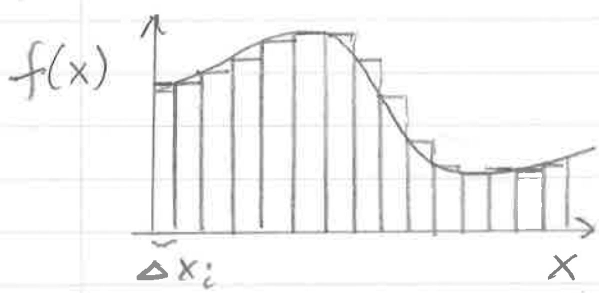
(Forbes 4.4)

# Dynamics of a Continuous Rigid Body (CRB)

In reality, rigid bodies are not made up of discrete particles (unless we go to the atomic level). Therefore, DRBs are in reality approximations of continuous rigid bodies.

Def: A continuous rigid body (CRB) is a continuum in which the distance between any two points on the body is constant.

Recall how to calculate the area under a curve in Calculus



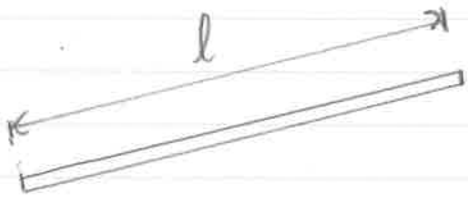

$$\sum_{i=1}^{N} f(x_i)\,\Delta x_i$$

As $\Delta x_i \to 0$,

$$\sum_{i=1}^{N} f(x_i)\,\Delta x_i \to \int f(x)\,dx$$

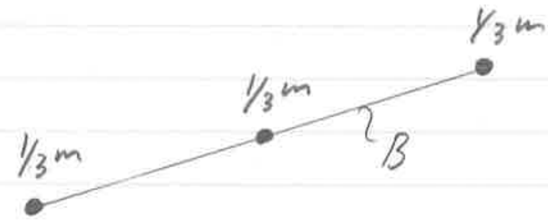
The same concept applies to DRBs / CRBs.

Consider a slender bar. In reality the bar is a continuous rigib body, but we can approximate it as a DRB.




Slender bar

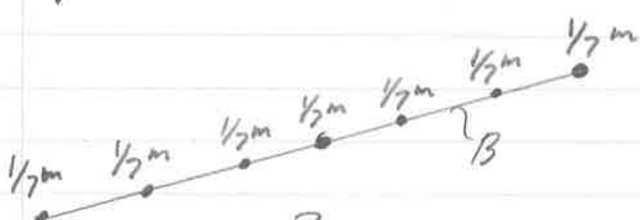
mass m




3 particle DRB model of slender bar

$$m_B = \sum_{i=1}^{3} \frac{1}{3} m = m$$

Let's improve our DRB by increasing number of particles

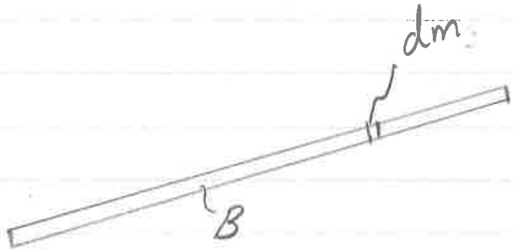



$$m_B = \sum_{i=1}^{7} \frac{1}{7} m = m$$

Our DRB approximation of the bar improves as we increase the number of particles, and proportionally decrease the mass of each particle.

What if we let $\ell$, the number of particles, approach infinity?

If we still want $m_B = m$, then the mass of each particle must approach zero.

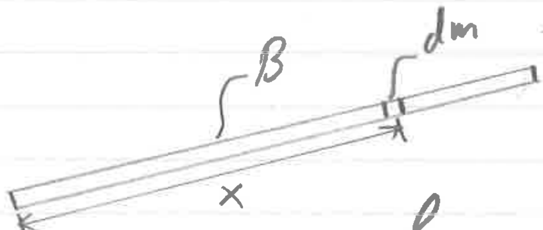



**Def:** Consider a CRB $B$. The <u>zeroth moment of mass</u> or <u>mass</u> of $B$ is

$$m_B = \int_B dm = \int_B \sigma \, dV$$

where $\sigma$ is the volumetric density of $B$, which in general does not need to be constant over the body.

For our slender bar, let $\rho$ be the density per unit length (i.e, $m = \rho \ell$)




$$dm = \rho \, dx$$

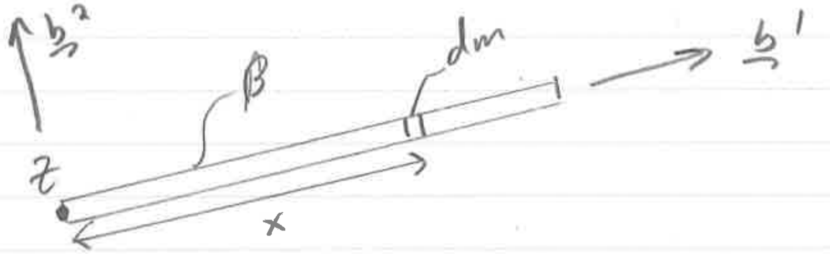
$$m_B = \int_0^\ell \rho \, dx = \rho x \Big|_0^\ell = \rho \ell = m$$

**Def:** Consider a CRB $\mathcal{B}$ and point $z$. The position of the <u>center of mass</u> of $\mathcal{B}$ relative to $z$ is

$$\vec{r}^{\,cz} = \frac{1}{m_{\mathcal{B}}} \int_{\mathcal{B}} \vec{r}^{\,dm\,z}\, dm = \frac{1}{m_{\mathcal{B}}} \int_{\mathcal{B}} \sigma\, \vec{r}^{\,dm\,z}\, dV,$$

wher $\sigma$ is the volumetric density of $\mathcal{B}$.

For slender bar




Again, $dm = \rho\, dx$ , $m_{\mathcal{B}} = m = \rho \ell$

$$\vec{r}^{\,dm\,z} = \vec{\mathcal{F}}_b^T \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{r}^{\,cz} = \frac{1}{\rho \ell} \int_0^{\ell} \rho\, \vec{\mathcal{F}}_b^T \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} dx$$

$$= \frac{1}{\ell} \vec{\mathcal{F}}_b^T \int_0^{\ell} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} dx = \frac{1}{\ell} \vec{\mathcal{F}}_b^T \begin{bmatrix} \frac{1}{2}x^2 \\ 0 \\ 0 \end{bmatrix} \Bigg|_0^{\ell}$$

$$= \frac{1}{\ell} \vec{\mathcal{F}}_b^T \begin{bmatrix} \frac{1}{2}\ell^2 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathcal{F}}_b^T \begin{bmatrix} \frac{1}{2}\ell \\ 0 \\ 0 \end{bmatrix}$$

Def: Consider a CRB $\mathcal{B}$ and point $z$. The first moment of mass of $\mathcal{B}$ relative to $z$ is

$$\underset{\rightharpoondown}{c}^{\mathcal{B}z} = \int_{\mathcal{B}} \underset{\rightharpoondown}{r}^{dm\,z}\, dm = \int_{\mathcal{B}} \sigma\, \underset{\rightharpoondown}{r}^{dm\,z}\, dV = m_{\mathcal{B}}\, \underset{\rightharpoondown}{r}^{cz}$$

where $\sigma$ is the volumetric density of $\mathcal{B}$.

For slender bar

$$\underset{\rightharpoondown}{c}^{\mathcal{B}z} = \int_0^{\ell} \rho\, \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} dx$$

$$= \rho\, \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \int_0^{\ell} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} dx = \rho\, \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} \frac{1}{2}x^2 \\ 0 \\ 0 \end{bmatrix} \Bigg|_0^{\ell}$$

since $m = \rho \ell$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} \frac{1}{2}\rho \ell^2 \\ 0 \\ 0 \end{bmatrix} = \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} \frac{1}{2} m \ell \\ 0 \\ 0 \end{bmatrix}$$

$$= m\, \underset{\rightharpoondown}{r}^{cz}$$

Def: Consider a CRB $\mathcal{B}$ and point $z$. The second moment of mass of $\mathcal{B}$ relative to $z$, resolved in $\mathcal{F}_b$ is

$$\underline{J}_b^{\mathcal{B}z} = -\int_{\mathcal{B}} \underline{r}_b^{dm z^\times} \underline{r}_b^{dm z^\times} \, dm$$

$$= -\int_{\mathcal{B}} \sigma \, \underline{r}_b^{dm z^\times} \underline{r}_b^{dm z^\times} \, dV,$$

where $\sigma$ is the volumetric density of $\mathcal{B}$.

For slender bar,

$$\underline{J}_b^{\mathcal{B}z} = -\int_0^{\ell} \rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x & 0 \end{bmatrix} dx$$

$$= -\rho \int_0^{\ell} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^2 & 0 \\ 0 & 0 & -x^2 \end{bmatrix} dx = -\rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{x^3}{3} & 0 \\ 0 & 0 & -\frac{x^3}{3} \end{bmatrix} \Bigg|_0^{\ell}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho\ell^3 & 0 \\ 0 & 0 & \frac{1}{3}\rho\ell^3 \end{bmatrix} \overset{m=\rho\ell}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{3}m\ell^2 \end{bmatrix}$$

If we want $\underline{J}_b^{BC}$, we need $\underline{r}_b^{dmc}$

$$\underset{\rightharpoonup}{r}^{dmc} = \underset{\rightharpoonup}{r}^{dmz} - \underset{\rightharpoonup}{r}^{cz}$$

$$= \underset{\rightharpoonup}{\mathcal{F}}_b^T \left( \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}\ell \\ 0 \\ 0 \end{bmatrix} \right) = \underset{\rightharpoonup}{\mathcal{F}}_b^T \begin{bmatrix} x - \frac{1}{2}\ell \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{J}_b^{BC} = -\int_0^\ell \rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x+\frac{\ell}{2} \\ 0 & x-\frac{\ell}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x+\frac{\ell}{2} \\ 0 & x-\frac{\ell}{2} & 0 \end{bmatrix} dx$$

$$= -\rho \int_0^\ell \begin{bmatrix} 0 & 0 & 0 \\ 0 & -x^2+x\ell-\frac{\ell^2}{4} & 0 \\ 0 & 0 & -x^2+x\ell-\frac{\ell^2}{4} \end{bmatrix} dx$$

$$= -\rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{x^3}{3}+\frac{1}{2}x^2\ell - x\frac{\ell^2}{4} & 0 \\ 0 & 0 & -\frac{x^3}{3}+\frac{1}{2}x^2\ell - x\frac{\ell^2}{4} \end{bmatrix} \Bigg|_0^\ell$$

$$= -\rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{\ell^3}{3}+\frac{1}{2}\ell^3-\frac{\ell^3}{4} & 0 \\ 0 & 0 & -\frac{\ell^3}{3}+\frac{1}{2}\ell^3-\frac{\ell^3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}\rho\ell^3 & 0 \\ 0 & 0 & \frac{1}{12}\rho\ell^3 \end{bmatrix} \overset{m=\rho\ell}{=} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{12}m\ell^2 \end{bmatrix}$$

Def: Consider frame $\mathcal{F}_a$, point $w$, and CRB $B$. The translational momentum of $B$ relative to $w$ wrt $\mathcal{F}_a$ is

$$\underset{\rightharpoonup}{p}^{Bw/a} = \int_B d\underset{\rightharpoonup}{p}^{dm\,w/a} = \int_B \underset{\rightharpoonup}{v}^{dm\,w/a}\, dm$$

Since $\underset{\rightharpoonup}{v}^{dm\,w/a} = \underset{\rightharpoonup}{r}^{dm\,w\,\cdot a}$

$$\underset{\rightharpoonup}{p}^{Bw/a} = \int_B \underset{\rightharpoonup}{r}^{dm\,w\,\cdot a}\, dm$$

$$= \left(\int_B \underset{\rightharpoonup}{r}^{dm\,w}\, dm\right)^{\cdot a}$$

$$= m_B\, \underset{\rightharpoonup}{r}^{cw\,\cdot a} = m_B\, \underset{\rightharpoonup}{v}^{cw/a}$$

Therefore,

$$\boxed{\underset{\rightharpoonup}{p}^{Bw/a} = m_B\, \underset{\rightharpoonup}{v}^{cw/a}}$$

## Euler's First Law (E1L)

Consider inertial frame $\mathcal{F}_a$, unforced particle $w$, CRB $\mathcal{B}$, and an external force per unit volume applied to mass element $dm$ of $d\underset{\rightharpoondown}{f}^{dm}$. Then,

$$\boxed{\underset{\rightharpoondown}{f}^{\mathcal{B}} = \underset{\rightharpoondown}{p}^{\mathcal{B}w/a \cdot a},}$$

where

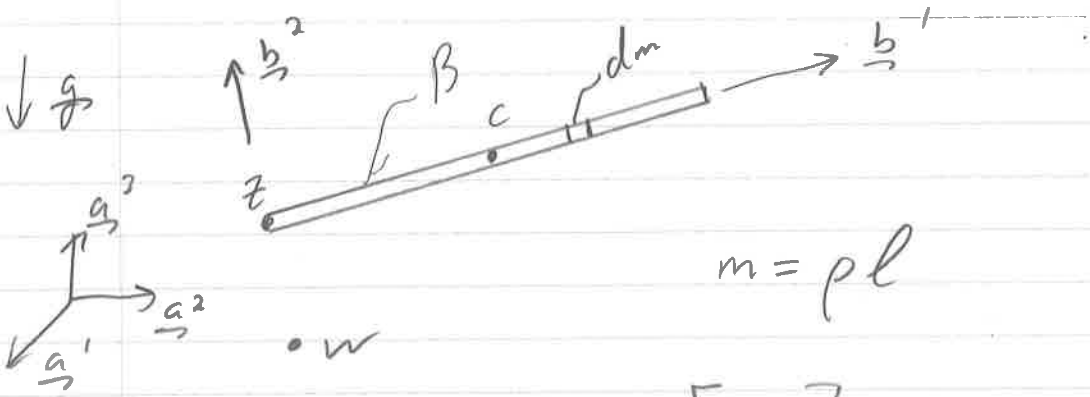
$$\underset{\rightharpoondown}{f}^{\mathcal{B}} = \int_{\mathcal{B}} d\underset{\rightharpoondown}{f}^{dm}.$$

If $m_{\mathcal{B}}$ is constant, then

$$\underset{\rightharpoondown}{f}^{\mathcal{B}} = \left(m_{\mathcal{B}} \underset{\rightharpoondown}{v}^{cw/a}\right)^{\cdot a} = m_{\mathcal{B}} \underset{\rightharpoondown}{v}^{cw/a \cdot a}$$

$$\boxed{\underset{\rightharpoondown}{f}^{\mathcal{B}} = m_{\mathcal{B}} \underset{\rightharpoondown}{a}^{cw/a/a}}$$

Return to slender bar




$$m = \rho l$$

Let $$\underrightarrow{r}^{cw} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$$

$$\underrightarrow{v}^{cw/a} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix}$$

$$\underrightarrow{a}^{cw/a/a} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{bmatrix}$$

$$\underrightarrow{p}^{Bw/a} = m_B \underrightarrow{v}^{cw/a/a} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} m\dot{x}_a \\ m\dot{y}_a \\ m\dot{z}_a \end{bmatrix}$$

Gravitational force on dm:

$$d\underrightarrow{f}^{dm} = \underrightarrow{g}\, dm = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} dm$$

$$\underrightarrow{f}^{B} = \int_B d\underrightarrow{f}^{dm} = \int_B \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} dm = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \int_0^l \rho\, dx$$

$$= \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \rho x \Big|_0^l = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -\rho l g \end{bmatrix} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

Using E1L:

$$\underset{\rightharpoondown}{f}^{B} = m_B \underset{\rightharpoondown}{a}^{cwlala}$$

$$\underset{\rightharpoondown}{\mathcal{I}}_a^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} = \underset{\rightharpoondown}{\mathcal{I}}_a^T \begin{bmatrix} m\ddot{x}_a \\ m\ddot{y}_a \\ m\ddot{z}_a \end{bmatrix}$$

EoMs:

$$\boxed{\begin{aligned} m\ddot{x}_a &= 0 \\ m\ddot{y}_a &= 0 \\ m(\ddot{z}_a + g) &= 0 \end{aligned}}$$

Def: Consider frame $\mathcal{F}_a$, point $z$, and CRB $B$. The angular momentum of $B$ relative to $z$ wrt $\mathcal{F}_a$ is

$$\underset{\rightharpoondown}{h}^{Bz/a} = \int_B d\underset{\rightharpoondown}{h}^{dmz/a} = \int_B \underset{\rightharpoondown}{r}^{dmz} \times d\underset{\rightharpoondown}{p}^{dmz/a}$$

$$= \int_B \underset{\rightharpoondown}{r}^{dmz} \times \underset{\rightharpoondown}{v}^{dmz/a}\, dm$$

Recall $\underset{\rightharpoondown}{v}^{dmz/a} = \underset{\rightharpoondown}{r}^{dmz^{\cdot a}} = \underset{\rightharpoondown}{r}^{dmz^{\cdot b}} + \underset{\rightharpoondown}{\omega}^{ba} \times \underset{\rightharpoondown}{r}^{dmz}$

If $z$ is a point on the CRB and frame $\mathcal{F}_b$ is fixed to the CRB, then $\underset{\rightharpoondown}{r}^{dmz^{\cdot b}} = \underset{\rightharpoondown}{0}$. Therefore,

$$\underset{\rightharpoondown}{v}^{dmz/a} = \underset{\rightharpoondown}{\omega}^{ba} \times \underset{\rightharpoondown}{r}^{dmz} = -\underset{\rightharpoondown}{r}^{dmz} \times \underset{\rightharpoondown}{\omega}^{ba}$$

and

$$\underset{\rightharpoondown}{h}^{Bz/a} = -\int_B \underset{\rightharpoondown}{r}^{dmz} \times (\underset{\rightharpoondown}{r}^{dmz} \times \underset{\rightharpoondown}{\omega}^{ba})\, dm$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \Big( \underbrace{-\int_B \underline{r}_b^{dmz^\times} \underline{r}_b^{dmz^\times}\, dm}_{\underline{J}_b^{Bz}}\; \underline{\omega}_b^{ba} \Big)$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^{T}\, \underline{J}_b^{Bz}\, \underline{\omega}_b^{ba} \quad \text{or} \quad \underline{h}_b^{Bz/a} = \underline{J}_b^{Bz}\, \underline{\omega}_b^{ba}$$

## Euler's Second Law (E2L)

Consider inertial frame $\mathcal{F}_a$, unforced particle $w$, point $z$, CRB $\mathcal{B}$, and an external force per unit volume applied to mass element $dm$ of $d\underset{\rightarrow}{f}^{dm}$. Then,

$$\boxed{\underset{\rightarrow}{h}^{\mathcal{B}z/a \cdot a} + \underset{\rightarrow}{c}^{\mathcal{B}z} \times \underset{\rightarrow}{v}^{zw/a \cdot a} = \underset{\rightarrow}{m}^{\mathcal{B}z}},$$

where

$$\underset{\rightarrow}{m}^{\mathcal{B}z} = \int_{\mathcal{B}} \underset{\rightarrow}{r}^{dm\, z} \times d\underset{\rightarrow}{f}^{dm}$$

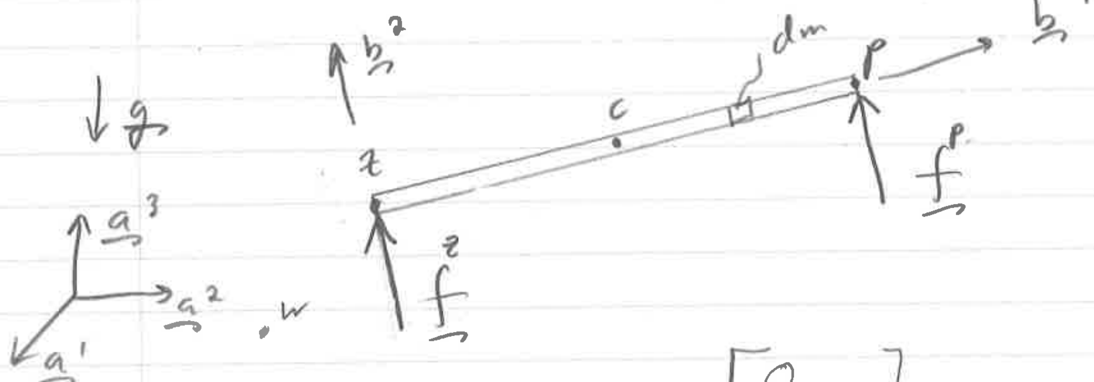
If $z = c$, then $\underset{\rightarrow}{c}^{\mathcal{B}c} = \underset{\rightarrow}{0}$ and

$$\boxed{\underset{\rightarrow}{h}^{\mathcal{B}c/a \cdot a} = \underset{\rightarrow}{m}^{\mathcal{B}c}}$$

Since $\underset{\rightarrow}{h}^{\mathcal{B}c/a} = \underset{\rightarrow}{\mathcal{F}}_b^T \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba}$, E2L in $\mathcal{F}_b$ is

Euler's Equation : $\boxed{\underline{J}_b^{\mathcal{B}c} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times} \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba} = \underline{m}_b^{\mathcal{B}c}}$

Return to slender bar




where $\underset{\rightharpoondown}{f}^{z} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ +f_{b2}^{z} \\ 0 \end{bmatrix}$, $\underset{\rightharpoondown}{f}^{P} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ f_{b2}^{P} \\ 0 \end{bmatrix}$

Recall $$\underline{J}_b^{Bc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

$$\underset{\rightharpoondown}{h}^{Bc/a} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} \begin{bmatrix} \omega_{b1}^{ba} \\ \omega_{b2}^{ba} \\ \omega_{b3}^{ba} \end{bmatrix}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ \frac{1}{12}ml^2\omega_{b2}^{ba} \\ \frac{1}{12}ml^2\omega_{b3}^{ba} \end{bmatrix}$$

$$\underset{\rightharpoondown}{m}^{Bc} = \int_B \underset{\rightharpoondown}{r}^{dmc} \times d\underset{\rightharpoondown}{f}^{dm}$$

$$= \int_B \underset{\rightharpoondown}{r}^{dmc} \times d\underset{\rightharpoondown}{f}^{dmg} + \underbrace{\underset{\rightharpoondown}{r}^{pc} \times \underset{\rightharpoondown}{f}^{P} + \underset{\rightharpoondown}{r}^{zc} \times \underset{\rightharpoondown}{f}^{z}}_{\text{Notice point forces appear without integral.}}$$

Recall $\underset{\rightarrow}{f}^{dmg} = \underset{\rightarrow}{g}\, dm = \underline{C}_{ba}\, \underline{g}_a\, dm$

$$\int_{\mathcal{B}} \underset{\rightarrow}{r}^{dmc \times} d\underset{\rightarrow}{f}^{dmg} = \int_0^{\ell} \rho\, \underset{\rightarrow}{\mathcal{F}}_b^{T} \begin{bmatrix} x - \ell/2 \\ 0 \\ 0 \end{bmatrix}^{\times} dx\, \underline{C}_{ba}\, \underline{g}_a$$

$$= \rho\, \underset{\rightarrow}{\mathcal{F}}_b^{T} \int_0^{\ell} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x + \frac{\ell}{2} \\ 0 & x - \frac{\ell}{2} & 0 \end{bmatrix} dx\, \underline{C}_{ba}\, \underline{g}_a$$

$$= \rho\, \underset{\rightarrow}{\mathcal{F}}_b^{T} \left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}x^2 + \frac{x}{2}\ell \\ 0 & \frac{1}{2}x^2 - \frac{x}{2}\ell & 0 \end{bmatrix} \right|_0^{\ell} \underline{C}_{ba}\, \underline{g}_a$$

$$= \rho\, \underset{\rightarrow}{\mathcal{F}}_b^{T} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\ell^2 - \frac{1}{2}\ell^2 \\ 0 & \cancel{\frac{1}{2}\ell^2} - \cancel{\frac{1}{2}\ell^2} & 0 \end{bmatrix} \underline{C}_{ba}\, \underline{g}_a$$

$$= \underset{\rightarrow}{0}$$

$$\underset{\rightarrow}{m}^{\mathcal{B}c} = \underset{\rightarrow}{0} + \underset{\rightarrow}{\mathcal{F}}_b^{T} \left( \underline{r}_b^{pc^{\times}} \underline{f}_b^{p} + \underline{r}_b^{zc^{\times}} \underline{f}_b^{z} \right)$$

$$= \underset{\rightarrow}{\mathcal{F}}_b^{T} \left( \begin{bmatrix} \ell/2 \\ 0 \\ 0 \end{bmatrix}^{\times} \begin{bmatrix} 0 \\ f_{b2}^{p} \\ 0 \end{bmatrix} + \begin{bmatrix} -\ell/2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ +f_{b2}^{z} \\ 0 \end{bmatrix} \right)$$

$$= \underset{\rightarrow}{\mathcal{F}}_b^{T} \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\ell/2 \\ 0 & \ell/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_{b2}^{p} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell/2 \\ 0 & -\ell/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_{b2}^{z} \\ 0 \end{bmatrix} \right)$$

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$$\underset{\rightharpoondown}{m}^{Bc} = \underline{\mathcal{I}}_b^{T} \begin{bmatrix} 0 \\ 0 \\ \ell/2\,(f^{P}_{b2} - f^{Z}_{b2}) \end{bmatrix}$$

E2L: $\underset{\rightharpoondown}{h}^{Bc/a\,\cdot a} = \underset{\rightharpoondown}{m}^{Bc}$

or $$\underline{J}_b^{Bc}\,\dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times}\,\underline{J}_b^{Bc}\,\underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m\ell^2 & 0 \\ 0 & 0 & \frac{1}{12}m\ell^2 \end{bmatrix} \begin{bmatrix} \dot{\omega}_{b1}^{ba} \\ \dot{\omega}_{b2}^{ba} \\ \dot{\omega}_{b3}^{ba} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_{b3}^{ba} & \omega_{b2}^{ba} \\ \omega_{b3}^{ba} & 0 & -\omega_{b1}^{ba} \\ -\omega_{b2}^{ba} & \omega_{b1}^{ba} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{12}m\ell^2\omega_{b2}^{ba} \\ \frac{1}{12}m\ell^2\omega_{b3}^{ba} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ell/2\,(f^{P}_{b2} - f^{Z}_{b2}) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{12}m\ell^2\dot{\omega}_{b2}^{ba} \\ \frac{1}{12}m\ell^2\dot{\omega}_{b3}^{ba} \end{bmatrix} + \begin{bmatrix} \frac{1}{12}m\ell^2\left(\cancel{-\omega_{b2}^{ba}\omega_{b3}^{ba}} + \cancel{\omega_{b2}^{ba}\omega_{b3}^{ba}}\right) \\ -\frac{1}{12}m\ell^2\omega_{b1}^{ba}\omega_{b3}^{ba} \\ \frac{1}{12}m\ell^2\omega_{b1}^{ba}\omega_{b2}^{ba} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ell/2\,(f^{P}_{b2} - f^{Z}_{b2}) \end{bmatrix}$$

$$\frac{1}{12}m\ell^2\left(\dot{\omega}_{b2}^{ba} - \omega_{b1}^{ba}\omega_{b3}^{ba}\right) = 0$$

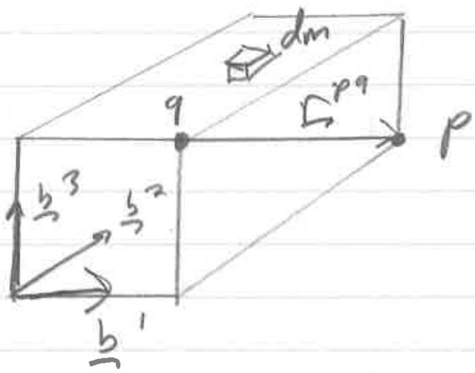
$$\frac{1}{12}m\ell^2\left(\dot{\omega}_{b3}^{ba} + \omega_{b1}^{ba}\omega_{b2}^{ba}\right) = \frac{\ell}{2}\left(f^{P}_{b2} - f^{Z}_{b2}\right)$$

$$\boxed{\begin{gathered} \dot{\omega}_{b2}^{ba} - \omega_{b1}^{ba}\omega_{b3}^{ba} = 0 \\ \frac{1}{6}m\ell\left(\dot{\omega}_{b3}^{ba} + \omega_{b1}^{ba}\omega_{b2}^{ba}\right) = f^{P}_{b2} - f^{Z}_{b2} \end{gathered}}$$

## Parallel Axis Theorem

Given $\underline{J}_b^{Bp} = -\int_B \underline{r}_b^{dmp\times}\,\underline{r}_b^{dmp\times}\,dm$,

can we find $\underline{J}_b^{Bq} = -\int_B \underline{r}_b^{dmq\times}\,\underline{r}_b^{dmq\times}$ ?




$$
\begin{aligned}
\underline{J}_b^{Bq} &= -\int_B \underline{r}_b^{dmq\times}\,\underline{r}_b^{dmq\times}\,dm \\
&= -\int_B \left(\underline{r}_b^{dmp} + \underline{r}_b^{pq}\right)^{\times}\left(\underline{r}_b^{dmp} + \underline{r}_b^{pq}\right)^{\times} dm \\
&= -\int_B \Big(\underline{r}_b^{dmp\times}\,\underline{r}_b^{dmp\times} + \underline{r}_b^{dmp\times}\,\underline{r}_b^{pq\times} + \underline{r}_b^{pq\times}\,\underline{r}_b^{dmp\times} \\
&\qquad + \underline{r}_b^{pq\times}\,\underline{r}_b^{pq\times}\Big)\,dm \\
&= -\int_B \underline{r}_b^{dmp\times}\,\underline{r}_b^{dmp\times}\,dm - \int_B \underline{r}_b^{dmp\times}\,dm\,\underline{r}_b^{pq\times} \\
&\quad - \underline{r}_b^{pq\times}\int_B \underline{r}_b^{dmp\times}\,dm - \int_B dm\,\underline{r}_b^{pq\times}\,\underline{r}_b^{pq\times}
\end{aligned}
$$

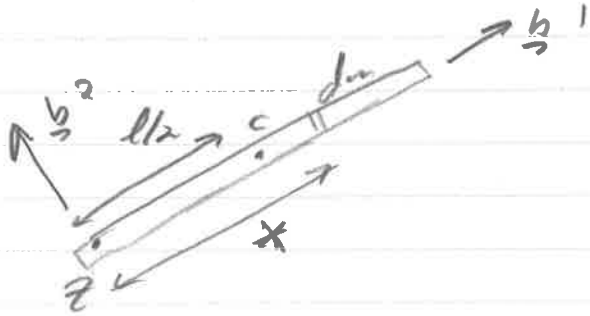
$$\boxed{\underline{J}_b^{Bq} = \underline{J}_b^{Bp} - \underline{c}_b^{Bp^\times}\underline{r}_b^{pq^\times} - \underline{r}_b^{pq^\times}\underline{c}_b^{Bp^\times} - m_B\,\underline{r}_b^{pq^\times}\underline{r}_b^{pq^\times}}$$

↳ Parallel Axis Theorem

If $p = c$, then $\underline{c}_b^{Bc} = \underline{0}$ and

$$\boxed{\underline{J}_b^{Bq} = \underline{J}_b^{Bc} - m_B\,\underline{r}_b^{cq^\times}\underline{r}_b^{cq^\times}}$$

Return to our slender bar example:




We previously calculated

$$\underline{J}_b^{Bz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}ml^2 & 0 \\ 0 & 0 & \frac{1}{3}ml^2 \end{bmatrix}$$

$$\underline{J}_b^{Bc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$

Notice that $\underset{\sim}{c}^{Bc} = \underset{\sim}{0}$ and $\underset{\sim}{r}^{cz} = \underset{\sim}{\mathcal{F}}_b^T \begin{bmatrix} l/2 \\ 0 \\ 0 \end{bmatrix}$

$$m_B \underline{r}_b^{cz\times} \underline{r}_b^{cz\times} = m \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l/2 \\ 0 & l/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l/2 \\ 0 & l/2 & 0 \end{bmatrix}$$

$$= m \begin{bmatrix} 0 & 0 & 0 \\ 0 & -l^2/4 & 0 \\ 0 & 0 & -l^2/4 \end{bmatrix}$$

$$\underline{J}_b^{Bc} - m_B \underline{r}_b^{cz\times} \underline{r}_b^{cz\times} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4}ml^2 & 0 \\ 0 & 0 & \frac{1}{4}ml^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4/12\, ml^2 & 0 \\ 0 & 0 & 4/12\, ml^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3\, ml^2 & 0 \\ 0 & 0 & 1/3\, ml^2 \end{bmatrix}}_{\underline{J}_b^{Bz}}$$

3 Steps to Success for a CRB (Newton-Euler Approach to Dynamics)

1) Kinematics

i) Frames and DCMs

ii) Angular Velocity

iii) Positions

- position of c (and z) $(\underrightarrow{r}^{cw})$ $(\underrightarrow{r}^{zw})$
- position of dm relative to c or z $(\underrightarrow{r}^{dmc}$ or $\underrightarrow{r}^{dmz})$

iv) Velocity

- velocity of c (and z) $(\underrightarrow{v}^{cw/a})$ $(\underrightarrow{v}^{zw/a})$

v) Acceleration

- acceleration of c $(\underrightarrow{a}^{cw/a/a})$

2) FBD of CRB

- forces
- moments about c or z $(\underrightarrow{m}^{Bc}$ or $\underrightarrow{m}^{Bz})$

3) Newton's / Euler's Laws

i) Angular momentum about c or z

$(\underrightarrow{h}^{Bc/a} = \underrightarrow{\mathcal{F}}_b^T \underline{J}_b^{Bc} \underline{\omega}_b^{ba}$ or $\underrightarrow{h}^{Bz/a} = \underrightarrow{\mathcal{F}}_b^T \underline{J}_b^{Bz} \underline{\omega}_b^{ba})$

ii) N2L / E1L $(\underrightarrow{f}^B = m_B \underrightarrow{a}^{cw/a/a})$

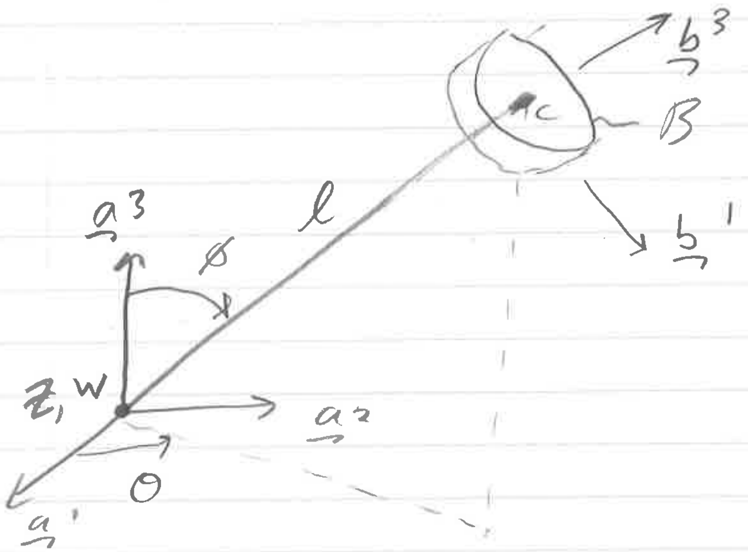
iii) N2LR / E2L $(\underrightarrow{h}^{Bc/a \cdot a} = \underrightarrow{m}^{Bc}$

or $\underrightarrow{h}^{Bz/a \cdot a} + \underrightarrow{c}^{Bz} \times \underrightarrow{v}^{zw/a \cdot a} = \underrightarrow{m}^{Bz})$

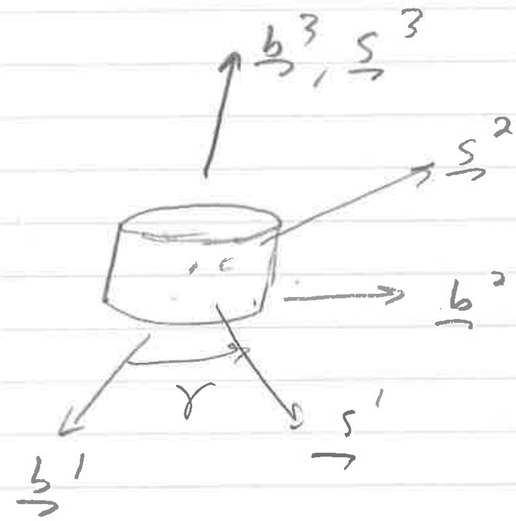
Euler's Equations $(\underline{J}_b^{Bc} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba \times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underline{m}_b^{Bc})$

$$\left( \underline{J}_b^{Bz} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba \times} \underline{J}_b^{Bz} \underline{\omega}_b^{ba} + \underline{c}_b^{Bz \times} \dot{\underline{v}}_b^{zw/a} + \underline{c}_b^{Bz \times} \underline{\omega}_b^{ba \times} \underline{v}_b^{zw/a} = \underline{m}_b^{Bz} \right)$$

Example: Consider a gyro pendulum.




Close up of gyro:




$$\mathcal{F}_a \xrightarrow{C_3(\theta)} \mathcal{F}_q \xrightarrow{C_2(\phi)} \mathcal{F}_b \xrightarrow{C_3(\gamma)} \mathcal{F}_s$$

$\dot{\gamma}$ is constant spin rate of gyro.

Find EoMs of gyro pendulum.

1) Kinematics

i) $\underline{C}_{sa} = \underline{C}_3(\gamma)\,\underline{C}_2(\phi)\,\underline{C}_3(\theta)$

ii) $\vec{\omega}^{sa} = \vec{\omega}^{sb} + \vec{\omega}^{bq} + \vec{\omega}^{qa}$

$$= \vec{\mathcal{F}}_s^T \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix} + \vec{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} + \vec{\mathcal{F}}_q^T \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

$$= \vec{\mathcal{F}}_s^T \left( \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix} + \underset{\underline{C}_3(\gamma)}{\underline{C}_{sb}} \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} + \underset{\underline{C}_3(\gamma)\underline{C}_2(\phi)}{\underline{C}_{sq}} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \right)$$

$$= \vec{\mathcal{F}}_s^T \left( \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix} + \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} + \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \right)$$

$$= \vec{\mathcal{F}}_s^T \left( \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix} + \begin{bmatrix} s\gamma\dot{\phi} \\ c\gamma\dot{\phi} \\ 0 \end{bmatrix} + \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s\phi\dot{\theta} \\ 0 \\ c\phi\dot{\theta} \end{bmatrix} \right)$$

$$= \vec{\mathcal{F}}_s^T \begin{bmatrix} s\gamma\dot{\phi} - c\gamma\, s\phi\, \dot{\theta} \\ c\gamma\dot{\phi} + s\gamma\, s\phi\, \dot{\theta} \\ \dot{\gamma} + c\phi\dot{\theta} \end{bmatrix} \quad \text{or} \quad \vec{\omega}^{sa} = \vec{\mathcal{F}}_b^T \begin{bmatrix} -s\phi\dot{\theta} \\ \dot{\phi} \\ c\phi\dot{\theta} + \dot{\gamma} \end{bmatrix}$$

iii) N/A

$$\underrightarrow{r}^{dmz} = \underrightarrow{r}^{dmc} + \underrightarrow{r}^{cz}$$

$$= \underrightarrow{F}_s^{T} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \\ \rho_{s3} \end{bmatrix} + \underrightarrow{F}_b^{T} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}$$

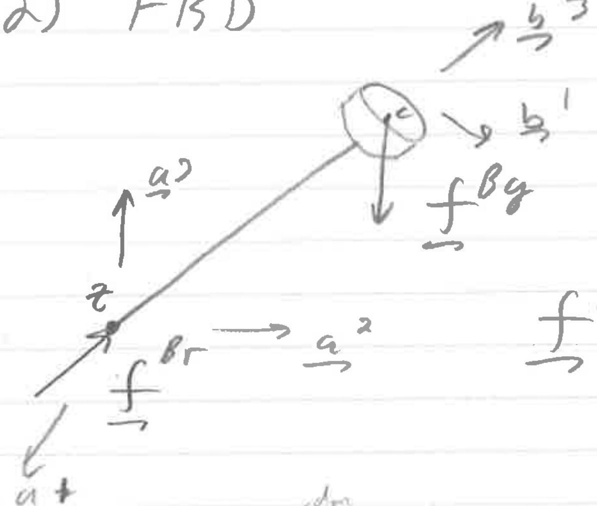
$$= \underrightarrow{F}_s^{T} \left( \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \\ \rho_{s3} \end{bmatrix} + \underline{C}_3(\gamma) \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} \right)$$

$$= \underrightarrow{F}_s^{T} \begin{bmatrix} \rho_{s1} \\ \rho_{s2} \\ \rho_{s3} + l \end{bmatrix}$$

$$\underrightarrow{r}^{zw} = \underrightarrow{0}$$

iv), v) N/A

2) FBD




$$\underrightarrow{f}^{Br} = \underrightarrow{F}_b^{T} \begin{bmatrix} f_{b1}^{Br} \\ f_{b2}^{Br} \\ f_{b3}^{Br} \end{bmatrix}$$

$$\underrightarrow{f}^{Bg} = \int_B d\underrightarrow{f}^{dmg} = \underrightarrow{F}_a^{T} \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$$

$$= \underrightarrow{F}_b^{T} \begin{bmatrix} s\phi\, mg \\ 0 \\ -c\phi\, mg \end{bmatrix}$$

$d\underrightarrow{f}^{dm} = dmg$

$$\underset{\rightharpoondown}{f}^{B} = \underset{\rightharpoondown}{f}^{Br} + \underset{\rightharpoondown}{f}^{Bg} = \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} f_{b1}^{Br} + S\phi mg \\ f_{b2}^{Br} \\ f_{b3}^{Br} - C\phi mg \end{bmatrix}$$

$$\underset{\rightharpoondown}{m}^{Bz} = \underset{\rightharpoondown}{r}^{cz} \times \underset{\rightharpoondown}{f}^{Bg}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \underline{r}_b^{cz\times} \underline{f}_b^{Bg} = \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} 0 & -l & 0 \\ l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S\phi mg \\ 0 \\ -C\phi mg \end{bmatrix}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \begin{bmatrix} 0 \\ lmg S\phi \\ 0 \end{bmatrix}$$

3) i) $\underset{\rightharpoondown}{h}^{Bz/a} = \underset{\rightharpoondown}{\mathcal{F}}_s^{T} \underline{J}_s^{Bz} \underline{\omega}_s^{sa}$

Use Parallel Axis Theorem

$$\underline{J}_s^{Bz} = \underline{J}_s^{Bc} - m_B \underline{r}_s^{cz\times} \underline{r}_s^{cz\times}$$

We are given

$$\underline{J}_s^{Bc} = \begin{bmatrix} \frac{1}{12} m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12} m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2} mr^2 \end{bmatrix}$$

For brevity let $J_{s11} = \frac{1}{12}(3r^2+h^2)$

$$J_{s22} = \frac{1}{12}(3r^2+h^2)$$

$$J_{s33} = \frac{1}{2} mr^2$$

$$\underline{J}_S^{Bc} = \begin{bmatrix} J_{S_{11}} & 0 & 0 \\ 0 & J_{S_{22}} & 0 \\ 0 & 0 & J_{S_{33}} \end{bmatrix} - m \begin{bmatrix} 0 & -\ell & 0 \\ \ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\ell & 0 \\ \ell & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} J_{S_{11}} & 0 & 0 \\ 0 & J_{S_{22}} & 0 \\ 0 & 0 & J_{S_{33}} \end{bmatrix} - m \begin{bmatrix} -\ell^2 & 0 & 0 \\ 0 & -\ell^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} J_{S_{11}} + m\ell^2 & 0 & 0 \\ 0 & J_{S_{22}} + m\ell^2 & 0 \\ 0 & 0 & J_{S_{33}} \end{bmatrix}$$

Therefore,

$$\underline{h}^{Bz/a} = \underline{\mathcal{F}}_S^T \begin{bmatrix} J_{S_{11}} + m\ell^2 & 0 & 0 \\ 0 & J_{S_{22}} + m\ell^2 & 0 \\ 0 & 0 & J_{S_{33}} \end{bmatrix} \begin{bmatrix} s\gamma\dot{\phi} - c\gamma s\phi\dot{\theta} \\ c\gamma\dot{\phi} + s\gamma s\phi\dot{\theta} \\ \dot{\gamma} + c\phi\dot{\theta} \end{bmatrix}$$

$$= \underline{\mathcal{F}}_S^T \begin{bmatrix} (J_{S_{11}} + m\ell^2)(s\gamma\dot{\phi} - c\gamma s\phi\dot{\theta}) \\ (J_{S_{22}} + m\ell^2)(c\gamma\dot{\phi} + s\gamma s\phi\dot{\theta}) \\ J_{S_{33}}(\dot{\gamma} + c\phi\dot{\theta}) \end{bmatrix}$$

ii) E1L

Not translating, so not interested

iii) E2L

$$\underline{h}^{Bz/a\,'a} + \underline{c}^{Bz} \times \cancelto{\underline{0}}{\underline{v}^{z/a\,'a}} = \underline{m}^{Bz}$$

$$\underline{h}^{Bz/a\,'s} + \underline{\omega}^{sa} \times \underline{h}^{Bz/a} = \underline{m}^{Bz}$$

$$\underline{\mathcal{F}}_s^T \left( \underline{J}_s^{Bz} \dot{\underline{\omega}}_s^{sa} + \underline{\omega}_s^{sa\times} \underline{J}_s^{Bz} \underline{\omega}_s^{sa} \right) = \underline{\mathcal{F}}_s^T \underline{m}_s^{Bz}$$

$$\underline{J}_s^{Bz} \dot{\underline{\omega}}_s^{sa} + \underline{\omega}_s^{sa\times} \underline{J}_s^{Bz} \underline{\omega}_s^{sa} = \underline{C}_{sb} \underline{m}_b^{Bz}$$

In first-order state-space form

$$\dot{\underline{\omega}}_s^{sa} = \underbrace{\underline{J}_s^{Bz^{-1}} \left( \underline{C}_{sb} \underline{m}_b^{Bz} - \underline{\omega}_s^{sa\times} \underline{J}_s^{Bz} \underline{\omega}_s^{sa} \right)}_{\underline{f}(\underline{\omega}_s^{ba})}$$

To make our derivation slightly easier, we will make some approximations. We could derive the equations of motion without the approximations, but the equations would be more complicated.

Assumption 1: $\dot{\gamma}$ is much larger than $\dot{\phi}$ and $\dot{\theta}$

Therefore $J_{s33}(\dot{\gamma} + c_\phi \dot{\theta}) \approx J_{s33}\dot{\gamma}$

Also,

$$\underbrace{J_{s11}(s_\gamma \dot{\phi} - c_\gamma s_\phi \dot{\theta})}_{\text{Assume } \approx 0} \lll J_{s33}(\dot{\gamma} + c_\phi \dot{\theta})$$

$$\underbrace{J_{s22}(c_\gamma \dot{\phi} + s_\gamma s_\phi \dot{\theta})}_{\text{assume } \approx 0} \lll J_{s33}(\dot{\gamma} + c_\phi \dot{\theta})$$

Therefore

$$\underset{\rightharpoondown}{h}^{Bzla} \approx \underset{\rightharpoondown}{\mathcal{F}}_s^T \begin{bmatrix} ml^2(s_\gamma \dot{\phi} - c_\gamma s_\phi \dot{\theta}) \\ ml^2(c_\gamma \dot{\phi} + s_\gamma s_\phi \dot{\theta}) \\ J_{s33}\dot{\gamma} \end{bmatrix}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \underline{C}_{sb}^T [\cdots]$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} -ml^2 s_\phi \dot{\theta} \\ ml^2 \dot{\phi} \\ J_{s33}\dot{\gamma} \end{bmatrix}$$

Return to E2L

$$\underline{h}^{Bz/a \cdot a} = \underline{m}^{Bz}$$

$$\underline{h}^{Bz/a \cdot b} + \underline{\omega}^{ba} \times \underline{h}^{Bz/a} = \underline{m}^{Bz}$$

$$\cancel{\underline{\mathcal{F}}_b^T}\left(\dot{\underline{h}}_b^{Bz/a} + \underline{\omega}_b^{ba\times}\, \underline{h}_b^{Bz/a}\right) = \underline{\mathcal{F}}_b^T\, \underline{m}_b^{Bz}$$

$$\begin{bmatrix} -ml^2(c_\phi \dot{\phi}\dot{\theta} + s_\phi \ddot{\theta}) \\ ml^2 \ddot{\phi} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -c_\phi \dot{\theta} & \dot{\phi} \\ c_\phi \dot{\theta} & 0 & s_\phi \dot{\theta} \\ -\dot{\phi} & -s_\phi \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} -ml^2 s_\phi \dot{\theta} \\ ml^2 \dot{\phi} \\ J_{s33} \dot{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ mgl s_\phi \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -ml^2(c_\phi \dot{\phi}\dot{\theta} + s_\phi \ddot{\theta}) - ml^2 \dot{\phi}\dot{\theta} c_\phi + J_{s33}\dot{\gamma}\dot{\phi} \\ ml^2\ddot{\phi} - ml^2 s_\phi c_\phi \dot{\theta}^2 + J_{s33}\dot{\gamma} s_\phi \dot{\theta} \\ ml^2 \cancel{s_\phi \dot{\theta}\dot{\phi}} - ml^2 \cancel{s_\phi \dot{\theta}\dot{\phi}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ mgl s_\phi \\ 0 \end{bmatrix}$$

EoMs: $ml^2 s_\phi \ddot{\theta} + 2ml^2 c_\phi \dot{\phi}\dot{\theta} = \dot{\phi} J_{s33}\dot{\gamma}$

$ml^2 \ddot{\phi} - ml^2 s_\phi c_\phi \dot{\theta}^2 = mgl s_\phi - s_\phi \dot{\theta} J_{s33} \dot{\gamma}$

When $\theta \doteq 0$, $\phi = 90°$, $\dot{\theta} = \dot{\phi} = 0$

$$ml^2\ddot{\theta} = 0$$

$$ml^2\ddot{\phi} = mgl$$

$\dot{\phi}$ increases, pendulum swings down and

$$ml^2\ddot{\theta} \approx \dot{\phi} J_{s33} \dot{\gamma}$$

$$ml^2\ddot{\phi} \approx mgl$$

$\dot{\theta}$ will increase, pendulum will precess, and eventually

$$ml^2 s_\phi \ddot{\theta} + 2ml^2 c_\phi \dot{\phi}\dot{\theta} = \dot{\phi} J_{s33}\dot{\gamma}$$

$$ml^2\ddot{\phi} - ml^2 s_\phi c_\phi \dot{\theta}^2 = s_\phi (mgl - \dot{\theta} J_{s33} \dot{\gamma})$$

Larger $\dot{\gamma}$ leads to slower preecession (smaller $\dot{\theta}$) and greater cancellation of the mgl term

# Energy and Work of a Continuous Rigid Body

Just like for a DRB, the kinetic energy associated with a CRB is

$$T_{B w/a} = \frac{1}{2} m_B \underline{V}_a^{cw/a^T} \underline{V}_c^{cw/a} + \frac{1}{2} \underline{\omega}_b^{ba^T} \underline{J}_b^{Bc} \underline{\omega}_b^{ba}$$

The gravitational potential associated with CRB $B$ is

$$V_{Bw} = -m_B \vec{g} \cdot \vec{r}^{cw}$$

All of the energy and work methods we saw for particles also apply to CRBs

For example, we can derive a Work-Energy Theorem for rigid bodies:

$$\frac{d}{dt}\left(T_{Bw/a}\right) = \frac{d}{dt}\left(\frac{1}{2} m_B \underline{V}_a^{cw/a^T} \underline{V}_a^{cw/a} + \frac{1}{2} \underline{\omega}_b^{ba^T} \underline{J}_b^{Bc} \underline{\omega}_b^{ba}\right)$$

$$= \frac{1}{2} m_B \left(\underline{\dot{V}}_a^{cw/a^T} \underline{V}_a^{cw/a} + \underline{V}_a^{cw/a^T} \underline{\dot{V}}_a^{cw/a}\right)$$

$$+ \frac{1}{2}\left(\underline{\dot{\omega}}_b^{ba^T} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} + \underline{\omega}_b^{ba^T} \underline{J}_b^{Bc} \underline{\dot{\omega}}_b^{ba}\right)$$

$$= m_B \underline{\dot{V}}_a^{cw/a^T} \underline{V}_a^{cw/a} + \underline{\omega}_b^{ba^T} \underline{J}_b^{Bc} \underline{\dot{\omega}}_b^{ba}$$

Knowing that $\underline{f}_a^{B} = m_B \dot{\underline{V}}_a^{cw/a}$

$$\underline{J}_b^{Bc} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$$

gives

$$\frac{d}{dt}(T_{Bw/a}) = \underline{f}_a^{B^T} \underline{V}_a^{cw/a} + \underline{\omega}_b^{ba^T}\left(\underline{m}_b^{Bc} - \underline{\omega}_b^{ba\times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba}\right)$$

$$= \vec{f}^{B} \cdot \vec{v}^{cw/a} + \vec{\omega}^{ba} \cdot \vec{m}^{Bc}$$

(recall $\underline{\omega}_b^{ba^T} \underline{\omega}_b^{ba\times} = \underline{0}$)

Therefore

$$\boxed{\frac{d}{dt}(T_{Bw/a}) = \vec{f}^{B} \cdot \vec{v}^{cw/a} + \vec{m}^{Bc} \cdot \vec{\omega}^{ba}}$$

↳ Work-Energy Theorem for a rigid body (DRB or CRB)

Total energy of rigid body:

$$E_{Bw/a} = T_{Bw/a} + V_{Bw}$$

If all forces are conservative, then

$$E_{Bw/a}(t_1) = E_{Bw/a}(t_0) \quad \text{or} \quad \dot{E}_{Bw/a} = 0$$

↳ Conservation of Energy.

FIVE STAR

We can also describe impulses acting on a rigid body.

$$\underset{\rightharpoondown}{\hat{f}}^{B} = \int_{t_1}^{t_2} \underset{\rightharpoondown}{f}^{B}\, dt = \int_{t_1}^{t_2} \underset{\rightharpoondown}{\dot{p}}^{Bw/a\cdot a}\, dt$$

$$\boxed{\underset{\rightharpoondown}{\hat{f}}^{B} = \underset{\rightharpoondown}{p}^{Bw/a}(t_2) - \underset{\rightharpoondown}{p}^{Bw/a}(t_1)}$$

$$\underset{\rightharpoondown}{\hat{m}}^{Bz} = \int_{t_1}^{t_2} \underset{\rightharpoondown}{m}^{Bz}\, dt = \int_{t_1}^{t_2} \left( \underset{\rightharpoondown}{h}^{Bz/a\cdot a} + \underset{\rightharpoondown}{c}^{Bz} \times \underset{\rightharpoondown}{v}^{zw/a\cdot a} \right) dt$$

$$\boxed{\underset{\rightharpoondown}{\hat{m}}^{Bz} = \underset{\rightharpoondown}{h}^{Bz/a}(t_2) - \underset{\rightharpoondown}{h}^{Bz/a}(t_1) + \int_{t_1}^{t_2} \underset{\rightharpoondown}{c}^{Bz} \times \underset{\rightharpoondown}{v}^{zw/a\cdot a}\, dt}$$

If $z = c$, then $\underset{\rightharpoondown}{c}^{Bz} = \underset{\rightharpoondown}{0}$

$$\boxed{\underset{\rightharpoondown}{\hat{m}}^{Bc} = \underset{\rightharpoondown}{h}^{Bc/a}(t_2) - \underset{\rightharpoondown}{h}^{Bc/a}(t_1)}$$

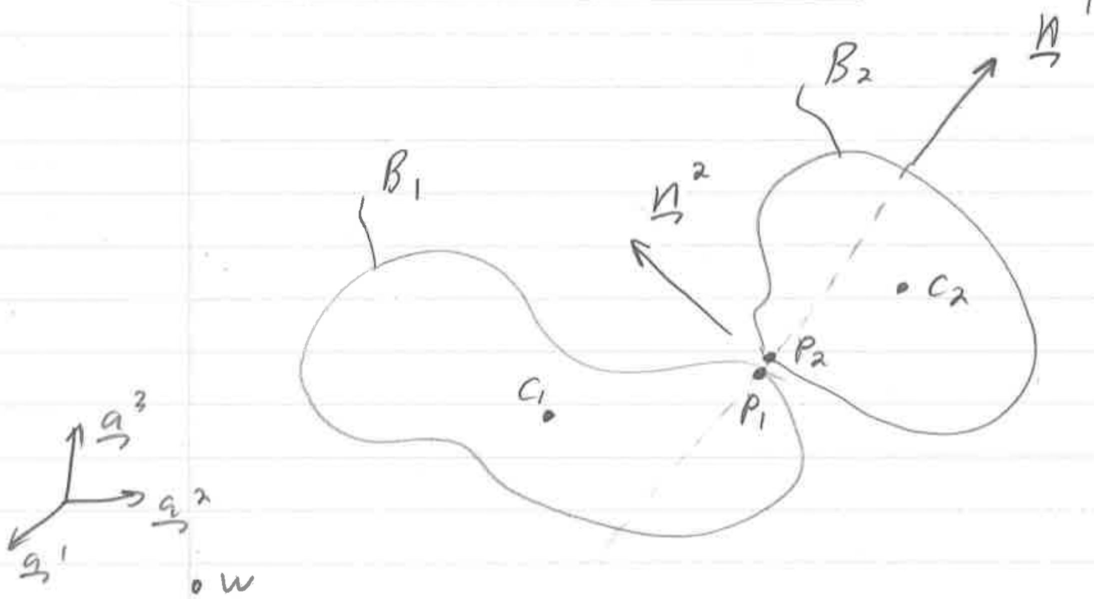
If $\underset{\rightharpoondown}{\hat{f}}^{B} = \underset{\rightharpoondown}{0}$, then $\underset{\rightharpoondown}{p}^{Bw/a}(t_1) = \underset{\rightharpoondown}{p}^{Bw/a}(t_2)$

↳ Conservation of translational momentum

If $\underset{\rightharpoondown}{\hat{m}}^{Bc} = \underset{\rightharpoondown}{0}$, then $\underset{\rightharpoondown}{h}^{Bc/a}(t_1) = \underset{\rightharpoondown}{h}^{Bc/a}(t_2)$

↳ conservation of angular momentum.

Collision of Rigid Bodies




$\underset{\sim}{V}^{C_1 w/a}$ : velocity of center of mass of $B_1$ rel. to w wrt $\mathcal{F}_a$

$\underset{\sim}{V}^{C_2 w/a}$ : velocity of center of mass of $B_2$ rel. to w wrt $\mathcal{F}_a$.

$\underset{\sim}{V}^{P_1 w/a}$ : velocity of impact point $P_1$ rel. to w wrt $\mathcal{F}_a$

$\underset{\sim}{V}^{P_2 w/a}$ : velocity of impact point $P_2$ rel. to w wrt $\mathcal{F}_a$.

$$\text{COR: } e = \frac{\left(\underset{\sim}{V}^{P_2 w/a}(t_2) - \underset{\sim}{V}^{P_1 w/a}(t_2)\right) \cdot \underset{\sim}{n}^1}{\left(\underset{\sim}{V}^{P_1 w/a}(t_1) - \underset{\sim}{V}^{P_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^1}$$

Conservation of Translational Momentum of System

$$m_{B_1} \underset{\sim}{V}^{C_1 w/a}(t_1) + m_{B_2} \underset{\sim}{V}^{C_2 w/a}(t_1)$$
$$= m_{B_1} \underset{\sim}{V}^{C_1 w/a}(t_2) + m_{B_2} \underset{\sim}{V}^{C_2 w/a}(t_2)$$

Conservation of Angular Momentum of System

$$\underset{\sim}{h}^{B_1 C_1/a}(t_1) + \underset{\sim}{h}^{B_2 C_2/a}(t_1) = \underset{\sim}{h}^{B_1 C_1/a}(t_2) + \underset{\sim}{h}^{B_2 C_2/a}(t_2)$$