A simple probability problem with an interesting interpretation.

Consider a game where two players take turns flipping a biased coin. The first player to flip heads wins the game, and the coin comes up heads with probability p. If a player flips tails, they simply pass the coin to the other player. The question then, is what is the total probability that the player going first wins the game?

Let the first player be A, and the other player is B. Obviously the probability that A wins on the first turn is p. And the probability that B wins on *their* first turn is thus $p(1-p)$ because A must flip tails followed by B flipping heads. It's obvious that, for a given turn t,

$$Prob(game\ ends\ on\ turn\ t) = p(1-p)^{t-1}$$

And the probability that A wins the game is simply the union of this probability for all odd values of t. Thus

$$Prob(A\ wins) = \sum_{t\ odd}^{\infty} p(1-p)^{t-1} = \sum_{k=0}^{\infty} p(1-p)^{2k+1-1}$$

$$Prob(B\ wins) = \sum_{t\ even}^{\infty} p(1-p)^{t-1} = \sum_{k=1}^{\infty} p(1-p)^{2k-1}$$

By writing out the first few terms of each sum, it can be seen that the first series actually contains the second, i.e.

$$Prob(A\ wins) = p + (1-p)S(p)$$

where

$$S(p) = \sum_{k=1}^{\infty} p(1-p)^{2k-1} = Prob(B\ wins)$$

And then the sum can be easily evaluated by requiring

$$Prob(A\ wins) + Prob(B\ wins) = 1$$

$$p + (1-p)S(p) + S(p) = 1$$

After doing algebra, we arrive at

$$\boxed{S(p) = \frac{1-p}{2-p}}$$

If we check the result for a fair coin with $p = ½$, we find that

$$Prob(B\ wins) = 1/3$$

$$Prob(A\ wins) = 2/3$$

Now for the interesting part. We might ask what value of p would make this coin-flipping game fair for both players? We simply set the above expression for $S(p) = ½$ and get that $p = 0$.

But wait, that means the coin will never flip heads, and the game can never end! One might take the position here that this is a valid mathematical result and it tells us that this game can never be fair. On the other hand, we arrived at the solution by requiring that the game *will* end when we forced the two probabilities to add to 1. This requirement actually allows our result to have meaning, and it is the reason this biased coin produces a fair game. Realistically speaking, if two players were taking turns flipping a coin that can never come up heads, the game will only end when one of them gives up and stops flipping, resulting in the other player winning by default. The math just works.