Introduction to Angular Momentum, Work, and Power in Rotational Motion

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1 Introduction

Angular momentum is a fundamental concept in physics that describes the motion of objects in rotational movement. It is particularly important in systems involving circular motion and is conserved in isolated systems, making it a powerful tool in analyzing physical phenomena.

2 Definition of Angular Momentum

Angular momentum, $\vec{L}$, of a particle about a point is defined as the cross product of the particle's position vector, $\vec{r}$, relative to the point, and its linear momentum, $\vec{p}$:

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{p} = m\vec{v}$ and m is the mass of the particle, and $\vec{v}$ is its velocity.

3 Conservation of Angular Momentum

The angular momentum of a system remains constant if no external torques are acting on it. This principle can be expressed as:

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

where $\vec{\tau}$ is the external torque applied to the system. In the absence of external torques, $\vec{\tau} = 0$ and therefore $\frac{d\vec{L}}{dt} = 0$, implying conservation of angular momentum.

4 Applications of Angular Momentum

4.1 Astronomy

The conservation of angular momentum explains why planets in our solar system rotate faster as they get closer to the sun in their elliptical orbits.

4.2 Sports

In figure skating, a skater pulls their arms in to spin faster on the ice. This is a practical demonstration of conservation of angular momentum.

5 Kinematic Relationships in Rotational Motion

5.1 Linear Momentum and Angular Momentum

Linear momentum $\vec{p}$ of a body is defined as the product of its mass m and its velocity $\vec{v}$:

$$\vec{p} = m\vec{v}$$

Angular momentum $\vec{L}$ for a particle can be calculated using the position vector $\vec{r}$ and the linear momentum $\vec{p}$ as follows:

$$\vec{L} = \vec{r} \times \vec{p} = rm\vec{v}\sin(\theta)$$

where θ is the angle between $\vec{r}$ and $\vec{v}$.

5.2 Right-Hand Rule for Angular Momentum

The direction of angular momentum is determined using the right-hand rule. Below is a diagram illustrating this rule:

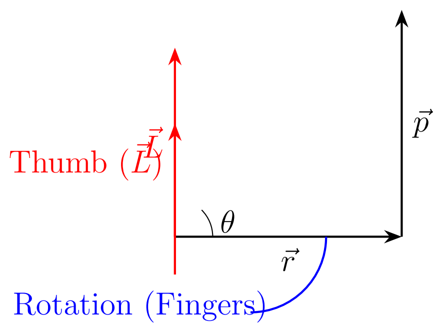

Figure 1: Applying the right-hand rule to determine the direction of angular momentum.

6 Angular Velocity and Angular Acceleration

Angular velocity ω is defined as the rate of change of angular displacement and is given by:

$$\omega = \frac{d\theta}{dt}$$

where θ is the angular displacement. Angular velocity is related to the linear velocity v by the equation:

$$v = r\omega$$

where r is the radius of the motion path.

Angular acceleration α is defined as the rate of change of angular velocity:

$$\alpha = \frac{d\omega}{dt}$$

It describes how quickly the angular velocity changes over time. Angular acceleration is related to the tangential acceleration a_t by:

$$a_t = r\alpha$$

7 Work and Power in Rotational Motion

7.1 Work Done by a Torque

Work done by a torque in rotational motion is given by the integral of torque τ with respect to angular displacement θ:

$$W = \int \tau \, d\theta$$

where τ is the torque and θ is the angle of rotation.

7.2 Power in Rotational Systems

Power in rotational motion is defined as the rate of doing work or the rate of change of angular energy:

$$P = \frac{dW}{dt} = \tau\omega$$

where P is the power, τ is the torque, and ω is the angular velocity. This relationship shows how effectively a rotating system can perform work over time.

8 Common Moments of Inertia

This section presents the moments of inertia for various standard geometries about their center of mass, unless otherwise noted. The moment of inertia (I) is a measure of an object's resistance to changes in its rotational motion.

8.1 Point Mass

For a point mass m at a distance r from the axis of rotation:

$$I = mr^2$$

8.2 Solid Sphere

For a solid sphere of radius R and mass m:

$$I = \frac{2}{5}mR^2$$

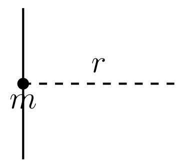

Figure 2: Point mass at a distance r from the rotation axis.

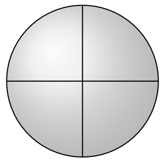

Figure 3: Solid sphere.

8.3 Hollow Sphere

For a thin-walled hollow sphere of radius R and mass m:

$$I = \frac{2}{3}mR^2$$

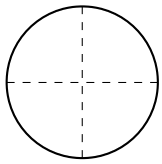

Figure 4: Hollow sphere.

8.4 Solid Cylinder

For a solid cylinder of radius R and mass m, about its axis:

$$I = \frac{1}{2}mR^2$$

8.5 Hollow Cylinder

For a hollow cylinder (thin-walled) of radius R and mass m, about its axis:

$$I = mR^2$$

8.6 Rectangular Plate

For a rectangular plate of mass m, height h, and width w, spinning around an axis through its center and perpendicular to the plate:

$$I = \frac{1}{12}m(h^2 + w^2)$$

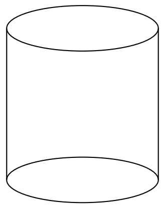

Figure 5: Solid cylinder.

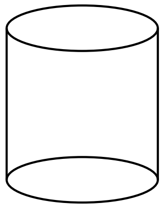

Figure 6: Hollow cylinder.

8.7 Thin Rod

For a thin rod of length L and mass m, spinning around an axis through its center perpendicular to the length:

$$I = \frac{1}{12}mL^2$$

9 Rolling Motion

Rolling motion is a type of motion that involves both translational and rotational movement of an object, such as a sphere. One of the key concepts in rolling motion is rolling without slipping, which occurs when the point of the object in contact with the ground does not slide.

9.1 Condition for Rolling Without Slipping

The condition for rolling without slipping is given by:

$$v = r\omega$$

where v is the translational velocity of the center of mass, r is the radius of the sphere, and ω is the angular velocity. This relationship ensures that the linear velocity at the point of contact with the ground is zero.

9.2 Kinetic Energy in Rolling Motion

The total kinetic energy (K) of a rolling object is the sum of its translational kinetic energy and rotational kinetic energy:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where m is the mass of the sphere, I is the moment of inertia about the axis of rotation ($I = \frac{2}{5}mR^2$ for a solid sphere), and v and ω are linked by the rolling without slipping condition.

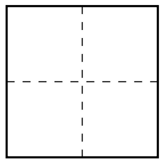

Figure 7: Rectangular plate.

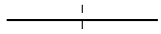

Figure 8: Thin rod.

9.3 Illustration of Rolling Motion

Below is a diagram illustrating a sphere rolling without slipping. The arrow indicates the direction of motion, while the rotation is shown with respect to the center of mass.

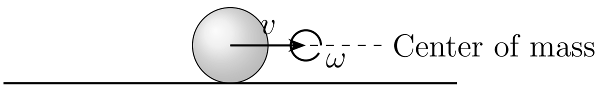

Figure 9: A sphere rolling without slipping on a flat surface.

This section on rolling motion explains the dynamics of spherical objects like balls when they roll without slipping, integrating both translational and rotational aspects of motion. The diagram helps visualize the motion and the forces involved.

10 Calculation of Moment of Inertia Using Integrals

The moment of inertia (I) of an object about an axis is calculated by integrating the product of the square of the distance from the axis (r^2) and the mass density over the volume of the object. The general formula for moment of inertia is:

$$I = \int r^2 \, dm$$

where dm is the mass element and r is the distance from the axis of rotation to the mass element.

10.1 Example: Moment of Inertia of a Disk

Consider a uniform circular disk of radius R and mass M, rotating about an axis through its center perpendicular to its face. The moment of inertia can be calculated by:

$$I = \int_0^R \int_0^{2\pi} r^3 \rho \, d\theta \, dr$$

where r is the radial distance from the axis, θ is the angular coordinate, and $\rho = \frac{M}{\pi R^2}$ is the mass per unit area. Simplifying, we find:

$$I = \frac{1}{2} M R^2$$

This calculation involves setting up an integral over the disk's area, multiplying each infinitesimal mass element by the square of its distance from the axis of rotation.

10.2 Visual Representation

Below is a diagram illustrating the division of a disk into infinitesimal elements for integration:

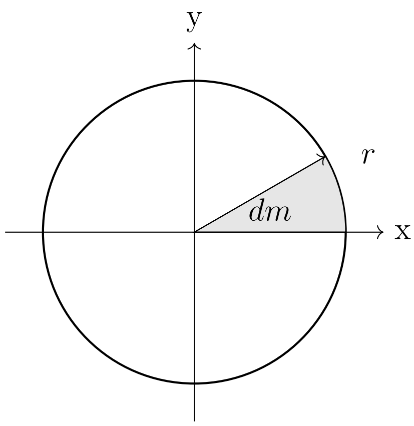

Figure 10: Division of a disk into differential mass elements for calculating moment of inertia.

11 Torque

Torque ($\vec{\tau}$) is a measure of the force causing the rotation of an object about an axis. It is defined as the cross product of the position vector ($\vec{r}$) and the force vector ($\vec{F}$), leading to:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of torque depends on three factors: the magnitude of the force, the distance of the force from the axis of rotation (lever arm), and the angle between the position vector and the force vector.

11.1 Example: Torque on a Door

If a force is applied perpendicularly at the edge of a door, the torque is maximized, as the lever arm is equal to the distance from the hinge to the point where the force is applied. The torque can be calculated by:

$$\tau = rF$$

where r is the distance from the hinge to the point of application of the force, and F is the force applied.

11.2 Illustration of Torque

Below is a diagram showing how torque is applied to a door:

12 Kinematic Equations for Angular Motion

Just as there are four key kinematic equations for linear motion under constant acceleration, there are analogous equations for rotational motion when angular acceleration is constant.

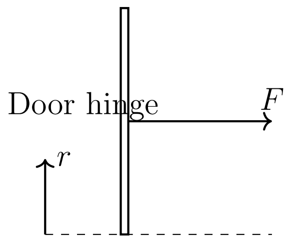

Figure 11: Application of force at the edge of a door to create torque.

These equations relate angular displacement (θ), angular velocity (ω), and angular acceleration (α), and are crucial for solving problems in rotational dynamics.

12.1 Angular Kinematic Equations

Assuming the angular acceleration α is constant, the following equations can be used to describe angular motion:

1. **Angular displacement with initial angular velocity and time:**

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

where θ_0 is the initial angular position, ω_0 is the initial angular velocity, and t is the time elapsed.

2. **Final angular velocity with initial angular velocity and time:**

$$\omega = \omega_0 + \alpha t$$

where ω_0 is the initial angular velocity.

3. **Angular displacement with initial and final angular velocity:**

$$\theta = \theta_0 + \frac{1}{2}(\omega_0 + \omega)t$$

This equation is derived by solving the average of the initial and final velocities over the time t.

4. **Final angular velocity squared:**

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

This equation results from eliminating t between the first two equations.

12.2 Example Application

Consider a rotating disk that starts from rest and has an angular acceleration of 3 rad/s^2. If it rotates for 5 seconds, we can use these equations to find the final angular velocity and the total angle it rotates through:

$$\omega = \omega_0 + \alpha t = 0 + 3 \times 5 = 15 \text{ rad/s}$$
$$\theta = \theta_0 + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2} \times 3 \times 5^2 = 37.5 \text{ rad}$$

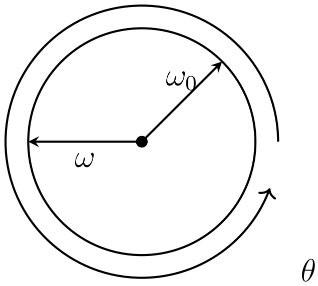

Figure 12: A diagram showing the rotational motion of a disk under constant angular acceleration.

12.3 Visualization of Angular Motion

Below is a diagram depicting the rotational motion with constant angular acceleration:

This section provides a comprehensive overview of the kinematic equations for angular motion under the assumption of constant angular acceleration, mirroring the well-known linear kinematic equations. The example and diagram help illustrate the application of these equations in real-world scenarios.