Cobb-Douglas Production and Returns to Scale

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Economists will often use Cobb-Douglas functions since they satisfy a number of crucial assumptions. The general form for a Cobb Douglas function of $\{x_1, x_2, ..., x_n\}$ is as follows:

$$F(\cdot) = \prod_{i=1}^{n} x_i^{\alpha_i}$$

This is a terrifying expression which most of you will hope to never see again. What most of have seen, is a Cobb-Douglas Production Function, with inputs capital (K), labour (L), and an exogenous productivity shock Z. If output is measured as Y, this is written:

$$Y = F(K, L) = ZK^{\alpha_1}L^{\alpha_2}$$

Note that we do not write Z as an input of our function. This is becuase we assume that firms do not choose Z. It is a random process that is influenced by random events in the world.

One of the properties that is desirabe from Cobb-Douglas production functions is constant elasticities of substitution. As an aside, I'm assuming that most of yuo have seen price elasticity of demand in your micro courses. Given a demand function $Q^d(p)$ the price elasticity of demand is written:

$$\epsilon = -\frac{\%\Delta Q}{\%\Delta p} = -\frac{\Delta Q}{\Delta p}\frac{p}{Q}$$

The elasticity measures the percent change in quantity demanded with respect to a percent change in price. This is easy enough to calculate when demand is linear, but what if demand is non-linear. For example what if $Q^d(p) = p^{-\lambda}$? The phrase "with respect to" should peak those of you who are familiar with calculus. We will take the derivative! For the non linear case:

$$\epsilon = -\frac{dQ}{dp}\frac{p}{Q}$$

Another application of your calculus course is recognizing that derivatives with respect to natural logarithms are equivalent to finding percent changes. That is:

$$\frac{d\ln Q}{dp} = \frac{\frac{dQ}{dp}}{Q}$$

We can use this to re-write our formula for price elasticoty of demand:

$$\epsilon = -\frac{d\ln Q}{d\ln p}$$

From here, the elasticity is easy to calculate:

$$Q = p^{-\lambda} \iff \ln Q = -\lambda \ln p \implies -\frac{d\ln Q}{d\ln p} = \lambda$$

Here we can see that the price elasticity of demand is constant!

Going back to constant elasticity of substitution of production we would define the elasticity of substitution of capital and labour as:

$$\sigma = \frac{d\ln K}{d\text{ln}\, L}$$

This looks easy to evaluate, but we need to consider that we are tealing with total differentials of a multivariate function. The math to solve for this is complicated, so for now let's just consider our log output:

$$\ln Y = \ln z + \alpha_1 \ln K + \alpha_2 \ln L$$

If we looks at

$$\frac{\partial \ln Y}{\partial \ln K} = \alpha_1$$

We are looking at the percent increase in production holding labour fixed with respect to a percent increase in capital.

When we look at returns to scale, we are exmaning the ratio between an percent increase in output versus a percent increase in the inputs. That is returns to scale is given by:

$$RTS = \frac{F(\gamma K, \gamma L)}{\gamma F(K, L)}$$

when we plug in our Cobb-Douglas production function, we get:

$$RTS = \frac{Z(\gamma K)^{\alpha_1}(\gamma L)^{\alpha_2}}{Z\gamma K^{\alpha_1} L^{\alpha_2}} = \gamma^{\alpha_1+\alpha_2-1}$$

Here we see that returns to scale is dependent on two things: the percent increase of our inputs, and the elasticities with respect to capital and labour. Intuitively, this follows from the elasticities representing the percent increase in production with repect to a percent increase in the corresponding input. The sum of these elasticities gives us the total percent increase when we increase both inputs by the same percent!

Notably, in economics, we choose α_1 and α_2 such that $alpha_1 + \alpha_2 = 1$. This gives us constant returns to scale. That is:

$$RTS \equiv 1 \iff \alpha_1 + \alpha_2 = 1$$

This is what we call the twin economies hypthesis. That is, we assume that if we had an economy and replicated all of it's inputs, we would get the same output in the new economy. This is because when we double out inputs (as would be demonstrated by a 100% increase) we have our elasticities of production that sum to 1. It gives us that our economy will increase by the same percent as the inputs increase!