

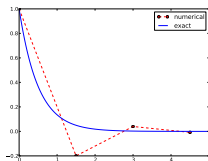
On Schemes for Exponential Decay

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Goal

The primary goal of this demo talk is to demonstrate how to write talks with [DocOnce](#) and get them rendered in numerous HTML formats.

Layout

This version utilizes beamer slides with the theme `red_plain`.

Problem setting and methods

Results

Problem setting and methods



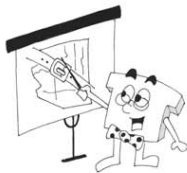
We aim to solve the (almost) simplest possible differential equation problem

$$u'(t) = -au(t) \quad (1)$$

$$u(0) = I \quad (2)$$

Here,

- ▶ $t \in (0, T]$
- ▶ a , I , and T are prescribed parameters
- ▶ $u(t)$ is the unknown function
- ▶ The ODE (1) has the initial condition (2)



The ODE problem is solved by a finite difference scheme

- ▶ Mesh in time: $0 = t_0 < t_1 < \dots < t_N = T$
- ▶ Assume constant $\Delta t = t_n - t_{n-1}$
- ▶ u^n : numerical approx to the exact solution at t_n

The θ rule,

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N-1$$

contains the Forward Euler ($\theta = 0$), the Backward Euler ($\theta = 1$), and the Crank-Nicolson ($\theta = 0.5$) schemes.

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The Forward Euler scheme explained

<http://youtube.com/PtJrPEIHNJw>

Implementation

Implementation in a Python function:

```
def solver(I, a, T, dt, theta):  
    """Solve  $u' = -a*u$ ,  $u(0) = I$ , for  $t$  in  $(0, T]$ ; step:  $dt$ ."""  
    dt = float(dt) # avoid integer division  
    N = int(round(old_div(T, dt))) # no of time intervals  
    T = N*dt # adjust T to fit time step dt  
    u = zeros(N+1) # array of  $u[n]$  values  
    t = linspace(0, T, N+1) # time mesh  
  
    u[0] = I # assign initial condition  
    for n in range(0, N): #  $n=0, 1, \dots, N-1$   
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]  
    return u, t
```

How to use the solver function

A complete main program

```
# Set problem parameters
I = 1.2
a = 0.2
T = 8
dt = 0.25
theta = 0.5
|\pause|

from solver import solver, exact_solution
u, t = solver(I, a, T, dt, theta)
|\pause|

import matplotlib.pyplot as plt
plt.plot(t, u, t, exact_solution)
plt.legend(['numerical', 'exact'])
plt.show()
```

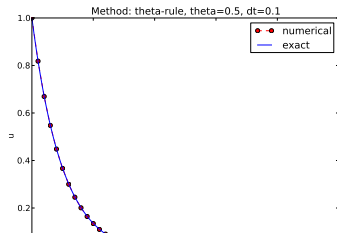
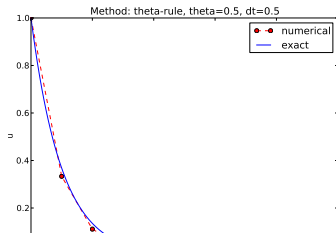
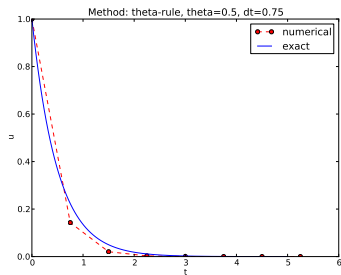
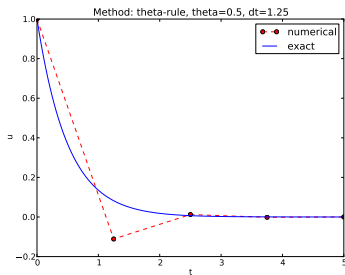
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The Crank-Nicolson method shows oscillatory behavior for not sufficiently small time steps, while the solution should be monotone



The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

Key results:

- ▶ Stability: $|A| < 1$
- ▶ No oscillations: $A > 0$
- ▶ $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- ▶ $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

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