Physics 3200

Mathematical Methods of Theoretical Physics

Recall Trig Identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Fourier Series

Any periodic function *f(t)* can be decomposed into a sum over an infinite series of sines and cosines, as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}$$

And the coefficients can be found via

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Complex Variables

A complex number is of the Cartesian form:

$$z = x \pm iy$$

Or of the Polar form:

$$z = re^{i\theta}$$

And one can convert between these two forms using the equations

$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

There is also the option to express as sines and cosines

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta$$

Complex numbers can also be integrated over using the Cauchy Integral Formula, also known as "method of residues"

$$f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{z - z_0} dz$$

Or generally, for the n-th derivative

$$f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Curvilinear coordinates

In general, can always write a position vector in Cartesian coordinates as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{1}$$

Or in spherical coordinates as:

$$\vec{r} = r\hat{r} \tag{2a}$$

Technically, (2a) can be written for any vector of any sort, by the definition of a vector, with:

$$r = |\vec{r}| \tag{2b}$$

Note also that:

$$\frac{\vec{r}}{|\vec{r}|} = \hat{r} = \frac{\partial \vec{r}}{\partial r} \tag{2c}$$

For general curvilinear coordinates, as in (1) can also write:

$$\vec{r} = u_1\widehat{u_1} + u_2\widehat{u_2} + u_3\widehat{u_3} \tag{3}$$

If we want to take a differential element of the position vector in Cartesian coordinates, we simply do so by first applying the general differential rule

$$\overrightarrow{dr} = \frac{\partial \vec{r}}{\partial x}dx + \frac{\partial \vec{r}}{\partial y}dy + \frac{\partial \vec{r}}{\partial z}dz \tag{4a}$$

But recognize from (1) that

$$\frac{\partial \vec{r}}{\partial x} = \hat{x} \tag{4b}$$

And similarly for y and z, giving us:

$$\overrightarrow{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z} \tag{4c}$$

This simplicity works because Cartesian coordinate basis vectors are already of unit length.

In general curvilinear coordinates:

$$\widehat{u_k} = \frac{\frac{\partial \vec{r}}{\partial u_k}}{\left|\frac{\partial \vec{r}}{\partial u_k}\right|} \quad k = 1,2,3 \tag{5a}$$

Where the denominator is called a scaling factor or "structure function":

$$U_k \stackrel{\text{def}}{=} \left|\frac{\partial \vec{r}}{\partial u_k}\right| \tag{5b}$$

Now following (4a) we can take the differential position vector in general coordinates as:

$$\overrightarrow{dr} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3 \tag{6a}$$

And take equations (5a) and (5b) together to get the generalized curvilinear differential position vector

$$\overrightarrow{dr} = U_1\widehat{u_1}du_1 + U_2\widehat{u_2}du_2 + U_3\widehat{u_3}du_3 \tag{6b}$$

Now to generalize gradient. In Cartesian coordinates, the gradient of a scalar function ψ is defined:

$$d\psi = grad\psi \cdot \overrightarrow{dr} \tag{7a}$$

Recall definition of dot product is:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{8a}$$

Where the components are themselves dot products of the form

$$A_x = \vec{A} \cdot \hat{x} \tag{8b}$$

So writing the right-hand side of (7a) in Cartesian coordinates can be done like so

$$grad\psi \cdot \overrightarrow{dr} = grad\psi_x dx + grad\psi_y dy + grad\psi_z dz \tag{7b}$$

And recognize that the left-hand side of (7a) is

$$d\psi = \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy + \frac{\partial\psi}{\partial z} dz \tag{7c}$$

Comparing with (8) implies

$$\frac{\partial\psi}{\partial x} = grad\psi_x = grad\psi \cdot \hat{x} \tag{9a}$$

Which leads us to the known equation for Cartesian coordinates

$$grad\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z} \tag{9b}$$

To extend this to generalized curvilinear coordinates, start with (6b) plugged into (7a)

$$d\psi = grad\psi \cdot (U_1 \widehat{u_1} du_1 + U_2 \widehat{u_2} du_2 + U_3 \widehat{u_3} du_3) \tag{10a}$$

And express (7c) in curvilinear coordinates

$$d\psi = \frac{\partial\psi}{\partial u_1} du_1 + \frac{\partial\psi}{\partial u_2} du_2 + \frac{\partial\psi}{\partial u_3} du_3 \tag{10b}$$

This implies

$$\frac{1}{U_k}\frac{\partial\psi}{\partial u_k} = grad\psi \cdot \widehat{u_k} \tag{10c}$$

Which gives all of the necessary components of the gradient for an arbitrary coordinate system

$$grad\psi = \frac{1}{U_1}\frac{\partial\psi}{\partial u_1}\widehat{u_1} + \frac{1}{U_2}\frac{\partial\psi}{\partial u_2}\widehat{u_2} + \frac{1}{U_3}\frac{\partial\psi}{\partial u_3}\widehat{u_3} \tag{11a}$$

Note that, in Cartesian, the gradient operator is often defined from (10c) as

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{11b}$$

This form is convenient for shorthand notation of other derivative operators which are often defined via (11b) like divergence, curl, and the Laplacian:

$$div\vec{F} = \vec{\nabla}\cdot\vec{F} \tag{12a}$$

$$curl\vec{F} = \vec{\nabla}\times\vec{F} \tag{12b}$$

$$divgrad\psi = \vec{\nabla}\cdot\vec{\nabla}\psi = \nabla^2\psi \tag{12c}$$

However, the idea of merely applying a curvilinear version of the gradient operator with a dot or cross product does not easily generalize, as the correct versions of these operators in curvilinear coordinates is given by

$$div\vec{F} = \frac{1}{U_1U_2U_3}\left[\frac{\partial}{\partial u_1}(U_2U_3F_1) + \frac{\partial}{\partial u_2}(U_1U_3F_2) + \frac{\partial}{\partial u_3}(U_1U_2F_3)\right] \tag{13}$$

$$curl\vec{F} = \frac{1}{U_1U_2U_3}\begin{vmatrix} U_1\widehat{u_1} & U_2\widehat{u_2} & U_3\widehat{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ U_1F_1 & U_2F_2 & U_3F_3 \end{vmatrix} \tag{14}$$

$$divgrad\psi = \frac{1}{U_1U_2U_3}\left[\frac{\partial}{\partial u_1}\left(\frac{U_2U_3}{U_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{U_1U_3}{U_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{U_1U_2}{U_3}\frac{\partial\psi}{\partial u_3}\right)\right] \tag{15}$$

Note that the Laplacian is still divergence of gradient as long as you are in the correct coordinates for both operations.