

Physics 3200

Mathematical Methods of Theoretical Physics

Recall Trig Identities

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

Fourier Series

Any periodic function $f(t)$ can be decomposed into a sum over an infinite series of sines and cosines, as

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L}$$

And the coefficients can be found via

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

Complex Variables

A complex number is of the Cartesian form:

$$z = x \pm iy$$

Or of the Polar form:

$$z = r e^{i\theta}$$

And one can convert between these two forms using the equations

$$\tan \theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

There is also the option to express as sines and cosines

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

Complex numbers can also be integrated over using the Cauchy Integral Formula, also known as “method of residues”

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz$$

Or generally, for the n-th derivative

$$f^n(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Curvilinear coordinates

In general, can always write a position vector in Cartesian coordinates as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{I}$$

Or in spherical coordinates as:

$$\vec{r} = r\hat{r} \tag{2a}$$

Technically, (2a) can be written for any vector of any sort, by the definition of a vector, with:

$$r = |\vec{r}| \tag{2b}$$

Note also that:

$$\frac{\vec{r}}{|\vec{r}|} = \hat{r} = \frac{\partial \vec{r}}{\partial r} \tag{2c}$$

For general curvilinear coordinates, as in (I) can also write:

$$\vec{r} = u_1 \widehat{u}_1 + u_2 \widehat{u}_2 + u_3 \widehat{u}_3 \quad (3)$$

If we want to take a differential element of the position vector in Cartesian coordinates, we simply do so by first applying the general differential rule

$$\overrightarrow{dr} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy + \frac{\partial \vec{r}}{\partial z} dz \quad (4a)$$

But recognize from (I) that

$$\frac{\partial \vec{r}}{\partial x} = \hat{x} \quad (4b)$$

And similarly for y and z, giving us:

$$\overrightarrow{dr} = dx \hat{x} + dy \hat{y} + dz \hat{z} \quad (4c)$$

This simplicity works because Cartesian coordinate basis vectors are already of unit length.

In general curvilinear coordinates:

$$\widehat{u}_k = \frac{\frac{\partial \vec{r}}{\partial u_k}}{\left| \frac{\partial \vec{r}}{\partial u_k} \right|} \quad k = 1, 2, 3 \quad (5a)$$

Where the denominator is called a scaling factor or “structure function”:

$$U_k \stackrel{\text{def}}{=} \left| \frac{\partial \vec{r}}{\partial u_k} \right| \quad (5b)$$

Now following (4a) we can take the differential position vector in general coordinates as:

$$\overrightarrow{dr} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \quad (6a)$$

And take equations (5a) and (5b) together to get the generalized curvilinear differential position vector

$$\overrightarrow{dr} = U_1 \widehat{u}_1 du_1 + U_2 \widehat{u}_2 du_2 + U_3 \widehat{u}_3 du_3 \quad (6b)$$

Now to generalize gradient. In Cartesian coordinates, the gradient of a scalar function ψ is defined:

$$d\psi = \text{grad}\psi \cdot \vec{dr} \quad (7a)$$

Recall definition of dot product is:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad (8a)$$

Where the components are themselves dot products of the form

$$A_x = \vec{A} \cdot \hat{x} \quad (8b)$$

So writing the right-hand side of (7a) in Cartesian coordinates can be done like so

$$\text{grad}\psi \cdot \vec{dr} = \text{grad}\psi_x dx + \text{grad}\psi_y dy + \text{grad}\psi_z dz \quad (7b)$$

And recognize that the left-hand side of (7a) is

$$d\psi = \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy + \frac{\partial\psi}{\partial z} dz \quad (7c)$$

Comparing with (8) implies

$$\frac{\partial\psi}{\partial x} = \text{grad}\psi_x = \text{grad}\psi \cdot \hat{x} \quad (9a)$$

Which leads us to the known equation for Cartesian coordinates

$$\text{grad}\psi = \frac{\partial\psi}{\partial x} \hat{x} + \frac{\partial\psi}{\partial y} \hat{y} + \frac{\partial\psi}{\partial z} \hat{z} \quad (9b)$$

To extend this to generalized curvilinear coordinates, start with (6b) plugged into (7a)

$$d\psi = \text{grad}\psi \cdot (U_1 \widehat{u}_1 du_1 + U_2 \widehat{u}_2 du_2 + U_3 \widehat{u}_3 du_3) \quad (10a)$$

And express (7c) in curvilinear coordinates

$$d\psi = \frac{\partial\psi}{\partial u_1} du_1 + \frac{\partial\psi}{\partial u_2} du_2 + \frac{\partial\psi}{\partial u_3} du_3 \quad (10b)$$

This implies

$$\frac{1}{u_k} \frac{\partial \psi}{\partial u_k} = \text{grad} \psi \cdot \widehat{u}_k \quad (10c)$$

Which gives all of the necessary components of the gradient for an arbitrary coordinate system

$$\text{grad} \psi = \frac{1}{u_1} \frac{\partial \psi}{\partial u_1} \widehat{u}_1 + \frac{1}{u_2} \frac{\partial \psi}{\partial u_2} \widehat{u}_2 + \frac{1}{u_3} \frac{\partial \psi}{\partial u_3} \widehat{u}_3 \quad (11a)$$

Note that, in Cartesian, the gradient operator is often defined from (10c) as

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad (11b)$$

This form is convenient for shorthand notation of other derivative operators which are often defined via (11b) like divergence, curl, and the Laplacian:

$$\text{div} \vec{F} = \vec{\nabla} \cdot \vec{F} \quad (12a)$$

$$\text{curl} \vec{F} = \vec{\nabla} \times \vec{F} \quad (12b)$$

$$\text{divgrad} \psi = \vec{\nabla} \cdot \vec{\nabla} \psi = \nabla^2 \psi \quad (12c)$$

However, the idea of merely applying a curvilinear version of the gradient operator with a dot or cross product does not easily generalize, as the correct versions of these operators in curvilinear coordinates is given by

$$\text{div} \vec{F} = \frac{1}{u_1 u_2 u_3} \left[\frac{\partial}{\partial u_1} (U_2 U_3 F_1) + \frac{\partial}{\partial u_2} (U_1 U_3 F_2) + \frac{\partial}{\partial u_3} (U_1 U_2 F_3) \right] \quad (13)$$

$$\text{curl} \vec{F} = \frac{1}{u_1 u_2 u_3} \begin{vmatrix} U_1 \widehat{u}_1 & U_2 \widehat{u}_2 & U_3 \widehat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ U_1 F_1 & U_2 F_2 & U_3 F_3 \end{vmatrix} \quad (14)$$

$$\text{divgrad} \psi = \frac{1}{u_1 u_2 u_3} \left[\frac{\partial}{\partial u_1} \left(\frac{U_2 U_3}{U_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{U_1 U_3}{U_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{U_1 U_2}{U_3} \frac{\partial \psi}{\partial u_3} \right) \right] \quad (15)$$

Note that the Laplacian is still divergence of gradient as long as you are in the correct coordinates.