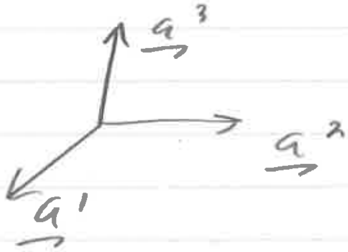
Review of AEM 2012

Physical Vectors:

$\underrightarrow{v}$ defined by magnitude and direction, <u>not</u> location.

Reference Frame:



Defined by 3 orthonormal (orthogonal and normalized) and dextral (right-handed) physical basis vectors.

Vectrix Notation:

$$\underrightarrow{v} = v_{a1}\underrightarrow{a}^1 + v_{a2}\underrightarrow{a}^2 + v_{a3}\underrightarrow{a}^3$$

$$= \underbrace{\begin{bmatrix} \underrightarrow{a}^1 & \underrightarrow{a}^2 & \underrightarrow{a}^3 \end{bmatrix}}_{\underrightarrow{\mathcal{F}}_a^T} \underbrace{\begin{bmatrix} v_{a1} \\ v_{a2} \\ v_{a3} \end{bmatrix}}_{\underline{v}_a} = \underrightarrow{\mathcal{F}}_a^T \underline{v}_a$$

$$= \begin{bmatrix} v_{a1} & v_{a2} & v_{a3} \end{bmatrix} \begin{bmatrix} \underrightarrow{a}^1 \\ \underrightarrow{a}^2 \\ \underrightarrow{a}^3 \end{bmatrix} = \underline{v}_a^T \underrightarrow{\mathcal{F}}_a$$

Dot and Cross Products

$$\underrightarrow{u} \cdot \underrightarrow{v} = \underline{u}_a^T \underline{v}_a \quad , \quad \underrightarrow{u} \times \underrightarrow{v} = \underrightarrow{\mathcal{F}}_a^T \underline{u}_a^{\times} \underline{v}_a$$

Direction Cosine Matrix (DCM)

$$\underline{C}_{ba} = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^T \qquad \underline{v}_b = \underline{C}_{ba} \underline{v}_a$$

Properties:
1) $\underline{C}_{ba} \in \mathbb{R}^{3\times 3}$
2) $\underline{C}_{ba}^T \underline{C}_{ba} = \underline{1}$
3) $\det(\underline{C}_{ba}) = +1$

Euler Angles

- Rotations about "principle axes" (1, 2, or 3 rotations)
- $\underline{C}_{ba}$ can be defined as Euler-angle sequence. E.g.,

$$\mathcal{F}_a \xrightarrow{\underline{C}_3(\alpha)} \mathcal{F}_i \xrightarrow{\underline{C}_2(\beta)} \mathcal{F}_j \xrightarrow{\underline{C}_1(\gamma)} \mathcal{F}_b$$

$$\underline{C}_{ba} = \underline{C}_1(\gamma)\,\underline{C}_2(\beta)\,\underline{C}_3(\alpha)$$

Kinematics

$$\underrightarrow{r}^{\cdot a} = \left(\underrightarrow{\mathcal{F}}_a^{T}\,\underline{r}_a\right)^{\cdot a} = \cancel{\underrightarrow{\mathcal{F}}_a^{T\,\cdot a}}^{\,\underline{0}}\,\underline{r}_a + \underrightarrow{\mathcal{F}}_a^{T}\,\dot{\underline{r}}_a = \underrightarrow{\mathcal{F}}_a^{T}\,\dot{\underline{r}}_a$$

3 R's: $\underline{r}_a$ is $\underrightarrow{r}$ resolved in $\mathcal{F}_a$

$\underrightarrow{r}^{pq}$ is the position of point p relative to point q.

$\underrightarrow{r}^{\cdot a}$ is the time derivative of $\underrightarrow{r}$ with respect to $\mathcal{F}_a$.

Transport Theorem: $\underrightarrow{r}^{\cdot a} = \underrightarrow{r}^{\cdot b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}$

Poisson's Equation: $\dot{\underline{C}}_{ba} = -\underline{\omega}_b^{ba\times}\,\underline{C}_{ba}$

$$\underline{\omega}_b^{ba} = \underline{S}_b^{ba}(\ldots)\,\dot{\underline{\theta}}^{ba}$$

Ex: $\underline{C}_{ba} = \underline{C}_3(\phi)\,\underline{C}_2(\varepsilon)\,\underline{C}_1(\delta)$

$$\underrightarrow{\omega}^{ba} = \underrightarrow{\mathcal{F}}_b^{T}\left(\underline{C}_3(\phi)\,\underline{C}_2(\varepsilon)\,\underline{1}_1\,\dot{\delta} + \underline{C}_3(\phi)\,\underline{1}_2\,\dot{\varepsilon} + \underline{1}_3\,\dot{\phi}\right)$$

$$= \underset{\rightharpoonup}{\mathcal{F}}_b^T \underbrace{\left[\underline{C}_3(\phi)\underline{C}_2(\varepsilon)\underline{1}_1 \quad \underline{C}_3(\phi)\underline{1}_2 \quad \underline{1}_3 \right]}_{\underline{S}_b^{ba}(\varepsilon,\phi)} \underbrace{\begin{bmatrix} \dot{\delta} \\ \dot{\varepsilon} \\ \dot{\phi} \end{bmatrix}}_{\dot{\underline{\theta}}^{ba}}$$

If $\underline{S}_b^{ba}$ is not invertible, then can't solve for

$\dot{\underline{\theta}}^{ba} = \underline{S}_b^{ba^{-1}} \underline{\omega}_b^{ba}$

Position : $\underset{\rightharpoonup}{r}^{yz}$

Velocity : $\underset{\rightharpoonup}{v}^{yz/a} = \underset{\rightharpoonup}{r}^{yz \cdot a}$

Acceleration : $\underset{\rightharpoonup}{a}^{yz/a/a} = \underset{\rightharpoonup}{v}^{yz/a \cdot a} = \underset{\rightharpoonup}{r}^{yz \cdot a \cdot a}$

Dynamics of a Particle

Inertial Frame: frame in which Newton's Laws hold (not rotating relative to another inertial frame)

Translational Momentum: $\underset{\rightharpoonup}{p}^{yw/a} = m \underset{\rightharpoonup}{v}^{yw/a}$

N2L: $\underset{\rightharpoonup}{f}^{y} = \underset{\rightharpoonup}{p}^{yw/a \cdot a} = m \underset{\rightharpoonup}{a}^{yw/a/a}$

↑ if m is constant

Angular Momentum: $\underset{\rightharpoonup}{h}^{yw/a} = \underset{\rightharpoonup}{r}^{yw} \times \underset{\rightharpoonup}{p}^{yw/a} = m \underset{\rightharpoonup}{r}^{yw} \times \underset{\rightharpoonup}{v}^{yw/a}$

N2LR: $\underset{\rightharpoonup}{h}^{yw/a \cdot a} = \underset{\rightharpoonup}{m}^{yw} = \underset{\rightharpoonup}{r}^{yw} \times \underset{\rightharpoonup}{f}^{y}$

N3L: $\underset{\rightharpoonup}{f}^{yx} = - \underset{\rightharpoonup}{f}^{xy}$

Forces:

Linear Spring Force:  $\underset{\rightharpoonup}{f}^{Y} = -K x_s \dfrac{\underset{\rightharpoonup}{r}^{Yw}}{\|\underset{\rightharpoonup}{r}^{Yw}\|_2}$

Linear Viscous Damping:  $\underset{\rightharpoonup}{f}^{Y} = -c\,\underset{\rightharpoonup}{v}^{Yw/a}$

Gravitational Force (close to Earth): $\underset{\rightharpoonup}{f}^{Y} = m\,\underset{\rightharpoonup}{g}$

3 Steps to Success in Dynamics

1) Kinematics
2) FBD
3) N2L

Energy of a Particle

Kinetic Energy: $T_{Yw/a} = \frac{1}{2} m\, \underset{\rightharpoonup}{v}^{Yw/a} \cdot \underset{\rightharpoonup}{v}^{Yw/a}$

Work: $W_{Yw}(\underset{\rightharpoonup}{f}^{Y}, C_Y) = \int_{C_Y} \underset{\rightharpoonup}{f}^{Y} \cdot d\underset{\rightharpoonup}{r}^{Yw}$

$$= T_{Yw/a}(t_2) - T_{Yw/a}(t_1)$$

If $\underset{\rightharpoonup}{f}^{Y}$ is conservative, then $W_{Yw}(\underset{\rightharpoonup}{f}^{Y}) = V_{Yw}(t_1) - V_{Yw}(t_2)$

Potential Energy
- linear spring: $V_{Yw} = \frac{1}{2} K x_s^2$
- gravity: $V_{Yw} = -m\,\underset{\rightharpoonup}{g} \cdot \underset{\rightharpoonup}{r}^{Yw}$

If forces are conservative, then $E_{Yw/a} = T_{Yw/a} + V_{Yw}$ is constant ($\dot{E}_{Yw/a} = 0$).

Power: $P_{Y w/a}(\underset{\rightharpoonup}{f}^{Y}) = \underset{\rightharpoonup}{f}^{Y} \cdot \underset{\rightharpoonup}{v}^{Y w/a}$

Work Energy Theorem: $\frac{d}{dt}(T_{Y w/a}) = \underset{\rightharpoonup}{f}^{Y} \cdot \underset{\rightharpoonup}{v}^{Y w/a}$

$\hookrightarrow$ gives EoMs of single DOF systems

4 'alternative' steps to success in dynamics.

Vibrations

Single DoF: $m\ddot{q} + Kq = f_p \cos(\omega_p t)$

Natural freq. $\omega = \sqrt{\frac{K}{m}}$

Resonance: $\omega_p \to \omega$, $|q(t)| \to \infty$

Multiple DoF:

$$\underline{M}\,\ddot{\underline{q}} + \underline{K}\,\underline{q} = \hat{\underline{b}}\,f$$

Natural freqs (ω): eigenvalues of $\underline{M}^{-1}\underline{K}$

Mode shapes: eigenvectors of $\underline{M}^{-1}\underline{K}$

Systems of Particles

$m_B = \sum_{i=1}^{l} m_i$ (mass)

$\underset{\rightharpoonup}{r}^{cw} = \frac{1}{m_B} \sum_{i=1}^{l} m_i \underset{\rightharpoonup}{r}^{Y_i w}$ (center of mass)

translational momentum: $\underset{\rightharpoonup}{p}^{B w/a} = m_B \underset{\rightharpoonup}{v}^{c w/a}$

N2L: $\underset{\rightharpoonup}{f}^{B} = \underset{\rightharpoonup}{p}^{B w/a \cdot a} = m_B \underset{\rightharpoonup}{a}^{c w/a/a}$

$\uparrow$ m_B constant

If $\underset{\rightharpoonup}{f}^{B} = 0$, then $\underset{\rightharpoonup}{p}^{B\,w/a}$ is constant

<u>Particle Impacts</u>

Impulse: $\hat{\underset{\rightharpoonup}{f}}^{Y} = \int_{t_1}^{t_2} \underset{\rightharpoonup}{f}^{Y} dt = \underset{\rightharpoonup}{p}^{Y\,w/a}(t_2) - \underset{\rightharpoonup}{p}^{Y\,w/a}(t_1)$

Angular Impulse: $\hat{\underset{\rightharpoonup}{m}}^{Y\,w} = \int_{t_1}^{t_2} \underset{\rightharpoonup}{m}^{Y\,w} dt = \underset{\rightharpoonup}{h}^{Y\,w/a}(t_2) - \underset{\rightharpoonup}{h}^{Y\,w/a}(t_1)$

If no external forces (impulses) during impact, then

$$m_1 \underset{\rightharpoonup}{v}^{Y_1 w/a}(t_1) + m_2 \underset{\rightharpoonup}{v}^{Y_2 w/a}(t_1) = m_1 \underset{\rightharpoonup}{v}^{Y_1 w/a}(t_2) + m_2 \underset{\rightharpoonup}{v}^{Y_2 w/a}(t_2)$$

↳ conservation of momentum of system

Coefficient of restitution (COR)

$$e = \frac{\left(\underset{\rightharpoonup}{v}^{Y_2 w/a}(t_2) - \underset{\rightharpoonup}{v}^{Y_1 w/a}(t_2)\right) \cdot \underset{\rightharpoonup}{n}^1}{\left(\underset{\rightharpoonup}{v}^{Y_1 w/a}(t_1) - \underset{\rightharpoonup}{v}^{Y_2 w/a}(t_1)\right) \cdot \underset{\rightharpoonup}{n}^1}$$

$e = 1$: perfectly elastic (Kinetic energy conserved)

$e = 0$: perfectly plastic (Kinetic energy decreases)
(perfectly inelastic)

Conservation of momentum of each particle in $\underset{\rightharpoonup}{n}^2, \underset{\rightharpoonup}{n}^3$ directions

$$\left.\begin{aligned} \left(\underset{\rightharpoonup}{v}^{Y_1 w/a}(t_2) - \underset{\rightharpoonup}{v}^{Y_1 w/a}(t_1)\right) \cdot \underset{\rightharpoonup}{n}^i &= 0 \\ \left(\underset{\rightharpoonup}{v}^{Y_2 w/a}(t_2) - \underset{\rightharpoonup}{v}^{Y_2 w/a}(t_1)\right) \cdot \underset{\rightharpoonup}{n}^i &= 0 \end{aligned}\right\} \; i = \{2, 3\}$$

Systems of Particles (continued)

Angular momentum: $\underset{\rightharpoondown}{h}^{Bz/a} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{r}^{y_i z} \times \underset{\rightharpoondown}{v}^{y_i w/a}$

First moment of mass: $\underset{\rightharpoondown}{c}^{Bz} = m_B \underset{\rightharpoondown}{r}^{cz}$

N2LR: $\underset{\rightharpoondown}{h}^{Bz/a \cdot a} + \underset{\rightharpoondown}{c}^{Bz} \times \underset{\rightharpoondown}{v}^{zw/a \cdot a} = \underset{\rightharpoondown}{m}^{Bz}$

If $z = c$, $\underset{\rightharpoondown}{c}^{Bz} = \underset{\rightharpoondown}{0}$ $\Rightarrow$ $\underset{\rightharpoondown}{h}^{Bc/a \cdot a} = \underset{\rightharpoondown}{m}^{Bc}$

Discrete Rigid Bodies (DRBs)

N2L: $\underset{\rightharpoondown}{f}^{B} = m_B \underset{\rightharpoondown}{a}^{cw/a/a}$ or $\underline{f}_a^{B} = m_B \underline{a}_a^{cw/a/a}$

Second moment of mass: $\underline{J}_b^{Bz} = -\sum_{i=1}^{\ell} m_i \underline{r}_b^{y_i z^\times} \underline{r}_b^{y_i z^\times}$

$\underset{\rightharpoondown}{h}^{Bz/a} = \underset{\rightharpoondown}{\mathcal{F}}_b^{T} \underline{J}_b^{Bz} \underline{\omega}_b^{ba}$

N2LR (about c) (Euler's Eqn): $\underline{J}_b^{Bc} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$

Kinetic Energy: $T_{Bw/a} = \frac{1}{2} m_B \underline{v}_a^{cw/a^T} \underline{v}_a^{cw/a}$

$+ \frac{1}{2} \underline{\omega}_b^{ba^T} \underline{J}_b^{Bc} \underline{\omega}_b^{ba}$

Gravitational Potential: $V_{Bw} = -m_B \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{cw}$

Continuous Rigid Bodies (CRBs)

Mass: $m_B = \int_B dm = \int_B \sigma \, dV$

Center of mass $\underset{\rightharpoondown}{r}^{cz} = \frac{1}{m_B} \int_B \underset{\rightharpoondown}{r}^{dmz} dm = \frac{1}{m_B} \int_B \sigma \underset{\rightharpoondown}{r}^{dmz} dV$

1st moment of mass: $\underset{\rightharpoondown}{c}^{Bz} = \int_B \underset{\rightharpoondown}{r}^{dmz} dm = m_B \underset{\rightharpoondown}{r}^{cz}$

2nd moment of mass: $\underline{J}_b^{Bz} = -\int_B \underline{r}_b^{dmz^\times} \underline{r}_b^{dmz^\times} dm$

$$= -\int_B \sigma \, \underline{r}_b^{dmz^\times} \underline{r}_b^{dmz^\times} dV$$

$\underset{\rightharpoondown}{p}^{Bw/a} = m_B \underset{\rightharpoondown}{v}^{cw/a}$

$\underset{\rightharpoondown}{h}^{Bz/a} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \underline{J}_b^{Bz} \underline{\omega}_b^{ba} \quad \left(= \underset{\rightharpoondown}{J} \cdot \underset{\rightharpoondown}{\omega}^{ba} \right)$

E1L: $\underset{\rightharpoondown}{f}^B = \underset{\rightharpoondown}{p}^{Bw/a \cdot a} = m_B \underset{\rightharpoondown}{a}^{cw/a/a}$

$\uparrow$ m_B constant

E2L: $\underset{\rightharpoondown}{h}^{Bz/a \cdot a} + \underset{\rightharpoondown}{c}^{Bz} \times \underset{\rightharpoondown}{v}^{zw/a \cdot a} = \underset{\rightharpoondown}{m}^{Bz}$

If $z = c$, $\quad \underset{\rightharpoondown}{h}^{Bc \cdot a} = \underset{\rightharpoondown}{m}^{Bc}$

$\hookrightarrow \underline{J}_b^{Bc} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$

$\left(\underset{\rightharpoondown}{J}^{Bc} \cdot \underset{\rightharpoondown}{\omega}^{ba \cdot b} + \underset{\rightharpoondown}{\omega}^{ba} \times \left(\underset{\rightharpoondown}{J}^{Bc} \cdot \underset{\rightharpoondown}{\omega}^{ba} \right) = \underset{\rightharpoondown}{m}^{Bc} \right)$

Parallel Axis Theorem

$$\underline{J}_b^{\mathcal{B}q} = \underline{J}_b^{\mathcal{B}p} - \underline{c}_b^{\mathcal{B}p^\times} \underline{r}_b^{pq^\times} - \underline{r}_b^{pq^\times} \underline{c}_b^{\mathcal{B}p^\times} - m_{\mathcal{B}} \underline{r}_b^{pq^\times} \underline{r}_b^{pq^\times}$$

If $p = c$, then $\underline{c}_b^{\mathcal{B}c} = \underline{0}$ and

$$\underline{J}_b^{\mathcal{B}q} = \underline{J}_b^{\mathcal{B}c} - m_{\mathcal{B}} \underline{r}_b^{cq^\times} \underline{r}_b^{cq^\times}.$$

Work-Energy Theorem for a Rigid Body

$$\frac{d}{dt}\left(T_{\mathcal{B}w/a}\right) = \underrightarrow{f}^{\mathcal{B}} \cdot \underrightarrow{v}^{cw/a} + \underrightarrow{m}^{\mathcal{B}c} \cdot \underrightarrow{\omega}^{ba}$$

Tensors

$$\underrightarrow{I} = \underrightarrow{\mathcal{F}}_b^{T} \underline{I}_b \underrightarrow{\mathcal{F}}_b \quad , \quad \underline{I}_b = \underline{C}_{ba} \underline{I}_a \underline{C}_{ab}$$