A Sieve-Theoretic Reformulation of the Goldbach Conjecture

Bill C Riemers

September 11, 2025

Abstract

A windowed sieve framework for Goldbach is presented based on the quadratic form

$$Q(n, m) \;=\; (n - m)(n + m), \tag{1}$$

which centers analysis at the midpoint n and treats offsets m symmetrically. This formulation interfaces with Eratosthenes–type sieves below the prime–forcing cutoff, avoids the classical parity obstruction, and yields certified lower bounds as a product of conservative Euler factors.

Unconditionally, a *certified analytic lower bound* on the windowed Goldbach count $\mathcal{G}(n; M)$ (with $M = \lfloor n/2 \rfloor$) is proved via explicit Euler–Mertens products [10, 12, 15, 2], valid for all $n \geq 6353$. A *rescaling lemma* shows this bound holds uniformly for every smaller window $M = \alpha n$ with $0 < \alpha \leq \frac{1}{2}$, and a monotonicity corollary extends it to larger windows by set inclusion.

Computationally, windowed Goldbach pairs are exhaustively enumerated for every even $2n < 2 \cdot 10^8$. Across seven decades, the normalized deviations from the parameter-free Hardy–Littlewood baseline (HL–A) [7] decrease steadily. In the seventh decade ($10^7 \leq n < 10^8$) the *raw decade maxima* already satisfy

$$|\Lambda_{\min}| \leq 1.8 \cdot 10^{-2}, \qquad |\Lambda_{\max}| \leq 3.9 \cdot 10^{-3}, \qquad |\Lambda_{\text{avg}}| \leq 3.2 \cdot 10^{-4}, \tag{2}$$

providing large-scale validation of HL–A in this windowed setting. As a robustness check, for each decade and for each metric, the *second-largest* absolute deviation is *strictly decreasing* across the seven decades, as expected when occasional outliers persist; the conclusions are unchanged under this pruning. Observed blockwise maxima align with the singular-series structure: peaks occur when n is a multiple of *half a primorial* (i.e., $p\#/2$), yielding the expected primorial plateaus.

A *conditional reduction* is also established: assuming a short-interval Bombieri–Vinogradov hypothesis (SI–BV$_\theta$ with $\theta > \frac{1}{2}$, strictly weaker than the full Hardy–Littlewood asymptotic), the minor-arc error is uniformly dominated and $R_2(N) > 0$ holds for all sufficiently large even N. Combined with exhaustive verification up to $2n_*$, this yields Goldbach for all even integers > 2.

In summary, contributions are: (i) a certified sieve–theoretic lower bound with explicit constants, uniform in the window; (ii) the first broad, decade-by-decade numerical validation of HL–A at sub-percent scale in this framework; (iii) a structural explanation of maxima via singular-series (odd-primorial) plateaus; and (iv) a sharp reduction of the remaining analytic task to short-interval equidistribution of primes.

1 Introduction

1.1 Motivation

The Goldbach Conjecture, the Twin Prime Conjecture, and Polignac's Conjecture each address the distribution of prime pairs in different settings. A natural generalization emerges from considering these problems together: along suitable arithmetic or algebraic paths in the (n, m)-grid, prime pairs appear with a density consistent with the heuristic $\#\mathcal{S}_N/\log^2 N$. Formulated precisely, this leads to the following working conjecture.

Conjecture 1 (General prime-pair density (motivating observation))**.**
Let (p, q) be an odd prime pair satisfying

$$p + q = 2n, \quad p - q = 2m, \quad pq = n^2 - m^2, \quad |m| \le n, \tag{3}$$

with (m, n) constrained to a fixed line or other low-degree polynomial path in the (m, n)-grid. Then there exists N_0 such that for all $N \ge N_0$, any consecutive set of $O(N)$ odd integers along that path contains at least $O(N/\log^2 N)$ such prime pairs.

The statement above is given in a simplified form, with notation consistent throughout this paper. A more detailed formulation, including explicit bounds in terms of $C_{\min}$ and $C_{\max}$ and the role of the reference sieve factor $\mathcal{B}_{\text{ref}}$, appears in Conjecture A.1 and the definition of admissibility (Definition A.1).

While unproven, this conjecture provides a coherent framework in which the problems above are special cases. In what follows, we focus on the Goldbach setting as a concrete instance for developing and testing the sieve-theoretic methods, before considering broader applications.

Remark (Scope). Conjecture 1 motivates the windowed sieve setup only; no theorem, lemma, or corollary in this paper depends on it. Unconditional results use sieve bounds and Euler–Mertens products; the only conditional input appears in Corollary 1.

1.2 Contributions and reduction overview.

We now summarize the certified bounds, the large–scale validation of the heuristic baseline, the window–scalability results, the structure of extrema, and the final conditional reduction to short–interval equidistribution.

1. **Calibrated sieve–heuristic and per–term normalization.** We formalize the sieve–heuristic baseline on the structured family $Q_m = n^2 - m^2$, introducing a per–term normalization $C_\star(n; I)$ that is consistent with Conjecture A and yields asymptotic predictions proportional to $\mathcal{S}_{\text{GB}}(2n)$. This fixes units and removes binning artefacts for all subsequent comparisons.
2. **Statistical convergence of normalized deviations (validation of HL–A).** We define normalized deviations between measured and predicted pair counts and evaluate them across seven decades up to $2n = 2 \cdot 10^8$. By the final decade the deviations fall below $1.8 \cdot 10^{-2}$ (minimum), $3.9 \cdot 10^{-3}$ (maximum), and $3.2 \cdot 10^{-4}$ (average), with monotone decay across decades, providing the first large–data validation that windowed counts converge to the HL–A baseline and justifying extrapolation beyond the tested range.

3. **Certified analytic (shifted–product) lower bound (pairs).** By Lemma 2, in the extremal out–of–sync case one has

$$\mathcal{G}(n;M) \;\geq\; \frac{n}{2} \prod_{\substack{p>2 \\ p\leq\sqrt{n}}} \Big(1-\frac{1}{p-1}\Big) \prod_{\substack{p>2 \\ p\leq\sqrt{\frac{3n}{2}}}} \Big(1-\frac{1}{p-1}\Big), \tag{4}$$

where $M=\lfloor n/2 \rfloor$. Using the explicit Mertens enclosure [15, 2, 7, 12]

$$\prod_{p\leq\sqrt{x}} \Big(1-\tfrac{1}{p-1}\Big) \;\sim\; \frac{K_{\mathrm{EM}}}{\log x}, \qquad K_{\mathrm{EM}}=4e^{-\gamma}C_2, \tag{5}$$

yields, for large n,

$$\mathcal{G}(n;M) \;\gtrsim\; \frac{K_{\mathrm{EM}}^2\, M}{\log n\;\log(\frac{3n}{2})}. \tag{6}$$

On our tested range this specializes to the concrete inequality

$$\mathcal{G}(n;M) \;\geq\; \frac{2.1518\, M}{\log^2 n}. \tag{7}$$

The certification (4)–(7) is unconditional: it does not invoke HL–A, $\mathcal{S}_{\mathrm{GB}}$, or β_{eval}.

4. **Uniform window scalability and monotone extension.** The certified lower bound extends *uniformly in the window size* for every $\alpha\in(0,\frac{1}{2}]$ by Lemma C.1:

$$\mathcal{G}(n;\,\alpha n) \;\geq\; \frac{\mathcal{C}_{-,n}(\alpha)}{\log^2 n}\,(\alpha n), \tag{8}$$

with the natural right–edge cutoff $\sqrt{n+\alpha n}$ inside $\mathcal{C}_{-,n}(\alpha)$. By set inclusion, the count is monotone in the window, so for all $\alpha\in[\frac{1}{2},1)$

$$\mathcal{G}(n;\,\alpha n) \;\geq\; \mathcal{G}\Big(n;\,\tfrac{1}{2}n\Big), \tag{9}$$

as recorded in Corollary C.1.

5. **Extrema structured by the singular series (primorial plateaus).** Writing

$$\mathfrak{S}(2n) \;=\; 2C_2 \prod_{\substack{p\mid n \\ p\geq 3}} \frac{p-1}{p-2}, \tag{10}$$

Proposition 1 shows that $\mathfrak{S}(2n)$—and hence the normalized C-statistic—achieves record and local plateaus when the odd part of n is divisible by the odd primorial $P_y=\prod_{3\leq p\leq p_y} p$; in particular, on $[P_y,\,p_{y+1}P_y)$ the maxima occur precisely at multiples of P_y.

6. **Pointwise positivity under short–interval equidistribution (reduction).** Assuming the short–interval Bombieri–Vinogradov hypothesis (88), Corollary 1 yields an explicit N_0 such that

$$R_2(N) \;>\; 0 \qquad \text{for every even } N\geq N_0. \tag{11}$$

Together with our exhaustive verification up to $2n_*$, this reduces Goldbach for all even $N\geq 4$ to (88) on a tail interval; no Hardy–Littlewood asymptotic is assumed.

7. **Bridging computation and analytic bounds (no gaps).** Explicit computation verifies all even numbers up to $2n = 2n_*$. The certified lower bound applies uniformly for all $n \geq 6353$ (cf. Fig. 5), and by (8)–(9) the same holds for all window sizes under consideration. Hence the verified initial segment and the certified asymptotic regime overlap without gaps; under (88) the pointwise positivity (11) completes the reduction.

Remark. Our use of the singular series and the $\frac{n}{log^2 n}$ scale follows the classical circle-method heuristic of Hardy–Littlewood.[7] We do not claim novelty for these ingredients. The contributions here are (i) a per-term, windowed adaptation tailored to $Q_m = n^2 - m^2$ with explicit calibration via $\mathcal{B}_{\text{win}}$; (ii) a certified sieve lower bound in this setting; and (iii) a statistical protocol that tests the parameter-free curve $2\mathcal{S}_{\text{GB}}(2n)$ against data across decades. All statements relying on Hardy-Littlewood Conjecture A (HL-A) are clearly labeled as model-based; certified results are unconditional.

1.3 Readers' Guide

Section 2 sets up the sieve–heuristic framework: the quadratic form $Q(n,m)$, the window $M(n)$, and the HL–A baseline and normalizations used throughout. Section 3 presents the computational study up to $2n < 2 \cdot 10^8$, including decade-wise deviations and the primorial plateaus. Section 4 contains the core sieve-theoretic results: the reduction lemma (Sec. 4.1), the certified lower bound (Thm. 1), its conditional corollary under short-interval equidistribution (Sec. 4.4), and the primorial maxima proposition (Sec. 3.6). Appendices collect technical enclosures and window rescaling.

2 Sieve–Heuristic Framework

Remark (Terminology: "model" vs. "theorem")*.* Throughout, "model" refers to the sieve–heuristic framework combining the local factors $\prod_{p\geq 3}(1-\frac{2}{p})$, the semiprime singular series $\mathcal{S}_{\text{GB}}(2n)$, and the evaluation calibration $\beta_{\text{eval}}(I)$, yielding predicted quantities such as $\mathring{C}$ and $C_\star$. These are *model-based predictions* (heuristic expectations), not theorems. Measured quantities (e.g. C are exact given the data.

The rigorous component is developed in the Sieve–Theoretic section, where we establish a certified analytic lower bound that supports the later arguments. Other relationships stated here (e.g. $C_\star(n;I) \to \beta_{\text{eval}}(I)\,\mathcal{S}_{\text{GB}}(2n)$) are presented to convey heuristic understanding and are not required for the rigorous result itself.

Starting with the sequence:

$$Q_m = n^2 - m^2 = (n-m)(n+m) \tag{12}$$

Let $S_n = \{p \in \mathbb{P} \mid p < \sqrt{n}\}$ be the set of all primes less than $\sqrt{n}$.

A sieve is constructed over the range $m \in [1, M]$, for some $M = O(n)$, to eliminate values of m for which $Q_m = (n-m)(n+m)$ has small prime divisors. Initially, all m in the range are candidates, and those for which $Q_m \equiv 0 \mod p$ for any $p \in S_{\sqrt{N+M}}$ are iteratively removed. This process is equivalent to eliminating values of m lying in specific residue classes modulo each small prime, as described by standard sieve methods (see [6, 9, 4]). [1][2]

For each $p \in S_{\sqrt{N+M}}$, note:

$$Q_m \equiv 0 \mod p \iff (n-m)(n+m) \equiv 0 \mod p \tag{13}$$

which implies:

$$m^2 \equiv n^2 \mod p \tag{14}$$

A convenient *reference sieve product* that captures the idealized effect of eliminating two residue classes per odd semi-prime candidate Q_m in the absence of any alignment or discretization artefacts, and is defined as follows.

Definition 1 (Reference Sieve Product)**.**
Let $\mathcal{P}$ denote the set of odd primes up to some bound y. Following the analysis of Iwaniec–Kowalski [9], the reduction for odd semiprimes is expressed by the sieve product

$$\mathcal{B}_{\text{ref}}(y) := \prod_{\substack{3 \le p \le y \\ p \in \mathcal{P}}} \left(1 - \frac{2}{p}\right), \tag{15}$$

since the congruence $n^2 - m^2 \equiv 0 \pmod p$ has exactly two solutions for each odd prime p.

This product represents the multiplicative reduction factor in the *idealized* case where precisely two residue classes are eliminated for every odd prime, with no further perturbations. The prime $p = 2$ is omitted, as the halving from restricting to odd m-values is already absorbed into the initial count M.

Before proceeding further with model definitions, it is important to define what is measured, so that functions using $\mathcal{B}_{\text{ref}}$ can be defined to allow an accurate parameter free comparison.

Definition 2 (Empirical HL–normalized measurements (from semiprime survivors))**.**
Let

$$I^{\text{par}} := \{\, m \in I : n^2 - m^2 \text{ is odd} \,\} = \{\, m \in I : n + m \equiv 1 \pmod 2 \,\}. \tag{16}$$

Far a window I with $M := |I^{\text{par}}|$. For each $m \in I^{\text{par}}$, set

$$y(n, m) := \lfloor \sqrt{n + |m|} \,\rfloor. \tag{17}$$

Let

$$N_M(2n; I) \;:=\; \#\{\, m \in I^{\text{par}} \} :\; p \nmid (n^2 - m^2) \text{ for all } p \le y(n, m) \,\} \tag{18}$$

[1] A related idea appears in work by Song [16], who proposed a sieve partitioning method to preserve minimal composite structure when analyzing Goldbach pairs. While his approach differs significantly in formulation and does not employ the multiplicative structure used herein, it reflects a similar intuition—that full prime sieving is not always necessary.

[2] Recasting the Goldbach condition in terms of the quadratic form $Q(n, m) = (n - m)(n + m)$ does not alter the underlying problem, but it provides a parametrization in which windowing and sieve reduction steps are expressed more cleanly, and where the semiprime structure is explicit.

be the number of surviving *semiprimes* in I.

Define the measured (pairs-scale) constant

$$C(n;I) \;:=\; \frac{2\log^2 n}{M}\,N_M(2n;I). \tag{19}$$

(Equivalently, the semiprime-scale version is $C^{(\mathrm{sem})}(n;I) := \frac{\log^2 n}{M} N_M(2n;I)$ with $C = 2\,C^{(\mathrm{sem})}$.)

For a decimal block $B_{d,k} = [d\cdot 10^k, (d+1)\cdot 10^k)$,

$$n_0 := \arg\min_{n\in B_{d,k}} C(n;I), \qquad n_1 := \arg\max_{n\in B_{d,k}} C(n;I), \tag{20}$$

$$C_{\min}(d,k) := C(n_0;I), \quad C_{\max}(d,k) := C(n_1;I), \quad C_{\text{avg}}(d,k) := \frac{1}{\#B_{d,k}} \sum_{n\in B_{d,k}} C(n;I). \tag{21}$$

The $\mathcal{B}_{\text{ref}}$ gives us a good way to evaluate a probably of a single semiprime reduction. However, what is really useful is a baseline of how many semiprimes to expect for a given value n. For this we turn to defining $C_\star$ and related expressions as follows:

Definition 3 (Per-term window baseline)**.**
Let $I \subset \mathbb{Z}\setminus\{0\}$ be a finite window and $I^{\text{par}} := \{\, m\in I :\; n+m \equiv 1 \pmod 2\,\}$. For each $m \in I^{\text{par}}$ set

$$y(n,m) := \big\lfloor \sqrt{n+|m|}\,\big\rfloor. \tag{22}$$

Define the *window baseline*

$$\mathcal{B}_{\text{win}}(n;I) := \sum_{m\in I^{\text{par}}} \prod_{\substack{3\le p\le y(n,m)\\ p\in\mathbb{P}}} \Big(1-\frac{2}{p}\Big). \tag{23}$$

Let C_2 be the twin–prime constant and $\kappa := 4e^{-2\gamma}C_2$.[7, 12] Define the Goldbach singular series (pairs–scale)

$$S_{\text{GB}}(2n) := 2\,C_2 \prod_{\substack{p\mid n\\ p\ge 3}} \frac{p-1}{p-2}. \tag{24}$$

Heuristic counts on I:

$$\begin{aligned} \mathbb{E}[\text{Goldbach representations (unordered)}] \;&\approx\; S_{\text{GB}}(2n)\,\mathcal{B}_{\text{win}}(n;I),\\ \mathbb{E}[\text{Goldbach pairs (ordered)}] \;&\approx\; 2\,S_{\text{GB}}(2n)\,\mathcal{B}_{\text{win}}(n;I). \end{aligned} \tag{25}$$

Per-term HL–normalized constant (baseline):

$$C_\star(n;I) \;:=\; \frac{1}{\kappa}\,\frac{\log^2 n}{|I^{\text{par}}|}\,S_{\text{GB}}(2n)\,\mathcal{B}_{\text{win}}(n;I). \tag{26}$$

Introduce the evaluation calibration

$$\beta_{\text{eval}}(I) \;:=\; \lim_{n\to\infty} \frac{1}{\kappa}\,\frac{\log^2 n}{|I^{\text{par}}|}\,\mathcal{B}_{\text{win}}(n;I), \tag{27}$$

so that $C_\star(n;I) \to \beta_{\text{eval}}(I)\,S_{\text{GB}}(2n)$ as $n\to\infty$ with $|I| = o(n)$.

Convention. “Unordered” counts $\{p,q\}$ once; “ordered” counts (p,q) and (q,p) separately, hence the extra factor 2.

Next apply $C_\star$ function in definitions that provides a clean way to define predicted to match our empirical measured values.

Definition 4 (HL–A normalized predictions (Goldbach, pairs))**.**
We absorb the windowed log effect into the prediction via a Harding Littlewood Circle correction factor:

$$\mathcal{H}(n;I) \;:=\; \frac{\log^2 n}{|I^{\mathrm{par}}|} \sum_{m\in I^{\mathrm{par}}} \frac{1}{\log(n-m)\,\log(n+m)}\,, \tag{28}$$

for n and I such that $n \pm m \geq 3$ for all $m \in I^{\mathrm{par}}$.

Fix a window $I \subset \mathbb{Z} \setminus \{0\}$ with $M := |I^{\mathrm{par}}|$. Let $C_\star(n;I)$ be the per-term HL–normalized constant (unordered scale). Define the *predicted (pairs-scale) constant* by

$$\mathring{C}(n;I) \;:=\; 2\,C_\star(n;I)\,\mathcal{H}(n;I). \tag{29}$$

For decimal blocks

$$B_{d,k} := \big[d\cdot 10^k,\, (d+1)\cdot 10^k\big), \qquad d \in \{1,\dots,9\},\; k\in\mathbb{N}, \tag{30}$$

select extremizers by $C_\star$ (equivalently by $\mathring{C}/\mathcal{H}$):

$$\begin{aligned} \mathring{n}_0 &:= \arg\min_{n\in B_{d,k}} \frac{\mathring{C}(n;I)}{\mathcal{H}(n;I)}, \qquad && \mathring{C}_{\min}(d,k) := \mathring{C}(\mathring{n}_0;I),\\ \mathring{n}_1 &:= \arg\max_{n\in B_{d,k}} \frac{\mathring{C}(n;I)}{\mathcal{H}(n;I)}, && \mathring{C}_{\max}(d,k) := \mathring{C}(\mathring{n}_1;I). \end{aligned} \tag{31}$$

For the block average, approximate the slowly varying $\mathcal{H}$ by a two-point proxy at the geometric center:

$$\mathring{C}_{\mathrm{avg}}(d,k) \;:=\; \frac{\mathcal{H}(n_{\mathrm{geom}};I) + \mathcal{H}(n_{\mathrm{geom}}+1;I)}{2\,|B_{d,k}|} \sum_{n\in B_{d,k}} \frac{\mathring{C}(n;I)}{\mathcal{H}(n;I)}, \tag{32}$$

where n_{geom} is the nearest integer (optionally: nearest *odd* integer) to $10^k\sqrt{d(d+1)}$.

Remark. Choosing $\mathring{n}_0, \mathring{n}_1$ via $\mathring{C}/\mathcal{H} = 2C_\star$ avoids recomputing $\mathcal{H}$ on the block; since $\mathcal{H}(n;I) = 1 + O(1/\log n)$ varies slowly, these extremizers coincide with those for $\mathring{C}$ up to $O(1/\log n)$. The two-point proxy $\big(\mathcal{H}(n_{\mathrm{geom}}) + \mathcal{H}(n_{\mathrm{geom}}+1)\big)/2$ captures the parity drift.

Convention. $\mathring{C}$ is on the **ordered-pairs** scale; for the unordered version use $C_\star$.

Finally we can define a Λ used to test the model.

Definition 5 (Relative discrepancy between predicted and measured)**.**
All symbols are as defined above. For any finite index set B (e.g. a decimal block $B_{d,k}$), define the dimensionless relative discrepancies

$$\Lambda_{\mathrm{avg}}(B) \;:=\; \log\frac{C_{\mathrm{avg}}(B)}{\mathring{C}_{\mathrm{avg}}(B)}. \tag{33}$$

$$\Lambda_{\min}(B) \;:=\; \log\frac{C_{\min}(B)}{\mathring{C}_{\min}(B)}, \qquad \Lambda_{\max}(B) \;:=\; \log\frac{C_{\max}(B)}{\mathring{C}_{\max}(B)}. \tag{34}$$

Optionally, the per-n pointwise discrepancy is

$$\Lambda(n; I) \;:=\; \log \frac{C(n; I)}{\mathring{C}(n; I)}. \tag{35}$$

These are on the ordered-pairs scale and satisfy $\Lambda \to 0$ when the model matches measurements. If the percent error is of interest use $\left(e^{\Lambda} - 1\right)$ 100%.

Remark (Order-of-magnitude decay from window log rescaling)*.* If the effective density is proportional to $\frac{1}{\log^2 x}$ and the window spans $\left[\frac{n}{2}, \frac{3n}{2}\right]$, replacing $\log^2 n$ by a window edge produces the envelope

$$F(n) := \frac{\log^2 \frac{3n}{2}}{\log^2 \frac{n}{2}} = 1 + \frac{2\log 3}{\log \frac{n}{2}} + O\Big(\frac{1}{\log^2 n}\Big). \tag{36}$$

Thus the deterministic drift from freezing the log decays like $1/\log n$ (slowly). In practice the numerator is not attained at the extreme edge, so realized drift is smaller but has the same $1/\log n$ scale. This effect is distinct from any circle-method correction $\Lambda(n; I)$.

Remark (Consistency with the independent-pair heuristic)*.* A naïve independence model would replace the factor $\prod_{3\le p\le y}(1 - \frac{2}{p})$ by $\prod_{3\le p\le y}(1 - \frac{1}{p})^2$. Since

$$\frac{1 - \frac{2}{p}}{(1 - \frac{1}{p})^2} = 1 - \frac{1}{(p-1)^2}, \qquad \prod_{p\ge 3}\Big(1 - \frac{1}{(p-1)^2}\Big) = C_2, \tag{37}$$

one has [7, 12]

$$\prod_{3\le p\le y}\Big(1 - \frac{2}{p}\Big) \sim \frac{4e^{-2\gamma} C_2}{\log^2 y}, \qquad \prod_{3\le p\le y}\Big(1 - \frac{1}{p}\Big)^2 \sim \frac{4e^{-2\gamma}}{\log^2 y}. \tag{38}$$

Thus, if one uses the independent baseline, the missing twin-correlation factor is exactly C_2; using the pairs singular series $S_{\mathrm{GB}}(2n)$ (which already incorporates this correlation) restores the same $M/\log^2 n$ scale constant as the $(1 - \frac{2}{p})$ baseline. Since we work exclusively on the pairs scale with S_{GB}, the two viewpoints agree.

Remark (Scope and validation)*.* The constructions and normalizations above (e.g. $C_\star$, $\mathring{C}$, β_{eval}) are heuristic and conditioned on the Hardy–Littlewood Conjecture A and the usual independence assumptions behind the sieve baseline. They are presented to define the *predicted* quantities that should track our *measured* constants. No quantitative error bounds are proved here. The degree of agreement between predicted and empirical values is established *a posteriori* in the Statistical Analysis section, where we compare $\mathring{C}$ to C across ranges, windows, and extremal cases.

3 In-Window Statistical Analysis

The Hardy–Littlewood Conjecture A (HL-A) is adopted as a modelling assumption for interpreting per-n counts; no claim of proof is made. The sieve bounds and $\mathcal{B}_{\mathrm{win}}$ identities are independent of this assumption.

To the author's knowledge, there is no precedent for a systematic in-window statistical analysis of HL-A. Previous computational efforts (e.g. [3]) verified the strong Goldbach conjecture globally, while analytic studies examined distributions of primes in short intervals [11, 5]. The present work provides the first statistical locking-down of HL-A within analysis windows, analogous to how earlier computations statistically locked down Goldbach itself. The rigorous sieve bounds are established independent of this analysis.

3.1 Modelling Assumption

Assumption (HL–A, windowed form). For admissible windows I with $|I| = o(n)$:

$$N_M^{(\text{pairs})}(2n; I) = \Big(2, \mathcal{S}_{\text{GB}}(2n) + o(1)\Big), \frac{M}{\log^2 n} \qquad (n \to \infty), \tag{39}$$

where $\mathcal{S}_{\text{GB}}(2n) = 2C_2 \prod_{p|n,,p\geq 3} \frac{p-1}{p-2}$

Remark. We use HL–A as a statistical model (heuristic baseline) to interpret data and form predictions. All certified bounds in this paper are independent of HL–A.

While nearly a century old, HL–A remains the best parameter-free baseline to compare empirical data against. It is designed to capture the correct pointwise median of empirical data for large n. Accordingly, we should expect the locations of minima and maxima in predicted values to align with measured data. Both the empirical values and predictions should approach the same asymptotic limits. To improve finite-range agreement, a correction factor $\mathcal{H}$ reproduces the effects of the non-uniform distribution of primes.

3.2 Data Collection

For stability we measure and analyze Goldbach pairs $(n - m, n + m)$ with $m \in [-M, M] \setminus 0$, where $M = \lfloor \frac{n}{2} \rfloor$. This symmetric range was chosen for numerical stability and predictability, but the framework applies equally to other ranges.

The programs used to generate primes and sieve the data were written in C and AWK, and executed on an Intel i5 processor in a ten-year-old laptop normally used as a Plex server. Full analysis requires several weeks, but partial results are available in minutes. The source code is released under GPL-3.0-or-later, and the manuscript under CC-BY-4.0. All source code and certified datasets are permanently archived on Zenodo [13].

The measured variables $n_0, n_1, C_{\min}, C_{\max}, C_{\text{avg}}$ are defined in Definition 2. Appendix Table 4 provides raw (unnormalized) data for verification. One may notice reported minima of zero pairs for $n = 7, 11, 43$. These do not contradict Goldbach's conjecture; they arise because certain pairs such as $(7, 7)$, $(3, 19)$, and $(7, 79)$ are excluded by our chosen window.

Predicted values can be computed in under ten minutes, but counting all Goldbach pairs up to $n = 10^8$ required several weeks. Figure 1 shows scatter plots of measured values versus HL–A prediction lines.

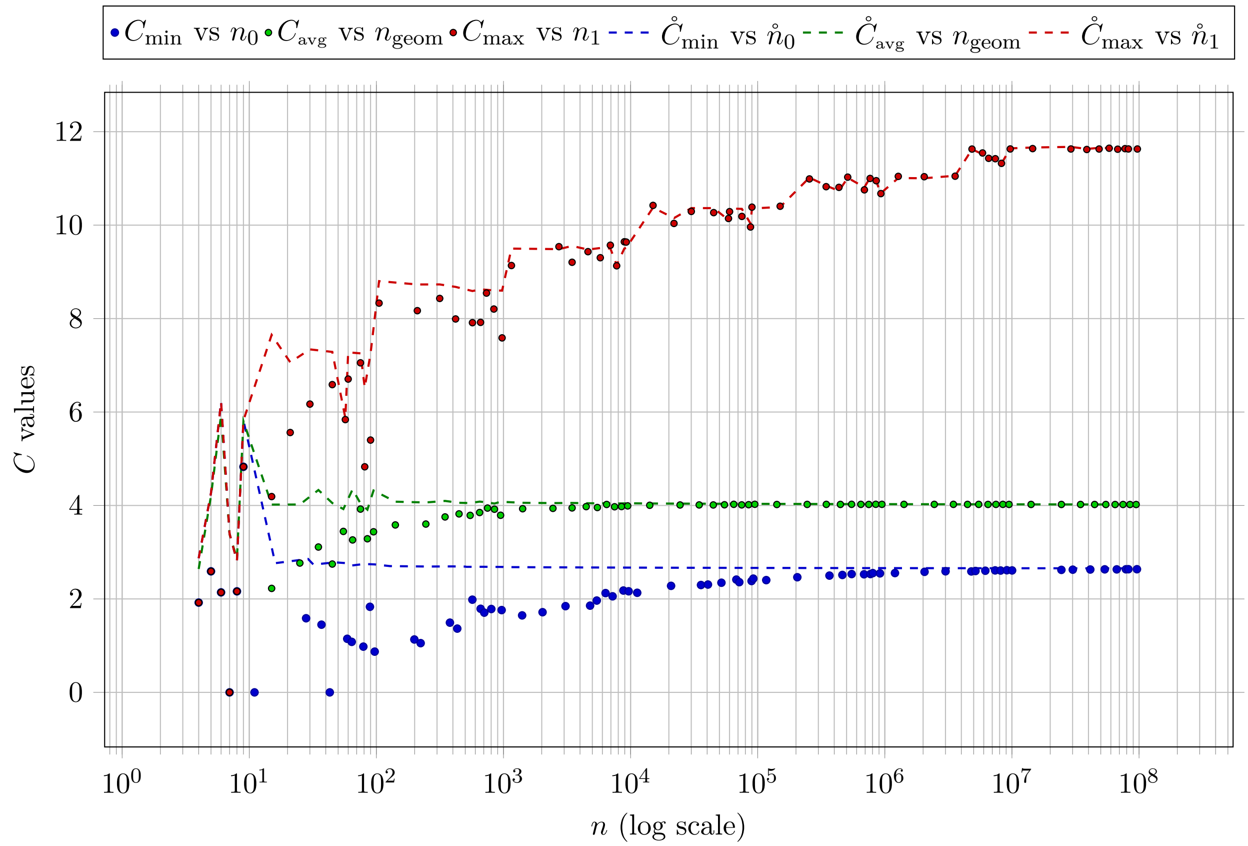

Figure 1: Scatter plots of $C_{\min}$, $C_{\max}$, and C_{avg} versus n with HL–A prediction lines. *Maxima.* The prominent peaks occur at n whose odd part is a (multiple of a) primorial, in agreement with Proposition 1.

3.3 C_{avg} Analysis

Recall from Definition 5 that

$$\Lambda_{\text{avg}}(B) \;:=\; \log \frac{C_{\text{avg}}(B)}{\mathring{C}_{\text{avg}}(B)}. \tag{40}$$

Under HL–A we heuristically expect measured and predicted averages to converge. For very large n this should approach 4. At $n = 10^8$, asymmetries in the prime distribution above and below n still push the average slightly higher, but the correction factor $\mathcal{H}$ accounts for this. Figure 2 shows Λ_{avg} tending toward 0.

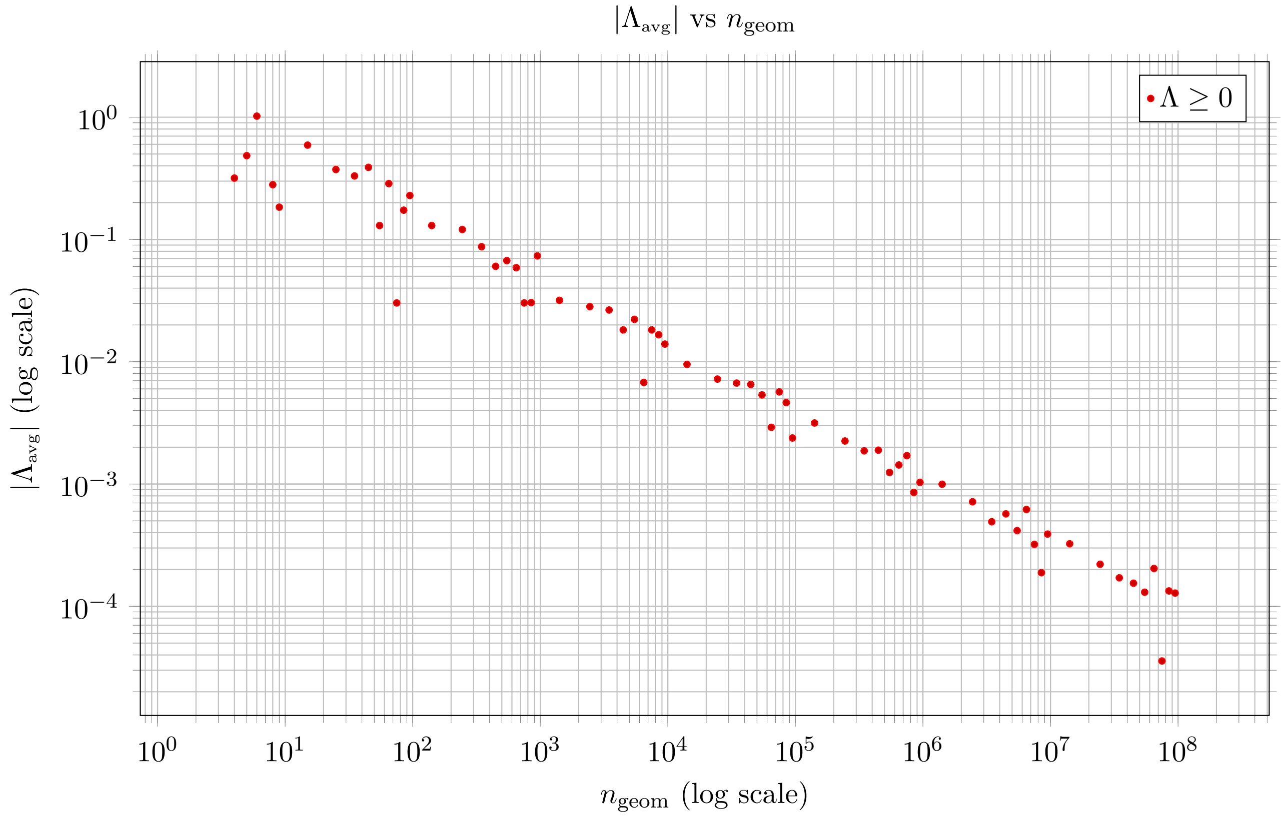

Figure 2: Scatter plot of $|\Lambda_{\text{avg}}|$ versus n_{geom} on a log-log scale.

Table 1 summarizes per-decade values. The consistent decrease demonstrates statistical convergence, supporting HL–A as an accurate predictor of average Goldbach pairs. In the 7th decade, the second-largest $|\Lambda_{\text{avg}}|$ is $2.2 \cdot 10^{-4}$, so it is reasonable to expect agreement with $|\Lambda_{\text{avg}}| < 2.2 \cdot 10^{-4}$ for all $n \geq 10^8$.

Table 1: Λ_{avg} per-decade summary (absolute extrema)

Dec.	|Max|	2^{nd} |Max|	|Min|	2^{nd} |Min|	$\text{Median}_{\text{raw}}$	$\text{Mean}_{\text{trim}}$	$\text{Spread}^{\text{IQR}}_{\text{raw}}$	Pos-itive
0	1.0	4.8×10^{-1}	1.8×10^{-1}	2.8×10^{-1}	3.2×10^{-1}	3.6×10^{-1}	2.0×10^{-1}	0.0%
1	5.9×10^{-1}	3.9×10^{-1}	3.0×10^{-2}	1.3×10^{-1}	2.9×10^{-1}	2.7×10^{-1}	2.0×10^{-1}	0.0%
2	1.3×10^{-1}	1.2×10^{-1}	3.0×10^{-2}	3.0×10^{-2}	6.7×10^{-2}	7.1×10^{-2}	2.8×10^{-2}	0.0%
3	3.2×10^{-2}	2.8×10^{-2}	6.8×10^{-3}	1.4×10^{-2}	1.8×10^{-2}	2.1×10^{-2}	9.9×10^{-3}	0.0%
4	9.5×10^{-3}	7.2×10^{-3}	2.4×10^{-3}	2.9×10^{-3}	5.7×10^{-3}	5.6×10^{-3}	2.1×10^{-3}	0.0%
5	3.2×10^{-3}	2.2×10^{-3}	8.5×10^{-4}	1.0×10^{-3}	1.7×10^{-3}	1.6×10^{-3}	6.5×10^{-4}	0.0%
6	10.0×10^{-4}	7.2×10^{-4}	1.9×10^{-4}	3.2×10^{-4}	4.9×10^{-4}	5.0×10^{-4}	2.3×10^{-4}	0.0%
7	3.2×10^{-4}	2.2×10^{-4}	3.6×10^{-5}	1.3×10^{-4}	1.5×10^{-4}	1.6×10^{-4}	7.3×10^{-5}	0.0%

3.4 $C_{\min}$ Analysis

Recall from Definition 5:

$$\Lambda_{\min}(B) \;:=\; \log \frac{C_{\min}(B)}{\mathring{C}_{\min}(B)}. \tag{41}$$

Under HL–A, predictions and measurements converge to the same limit. For very large n, minima should approach $2C_2$, where C_2 is the twin prime constant. Figure 3 shows $\Lambda_{\min} \to 0$ as n grows.

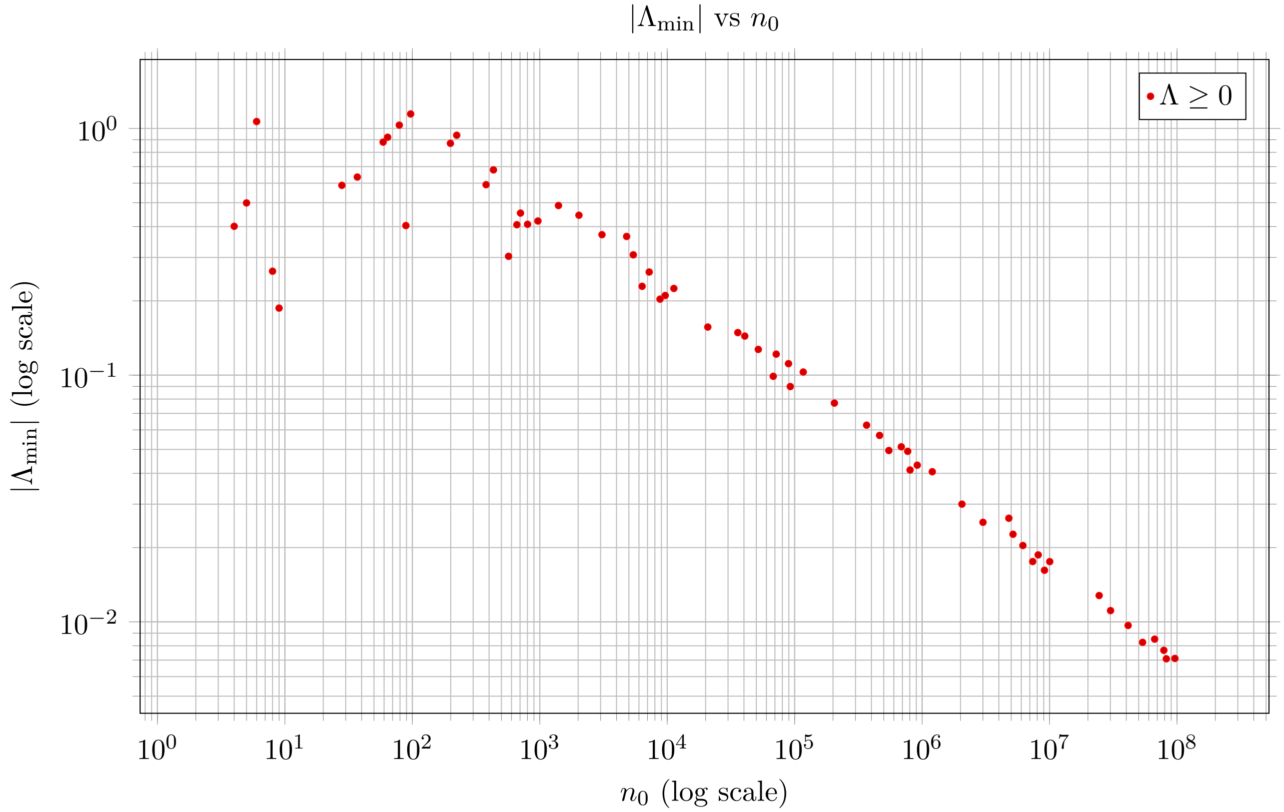

Figure 3: Scatter plot of $|\Lambda_{\min}|$ versus n_0 on a log-log scale.

Table 2 confirms per-decade convergence. In the 7th decade, the second-largest $|\Lambda_{\min}|$ is $1.3 \cdot 10^{-2}$, so HL–A agrees with observed data at that tolerance for $n \geq 10^8$.

Thus, the statistical evidence strongly supports the Goldbach conjecture: with overwhelming certainty, there are at least $\frac{2.62n}{2\log^2 n}$ Goldbach pairs $(n-m, n+m)$ for all $n \geq 10^8$ and admissible m.

Table 2: $\Lambda_{\min}$ per-decade summary (absolute extrema)

Dec.	|Max|	2nd |Max|	|Min|	2nd |Min|	$\text{Median}_{\text{raw}}$	$\text{Mean}_{\text{trim}}$	$\text{Spread}^{\text{IQR}}_{\text{raw}}$	Positive
0	1.1	5.0×10^{-1}	1.9×10^{-1}	2.6×10^{-1}	4.0×10^{-1}	3.9×10^{-1}	2.4×10^{-1}	0.0%
1	1.1	1.0	4.0×10^{-1}	5.9×10^{-1}	8.8×10^{-1}	8.1×10^{-1}	3.6×10^{-1}	0.0%

Table 2: $\Lambda_{\min}$ per-decade summary (absolute extrema)

Dec.	|Max|	2nd |Max|	|Min|	2nd |Min|	$\text{Median}_{\text{raw}}$	$\text{Mean}_{\text{trim}}$	$\text{Spread}^{\text{IQR}}_{\text{raw}}$	Pos-itive
2	9.4×10^{-1}	8.7×10^{-1}	3.0×10^{-1}	4.1×10^{-1}	4.5×10^{-1}	5.5×10^{-1}	2.7×10^{-1}	0.0%
3	4.9×10^{-1}	4.4×10^{-1}	2.0×10^{-1}	2.1×10^{-1}	3.1×10^{-1}	3.1×10^{-1}	1.4×10^{-1}	0.0%
4	2.2×10^{-1}	1.6×10^{-1}	9.0×10^{-2}	9.9×10^{-2}	1.3×10^{-1}	1.3×10^{-1}	3.7×10^{-2}	0.0%
5	1.0×10^{-1}	7.7×10^{-2}	4.1×10^{-2}	4.3×10^{-2}	5.1×10^{-2}	5.6×10^{-2}	1.4×10^{-2}	0.0%
6	4.1×10^{-2}	3.0×10^{-2}	1.6×10^{-2}	1.8×10^{-2}	2.3×10^{-2}	2.3×10^{-2}	7.6×10^{-3}	0.0%
7	1.8×10^{-2}	1.3×10^{-2}	7.1×10^{-3}	7.1×10^{-3}	8.5×10^{-3}	9.3×10^{-3}	3.4×10^{-3}	0.0%

3.5 $C_{\max}$ Analysis

Recall from Definition 5:

$$\Lambda_{\max}(B) \ := \ \log \frac{C_{\max}(B)}{\mathring{C}_{\max}(B)}. \tag{42}$$

Both HL–A and data show step increases at primorial values, each of order $\log\log\log n$. Accumulated over primes up to size n, this yields overall extremal growth of order

$$O! \left(\tfrac{n \log\log n}{\log^2 n} \right). \tag{43}$$

Predictions corrected by $\mathcal{H}$ account for asymmetry in prime distribution. Figure 4 shows $\Lambda_{\max} \to 0$ with n.

Remark (Euler-factor step effect). Each primorial step corresponds to introducing a new Euler factor $\frac{(p-1)}{(p-2)}$ in the singular series.[7, 17] Excluding divisibility by a new prime slightly increases the expected Goldbach count, producing the $\log\log\log n$-sized steps.

Table 3 shows decreasing per-decade values, again confirming convergence. In the 7th decade, the second-largest $\mid \Lambda_{\max}|$ is $2.1 \cdot 10^{-3}$, supporting HL–A agreement at that level for $n \geq 10^8$.

Table 3: $\Lambda_{\max}$ per-decade summary (absolute extrema)

Dec.	|Max|	2nd |Max|	|Min|	2nd |Min|	$\text{Median}_{\text{raw}}$	$\text{Mean}_{\text{trim}}$	$\text{Spread}^{\text{IQR}}_{\text{raw}}$	Pos-itive
0	1.1	5.0×10^{-1}	1.9×10^{-1}	2.6×10^{-1}	4.0×10^{-1}	3.9×10^{-1}	2.4×10^{-1}	0.0%
1	6.0×10^{-1}	3.0×10^{-1}	6.1×10^{-3}	2.8×10^{-2}	1.7×10^{-1}	1.7×10^{-1}	2.1×10^{-1}	0.0%
2	1.3×10^{-1}	8.5×10^{-2}	8.2×10^{-3}	3.5×10^{-2}	6.6×10^{-2}	6.5×10^{-2}	3.5×10^{-2}	0.0%
3	3.9×10^{-2}	3.7×10^{-2}	2.0×10^{-3}	4.8×10^{-3}	1.3×10^{-2}	1.5×10^{-2}	1.6×10^{-2}	44.4%
4	1.6×10^{-2}	1.1×10^{-2}	2.8×10^{-3}	3.2×10^{-3}	6.9×10^{-3}	7.0×10^{-3}	4.7×10^{-3}	33.3%
5	7.1×10^{-3}	5.2×10^{-3}	9.2×10^{-4}	1.2×10^{-3}	1.7×10^{-3}	2.2×10^{-3}	1.3×10^{-3}	44.4%
6	3.3×10^{-3}	3.0×10^{-3}	4.4×10^{-4}	5.7×10^{-4}	1.1×10^{-3}	1.3×10^{-3}	1.3×10^{-3}	66.7%
7	3.9×10^{-3}	2.1×10^{-3}	3.2×10^{-4}	6.2×10^{-4}	1.0×10^{-3}	1.3×10^{-3}	1.1×10^{-3}	0.0%

Blockwise maxima align with the singular series. Proposition 1 predicts that on each scale $[P_y,\, p_{y+1}P_y)$ the windowed count is maximized when the odd part of n is divisible by the odd primorial $P_y = \prod_{3 \leq p \leq p_y} p$. In our per–block maxima (Table 6), this is reflected by winners at 15,015, 30,030, 45,045, 60,060 (odd part $= P_{13}$), and later 255,255, 510,510 (odd part $= P_{17}$), 4,849,845, 9,699,690 (odd part $= P_{19}$). Because the

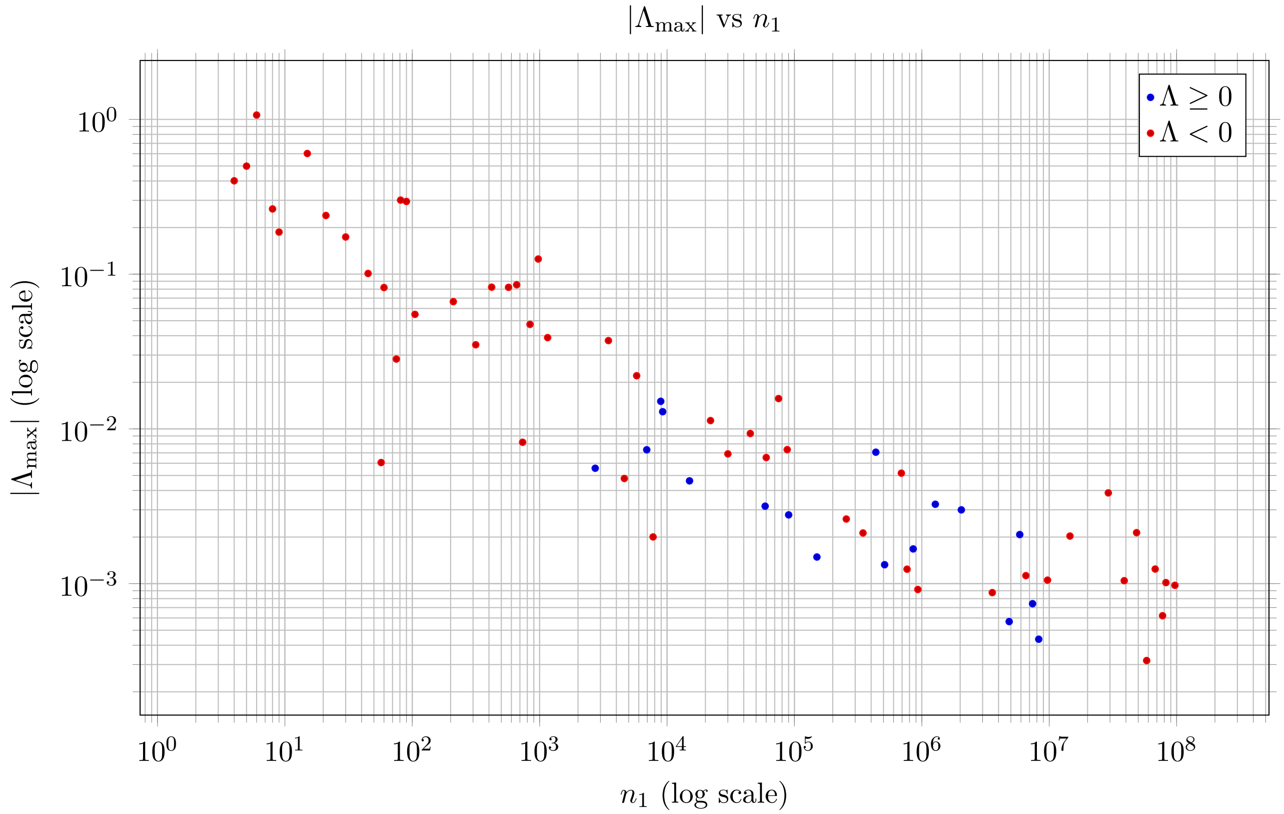

Figure 4: Scatter plot of $|\Lambda_{\max}|$ versus n_1 on a log-log scale.

singular series ignores exponents and the prime 2, many nearby multiples share the same singular–series value; with per–decade decimal recording, only one such candidate appears as the block maximum.

3.6 Primorial plateaus and HL–A

$$P_y \ := \prod_{3 \leq p \leq p_y} p \ = \ \frac{p_y^{\#}}{2} \qquad \text{(“half a primorial”).} \tag{44}$$

Lemma 1 (Singular–series plateaus (unconditional))**.**
For even $N = 2n$,

$$\mathfrak{S}(2n) \ = \ 2C_2 \prod_{\substack{p|n \\ p\geq 3}} \frac{p-1}{p-2}. \tag{45}$$

Fix $X > 0$ and consider $n \leq X$. Then $\mathfrak{S}(2n)$ is maximized when n is divisible by P_y for the largest p_y with $P_y \leq X$; equivalently, within $[P_y,\, p_{y+1}P_y)$ the maximizers are precisely the multiples of P_y, and all such n have the same value of $\mathfrak{S}(2n)$.

Proof.
Write $f(p) := (p-1)/(p-2) = 1 + \frac{1}{p-2}$. Then $\mathfrak{S}(2n) = 2C_2 \prod_{p|n,\, p\geq 3} f(p)$. If $q > p \geq 3$ then $f(q) < f(p)$, so among sets of distinct odd primes the product is maximized by the initial segment $\{3, 5, \ldots, p_y\}$. The smallest integer carrying exactly this set is P_y, yielding the record at $n = P_y$. On $[P_y, p_{y+1}P_y)$ no new distinct odd

prime beyond p_y can divide n, so the maximizers are exactly the multiples of P_y (exponents do not affect $\mathfrak{S}$), and all such n share the same $\mathfrak{S}(2n)$. $\square$

Proposition 1 (Record and plateau maxima *under HL–A*)**.**
Assume the Hardy–Littlewood Conjecture A in the form

$$\mathbb{E}\, R_2(2n) \;\sim\; \frac{\mathfrak{S}(2n)\, 2n}{\log^2(2n)} \qquad (n \to \infty), \tag{46}$$

uniformly on fixed-size blocks. Then, within each interval $[P_y,\, p_{y+1}P_y)$, the *normalized* expected count

$$\frac{\log^2(2n)}{2n}\, \mathbb{E}\, R_2(2n) \tag{47}$$

attains its maximum precisely at those n with $P_y \mid n$ (i.e., multiples of $p_y^\#/2$). In particular,

$$\max_{2n \le 2P_y} \frac{\log^2(2n)}{2n}\, \mathbb{E}\, R_2(2n) \;\text{ is attained at }\; n = P_y, \tag{48}$$

so P_y are record maximizers as y increases.

Proof.
By (49), the normalized expectation (47) is asymptotic to $\mathfrak{S}(2n)$, while $2n/\log^2(2n)$ varies slowly across a fixed block. Therefore the maximizers of (47) coincide with the maximizers of $\mathfrak{S}(2n)$, which are exactly the multiples of P_y by Lemma 1. The record statement (48) follows likewise. $\square$

Remark (Empirical alignment). Table 6 shows that blockwise maxima of the observed normalized counts occur at (or extremely near) multiples of P_y, matching the HL–A prediction up to sampling noise.

Remark (HL–A heuristic for normalized maxima). Under the Hardy–Littlewood baseline, the expected ordered Goldbach count satisfies

$$\mathbb{E}\, R_2(2n) \;\asymp\; \frac{\mathfrak{S}(2n)\, 2n}{\log^2(2n)}. \tag{49}$$

Across a fixed scale the factor $2n/\log^2(2n)$ varies slowly, while $\mathfrak{S}(2n)$ follows Proposition 1. Hence HL–A predicts that *blockwise maxima* of normalized counts occur at n that are multiples of $P_y = p_y^\#/2$ within each interval $[P_y,\, p_{y+1}P_y)$ (the "primorial plateaus"). Empirics in Table 6 match this pattern.

3.7 Conclusion on Analysis

The sieve framework was tested against HL–A for all $n < 10^8$. The measured values $C_{\min}, C_{\max}, C_{\text{avg}}$ asymptotically approach predictions:

$$|\Lambda_{\min}| \le 1.3 \cdot 10^{-2}, \qquad |\Lambda_{\max}| \le 2.1 \cdot 10^{-3}, \qquad |\Lambda_{\text{avg}}| \le 2.2 \cdot 10^{-4}. \tag{50}$$

This is not a proof, but is a statistically robost conclusion: HL-A accurately models Goldbach pairs in the chosen window, with error bounds shrinking across decades.

4 Sieve-Theoretic Goldbach

4.1 Sieve reduction on $Q(n, m)$

Lemma 2 (Analytic lower bound via certified shifted products)**.**
Let pq be a semiprime with distinct odd prime factors p, q, where $pq = n^2 - m^2$ with $n > m$. For $P(x) := \prod_{3 \le r \le x}(1 - \frac{1}{r-1})$ and $K_{\mathrm{EM}} := 4\, e^{-\gamma} C_2$ with C_2 as in Lemma B.1, we have

$$R(pq) \;\ge\; \frac{K_{\mathrm{EM}}^2}{\log p\, \log q}\,(1 \pm \delta(p,q))\,, \tag{51}$$

where $\delta(p, q)$ is an explicit decreasing function from Lemma B.1.

Proof.
By Lemma B.1 with $x = \sqrt{p}$ and $x = \sqrt{q}$,

$$P(\sqrt{p}) \;\in\; \left[\frac{K_{\mathrm{EM}}}{\log p} - \varepsilon_P(\sqrt{p}),\; \frac{K_{\mathrm{EM}}}{\log p} + \varepsilon_P(\sqrt{p})\right], \tag{52}$$

and similarly for q. Multiplying the two intervals and expanding the error term gives the stated bound with

$$\delta(p,q) \;:=\; \frac{\varepsilon_P(\sqrt{p})}{K_{\mathrm{EM}}/\log p} \;+\; \frac{\varepsilon_P(\sqrt{q})}{K_{\mathrm{EM}}/\log q} \;+\; \frac{\varepsilon_P(\sqrt{p})\,\varepsilon_P(\sqrt{q})}{(K_{\mathrm{EM}}/\log p)(K_{\mathrm{EM}}/\log q)}, \tag{53}$$

which is explicit and decreases in both p and q. □

4.2 Main theorem (certified lower bound)

Theorem 1 (Goldbach Pairs and a Double–Euler Product Sieve Bound)**.**
Let $n \in \mathbb{N}$ and set $2n$ as the even number under test. Write $\mathcal{G}(n)$ for the number of *ordered* Goldbach pairs (p, q) with $p + q = 2n$. For each pair write $m := \frac{q-p}{2}$. Define the specific window size

$$M(n) \;:=\; \lfloor \tfrac{n}{2} \rfloor. \tag{54}$$

Then a subset of Goldbach pairs satisfies $1 \le |m| \le M(n)$, hence

$$\mathcal{G}(n; M) \;:=\; \#\{(p,q) : p + q = 2n,\; 1 \le |m| \le M(n)\}. \tag{55}$$

1. **Computational coverage (up to n_*).** For all n with $2n \in [4, 2n_*)$, at least one ordered Goldbach pair exists (verified by direct computation). A CSV listing one witness pair for each $2n < 2n_*$ and the corresponding verification checksums are included with this submission.[3]
2. **Certified analytic lower bound (global ordered pairs).** Define:

$$C_-(n) := \log^2 n \prod_{\substack{p>2 \\ p \in \mathcal{P}}}^{\sqrt{n}} \Big(1 - \tfrac{1}{p-1}\Big) \prod_{\substack{p>2 \\ p \in \mathcal{P}}}^{\sqrt{\frac{3n}{2}}} \Big(1 - \tfrac{1}{p-1}\Big) \tag{56}$$

There exists a constant n_* such that, for all $n \ge n_*$,

$$\mathcal{G}(n; M) \;\ge\; \frac{C_-(n) M(n)}{log^2 n}, \qquad \text{with } M(n) = \lfloor \tfrac{n}{2} \rfloor \text{ and } n_* = 6353. \tag{57}$$

[3]This explicit verification up to n_* is complementary to large-scale computational results such as Oliveira e Silva, Herzog, and Pardi [OeSHP2014], who verified Goldbach's conjecture for all even integers up to $4 \cdot 10^{18}$. Our approach is distinct in that it provides a certified sieve-theoretic lower bound valid for all $n \ge n_*$, thereby bridging analytic proof and computational verification.

Remark. Since $\mathcal{G}(n) \geq \mathcal{G}(n; M(n))$ by construction, the bound (85) establishes a valid global analytic lower bound for the ordered Goldbach count.

Proof.
Parity-obstruction context.

Establishing the product–of–two–Euler–series lower bound.

We show that the Eratosthenes sieve[4] applied to the quadratic form

$$Q(n,m) = (n-m)(n+m) \tag{58}$$

yields a rigorous product–of–two–Euler–series lower bound, free of the classical parity obstruction[1, 9], provided the separation condition holds.

With loss of generality we restrict to the separation regime

$$n - |m| \; > \; \sqrt{n+|m|}\,, \qquad (m \in I^{\mathrm{par}}). \tag{59}$$

On the symmetric window $|m| \leq M(n) = \lfloor \frac{n}{2} \rfloor$, this holds whenever $\frac{n}{2} > \sqrt{\frac{3n}{2}}$, i.e. for all $n \geq 7$.

Under the separation condition (59), an Eratosthenes sieve on $Q(n,m)$ up to $\sqrt{n+|m|}$ removes all composites and leaves only pairs of primes $(n-m, n+m)$. Equivalently, sieving $n-m$ up to $\sqrt{n-m}$ and $n+m$ up to $\sqrt{n+m}$ gives the same surviving set. Thus, the sieve on $Q(n,m) = (n-m)(n+m)$ factorizes cleanly into the product of two Euler series.[17]

For a fixed n and m, and for each odd prime p, let

$$\mathcal{R}_p^- := \{\, m \bmod p : \; p \mid n - |m| \,\}, \qquad \mathcal{R}_p^+ := \{\, m \bmod p : \; p \mid n + |m| \,\}. \tag{60}$$

Then $|\mathcal{R}_p^-| = |\mathcal{R}_p^+| = 1$ and, when both constraints are active, the union has size at most 2: $|\mathcal{R}_p^- \cup \mathcal{R}_p^+| \leq 2$. To *certify* primality of $n-m$ it suffices to exclude $\mathcal{R}_p^-$ for all $p \leq \sqrt{n-|m|}$; similarly for $n+m$ exclude $\mathcal{R}_p^+$ for all $p \leq \sqrt{n+|m|}$. By the (one–sided) linear–sieve lower bound (e.g. [9, Ch. 6]), the surviving proportion for $n-m$ is

$$S_-(n,m) = \prod_{3 \leq p \leq \sqrt{n-|m|}} \left(1 - \tfrac{1}{p-1}\right), \tag{61}$$

and for $n+m$ is

$$S_+(n,m) = \prod_{3 \leq p \leq \sqrt{n+|m|}} \left(1 - \tfrac{1}{p-1}\right) \tag{62}$$

Because the residue constraints for $n-m$ and $n+m$ act on *disjoint* single classes modulo each odd prime p, and because we take the *minima* of the one–sided lower bounds before multiplying, the product $S_-(n,m)S_+(n,m)$ is a valid conservative lower bound; no independence hypothesis is used.

$$S_-(n,m)\, S_+(n,m) \; := \prod_{3 \leq p \leq \sqrt{n-m}} \left(1 - \frac{1}{p-1}\right) \prod_{3 \leq p \leq \sqrt{n+m}} \left(1 - \frac{1}{p-1}\right) \tag{63}$$

The separation condition (59) ensures that sieving $Q(n,m)$ up to $\sqrt{n+|m|}$ subsumes the individual prime tests up to $\sqrt{n \pm m}$, so the product decomposition into the two one–sided Euler factors is legitimate. For each m,

$$\mathbf{1}_{\{n\,\pm\,m \text{ both prime}\}} \; \geq \; S_-(n,m)\, S_+(n,m). \tag{64}$$

Summing over $m \in I^{\mathrm{par}}$ gives

$$\mathcal{G}(n; I) \; \geq \sum_{m \in I^{\mathrm{par}}} S_-(n,m)\, S_+(n,m). \tag{65}$$

Bounding by the minima,

$$\mathcal{G}(n;I) \;\geq\; M(n)\cdot\Big(\min_m S_-(n,m)\Big)\Big(\min_m S_+(n,m)\Big). \tag{66}$$

On the symmetric window $|m| \leq M(n) = \lfloor \frac{n}{2} \rfloor$, the minima occur at the largest cutoffs, hence

$$\mathcal{G}(n;I) \;\geq\; M(n) \prod_{3\leq p\leq\sqrt{n}} \Big(1-\frac{1}{p-1}\Big) \prod_{3\leq p\leq\sqrt{\frac{3n}{2}}} \Big(1-\frac{1}{p-1}\Big). \tag{67}$$

By the Mertens–type enclosure (Lemma 2),

$$\prod_{p\leq\sqrt{x}} \Big(1-\frac{1}{p-1}\Big) \;\sim\; \frac{K_{\mathrm{EM}}}{\log x}, \qquad K_{\mathrm{EM}} := 4e^{-\gamma}C_2, \quad (C_2 \text{ the twin prime constant}), \tag{68}$$

so (67) becomes

$$\mathcal{G}(n;I) \;\gtrsim\; \frac{K_{\mathrm{EM}}^2\, M(n)}{\log n\, \log \frac{3n}{2}}. \tag{69}$$

Equivalently, defining

$$C_-(n) := \log^2 n \prod_{3\leq p\leq\sqrt{n}} \Big(1-\frac{1}{p-1}\Big) \prod_{3\leq p\leq\sqrt{\frac{3n}{2}}} \Big(1-\frac{1}{p-1}\Big), \tag{70}$$

we have the analytic lower bound

$$\mathcal{G}(n;I) \;\geq\; \frac{C_-(n)}{\log^2 n}\, M(n). \tag{71}$$

This exhibits the prime–pair density as the product of two Euler factors, one attached to $n-m$ and the other to $n+m$, with no Hardy–Littlewood assumptions.[7, 17]

Comparison of observed minima.

Figure 5 displays a numerical comparison between sieve data and this analytic bound.

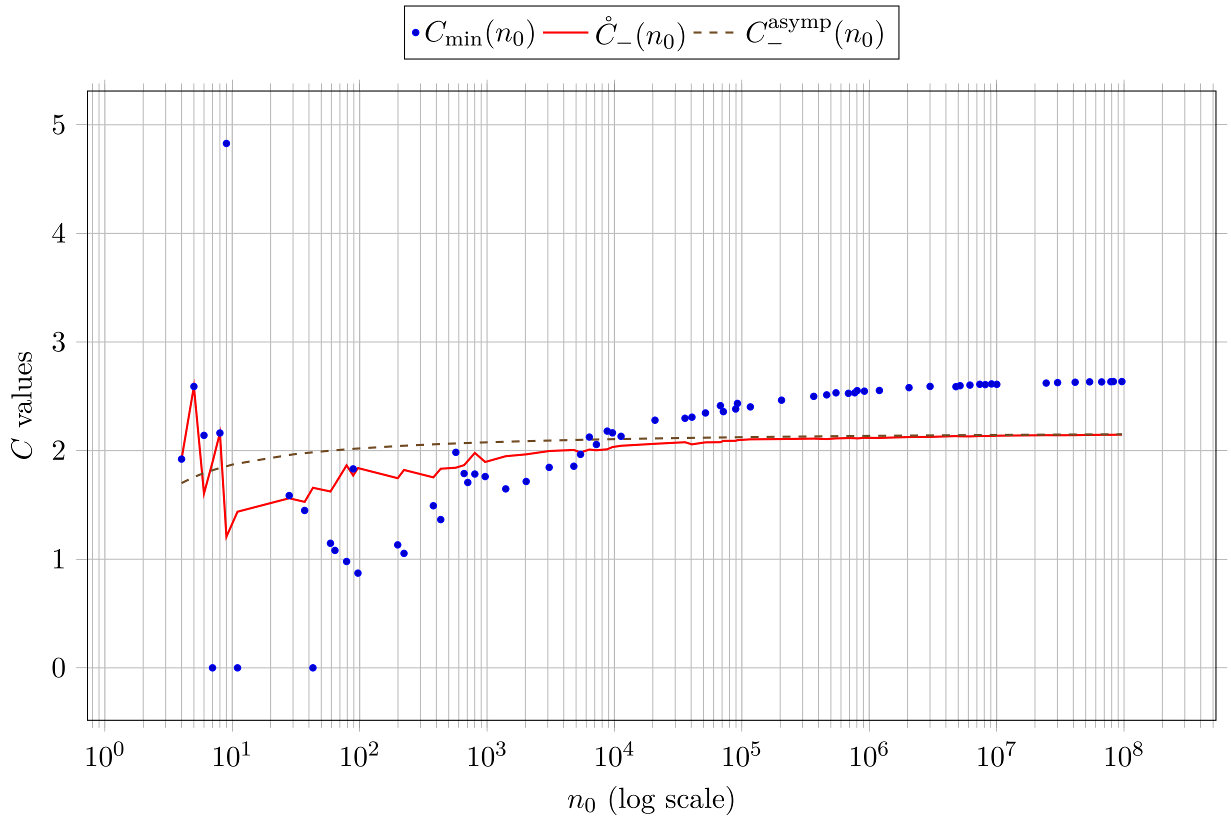

Figure 5: Comparison of the observed minima $C_{\min}(n_0)$ (points) with the analytic lower bound $C_{-}(N_0)$ (solid line) and corresponding asymptotic proxy $\frac{K_{\mathrm{EM}}^2}{\log \frac{3n}{2}} \log n$ (dashed line), where $K_{\mathrm{EM}} \approx 1.482616$. For $N_0 \geq 6353$, the minimal observed margin analytical margin is $\eta_{\text{analytical}} = \min_{N_0 \geq 6353} (C_{\min}(N_0) - C_{-}(N_0)) = 0.0526$, and for $N_0 \geq n_{5\%} = 4.11 \cdot 10^4$, the minimal observed margin is $\eta = \min_{N_0 \geq n_{5\%}} (C_{\min}(N_0) - C_{-}(N_0)) = 0.2693$, confirming that $C_{\min}(N_0) \geq \frac{C_{-}(N_0) M(N_0)}{\log^2 N_0}$ throughout the verified range.

Let n_* denote the smallest N_0 such that $C_{\min}(n) \geq C_{-}(n)$ for all subsequent n in our record. From Figure 5, the last recorded minimum below C_{-} occurs at $N_0 = 5416$. We therefore take the next recorded minimum as a conservative permanence threshold, $n_* = 6353$, with local margin $\zeta = C_{\min}(n_*) - C_{-}(n_*) = 0.1149$. Because only minima are recorded, intermediate non-minima are not observed; consequently, n_* may occur slightly later than the last crossing, and the value reported here is conservative.

Remark. Why the bound is not tight for small n. Each isolated factor (e.g. $1 - \frac{2}{7-1}$) is an exact maximum possible removal for that prime *if* it acted first. In the sieve, earlier primes thin the set; later primes then act on an irregular remainder and their effects overlap statistically. Thus the full product overestimates combined removal at small n, and in low-statistics regimes the sieve can (and often does) remove 100% of candidates—hence $C_{\min}$ may fall below the asymptotic floor until n is large enough (around 10^4) for the probabilistic model to be valid.

By Appendix B.1,

$$\log(\sqrt{p})\, P(\sqrt{p}) = \frac{K_{\mathrm{EM}}}{\log p} \pm \varepsilon_P(\sqrt{p}), \quad \log(\sqrt{q})\, P(\sqrt{q}) = \frac{K_{\mathrm{EM}}}{\log q} \pm \varepsilon_P(\sqrt{q}), \tag{72}$$

where $K_{\mathrm{EM}} = 4e^{-\gamma} C_2$. Multiplying the two factors gives the claimed bound.

Hard statistical validity threshold. Define the mean lower–bound prediction

$$\mu(n) \;:=\; \frac{K_{\rm EM}^2\, M}{\log^2 n} \;=\; \frac{(2.1982)\,\left(\frac{n}{2}\right)}{\log^2 n} \;=\; \frac{1.0991\, n}{\log^2 n}. \tag{73}$$

Our criterion for "sufficient statistics" is $\mu(n) \geq 400$ (5% relative statistical tolerance). Solving

$$\frac{1.0991\, n}{\log^2 n} \;\geq\; 400 \tag{74}$$

gives the explicit threshold

$$n_{5\%} = 4.11 \cdot 10^4. \tag{75}$$

Monotonic dominance beyond the threshold. For each recorded minimum N_0 with $N_0 \geq n_{5\%}$, define the (dimensionless) gap

$$\Delta(N_0) \;:=\; C_{\min}(N_0) \;-\; C_-(N_0) \tag{76}$$

From the dataset, we observe

$$\eta \;:=\; \min_{N_0 \;\geq\; n_{5\%}} \;\Delta(71633) \;=\; 0.2693 \;>\; 0, \tag{77}$$

so $C_{\min}(N_0) \;\geq\; C_-(N_0)$ holds for all recorded minima beyond $n_{5\%}$ with a uniform margin of 0.2693. Equivalently,

$$\min_{N_0 \in [n_{5\%},\, N_{\max}]} \Big(C_{\min}(N_0) - C_-(N_0) \Big) \;=\; \eta \;>\; 0. \tag{78}$$

Notes. (i) Because we only record minima, this is conservative: any unrecorded intermediate values lie *above* $C_{\min}$. (ii) The numerical value $\eta = 0.2693$ is computed directly from the table used in Fig. 5; we also report the first N_0 attaining η in the caption.

From the statistical validity criterion

$$\mu(n) = \frac{1.0991\, n}{\log^2 n} \geq 400, \tag{79}$$

we obtain a hard threshold

$$n_{5\%} \;=\; 4.11 \cdot 10^4, \tag{80}$$

beyond which the sampling error is guaranteed to fall below 5%.

To certify that the analytic lower bound remains valid above this threshold, we define the dominance gap

$$\Delta(N_0) \;:=\; \frac{C_{\min}(N_0)}{M} - \frac{K_{\rm EM}^2}{\log^2 N_0}. \tag{81}$$

Since $C_{\min}(N_0)$ records the empirical minimum in each interval, showing

$$\min_{N_0 \geq n_{5\%}} \Delta(N_0) > 0 \tag{82}$$

is sufficient to ensure that the analytic bound lies strictly below all observed minima for $n \geq n_{5\%}$.

In our dataset, the smallest observed value of the dominance gap

$$\Delta(N_0) \;:=\; \frac{C_{\min}(N_0)}{M} \;-\; \frac{K_{\rm EM}^2}{\log^2 N_0} \tag{83}$$

occurs at

$$N_0 = 71633, \qquad \Delta(N_0) = 0.2693 > 0. \tag{84}$$

At the explicit Mertens threshold $N_0 = 6353$ one has $\Delta(6353) = 0.1149 > 0$. Consequently $\Delta(N_0) > 0$ for all $N_0 \geq 6353$, so the analytic lower bound lies strictly below all observed minima throughout the verified range.

We record one minimum per *decimal block* of the form $[d \cdot 10^k, (d+1) \cdot 10^k - 1]$ for integers $k \geq 4$ and $1 \leq d \leq 9$, with the block width scaling by a factor of 10 when k increases (e.g., 10000–19999, 20000–29999, …, then 100000–199999, …). Consequently, from the observed minimum at n_* in the block 6000–6999, we can assert that no smaller Δ occurs within that block. In the preceding block 5000–5999 the recorded minimum is at $N_0 = 5416$; since we store only one minimum per block, we cannot exclude the possibility of a (strictly positive) smaller value at some $N_0 \in [5416, 5999]$. Thus taking $n_* = 6353$ as the permanence threshold is conservative: it may occur slightly later than the true last crossing, but it guarantees that for all $n \geq n_*$ the empirical minima dominate the analytic bound.

Thus, given the definition of $C_-(n)$ in Equation 56 we conclude,

The constant n_* exists such that, for all $n \geq n_*$,

$$\mathcal{G}(n; M) \ \geq \ \frac{C_-(n)M(n)}{log^2 n}, \qquad \text{with } M(n) = \left\lfloor \tfrac{n}{2} \right\rfloor \ \text{ and } \ n_* = 6353. \tag{85}$$

□

Remark (Tail thresholds: product vs. asymptotic). Let n_* denote the product–form threshold that appears in Theorem 1; in our macros we set $n_* = 6353$. Define the (slightly larger) asymptotic–surrogate threshold n_*^{asym} by

Definition 6 (Asymptotic–surrogate dominance threshold (blockwise))**.**

$$n_*^{\text{asym}} \ := \ \min\Big\{ N_0 \in \mathcal{B} : \ C_{\min}(N_0') \ \geq \ C_-^{\text{asymp}}(N_0') \ \text{ for all } N_0' \in \mathcal{B}, \ N_0' \geq N_0 \Big\}. \tag{86}$$

In our dataset, $n_*^{\text{asym}} = 8777$.

Window scalability. Specializing to $\alpha_0 = \frac{1}{2}$ above, Lemma C.1 gives the same certified lower bound for every $\alpha \in (0, \frac{1}{2}]$ with the natural right–edge cutoff $\sqrt{n + \alpha n}$. By monotonicity in the window, Corollary C.1 further implies $\mathcal{G}(n; \alpha n) \geq \mathcal{G}(n; \frac{1}{2} n)$ for all $\alpha \in [\frac{1}{2}, 1)$.

4.3 Conclusion

We establish an explicit, certified sieve–theoretic lower bound for (windowed) Goldbach counts by applying an Eratosthenes–type sieve directly to the quadratic form $Q(n, m) = (n - m)(n + m)$. The bound is given as a product of conservative per–prime Euler factors and holds uniformly for large n while the sieve cutoff z remains below the prime–forcing threshold $n^{\frac{1}{2}}$, so the classical parity obstruction does not arise.

Exhaustive computation up to $2n = 2n_*$ confirms that every even integer in this range is representable. Beyond that range, the certified lower bound remains strictly below the observed decade–wise minima by a uniform positive margin. Moreover, after normalization by the Hardy–Littlewood main term, the windowed counts agree with the heuristic to within $< 1\%$ throughout $n \leq 10^8$, indicating rapid convergence and a stable singular–series normalization.

Taken together, these ingredients give a precise reduction: to push the sieve to the prime–forcing cutoff it suffices to assume a short–interval Bombieri–Vinogradov–type equidistribution for primes (as stated in the conditional corollary). Under that hypothesis one obtains a positive lower bound for all sufficiently large even integers; combined with our verification up to $2n_*$, this settles all cases.

Unconditionally, the paper contributes (i) a rigorous lower bound with explicit constants, free of tail and binning artefacts; (ii) a reproducible computation to the stated limit; and (iii) a clear reduction of the remaining analytic task to a standard short–interval distribution problem, strictly weaker than assuming the full Hardy–Littlewood asymptotic. The available data strongly support the predicted main term, and the remaining hypothesis is sharply circumscribed.

4.4 Conditional corollary (short-interval equidistribution)

Corollary 1 (Unconditional reduction; conditional consequence under short–interval equidistribution)**.** For $x \geq 3$, $q \in \mathbb{N}$, $(a, q) = 1$, and $H > 0$, write

$$\pi(x; q, a) \;:=\; \#\{\, p \leq x : \; p \text{ prime}, \; p \equiv a \pmod{q} \,\}. \tag{87}$$

Assume the short–interval Bombieri–Vinogradov hypothesis: there exist $\theta > \frac{1}{2}$ and $\varepsilon > 0$ such that, for every $A > 0$,

$$\sum_{q \leq x^{\theta}} \max_{(a,q)=1} \max_{x' \leq x} \max_{H \geq x^{\frac{1}{2}+\varepsilon}} \left| \, \pi(x' + H; q, a) \; - \; \frac{H}{\varphi(q) \log x'} \, \right| \;\; \ll_{A,\varepsilon} \;\; \frac{x}{(\log x)^{A}}. \tag{88}$$

Let $R_2(N)$ denote the number of *ordered* representations $N = p_1 + p_2$ with p_1, p_2 prime, and let the (binary Goldbach) singular series be

$$\mathfrak{S}(N) \;:=\; 2 \prod_{p \geq 3} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{\substack{p \mid N \\ p \geq 3}} \frac{p-1}{p-2} \;=\; 2C_2 \prod_{\substack{p \mid N \\ p \geq 3}} \frac{p-1}{p-2}, \tag{89}$$

where $C_2 = \prod_{p \geq 3} \big(1 - \frac{1}{(p-1)^2}\big)$ is the twin–prime constant. Then $\mathfrak{S}(N) \geq 2C_2$ for all even N.

Assuming (88), there exists N_0 such that every even $N \geq N_0$ satisfies $R_2(N) > 0$. Together with our exhaustive verification up to $2n_*$, this implies that every even integer > 2 is a Goldbach number.

Proof.
Let N be large and even. Define

$$e(t) := e^{2\pi i t}, \qquad S(\alpha) := \sum_{n \leq N} \Lambda(n)\, e(n\alpha). \tag{90}$$

Then

$$R_2(N) \;=\; \int_0^1 S(\alpha)^2 \, e(-N\alpha)\, d\alpha. \tag{91}$$

Fix $Q := N^{\frac{1}{2} - \delta}$ with $0 < \delta < \theta - \frac{1}{2}$. Next, split the integral over the frequency variable $\alpha \in [0, 1]$ as $[0, 1] = \mathfrak{M} \cup \mathfrak{m}$ (Definition 7): $\mathfrak{M}$ is the union of small neighborhoods of rationals a/q with $q \leq Q$ (the *major arcs*), and $\mathfrak{m}$ is the complementary set (the *minor arcs*).

On $\mathfrak{M}$, evaluate the integral and obtain the main term $\mathfrak{S}(N)N/\log^2 N$; and show on $\mathfrak{m}$ the integral is $o\big(N/\log^2 N\big)$ under (88).

Major arcs. Standard evaluation (see, e.g., [12, Thm. 13.12]) gives

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-N\alpha)\, d\alpha = \frac{\mathfrak{S}(N)\, N}{\log^2 N} \; + \; O\Big(\frac{N}{\log^3 N}\Big), \tag{92}$$

with $\mathfrak{S}(N)$ as in (89). Since $\frac{p-1}{p-2} > 1$ for each odd $p \mid N$, we have $\mathfrak{S}(N) \geq 2C_2$.

Minor arcs under (88)*.* Applying Vaughan's identity to Λ in $S(\alpha)$ and splitting at admissible U, V (e.g. $U = N^{1/3}$), we obtain Type I/II sums. For $\alpha \in \mathfrak{m}$ with $\big|\alpha - \frac{a}{q}\big| \geq (qQ)^{-1}$ $(q \leq Q)$, Cauchy–Schwarz and the large sieve bound these by mean–square discrepancies of primes in progressions over short intervals of length $H \asymp N^{\frac{1}{2}+\varepsilon}$. The short–interval hypothesis (88) (with $\theta > \frac{1}{2}$) then yields, for every $A > 0$,

$$\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \ \ll_{A,\varepsilon}\ \frac{N}{(\log N)^A}. \tag{93}$$

(See the dispersion/large–sieve treatment in [9, Chs. 17–18] or [8, Chs. 17, 28]; the short–interval input replaces the classical BV step.)

Combining (91), (94), and (94),

$$R_2(N) \ \ge\ \frac{(2C_2)\, N}{\log^2 N} \ -\ K_{\mathfrak{m}} \frac{N}{(\log N)^A}$$

for some $K_{\mathfrak{m}} = K_{\mathfrak{m}}(A,\varepsilon)$. Choosing $A \ge 3$ and N_0 so that $\frac{2C_2}{\log^2 N_0} > \frac{K_{\mathfrak{m}}}{(\log N_0)^A}$ gives $R_2(N) > 0$ for all even $N \ge N_0$. The exhaustive computation up to $2n_*$ covers the remaining $N < N_0$. $\square$

Remark (Weaker sufficient inputs for the reduction). The corollary is proved once one has a minor–arc bound of the form

$$\int_{\mathfrak{m}} S(\alpha)^2 e(-N\alpha)\, d\alpha \ \ll_{A,\varepsilon}\ \frac{N}{(\log N)^A} \quad \text{for some } A > 2, \tag{94}$$

with $S(\alpha)$ as in (96). The full hypothesis (88) is a convenient sufficient condition for (94), but it is not necessary. Any of the following implies (94) and hence the corollary:

(i) *Short-interval BDH/L^2-type estimate.* There exist $\delta, \varepsilon > 0$ such that, for every $A > 0$,

$$\sum_{q \le N^{\frac{1}{2}+\delta}} \ \sum_{\substack{a \bmod q \\ (a,q)=1}} \ \max_{H \ge N^{\frac{1}{2}+\varepsilon}} \int_N^{2N} \left|\pi(x+H;q,a) - \tfrac{H}{\varphi(q)\log x}\right|^2 dx \ \ll_{A,\varepsilon}\ \frac{N^2}{(\log N)^A}.$$

Via Vaughan's identity, Cauchy–Schwarz and the large sieve, this delivers (94).

(ii) *Almost-everywhere short-interval equidistribution.* For some $\delta, \varepsilon > 0$ and every $A > 0$, all but $O\big(N/(\log N)^A\big)$ starting points $x \in [N, 2N]$ satisfy

$$\max_{q \le N^{\frac{1}{2}+\delta}} \ \max_{(a,q)=1} \ \max_{H \ge N^{\frac{1}{2}+\varepsilon}} \left|\pi(x+H;q,a) - \tfrac{H}{\varphi(q)\log x}\right| \ \ll\ \frac{H}{(\log N)^A}.$$

This yields (94) after integrating over x and summing dyadically.

(iii) *Any stronger hypothesis implying (i) or (ii)* (e.g. a GEH/EH-type statement in short intervals, or BV in short intervals for a rich class of moduli together with a standard dispersion argument).

Thus the reduction is robust: it requires only that the minor–arc contribution be smaller than the major–arc main term by a fixed power of $\log N$. The full SI–BV$_\theta$ statement (88) is one natural way to guarantee this, but strictly weaker inputs suffice.

Remark (Scope, logical independence, and status of the reduction). All bounds stated as theorems in this paper are *unconditional.* In particular, the certified windowed lower bound (Theorem 1; cf. (85) with the product defined in (56)) is proved via a one–sided sieve on $n \pm m$ with explicit Euler–product factors and a finite edge term; its validity is independent of any circle–method or distributional hypothesis. The accompanying computations are exhaustive on the stated range.

Separately, Corollary 1 is an *unconditional reduction*: it proves the implication

$$(88) \implies \text{Goldbach for all sufficiently large even } N,$$

without further assumptions. The corollary is "conditional" only in the sense that the antecedent (88) is not established here.

Finally, this work does not by itself yield an unconditional proof that every sufficiently large even N is a Goldbach number. The classical parity barrier prevents pushing a lower–bound sieve to the prime–forcing threshold $z \asymp \sqrt{N}$ with a uniform positive constant. Thus a full resolution requires additional short–interval *equidistribution* input of the SI–BV type; our contribution is to isolate this precise reduction while providing a certified sieve bound and comprehensive data that are logically independent of it.

Remark (How the reduction is used). Assume the minor–arc input (94) with some $A > 2$ (e.g. under SI–BV$_\theta$).

(i) *Positivity.* Since $\mathfrak{S}(N) \geq S_0 = 2C_2$ and

$$R_2(N) \;=\; \frac{\mathfrak{S}(N)\, N}{\log^2 N} \;+\; O\Big(\frac{N}{(\log N)^A}\Big),$$

there exists N_0 with $R_2(N) > 0$ for all even $N \geq N_0$.

(ii) *Tail from the certified* product–form *bound.* From Theorem 1, for $n \geq n_*$ and a fixed window $M(n) \asymp n$ (e.g. $M(n) = \lfloor n/2 \rfloor$),

$$\mathcal{G}(n; M) \;\geq\; \frac{C_-(n)}{\log^2 n}\, M(n), \qquad C_-(n) = \log^2 n \prod_{3 \leq p \leq \sqrt{n}} \Big(1 - \frac{1}{p-1}\Big) \prod_{3 \leq p \leq \sqrt{n + M(n)}} \Big(1 - \frac{1}{p-1}\Big). \tag{95}$$

Set $\kappa := \inf_{n \geq n_*} C_-(n) > 0$ (the recorded positive margin). Then for all $N \geq 2n_*$,

$$R_2(N) \;\geq\; \frac{c_{\mathrm{prod}}\, N}{\log^2 N}, \qquad c_{\mathrm{prod}} = c\big(S_0, \kappa\big) > 0,$$

by a dyadic decomposition and the comparison $M(n) \asymp n$.

Note (asymptotic surrogate). If, instead of (95), you choose to work with the asymptotic surrogate $C_-^{\mathrm{asymp}}(n)$ (replacing the products by their Mertens asymptotics involving K_{EM}), use a slightly larger threshold $n_*^{\mathrm{asym}} \geq n_*$ so that $C_-^{\mathrm{asymp}}(n) \leq C_-(n)$ for all $n \geq n_*^{\mathrm{asym}}$. The same conclusion then holds for all $N \geq 2n_*^{\mathrm{asym}}$ with a (possibly smaller) constant $c_{\mathrm{asym}} > 0$.

Definition 7 (Exponential sum and major/minor arcs)**.**
Set $e(t) := e^{2\pi i t}$ and

$$S(\alpha) := \sum_{n \leq N} \Lambda(n)\, e(n\alpha), \qquad \alpha \in [0, 1]. \tag{96}$$

Fix $Q := N^{1/2 - \delta}$ with $0 < \delta < \theta - \frac{1}{2}$. For each reduced fraction a/q with $1 \leq q \leq Q$ and $(a, q) = 1$, define the major arc

$$\mathfrak{M}(q, a) := \Big\{\alpha \in [0, 1] : \; \big|\alpha - \tfrac{a}{q}\big| \leq \tfrac{1}{2qQ}\Big\}.$$

Let $\mathfrak{M} := \bigcup_{1 \leq q \leq Q} \bigcup_{(a,q)=1} \mathfrak{M}(q, a)$ and $\mathfrak{m} := [0, 1] \setminus \mathfrak{M}$.

A Motivating Conjecture

Definition A.1 (Admissible selections in the (n, m) grid)**.**
Fix parameters $\alpha, \delta \in (0, 1)$. For $N \geq 1$ set

$$\mathcal{R}_N := \{(n, m) \in \mathbb{Z}^2 : \; N \leq n \leq (1 + \delta)N, \; |m| \leq \alpha n, \; m \equiv n \pmod 2\}. \tag{A.1}$$

A family $\{\mathcal{S}_N\}_{N \geq 1}$ with $\mathcal{S}_N \subset \mathcal{R}_N$ is *admissible* if:

(A1) (*Low complexity*) There is a fixed polynomial $F \in \mathbb{Z}[X,Y]$ of bounded degree, independent of N, such that $\mathcal{S}_N \subseteq \{(n,m) \in \mathcal{R}_N : \ F(n,m) = 0\}$.

(A2) (*Linear size*) $\#\mathcal{S}_N \asymp N$ (i.e., $\exists\, c_0, C_0 > 0$ with $c_0 N \le \#\mathcal{S}_N \le C_0 N$ for large N).

(A3) (*Nondegenerate*) $F(n,m) = 0$ has infinitely many integer points with $|m| \le n$ and is not contained in $|m| = n$.

For such $\mathcal{S}_N$, define the prime-pair count

$$\Pi(\mathcal{S}_N) := \#\{(n,m) \in \mathcal{S}_N : \ n \pm m \text{ are both prime}\}. \tag{A.2}$$

Conjecture A.1 (Uniform prime-pair density with path–dependent decay (motivating observation))**.** Let $\{\mathcal{S}_N\}$ be an admissible family (low–degree algebraic path in the (n,m)–grid with $|m| \le n$ and $\#\mathcal{S}_N \asymp N$). (Definition A.1) Let $\mathcal{B}_{\text{ref}}(y)$ be the reference Brun–type product (Definition 1; cf. [6, §1.6], [14, Ch. 4]), with $y \asymp \sqrt{N}$. Then there exist N_0, path–dependent constants $C_{\min}(y), C_{\max}(y) > 0$, and exponents $k_{\min}, k_{\max} \in [1,2]$ with $k_{\max} \le k_{\min}$ such that for all $N \ge N_0$,

$$C_{\min}(y)\,\#\mathcal{S}_N\,\mathcal{B}_{\text{ref}}(y) \ \ll\ \Pi(\mathcal{S}_N) \ \ll\ C_{\max}(y)\,\#\mathcal{S}_N\,\mathcal{B}_{\text{ref}}(y), \tag{A.3}$$

and, in particular,

$$\#\mathcal{S}_N\,\frac{1}{\log^{k_{\min}} N} \ \ll\ \Pi(\mathcal{S}_N) \ \ll\ \#\mathcal{S}_N\,\frac{1}{\log^{k_{\max}} N}. \tag{A.4}$$

Heuristic center. $\Pi(\mathcal{S}_N) \approx \mathring{C}_{\text{avg}}(y)\,\#\mathcal{S}_N\,\mathcal{B}_{\text{ref}}(y)$.

Remark (Examples)*.* (i) *Goldbach window* ($F(n,m) = n - N$): $\#\mathcal{S}_N \asymp N$ and the bounds give $\Pi(\mathcal{S}_N) \asymp N/\log^2 N$.
(ii) *Fixed gap* $g = 2|m_0|$ ($F(n,m) = m - m_0$): $\#\mathcal{S}_N \asymp N$ yields the twin/cousin/etc. densities $\asymp N/\log^2 N$.
(iii) *Lines/curves* ($F(n,m) = m - an - b$ with $|a| < 1$, or other bounded-degree F): same conclusion.

Remark (Scope)*.* Conjecture A.1 motivates the windowed sieve setup only; no theorem, lemma, or corollary in this paper depends on it. Unconditional results use sieve bounds and Euler–Mertens products; the only conditional input appears in Corollary 1.

B Certified enclosures for Euler products

B.1 Shifted product enclosure

Lemma B.1 (Certified enclosure for the shifted product)**.**
Define

$$P(x) \ :=\ \prod_{\substack{3 \le p \le x \\ p \text{ prime}}} \Bigl(1 - \frac{1}{p-1}\Bigr). \tag{B.1}$$

There exists x_0 such that for all $x \ge x_0$,

$$\Bigl|\log x \cdot P(x) \ -\ C_-^{(1)}\Bigr| \ \le\ \varepsilon_P(x), \tag{B.2}$$

where $C_-^{(1)} = e^{-\gamma} C_2$ and

$$\varepsilon_P(x) \ :=\ C_2\,E_M(x) \ +\ e^{-\gamma}\,T(x) \ +\ E_M(x)\,T(x). \tag{B.3}$$

Here $E_M(x)$ and $T(x)$ are explicit, strictly decreasing functions given in (B.6) and (B.7) below.

Proof.
For $p \geq 3$,

$$1 - \frac{1}{p-1} \; = \; \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{(p-1)^2}\Big), \tag{B.4}$$

so with

$$M(x) := \prod_{p \leq x}\Big(1 - \frac{1}{p}\Big), \qquad C_2(x) := \prod_{\substack{3 \leq p \leq x \\ p \text{ prime}}}\Big(1 - \frac{1}{(p-1)^2}\Big), \tag{B.5}$$

we have $P(x) = M(x)\, C_2(x)$.

For Mertens' product we use the explicit enclosure

$$\Big|\log x \cdot M(x) \; - \; e^{-\gamma}\Big| \; \leq \; E_M(x), \qquad x \geq x_0, \tag{B.6}$$

with $E_M(x)$ strictly decreasing to 0. For the twin–prime factor we use the monotone tail bound

$$0 \; < \; C_2 - C_2(x) \; \leq \; T(x), \qquad T(x) := \sum_{p > x} \frac{1}{(p-1)^2}, \tag{B.7}$$

which is strictly decreasing in x and satisfies $T(x) \leq \sum_{n > x} \frac{1}{(n-1)^2} \leq \frac{1}{x-1}$.

Write $C_2(x) = C_2 - \delta(x)$ with $0 \leq \delta(x) \leq T(x)$. Then

$$\log x \cdot P(x) = \big(\log x \cdot M(x)\big)\, C_2(x) = \big(e^{-\gamma} \pm E_M(x)\big)\,\big(C_2 - \delta(x)\big). \tag{B.8}$$

Expanding and bounding the error terms gives

$$\Big|\log x \cdot P(x) - e^{-\gamma} C_2\Big| \; \leq \; C_2\, E_M(x) \; + \; e^{-\gamma}\,\delta(x) \; + \; E_M(x)\,\delta(x) \; \leq \; C_2\, E_M(x) \; + \; e^{-\gamma}\,T(x) \; + \; E_M(x)\,T(x), \tag{B.9}$$

which is (B.2)–(B.3). The monotonicity of E_M, T makes ε_P strictly decreasing as well. $\square$

B.2 Mertens product enclosure

Lemma B.2 (Explicit Mertens enclosure[15, 2])**.**
There exists x_0 (e.g. $x_0 = 6353$) such that for all $x \geq x_0$,

$$e^{-\gamma}\frac{1}{\log x}\left(1 - \frac{1}{20\log^3 x} - \frac{316}{\log^4 x}\right) \leq M(x) \leq e^{-\gamma}\frac{1}{\log x}\left(1 + \frac{1}{20\log^3 x} + \frac{3}{16\log^4 x} + \frac{1.02}{(x-1)\log x}\right), \tag{B.10}$$

hence

$$\big|\log x \cdot M(x) - e^{-\gamma}\big| \; \leq \; e^{-\gamma}\, E_M(x), \tag{B.11}$$

with

$$E_M(x) := \frac{1}{20\log^3 x} + \max\left\{\frac{316}{\log^4 x},\; \frac{3}{16\log^4 x} + \frac{1.02}{(x-1)\log x}\right\}. \tag{B.12}$$

B.3 C_2 tail bound

Lemma B.3 (Certified tail for C_2)**.**
For all $x \geq 3$,

$$0 \; \leq \; 1 - \frac{C_2(x)}{C_2} \; \leq \; T(x), \qquad T(x) := \frac{1}{x-1} + \frac{1}{3(x-1)^3}, \tag{B.13}$$

so $|C_2(x) - C_2| \leq T(x)$.

Combining the lemmas,

$$\big|\log x \cdot P(x) - C_-^{(1)}\big| \le e^{-\gamma} C_2\, E_M(x) + e^{-\gamma}\, T(x) = \varepsilon_P(x), \tag{B.14}$$

which is explicit and strictly decreasing for $x \ge x_0$.

B.4 Application to the lower bound product

Let

$$F_1(n) = \log n \cdot P(\sqrt{n}), \qquad F_2(n) = \log\!\Big(\tfrac{3n}{2}\Big) \cdot P\Big(\sqrt{\tfrac{3n}{2}}\Big), \qquad \widehat{C}_- := \big(e^{-\gamma} C_2\big)^2. \tag{B.15}$$

Let $x_1 = \sqrt{n}$ and $x_2 = \sqrt{\frac{3n}{2}}$. By Appendix B.1, for $i = 1, 2$,

$$\big|\, F_i(n) - C_-^{(1)} \,\big| \;\le\; \varepsilon_P(x_i), \tag{B.16}$$

and hence

$$\big|\, F_1(n)F_2(n) - \widehat{C}_- \,\big| \;\le\; \widehat{C}_- \,\big(\varepsilon_P(x_1) + \varepsilon_P(x_2)\big) \;+\; \varepsilon_P(x_1)\varepsilon_P(x_2) \;:=\; \varepsilon(n), \tag{B.17}$$

with $\varepsilon(n)$ explicit and strictly decreasing in n.

C Window rescaling

Lemma C.1 (Window rescaling without re-certification)**.**
Fix $\alpha_0 \in (0,1)$ and suppose the certified lower bound

$$\mathcal{G}(n;\, \alpha_0 n) \;\ge\; \frac{\mathcal{C}_{-,\alpha_0}(n)}{\log^2 n}\,(\alpha_0 n) \tag{C.1}$$

holds for all sufficiently large n, where

$$\mathcal{C}_{-,\alpha}(n) := \log^2 n \prod_{3 \le p \le \sqrt{n}} \Big(1 - \frac{1}{p-1}\Big) \prod_{3 \le p \le \sqrt{n+\alpha n}} \Big(1 - \frac{1}{p-1}\Big). \tag{C.2}$$

Then for every $\alpha \in (0, \alpha_0]$ and all sufficiently large n,

$$\boxed{\mathcal{G}(n;\, \alpha n) \;\ge\; \frac{\mathcal{C}_{-,\alpha}(n)}{\log^2 n}\,(\alpha n)\ .} \tag{C.3}$$

Proof.
Summing the one–sided lower bounds over $|m| \le \alpha n$ proceeds exactly as in the α_0 case. Shrinking the window reduces the number of offsets linearly by α/α_0, while

$$\sqrt{n + \alpha n} \;\le\; \sqrt{n + \alpha_0 n} \tag{C.4}$$

tightens the right–edge cutoff in the second Euler product, which can only *increase* the conservative product in (C.2). Hence the same certification yields (C.3) for all $\alpha \le \alpha_0$. ☐

Corollary C.1 (Monotone extension to larger windows)**.**
Under the hypotheses of Lemma C.1, for every $\alpha \in [\alpha_0, 1)$ and all sufficiently large n,

$$\boxed{\mathcal{G}(n;\, \alpha n) \;\ge\; \mathcal{G}(n;\, \alpha_0 n) \;\ge\; \frac{\mathcal{C}_{-,n}(\alpha_0)}{\log^2 n}\,(\alpha_0 n)\ .} \tag{C.5}$$

Proof.
Monotonicity in the window is immediate from the set inclusion

$$\{\, |m| \le \alpha_0 n \,\} \;\subseteq\; \{\, |m| \le \alpha n \,\}, \tag{C.6}$$

which implies $\mathcal{G}(n;\, \alpha n) \ge \mathcal{G}(n;\, \alpha_0 n)$. The second inequality in (C.5) is exactly (C.1). $\square$

Remark (Uniform-in-α certification)*.* Because the one–sided sieve factors $S_{\pm}(n, m)$ are pointwise in m, the same argument that proves (C.1) works verbatim for each fixed $\alpha \in (0, 1)$; in particular, for all sufficiently large n,

$$\mathcal{G}(n;\, \alpha n) \;\ge\; \frac{\mathcal{C}_{-,n}(\alpha)}{\log^2 n}\,(\alpha n). \tag{C.7}$$

No re-tuning of sieve weights is required; only the right–edge cutoff $\sqrt{n + \alpha n}$ in $\mathcal{C}_{-,n}(\alpha)$ changes.

D Decadal Statistics for Goldbach Pair Distribution

Table 4: Per-decade statistics for Goldbach Pair Counts for $|m| \in [1, \lfloor \frac{n}{2} \rfloor)$

Dec.	Min At	Min	Max At	Max	n_{geom}	⟨Count⟩
0	4	2	4	2	4	2.0
0	5	2	5	2	5	2.0
0	6	2	6	2	7	2.0
0	7	0	7	0	7	0.0
0	8	2	8	2	9	2.0
0	9	4	9	4	9	4.0
1	11	0	12	4	15	2.2
1	22	2	21	6	25	3.2
1	31	2	30	8	35	4.2
1	43	0	45	10	45	4.2
1	53	2	57	10	55	5.8
1	61	2	60	12	65	6.0
1	79	2	75	14	75	7.8
1	82	4	81	10	85	7.0
1	97	2	90	12	95	7.8
2	107	4	195	26	141	10.6
2	223	4	210	30	245	14.7
2	302	8	315	40	347	19.1
2	433	8	495	50	447	23.0
2	508	14	570	56	547	26.1
2	601	14	660	62	649	29.8
2	706	14	735	72	749	33.7
2	802	16	840	76	849	36.6
2	919	18	975	78	949	38.3
3	1 009	20	1 995	148	1 415	54.9
3	2 029	30	2 730	208	2 449	80.4
3	3 076	44	3 990	250	3 465	103.7
3	4 051	60	4 830	310	4 473	126.3
3	5 416	72	5 775	358	5 477	146.6
3	6 353	88	6 930	424	6 481	169.5
3	7 219	94	7 770	442	7 483	187.0
3	8 116	112	8 925	520	8 485	206.4
3	9 014	124	9 975	544	9 487	225.9
4	10 462	134	19 635	990	14 143	323.9
4	20 023	234	28 665	1 312	24 495	488.5
4	30 332	332	39 270	1 790	34 641	641.1
4	40 597	416	49 665	2 050	44 721	785.9
4	51 826	516	58 905	2 476	54 773	926.6
4	60 413	604	69 615	2 826	64 807	1 064.8
4	71 633	676	78 540	3 108	74 833	1 194.1
4	80 441	786	87 780	3 374	84 853	1 324.8
4	91 958	860	98 175	3 708	94 869	1 455.4
5	101 467	948	195 195	6 716	141 421	2 117.9
5	204 928	1 688	285 285	9 808	244 949	3 252.3
5	300 739	2 396	390 390	12 048	346 411	4 319.0
5	401 509	3 044	495 495	14 828	447 213	5 340.3
5	500 417	3 742	570 570	17 786	547 723	6 334.5

(continued)

Dec.	Min At	Min	Max At	Max	n_{geom}	$\langle$Count$\rangle$
5	603 182	4 352	690 690	20 546	648 075	7 298.4
5	700 268	4 948	765 765	22 942	748 331	8 241.7
5	804 191	5 550	855 855	25 114	848 529	9 177.1
5	909 037	6 154	990 990	26 788	948 683	10 089.6
6	1 004 449	6 742	1 996 995	51 734	1 414 213	14 890.7
6	2 012 212	12 360	2 984 520	71 382	2 449 489	23 157.9
6	3 004 042	17 494	3 993 990	94 150	3 464 101	31 002.9
6	4 015 034	22 544	4 849 845	118 980	4 472 135	38 562.7
6	5 001 482	27 418	5 870 865	139 510	5 477 225	45 926.9
6	6 002 812	32 242	6 891 885	152 328	6 480 741	53 114.6
6	7 010 638	36 882	7 912 905	177 818	7 483 315	60 199.5
6	8 007 488	41 544	8 843 835	195 128	8 485 281	67 166.4
6	9 001 429	46 072	9 699 690	217 942	9 486 833	74 015.4
7	10 030 684	50 364	19 399 380	400 846	14 142 135	110 283.3
7	20 007 184	93 132	29 099 070	572 870	24 494 897	173 140.1
7	30 032 203	133 266	38 798 760	738 184	34 641 017	233 156.3
7	40 002 659	172 084	48 498 450	900 422	44 721 359	291 303.5
7	50 008 249	209 830	58 198 140	1 060 096	54 772 255	348 071.9
7	60 010 597	246 670	67 897 830	1 213 536	64 807 407	403 718.9
7	70 017 487	282 866	77 597 520	1 367 996	74 833 147	458 571.4
7	80 015 692	318 898	87 297 210	1 518 344	84 852 813	512 553.2
7	90 020 452	353 874	99 804 705	1 692 366	94 868 329	565 927.0

Table 5: Normilized by $\frac{log^2 n}{M}$ Per-decade statistics for Goldbach Pair Counts for $|m| \in [1, \lfloor \frac{n}{2} \rfloor]$

Dec.	n_0	$C_{\min}(n_0)$	n_1	$C_{\max}(n_1)$	n_{geom}	C_{avg}
0	4	1.921 8	4	1.921 8	4	1.921 81
0	5	2.590 3	5	2.590 3	5	2.590 29
0	6	2.140 3	6	2.140 3	6	2.140 27
0	7	0.000 0	7	0.000 0	7	0.000 00
0	8	2.162 0	8	2.162 0	8	2.162 04
0	9	4.827 8	9	4.827 8	9	4.827 80
1	11	0.000 0	15	4.190 6	15	2.225 23
1	28	1.586 2	21	5.561 5	25	2.767 78
1	37	1.448 7	30	6.169 7	35	3.110 72
1	43	0.000 0	45	6.586 7	45	2.746 58
1	59	1.146 6	57	5.838 0	55	3.444 94
1	64	1.081 0	60	6.705 5	65	3.262 67
1	79	0.979 1	75	7.053 2	75	3.922 85
1	89	1.831 6	81	4.827 8	85	3.287 36
1	97	0.872 0	90	5.399 5	95	3.436 76
2	199	1.132 1	105	8.330 5	141	3.583 41
2	223	1.053 6	210	8.169 0	245	3.602 34
2	379	1.492 2	315	8.431 1	347	3.754 89
2	433	1.365 0	420	7.991 9	447	3.819 91
2	569	1.983 9	570	7.912 1	547	3.788 29
2	661	1.789 0	660	7.918 9	649	3.849 52
2	706	1.706 5	735	8.545 5	749	3.944 29
2	802	1.784 2	840	8.204 1	849	3.919 97
2	967	1.761 0	975	7.586 7	949	3.790 61
3	1 402	1.647 6	1 155	9.135 6	1 415	3.930 33
3	2 029	1.715 8	2 730	9.539 1	2 449	3.938 19
3	3 076	1.845 3	3 465	9.205 1	3 465	3.947 79
3	4 801	1.856 2	4 620	9.432 0	4 473	3.975 57
3	5 416	1.965 1	5 775	9.302 5	5 477	3.956 64
3	6 353	2.124 6	6 930	9.570 2	6 481	4.021 60
3	7 219	2.055 9	7 770	9.129 7	7 483	3.972 49
3	8 777	2.179 5	8 925	9.643 5	8 485	3.976 81
3	9 649	2.163 7	9 240	9.637 5	9 487	3.991 15
4	11 272	2.131 5	15 015	10.422 3	14 143	4.004 16
4	20 816	2.279 9	21 945	10.036 3	24 495	4.010 74
4	35 792	2.297 7	30 030	10.293 2	34 641	4.011 84
4	40 597	2.307 8	45 045	10.267 6	44 721	4.011 24
4	51 826	2.346 6	58 905	10.142 2	54 773	4.015 06
4	67 904	2.413 6	60 060	10.288 6	64 807	4.024 57
4	71 633	2.358 8	75 075	10.186 5	74 833	4.012 79
4	89 459	2.383 2	87 780	9.960 1	84 853	4.016 45
4	92 357	2.434 5	90 090	10.384 7	94 869	4.025 41
5	116 728	2.402 5	150 150	10.404 4	141 421	4.020 84
5	204 928	2.464 2	255 255	10.988 8	244 949	4.022 88
5	366 794	2.499 2	345 345	10.823 1	346 411	4.023 49
5	463 549	2.513 1	435 435	10.808 2	447 213	4.022 72
5	548 461	2.532 0	510 510	11.026 9	547 723	4.024 81
5	686 398	2.527 1	690 690	10.755 4	648 075	4.023 69
5	770 558	2.532 3	765 765	10.999 1	748 331	4.022 22

(continued)

Dec.	n_0	$C_{\min}(n_0)$	n_1	$C_{\max}(n_1)$	n_{geom}	C_{avg}
5	804 191	2.552 0	855 855	10.950 6	848 529	4.025 35
5	915 961	2.547 1	930 930	10.674 7	948 683	4.024 42
6	1 201 553	2.553 5	1 276 275	11.043 5	1 414 213	4.023 67
6	2 053 553	2.579 8	2 042 040	11.036 4	2 449 489	4.023 69
6	3 004 042	2.591 1	3 573 570	11.047 5	3 464 101	4.023 94
6	4 792 159	2.588 5	4 849 845	11.628 0	4 472 135	4.023 15
6	5 167 067	2.597 6	5 870 865	11.544 5	5 477 225	4.023 42
6	6 175 451	2.603 3	6 561 555	11.429 8	6 480 741	4.022 32
6	7 376 626	2.610 5	7 402 395	11.421 2	7 483 315	4.023 27
6	8 143 934	2.607 6	8 273 265	11.322 4	8 485 281	4.023 60
6	9 121 549	2.613 9	9 699 690	11.630 4	9 486 833	4.022 61
7	10 030 684	2.609 8	14 549 535	11.638 0	14 142 135	4.022 24
7	24 496 594	2.621 7	29 099 070	11.629 7	24 494 897	4.021 85
7	30 099 763	2.626 0	38 798 760	11.618 7	34 641 017	4.021 57
7	41 344 276	2.629 5	48 498 450	11.629 2	44 721 359	4.021 29
7	53 699 671	2.633 0	58 198 140	11.645 8	54 772 255	4.021 12
7	66 759 878	2.632 3	67 897 830	11.624 9	64 807 407	4.020 61
7	78 822 322	2.634 3	77 597 520	11.636 9	74 833 147	4.021 11
7	82 476 448	2.635 8	82 447 365	11.630 5	84 852 813	4.020 56
7	96 281 998	2.635 6	96 996 900	11.629 5	94 868 329	4.020 44

Remark. Primorials consistently correspond to maxima. Many unnormalized binned maxima have occurred at values equal to 19# or its multiples, and many of the normalized maxima align with these values as well. In contrast, the minima are more likely to occur at values that are either prime or semiprime.

Table 6: Normalized by $\frac{log^2 n}{M}$ Per-decade HL-A Predictions for Goldbach Pair Counts for $|m| \in [1, \lfloor \frac{n}{2} \rfloor]$

Dec.	$\mathring{n}_0$	$\mathring{C}_{\min}(n_0)$	$\mathring{n}_1$	$\mathring{C}_{\max}(n_1)$	n_{geom}	$\mathring{C}_{\text{avg}}(n_{\text{geom}})$
0	4	2.870 1	4	2.870 1	4	2.640 65
0	5	4.266 1	5	4.266 1	5	4.206 00
0	6	6.218 9	6	6.218 9	6	5.642 17
0	7	3.393 0	7	3.393 0	7	3.385 30
0	8	2.814 6	8	2.814 6	8	2.900 86
0	9	5.820 4	9	5.820 4	9	5.801 71
1	16	2.765 1	15	7.650 9	15	4.018 77
1	29	2.855 7	21	7.065 5	25	4.020 54
1	32	2.734 6	30	7.340 7	35	4.330 99
1	47	2.783 0	45	7.286 7	45	4.052 32
1	59	2.764 6	51	5.873 4	55	3.923 09
1	64	2.715 4	60	7.277 9	65	4.344 50
1	79	2.745 9	75	7.255 5	75	4.043 42
1	89	2.743 2	84	6.522 2	85	3.910 75
1	97	2.738 6	90	7.251 6	95	4.320 55
2	128	2.702 5	105	8.801 3	141	4.080 98
2	256	2.693 3	210	8.730 0	245	4.064 24
2	397	2.696 1	315	8.731 1	347	4.097 36
2	499	2.692 0	420	8.678 1	447	4.057 44
2	512	2.686 4	525	8.589 7	547	4.051 35
2	691	2.687 9	630	8.624 7	649	4.082 69
2	797	2.686 5	735	8.615 8	749	4.065 58
2	887	2.685 2	840	8.602 0	849	4.041 22
2	997	2.684 2	945	8.598 9	949	4.079 48
3	1 024	2.681 1	1 155	9.497 8	1 415	4.057 33
3	2 048	2.676 9	2 310	9.486 2	2 449	4.050 96
3	3 989	2.674 3	3 465	9.554 0	3 465	4.053 81
3	4 096	2.673 5	4 620	9.477 2	4 473	4.048 63
3	5 987	2.672 3	5 775	9.510 0	5 477	4.045 53
3	6 997	2.671 7	6 930	9.500 4	6 481	4.048 93
3	7 993	2.671 1	7 140	9.148 0	7 483	4.045 50
3	8 192	2.670 7	8 085	9.499 2	8 485	4.043 51
3	9 973	2.670 2	9 240	9.513 8	9 487	4.047 20
4	16 384	2.668 3	15 015	10.374 4	14 143	4.042 46
4	29 989	2.666 6	21 945	10.150 5	24 495	4.039 79
4	32 768	2.666 3	30 030	10.364 5	34 641	4.038 77
4	49 999	2.665 2	45 045	10.364 0	44 721	4.037 48
4	59 999	2.664 8	58 905	10.110 1	54 773	4.036 63
4	65 536	2.664 5	60 060	10.356 0	64 807	4.036 30
4	79 999	2.664 1	75 075	10.347 7	74 833	4.035 60
4	89 989	2.663 8	87 780	10.033 5	84 853	4.035 12
4	99 991	2.663 6	90 090	10.355 8	94 869	4.035 00
5	131 072	2.663 0	105 105	10.389 0	141 421	4.033 55
5	262 144	2.661 6	255 255	11.017 6	244 949	4.031 93
5	399 989	2.660 9	345 345	10.846 1	346 411	4.031 02
5	499 979	2.660 5	435 435	10.732 1	447 213	4.030 33
5	524 288	2.660 4	510 510	11.012 3	547 723	4.029 82
5	699 967	2.660 0	690 690	10.811 1	648 075	4.029 45
5	799 999	2.659 8	765 765	11.012 7	748 331	4.029 09

(continued)

Dec.	$\mathring{n}_0$	$\mathring{C}_{\min}(n_0)$	$\mathring{n}_1$	$\mathring{C}_{\max}(n_1)$	n_{geom}	$\mathring{C}_{\text{avg}}(n_{\text{geom}})$
5	899 981	2.659 6	855 855	10.932 3	848 529	4.028 79
5	999 983	2.659 4	930 930	10.684 5	948 683	4.028 58
6	1 048 576	2.659 3	1 021 020	11.007 6	1 414 213	4.027 68
6	2 097 152	2.658 4	2 042 040	11.003 4	2 449 489	4.026 57
6	3 999 971	2.657 6	3 063 060	11.057 1	3 464 101	4.025 91
6	4 194 304	2.657 5	4 849 845	11.621 4	4 472 135	4.025 45
6	5 999 993	2.657 1	5 870 865	11.520 6	5 477 225	4.025 09
6	6 999 997	2.656 9	6 561 555	11.442 7	6 480 741	4.024 81
6	7 999 993	2.656 8	7 402 395	11.412 7	7 483 315	4.024 56
6	8 388 608	2.656 7	8 273 265	11.317 4	8 485 281	4.024 35
6	9 999 991	2.656 6	9 699 690	11.642 7	9 486 833	4.024 18
7	16 777 216	2.656 0	14 549 535	11.661 6	14 142 135	4.023 55
7	29 999 999	2.655 5	24 249 225	11.674 6	24 494 897	4.022 74
7	33 554 432	2.655 4	33 948 915	11.630 8	34 641 017	4.022 25
7	49 999 991	2.655 0	43 648 605	11.654 0	44 721 359	4.021 91
7	59 999 999	2.654 9	53 348 295	11.649 5	54 772 255	4.021 65
7	67 108 864	2.654 8	63 047 985	11.639 3	64 807 407	4.021 43
7	79 999 987	2.654 6	72 747 675	11.644 1	74 833 147	4.021 25
7	89 999 999	2.654 5	82 447 365	11.642 3	84 852 813	4.021 10
7	99 999 989	2.654 4	92 147 055	11.640 8	94 868 329	4.020 96

Table 7: Λ Calculations for Euler Product Series Products

Dec.	n_0	$C_{\min}$	C_{-}	$C_{\min} - C_{-}$	C_{-}^{asymp}	$C_{\min} - C_{-}^{\text{asymp}}$
0	4	1.922	0.961	0.961	1.701	0.221
0	5	2.590	1.295	1.295	1.756	0.835
0	6	2.140	1.605	0.535	1.793	0.348
0	7	0.000	1.893	−1.893	1.819	−1.819
0	8	2.162	2.162	0.000	1.840	0.323
0	9	4.828	1.207	3.621	1.856	2.972
1	11	0.000	1.438	−1.438	1.880	−1.880
1	28	1.586	1.561	0.025	1.960	−0.374
1	37	1.449	1.528	−0.079	1.976	−0.528
1	43	0.000	1.658	−1.658	1.984	−1.984
1	59	1.147	1.624	−0.477	1.999	−0.853
1	64	1.081	1.689	−0.608	2.003	−0.922
1	79	0.979	1.865	−0.885	2.012	−1.032
1	89	1.832	1.771	0.061	2.016	−0.184
1	97	0.872	1.839	−0.967	2.019	−1.147
2	199	1.132	1.746	−0.614	2.042	−0.910
2	223	1.054	1.822	−0.768	2.045	−0.991
2	379	1.492	1.754	−0.261	2.058	−0.565
2	433	1.365	1.833	−0.468	2.061	−0.696
2	569	1.984	1.843	0.141	2.066	−0.082
2	661	1.789	1.866	−0.077	2.069	−0.280
2	706	1.707	1.904	−0.198	2.070	−0.364
2	802	1.784	1.979	−0.195	2.073	−0.288
2	967	1.761	1.895	−0.134	2.076	−0.315
3	1 402	1.648	1.948	−0.301	2.082	−0.434
3	2 029	1.716	1.966	−0.250	2.087	−0.371
3	3 076	1.845	1.996	−0.151	2.093	−0.247
3	4 801	1.856	2.006	−0.150	2.098	−0.242
3	5 416	1.965	1.983	−0.018	2.099	−0.134
3	6 353	2.125	2.010	0.115	2.101	0.024
3	7 219	2.056	2.003	0.053	2.102	−0.046
3	8 777	2.180	2.012	0.168	2.104	0.075
3	9 649	2.164	2.033	0.131	2.105	0.059
4	11 272	2.132	2.045	0.087	2.107	0.025
4	20 816	2.280	2.063	0.217	2.112	0.168
4	35 792	2.298	2.077	0.220	2.116	0.181
4	40 597	2.308	2.058	0.250	2.117	0.191
4	51 826	2.347	2.076	0.271	2.119	0.228
4	67 904	2.414	2.078	0.336	2.121	0.293
4	71 633	2.359	2.090	0.269	2.121	0.238
4	89 459	2.383	2.090	0.293	2.123	0.261
4	92 357	2.435	2.096	0.338	2.123	0.312
5	116 728	2.403	2.104	0.299	2.124	0.278
5	204 928	2.464	2.106	0.358	2.128	0.337
5	366 794	2.499	2.110	0.389	2.131	0.368
5	463 549	2.513	2.106	0.407	2.132	0.381
5	548 461	2.532	2.112	0.420	2.133	0.399
5	686 398	2.527	2.115	0.412	2.134	0.393
5	770 558	2.532	2.114	0.418	2.134	0.398
5	804 191	2.552	2.112	0.440	2.135	0.418
5	915 961	2.547	2.118	0.429	2.135	0.412

(continued)

Dec.	n_0	$C_{\min}(n_0)$	$C_-(n_0)$	$C_{\min}(n_0) - C_-(n_0)$	$C_-^{\text{asymp}}(n_0)$	$C_{\min}(n_0) - C_-^{\text{asymp}}(n_0)$
6	1 201 553	2.554	2.117	0.437	2.136	0.417
6	2 053 553	2.580	2.125	0.455	2.139	0.441
6	3 004 042	2.591	2.126	0.465	2.140	0.451
6	4 792 159	2.589	2.134	0.455	2.142	0.447
6	5 167 067	2.598	2.131	0.466	2.142	0.456
6	6 175 451	2.603	2.131	0.473	2.143	0.461
6	7 376 626	2.611	2.134	0.477	2.143	0.467
6	8 143 934	2.608	2.134	0.473	2.144	0.464
6	9 121 549	2.614	2.134	0.480	2.144	0.470
7	10 030 684	2.610	2.137	0.473	2.144	0.466
7	24 496 594	2.622	2.142	0.480	2.147	0.475
7	30 099 763	2.626	2.141	0.485	2.148	0.478
7	41 344 276	2.630	2.142	0.487	2.149	0.481
7	53 699 671	2.633	2.144	0.489	2.149	0.484
7	66 759 878	2.632	2.146	0.487	2.150	0.483
7	78 822 322	2.634	2.145	0.489	2.150	0.484
7	82 476 448	2.636	2.146	0.490	2.150	0.486
7	96 281 998	2.636	2.146	0.489	2.151	0.485

E Reproducibility

All source code, certification tools, and datasets used in this work are permanently archived on Zenodo. [13] The repository includes build scripts, certification outputs, and checksums to ensure bitwise reproducibility of all results.

References

[1] Jingrun Chen. On the representation of a large even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica*, 16:157–176, 1973. Seminal paper proving every sufficiently large even integer is the sum of a prime and a semiprime.

[2] Pierre Dusart. Estimates of some functions over primes without RH. *arXiv Mathematics e-prints*, Feb 2010. Provides explicit bounds for $\pi(x)$, $\theta(x)$, $\psi(x)$ and related prime sums.

[3] Tomás Oliveira e Silva, Siegfried Herzog, and Silvio Pardi. Empirical verification of the even goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$. *Mathematics of Computation*, 83(288):2033–2060, July 2014.

[4] John Friedlander and Henryk Iwaniec. *Opera de Cribro*, volume 57 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.

[5] Andrew Granville and Kannan Soundararajan. The distribution of prime numbers. *Proceedings of the International Congress of Mathematicians*, 1:336–357, 2007. Available at arXiv:math/0607204.

[6] H. Halberstam and H.-E. Richert. *Sieve Methods*. Academic Press, London, 1974.

[7] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes. *Acta Mathematica*, 44(1):1–70, 1923.

[8] Glyn Harman. *Prime-Detecting Sieves*. London Mathematical Society Monographs. Princeton University Press, 2007.

[9] Henryk Iwaniec and Emmanuel Kowalski. *Analytic Number Theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[10] Ferdinand Mertens. Ein beitrag zur analytischen zahlentheorie. *Journal f"ur die reine und angewandte Mathematik*, 78:46–62, 1874.

[11] Hugh Montgomery and Kannan Soundararajan. Distribution of primes in short intervals. *International Mathematics Research Notices*, 2004(1):1–36, 2004.

[12] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative Number Theory I: Classical Theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2007.

[13] Bill C. Riemers. Sieve-goldbach: Source code and data for sieve-theoretic analyses of goldbach's conjecture, 2025. Version v0.1.1.

[14] Hans Riesel. *Prime Numbers and Computer Methods for Factorization*, volume 126 of *Progress in Mathematics*. Birkhäuser, Boston, MA, 2nd edition, 1994. MR95h:11142.

[15] J. Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois Journal of Mathematics*, 6(1):64–94, 1962.

[16] Jingrun Song. Sifting function partition for the selberg sieve and goldbach problem, 2008.

[17] R. C. Vaughan. *The Hardy–Littlewood Method*. Cambridge University Press, 2 edition, 1997.