and then $n$ times summation algorithm presented in [17] by W. Hilberg is obtained.

### 3.2. Family of Atomic Functions $\mathrm{h_a(x)}$

Family of atomic functions $\mathrm{h}_a(x)$ presents native generalization of $\mathrm{up}(x)$. It is defined as solution of equation

$$y'(x)=\frac{a^2}{2}\left(y(ax+1)-y(ax-1)\right) \tag{9}$$

with such properties

$$\mathrm{supp}(\mathrm{h}_a(x))=\left[-\frac{1}{a-1};\frac{1}{a-1}\right].\quad \text{If } a>2 \text{ then } \mathrm{h}_a(x)=\frac{a}{2} \text{ for } x\in\left[-\frac{a-2}{a(a-1)};\frac{a-2}{a(a-1)}\right]. \tag{10}$$

Function $\mathrm{h}_a(x)$ is normalized to unit square. Its Fourier transform

$$F_a(\omega)=\prod_{k=1}^{\infty}\mathrm{sinc}\left(\frac{\omega}{a^k}\right) \tag{11}$$

is applied as basis function for Kravchenko-Kotelnikov theorem [1, 18–22], which presents generalization of famous sampling theorem. According to the theorem if function $f(x)$ has compactly supported spectrum ($\mathrm{supp}\hat{f}(\omega)=[-\Omega,\Omega]$) then it is exactly represented by the series

$$f(x)=\sum_{k\in\mathbb{Z}}f(k\Delta)F_a\left(\frac{a\pi}{\Delta}(x-k\Delta)\right), \tag{12}$$

where $F_a(\omega)$ is Fourier transform (11) of $\mathrm{h}_a(x)$. Expression (12) usually called Kravchenko-Kotelnikov series. Fundamental research on ways of practical application of Kravchenko-Kotelnikov theorem is now performed in Kotelnikov Institute of Radioengineering and Electronics of RAS. In [21] it is found that truncation error of Kravchenko-Kotelkinov expansion

$$\epsilon_N(t)=\sum_{k=N+1}^{+\infty}f(k\Delta)F_a\left(\frac{a\pi}{\Delta}(x-k\Delta)\right)+\sum_{k=-\infty}^{-N-1}f(k\Delta)F_a\left(\frac{a\pi}{\Delta}(x-k\Delta)\right), \tag{13}$$

is bounded by the value decreasing faster then $N^r$, where $r$ is an arbitrary positive number. Generalization of Kravchenko-Kotelnikov theorem is presented in [23].

Properties (10) allows to consider function $\mathrm{h}_a(x)$ as a magnitude response of some ideal low-pass filter (LPF). FIR filters constructed from function $\mathrm{h}_a(x)$ are presented in [22, 24]. Kravchenko-Kotelnikov theorem provides correct impulse response of the filters. Development of methods of squares of atomic functions approximation by nonnegative rational fractions allows creation of analogue filters suitable for digital to analog conversion [25–27]. Construction of IIR filters based on $\mathrm{h}_a(x)$ is discussed in the report [28] presented at this conference.

### 3.3. Families of Atomic Functions $\mathrm{fup_n(x)}$ and $\mathrm{Fup_n(x)}$

Atomic functions $\mathrm{fup}_n(x)$ are defined as convolutions of $\mathrm{up}(x)$ and B-spline $\Theta_{n-1}$

$$\mathrm{fup}_n(x) = \mathrm{up}(x) * \Theta_{n-1}(x), \quad \text{where } \Theta_{n-1}(x) = \underbrace{\phi(x) * \phi(x) * \cdots * \phi(x)}_{n}, \quad \mathrm{fup}_0(x) = \mathrm{up}(x). \quad (14)$$

They are solutions of equation

$$y'(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{n+2} \left( C_{n+1}^{k} - C_{n+1}^{k-1} \right) y\left( 2x + \frac{n+2}{2} - k \right) \quad (15)$$

supported on $\left[-\frac{n+2}{2}; \frac{n+2}{2}\right]$ and normalized to unit square. If (14) is modified to $\mathrm{fup}_n(x) = 2\mathrm{up}(2x) * \Theta_n(x)$ then one can find that $\mathrm{fup}_n(x)$ contains B-spline $\Theta_n(x)$ and, according to Strang-Fix condition [10] presents effective approximation basis. Interconnections between functions

$$\mathrm{fup}_n'(x) = \mathrm{fup}_{n-1}\left( x + \frac{1}{2} \right) - \mathrm{fup}_{n-1}\left( x - \frac{1}{2} \right), \quad (16)$$

$$\mathrm{fup}_n(x) = \frac{1}{2^n} \sum_{k=0}^{n+1} C_{n+1}^{k} \mathrm{fup}_{n+1}\left( 2x + \frac{n+1}{2} - k \right) \quad (17)$$

allow effective computation of derivatives. Relation (17) allows construction of hierarchic basis [29], where in the neighborhood of singularity points basis elements may be replaced with elements of higher order providing increase of grid density and more precise approximation.

Family of atomic functions $\mathrm{Fup}_n(x)$ was constructed earlier. Functions $\mathrm{Fup}_n(x)$ are similar to $\mathrm{fup}_n(x)$ but have different support and normalization. In most works concerned on these functions $\mathrm{Fup}_n(x) = 2^n \mathrm{fup}_n(2^n x)$ is considered. In some early works $\mathrm{Fup}_n(x)$ has different normalization. In most modern works $\mathrm{fup}_n(x)$ is considered due to it allows shorter and more clear notation, but some authors still work with $\mathrm{Fup}_n(x)$, [29] for example.

### 3.4. Family of Atomic Functions $\mathrm{ch_{a,n}(x)}$

Atomic functions $\mathrm{ch}_{a,n}(x)$ [30, 31] are defined as iterated autoconvolutions of atomic function $\mathrm{h}_a(x)$

$$\mathrm{ch}_{a,n} = \underbrace{\mathrm{h}_a * \cdots * \mathrm{h}_a}_{n}. \quad (18)$$

For each real $a > 1$ and positive integer $n$ $\mathrm{ch}_{a,n}(x)$ is supported on the segment $\left[-\frac{n}{a-1}, \frac{n}{a-1}\right]$, normalized to unit square and satisfies equation

$$y^{(n)}(x) = a^{n+1} 2^{-n} \sum_{k=0}^{n} C_n^k (-1)^k y(ax + n - 2k). \quad (19)$$

Family $\mathrm{ch}_{a,n}(x)$ includes previously defined by Rvachev atomic functions

$$\mathrm{ch}_{2,1}(x) = \mathrm{up}(x), \quad \mathrm{ch}_{2,2}(x) = \mathrm{cup}(x), \quad \mathrm{ch}_{a,1}(x) = \mathrm{h}_a, \quad \mathrm{ch}_{n+1,n}(x) = \Xi_n. \quad (20)$$

### 3.5. General Definition of Atomic Functions

In general case atomic functions are defined as compactly supported solutions of differential equations of special form

$$L(y) = \sum_{n=0}^{N} c_n y(ax + b_n), \quad (21)$$

where $L(y)$ is linear differential operator. In the simplest case atomic functions are infinite convolutions of rectangular pulses (or their linear combinations) of decreasing length. Shifts of such functions present basis approximating polynomials. Already considered functions $\mathrm{ch}_{a,n}(x)$ and $\mathrm{fup}_n(x)$ are typical examples. There are other families of functions presenting approximation of exponents [32] and harmonic functions. They are presented as convolutions of exponential and harmonic pulses respectively.

Fundamental theorem [33] about convolution of two atomic functions with same scale factor was formulated by V. A. Rvachev. If $f(x)$ and $g(x)$ are atomic functions with equations

$$f^{(n_f)} = \sum_{m=1}^{M} c_{f_m} f(ax - b_{f_m}) \quad \text{and} \quad g^{(n_g)} = \sum_{k=1}^{K} c_{g_k} f(ax - b_{g_k}) \tag{22}$$

then their convolution $h(x) = f(x) * g(x)$ is atomic function with equation presented in such form

$$h^{(n_f+n_g)} = \sum_{m} \sum_{k} \frac{c_{f_m} c_{g_k}}{a} h(ax - b_{f_m} - b_{g_k}). \tag{23}$$

Conditions of the theorem require $n_f \geq 0$ and $n_g \geq 0$, then this theorem is applicable not only to atomic functions, but also to solutions of functional equations too. Among them we note rectangular pulse and B-splines. Equations of $\mathrm{ch}_{a,n}(x)$ and $\mathrm{fup}_n(x)$ are constructed basing on this theorem.

All atomic functions are infinitely smooth and useful for construction of weight functions [34]. They also present effective basis for interpolation and collocation [35]. Orthogonalization of $\mathrm{up}(x)$ basis performed in [36].

Atomic functions are contained in intersection of two fundamental families of functions: infinite convolutions of pulses and solutions of equations (21). Now we like to discuss some cases outside of this intersection.

### 3.6. Equations with Different Scale Factors

If equations of atomic functions contain only one scale factor then Fourier transform allows explicit form of spectrum of solution as infinite product corresponding to infinite convolution. Consider equations [16] with two different scale factors.

$$y' = 2\left(y(2x-1) - y(4x-1) - y(4x-3)\right) \tag{24}$$

$$y' = \frac{1}{3}\left(2y\left(\frac{5}{3}x\right) - 3y\left(\frac{5}{2}x - \frac{15}{2}\right)\right) \tag{25}$$

Arbitrary exact approximations of their compactly supported solutions may be simply found with iterative algorithm [16] but expressions for their spectra are not found yet.

### 3.7. Family $\mathrm{fip}_{\mathbf{a,n}}(\mathbf{x})$ as Generalization of Atomic Functions

Consider new family of infinitely smooth functions $\mathrm{fip}_{a,n}(x)$ [37, 38]. For each real $a > 1$ and positive integer $n$ $\mathrm{fip}_{a,n}(x)$ is defined as

$$\mathrm{fip}_{a,n}(x) = \mathrm{h}_a(x) * \Theta_{n-1}(x). \tag{26}$$

Functions $\mathrm{fup}_n(x)$ and $\mathrm{h}_a(x)$ are a special cases of $\mathrm{fip}_{a,n}(x)$

$$\mathrm{fip}_{2,n}(x) = \mathrm{fup}_n(x) \quad \text{and} \quad \mathrm{fip}_{a,0}(x) = \mathrm{h}_a(x). \tag{27}$$

Similar to (16) functions of the family are connected by relation

$$\mathrm{fip}'_{a,n}(x) = \mathrm{fip}_{a,n-1}\left(x + \frac{1}{2}\right) - \mathrm{fip}_{a,n-1}\left(x - \frac{1}{2}\right). \tag{28}$$

Family $\mathrm{fip}_{a,n}(x)$ presents effective infinitely smooth spline-like basis for interpolation and collocation.

The problem of construction of equation for each $\mathrm{fip}_{a,n}(x)$ is complicated. The form of equation qualitatively different for special cases.

If $n = 0$ then $\mathrm{fip}_{a,0}(x) = \mathrm{h}_a(x)$ fits (9). If $n = 1$ then for each $a > 1$ $\mathrm{fip}_{a,1}(x)$ satisfies equation

$$y' = \frac{a}{2}\left(y\left(ax + \frac{a+1}{2}\right) + y\left(ax + \frac{a-1}{2}\right) - y\left(ax - \frac{a-1}{2}\right) - y\left(ax - \frac{a+1}{2}\right)\right). \tag{29}$$

General equations for higher values of $n$ are not found.

For next consideration we note that rectangular pulse for each positive integer $n$ satisfies equation

$$y(x) = \sum_{k=1}^{n} y(nx + n + 1 - 2k). \tag{30}$$

Then functional-differential equation of $\mathrm{fip}_{a,n}(x)$ with integer $a$ for each $n$ may be constructed by iterated application of presented above convolution theorem [33] to Equations (9) and (30).

Consider $a = \sqrt[k]{m}$, where $k$ and $m$ are positive integers. By iterated derivation of (9) and substitution (9) to right-hand part we obtain equation for $\mathrm{h}_a(x)$ of order $k$ and scale $a^k = m$

$$y^{(k)} = \frac{a^{2k+\frac{k(k-1)}{2}}}{2^k} \sum_{s=0}^{2^k-1} (-1)^{\sum_{l=0}^{k-1} p_l} y\left(a^k x + \sum_{l=0}^{k-1} (-1)^{p_l} a^l\right), \tag{31}$$

where $p_l$ are binary digits of integer $s$. Then the theorem of convolution of atomic functions may be applied to (31) and (30) for iterative construction of equations of $\mathrm{fip}_{\sqrt[k]{m},n}(x)$ with increasing $n$.

Large variety of forms of equations for $\mathrm{fip}_{a,n}(x)$ move us to conclusion that general equation for such functions does not exist and most functions of the family are not atomic.

## 4. CONNECTION BETWEEN ATOMIC FUNCTIONS AND WAVELETS

### 4.1. Some Notes on General Wavelet Construction

Wavelets are well-known basis systems presenting basis constructed from scaled shifts of one function called mother wavelet. Usually construction of wavelets starts from construction of multiresolution analysis [39–41], a system of closed embedded subspaces $V_j \subset L^2(\mathbb{R})$ with following properties:

$$V_j \subset V_{j+1}, \quad f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}) \quad \text{and} \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}. \tag{32}$$

Shifts of scaling function $\varphi(x) \in V_0$ form a Riesz or orthonormal basis in $V_0$. The first two properties make scaling function be represented as a sum of its double scaled shifts. This representation is called scaling equation

$$\varphi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \varphi(2x - n). \tag{33}$$

Some times besides (33) its Fourier transform is considered in the form

$$\hat{\varphi}(\omega) = H_0\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right). \tag{34}$$

If $\hat{\varphi}(\omega)$ is known then $H_0(\omega)$ found from (34). Otherwise $\hat{\varphi}(\omega)$ may be obtained from $H_0(\omega)$

$$\hat{\varphi}(\omega) = \prod_{k=1}^{\infty} H_0\left(\frac{\omega}{2^k}\right). \tag{35}$$

Similarity between (3), (11) and (35) should be noted. If scaling function is found than corresponding wavelet function may be obtained in the form

$$\psi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \overline{h_{-n-1}} \varphi(2x - n). \tag{36}$$

If Fourier transform of scaling function is known another form of (36) may be used

$$\hat{\psi}(\omega) = e^{\frac{i\omega}{2}} \overline{H_0\left(\frac{\omega}{2} + \pi\right)} \hat{\varphi}\left(\frac{\omega}{2}\right). \tag{37}$$

Note that formulae (36) and (37) present the most common but not exclusive way of wavelet construction.

Let's consider several constructions of wavelets connected to atomic functions.

### 4.2. Derivatives of Atomic Functions up(x) and $h_a(x)$ as Wavelets

Interesting construction of wavelet based on atomic function up$(x)$ is presented in [42]. Function up$(x)$ is considered as a scaling function and up$'(x)$ is considered as wavelet. Authors demonstrated efficiency of wavelet in image processing. In the last section of the paper generalization of up$(x)$ function $\phi(x)$ is introduced as supported on $[-1;1]$ solution of equation

$$\frac{d\phi(x)}{dx} = \frac{L^2}{2(L-1)}\left(\phi(Lt+L-1) - \phi(Lt-L+1)\right), \tag{38}$$

where $L$ is real parameter. One can compare (38) and (9) and find that $\phi(x) = \frac{1}{L-1}\mathrm{h}_L\left(\frac{x}{L-1}\right)$. Obtained wavelets are infinitely smooth and compactly supported. Scaled shifts of wavelets do not form classical orthonormal basis but constitute a frame.

### 4.3. Hutlets Family

Article [43] presented by the disciple of Wolfgang Hilberg Uwe Meyer-Base and Fred J. Taylor is imbued with spirit of iterative algorithms. In the introduction some properties of up$(x)$ are considered. Then authors presented a family of scaling functions defined as compactly supported solutions of scaling equation

$$\varphi(x) = \sum_{k=0}^{N-1} c_k \varphi(2x-k), \quad \text{where} \quad c_k = \frac{2}{N} \tag{39}$$

and $N$ is even positive integer. Note that (39) is a special case of (33) and its solution may be found with iterative algorithm. With increasing $N$ $\varphi(x)$ tends to up$(x)$. For each scaling function corresponding wavelet is found according to (36). Obtained functions are compactly supported smooth biorthogonal wavelets. An efficient implementation of the digital wavelet transform based on obtained wavelets is discussed. Use of constructed wavelets in application to the detection of discontinuities in signals problem is demonstrated. If authors of [43] denoted up$(x)$ as hut$(x)$ according to [12, 17] then they called their wavelets "hutlets".

### 4.4. Kravchenko-Meyer Wavelets

Consider Meyer wavelet construction. At first Fourier transform of scaling function $\hat{\varphi}(\omega)$ is constructed. It is known [39–41] that following conditions:

$$\begin{aligned}
&\hat{\varphi}(\omega) = \hat{\varphi}(-\omega),\\
&\operatorname{supp}(\hat{\varphi}(\omega)) = \left[-\frac{4\pi}{3};\frac{4\pi}{3}\right] \quad \text{and} \quad \hat{\varphi}(\omega) = 1 \quad \text{for} \quad \omega \in \left[-\frac{2\pi}{3};\frac{2\pi}{3}\right],\\
&\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega+2\pi n)|^2 = 1
\end{aligned} \tag{40}$$

satisfied so it is sufficient to (37) produce orthogonal wavelets. To construct Fourier transform of scaling function meeting conditions (40) Meyer used trigonometric polynomials of special form.

Fourier transforms of scaling function of Kravchenko-Meyer wavelets [1, 2, 31, 44, 45] are square roots of sums of shifts of atomic functions. Original construction was first presented in [45] and [46]. For more clear consideration we introduce function $\chi(\omega)$ with following properties:

$$\begin{aligned}
&\chi(\omega) = \chi(-\omega),\\
&\operatorname{supp}(\chi(\omega)) = \left[-\frac{4\pi}{3};\frac{4\pi}{3}\right] \quad \text{and} \quad \chi(\omega) = 1 \quad \text{for} \quad \omega \in \left[-\frac{2\pi}{3};\frac{2\pi}{3}\right],\\
&\sum_{n\in\mathbb{Z}} \chi(\omega+2\pi n) = 1.
\end{aligned} \tag{41}$$

Such function may be constructed as a partial sum of partition of unity by atomic functions [31].

Consider construction of $\chi(\omega)$ in the form of convolution of rectangular pulse $\phi\left(\frac{\omega}{2\pi}\right)$ supported on $[-\pi;\pi]$ and arbitrary even function $g(\omega)$ with unit square supported on $\left[-\frac{\pi}{3};\frac{\pi}{3}\right]$. Due to the construction it meets all requirements (41) and suitable for construction of wavelets. To obtain good

wavelets smooth $\chi(\omega)$ is recommended. We use for $g(\omega)$ properly scaled atomic functions. This new approach to construction of $\chi(\omega)$ is discussed in detail in [47] presented at current conference.

Both modifications of construction leads to smooth orthonormal wavelets, which are not compactly supported but have short effective support. They are useful in noise reduction, sound and image compression and wavelet differentiation.

### 4.5. Kravchenko-Daubechies wavelets

Construction of Daubechies wavelets [39–41] starts from $H_0(\omega) = \cos^n\left(\frac{\omega}{2}\right) T(\omega)$. Then construction of scaling function and wavelets are performed according to (35) and (37). In Modification of Daubechies wavelets construction [48] consists in replacement trigonometric polynomial $T(\omega)$ with spectra of atomic functions. Obtained wavelets are smooth, orthogonal and compactly supported.

## 5. NON-ARCHIMEDEAN CALCULUS

Now consider one special concept presented by V. L. Rvachev. All classical calculus is based on the notion of natural numbers. Starting with 1 and applying addition operation we obtain $2 = 1 + 1$, then $3 = 2 + 1$, etc. This process could not be stopped and arbitrary large numbers could be obtained. Archimedean axiom states that for each arbitrary large number the larger one cold be found. Then there is no the largest number. Rvachev considered alternative addition operation

$$x \mathring{+} y = \frac{x + y}{1 + \frac{xy}{c^2}}, \tag{42}$$

where $c$ is given fixed number. Starting with ordinary $\mathring{1} = 1$ one can construct sequence of relativistic natural numbers $\mathring{2} = \mathring{1} \mathring{+} \mathring{1}$, $\mathring{3} = \mathring{2} \mathring{+} \mathring{1}$ etc. Then other operations based on addition (42) are derived and alternative calculus is developed [49]. Note similarity of (42) to composition law for velocities in relativity theory. If $c$ considered as speed of light then new calculus may be applied to problems of mathematical physics and cosmology.

## 6. CONCLUSION

Rvachev was a talented mathematician and computer scientist, a hard worker and an effective leader who left behind a rich legacy. The ideas and concepts proposed by him in the 60–70 s haven't lost their relevance to this day. The scientific directions founded by him are developing all over the world.

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