Systems of Particles

Def: A discrete body is a collection of particles. This is also known as a system of particles

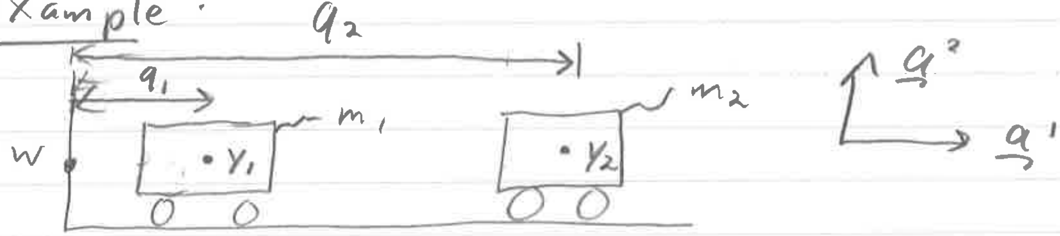
Def: The zeroth moment of mass, or simply the mass, of a body $\mathcal{B}$ composed of particles $y_1, y_2, \ldots, y_\ell$ whose masses are $m_1, m_2, \ldots, m_\ell$ is defined as

$$m_{\mathcal{B}} = \sum_{i=1}^{\ell} m_i$$

Def: Consider a body $\mathcal{B}$. Let $m_{\mathcal{B}}$ be the mass and consider a pt. w. Then the position of the center of mass relative to w is

$$\underline{r}^{cw} = \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{\ell} m_i \, \underline{r}^{y_i w}$$

Example:



$$\underset{\rightharpoonup}{r}^{y_1 w} = \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} q_1 \\ 0 \\ 0 \end{bmatrix}, \quad \underset{\rightharpoonup}{r}^{y_2 w} = \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} q_2 \\ 0 \\ 0 \end{bmatrix}$$

$$m_B = \sum_{i=1}^{2} m_i = m_1 + m_2$$

$$\underset{\rightharpoonup}{r}^{cw} = \frac{1}{m_B} \sum_{i=1}^{l} m_i \underset{\rightharpoonup}{r}^{y_i w}$$

$$= \frac{1}{m_1+m_2}\left(m_1 \underset{\rightharpoonup}{r}^{y_1 w} + m_2 \underset{\rightharpoonup}{r}^{y_2 w} \right)$$

$$= \frac{1}{m_1+m_2}\left(\underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} m_1 q_1 \\ 0 \\ 0 \end{bmatrix} + \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} m_2 q_2 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} \frac{1}{m_1+m_2}(m_1 q_1 + m_2 q_2) \\ 0 \\ 0 \end{bmatrix}$$

Newton's 2nd Law for Multi Particle Systems

Def: Let $\mathcal{F}_a$ be a frame and consider body B and a pt. w. The translational momentum of B relative to w wrt $\mathcal{F}_a$ is

$$\underset{\rightharpoonup}{p}^{Bw/a} = \sum_{i=1}^{\ell} \underset{\rightharpoonup}{p}^{y_i w/a} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{v}^{y_i w/a}$$

We just defined the center of mass as

$$\underset{\rightharpoonup}{r}^{cw} = \frac{1}{m_B} \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{y_i w}$$

Therefore $\quad m_B \underset{\rightharpoonup}{r}^{cw} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{y_i w}$

Take the time derivative wrt $\mathcal{F}_a$

$$m_B \underset{\rightharpoonup}{r}^{cw \cdot a} = m_B \underset{\rightharpoonup}{v}^{cw/a} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{y_i w \cdot a} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{v}^{y_i w/a}$$

Thus $\quad \boxed{\underset{\rightharpoonup}{p}^{Bw/a} = m_B \underset{\rightharpoonup}{v}^{cw/a}}$

Consider l particles

$\underline{f}^{i}, \underline{f}^{j}$: external forces

$\underline{f}^{Y_iY_j}, \underline{f}^{Y_jY_i}$: internal forces

From N3L, we know that $\underline{f}^{Y_iY_j} = -\underline{f}^{Y_jY_i}$

Use N2L on each particle

$$\underline{p}^{Y_i W/a \cdot a} = \underline{f}^{Y_i} + \sum_{j=1}^{l} \underline{f}^{Y_iY_j} \quad , \underline{f}^{Y_iY_i} = \underline{0}$$

Let's sum up thes equation

$$\underbrace{\sum_{i=1}^{l} \underline{p}^{Y_i W/a \cdot a}}_{\underline{p}^{BW/a \cdot a}} = \underbrace{\sum_{i=1}^{l} \underline{f}^{Y_i}}_{\underline{f}^{B}} + \sum_{i=1}^{l}\sum_{j=1}^{l} \underline{f}^{Y_iY_j}$$

Since $\underline{f}^{Y_iY_j} = -\underline{f}^{Y_jY_i}$, then $\sum_{i=1}^{l}\sum_{j=1}^{l} \underline{f}^{Y_iY_j} = \underline{0}$

Therefore,

$$\boxed{\underline{p}^{BW/a \cdot a} = \underline{f}^{B}}$$

This is N2L for a body composed of multiple particles.

If m_B is constant, then

$$\left(m_B \underset{\rightharpoondown}{v}^{cw/a}\right)^{\cdot a} = \underset{\rightharpoondown}{f}^{B}$$

$$m_B \underset{\rightharpoondown}{a}^{cw/a/a} = \underset{\rightharpoondown}{f}^{B}$$

Notice that if $\underset{\rightharpoondown}{f}^{B} = \underset{\rightharpoondown}{0}$, then

$$\underset{\rightharpoondown}{p}^{Bw/a \cdot a} = \left(m_B \underset{\rightharpoondown}{v}^{cw/a}\right)^{\cdot a} = \underset{\rightharpoondown}{0}$$

This implies that $\underset{\rightharpoondown}{p}^{Bw/a} = m_B \underset{\rightharpoondown}{v}^{cw/a}$ is constant wrt $\mathcal{F}_a$.

This is known as <u>conservation of momentum</u>

Recall

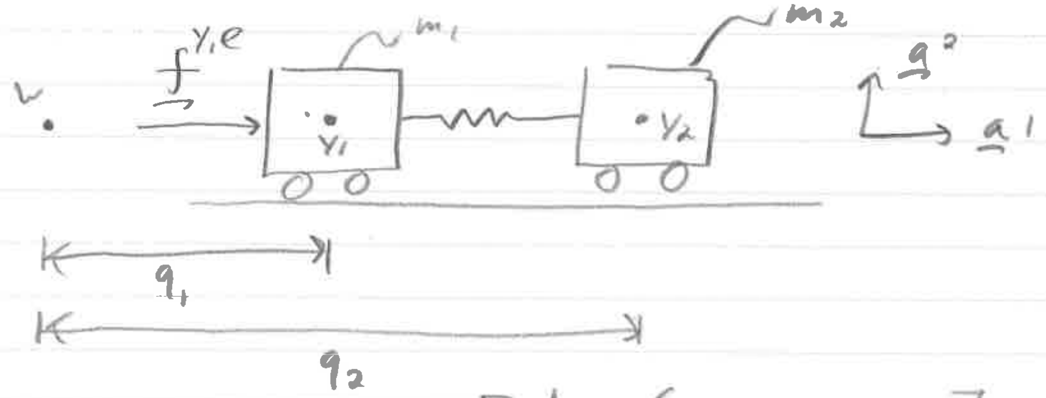
$$\underset{\rightharpoondown}{p}^{Bw/a} = \sum_{i=1}^{\ell} \underset{\rightharpoondown}{p}^{y_i w/a} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i w/a}$$

Therefore if there are no external forces acting on the system of particles, we have

$$\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i w/a} = \text{constant}$$

This will be useful when we study impacts.

Example: Consider two particles with spring in between them



Recall $\underline{r}^{cw} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \frac{1}{m_1+m_2}(m_1 q_1 + m_2 q_2) \\ 0 \\ 0 \end{bmatrix}$, $m_B = m_1 + m_2$

Let's perform 3 steps to success on center of mass.

1) Kinematics

i), ii) N/A

iii) Given

iv)
$$\underline{v}^{cw/a} = \underline{r}^{cw \cdot a} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \frac{1}{m_1+m_2}(m_1 \dot{q}_1 + m_2 \dot{q}_2) \\ 0 \\ 0 \end{bmatrix}$$

v) $$\underline{a}^{cw/a/a} = \underline{v}^{cw/a \cdot a} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \frac{1}{m_1+m_2}(m_1 \ddot{q}_1 + m_2 \ddot{q}_2) \\ 0 \\ 0 \end{bmatrix}$$

2) FBD

$\underline{f}^{y_1 e}$ center of mass

$$\underline{f}^{y_1 e} = \underline{\mathcal{F}}_a^T \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

3) N2L

$$\underset{\rightharpoonup}{f}^{B} = m_B \underset{\rightharpoonup}{a}^{cw/a/a}$$

$$\underset{\rightharpoonup}{f}^{Y_1 e} = (m_1 + m_2)\, \underset{\rightharpoonup}{a}^{cw/a/a}$$

$$\underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} = \cancel{(m_1 + m_2)}\, \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} \cancel{\frac{1}{m_1+m_2}} (m_1 \ddot{q}_1 + m_2 \ddot{q}_2) \\ 0 \\ 0 \end{bmatrix}$$

$$m_1 \ddot{q}_1 + m_2 \ddot{q}_2 = f \quad ①$$

This gives us one EoM, but we know we should obtain 2 EoMs since this is a 2DoF system. We can get the "missing" EoM from applying the 3 steps to success to either one of the particles.

Recall we previously found

$$m_1 \ddot{q}_1 + K(q_1 - q_2) = f \quad ②$$
$$m_2 \ddot{q}_2 + K(q_2 - q_1) = 0 \quad ③$$

Let's say we performed 3 steps to success on particle Y_2, giving ③. Combining this with ① gives

$$m_1 \ddot{q}_1 + m_2 \ddot{q}_2 = f \quad ①$$
$$m_2 \ddot{q}_2 + K(q_2 - q_1) = 0 \quad ③$$

These are the EoMs. To show they are equivant to ② and ③, subtract ③ from ① to get

$$m_1 \ddot{q}_1 - K(q_2 - q_1) = f \Rightarrow m_1 \ddot{q}_1 + K(q_1 - q_2) = f$$

FIVE STAR

Combining this with (3) gives

$$m_1 \ddot{q}_1 + k(q_1 - q_2) = f$$
$$m_2 \ddot{q}_2 + k(q_2 - q_1) = 0$$

which matches (1) and (3)

Particle Impact Dynamics

Def: Consider force $\underset{\rightharpoonup}{f}^y$ acting on particle y on the time interval $t \in [t_1, t_2]$. The translational impulse of the force $\underset{\rightharpoonup}{f}^y$ is

$$\underset{\rightharpoonup}{\hat{f}}^y = \int_{t_1}^{t_2} \underset{\rightharpoonup}{f}^y \, dt$$

If $\mathcal{F}_a$ is an inertial frame and w is an unforced particle, then we know that

$$\underset{\rightharpoonup}{f}^y = \underset{\rightharpoonup}{p}^{yw/a \, \cdot a} \qquad (N2L)$$

Substituting this into the definition of translational impulse gives

$$\underset{\rightharpoonup}{\hat{f}}^y = \int_{t_1}^{t_2} \underset{\rightharpoonup}{p}^{yw/a \, \cdot a} \, dt = \underset{\rightharpoonup}{p}^{yw/a}(t_2) - \underset{\rightharpoonup}{p}^{yw/a}(t_1)$$

$$\underset{\rightharpoonup}{\hat{f}}^y = \underset{\rightharpoonup}{p}^{yw/a}(t_2) - \underset{\rightharpoonup}{p}^{yw/a}(t_1)$$ is known as the principle of translational impulse and translational momentum

Recall the definition of angular momentum and N2LR

$$\underset{\rightharpoonup}{h}^{yw/a} = \underset{\rightharpoonup}{r}^{yw} \times \underset{\rightharpoonup}{p}^{yw/a} \quad \text{(angular momentum)}$$

$$\underset{\rightharpoonup}{h}^{yw/a \cdot a} = \underset{\rightharpoonup}{m}^{yw} = \underset{\rightharpoonup}{r}^{yw} \times \underset{\rightharpoonup}{f}^{y} \quad \text{(N2LR)}$$

Def: Consider moment $\underset{\rightharpoonup}{m}^{yw}$, particle y, unforced particle w and inertial frame $\mathcal{F}_a$. The angular impulse of $\underset{\rightharpoonup}{m}^{yw}$ is

$$\hat{\underset{\rightharpoonup}{m}}^{yw} = \int_{t_1}^{t_2} \underset{\rightharpoonup}{m}^{yw} \, dt \quad \text{and}$$

$$\hat{\underset{\rightharpoonup}{m}}^{yw} = \int_{t_1}^{t_2} \underset{\rightharpoonup}{h}^{yw/a \cdot a} \, dt = \underset{\rightharpoonup}{h}^{yw/a}(t_2) - \underset{\rightharpoonup}{h}^{yw/a}(t_1)$$

is the principle of angular impulse and angular momentum

Impulses are useful when describing forces that are applied instantaneously, or over very short periods of time that can be approximated as instantaneous. In this case we call the impulse an

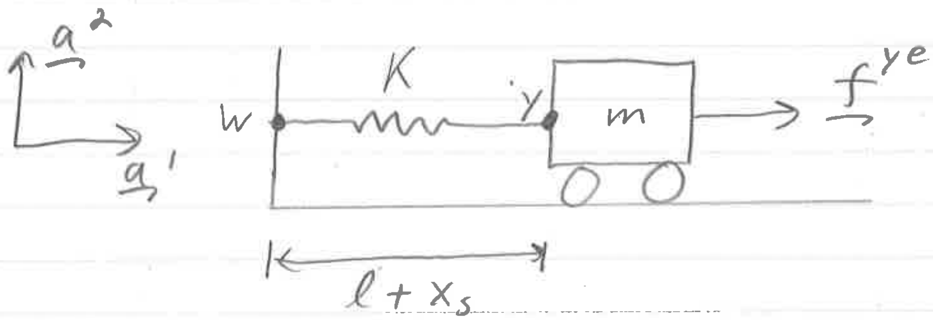
instantaneous impulse

It is also worth noting that due to N3L ($\underset{\rightharpoondown}{f}^{yx} = -\underset{\rightharpoondown}{f}^{xy}$), the impulse $\underset{\rightharpoondown}{\hat{f}}^{yx}$ acting on y due to x is equal to $-\underset{\rightharpoondown}{\hat{f}}^{xy}$, the opposite of the impulse acting on x due to y

$$\underset{\rightharpoondown}{\hat{f}}^{yx} = \int_{t_1}^{t_2} \underset{\rightharpoondown}{f}^{yx} \, dt = \int_{t_1}^{t_2} -\underset{\rightharpoondown}{f}^{xy} \, dt = -\underset{\rightharpoondown}{\hat{f}}^{xy}$$

We can also incorporate impulses into our FBDs.

Example: Consider a mass-spring system



$\underline{f}^{ye}$ is applied instanteneously and is described by the instantaneous impulse $\hat{\underline{f}}^{ye} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \hat{f} \\ 0 \\ 0 \end{bmatrix}$,

where $\hat{f}$ is constant.

Assuming particle y is initially at rest, what is the velocity of y relative to w after the impulse is applied?

Solution:

1) Kinematics
   i) Frames and DCM : N/A
   ii) Angular Velocity : N/A
   iii) Position

$$\underline{r}^{yw} = \underline{\mathcal{F}}_a^T \begin{bmatrix} l + x_s \\ 0 \\ 0 \end{bmatrix}$$

iv) Velocity

$$\underset{\rightharpoondown}{v}^{yw/a} = \underset{\rightharpoondown}{\dot{r}}^{yw/a} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} \dot{x}_s \\ 0 \\ 0 \end{bmatrix}$$

2) Principle of Translational Impulse / Momentum

$$\underset{\rightharpoondown}{\hat{f}}^{y} = \underset{\rightharpoondown}{p}^{yw/a}(t_2) - \underset{\rightharpoondown}{p}^{yw/a}(t_1)$$

$$= m\underset{\rightharpoondown}{v}^{yw/a}(t_2) - m\,\underset{\rightharpoondown}{v}^{yw/a}(t_1)$$

t_1: before impulse is applied

t_2: right after impulse is applied

$$\underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} \hat{f} \\ 0 \\ 0 \end{bmatrix} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} m\,\dot{x}_s(t_2) \\ 0 \\ 0 \end{bmatrix} + \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} m\,\cancelto{0}{\dot{x}_s(t_1)} \\ 0 \\ 0 \end{bmatrix}$$

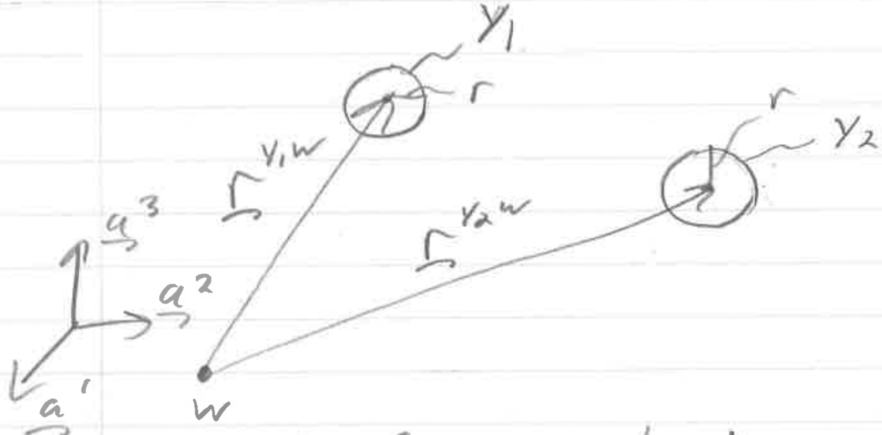
at rest

$$\hat{f} = m\,\dot{x}_s(t_2) \Rightarrow \dot{x}_s(t_2) = \frac{\hat{f}}{m}$$

Therefore $\underset{\rightharpoondown}{v}^{yw/a}(t_2) = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} \hat{f}/m \\ 0 \\ 0 \end{bmatrix}$

Collision of Particles

Consider particles y_1 and y_2 with masses m_1 and m_2, an unforced particle w and an inertial frame $\mathcal{F}_a$. For simplicity, assume the particles have a radius $r>0$ that is infinitessimally small



Let S represent the system of particles.

When particles y_1 and y_2 collide, there will be a force (or impulse) acting on y_1 due to y_2 and an equal an opposite force (or impulse) acting on y_2 due to y_1:

$$\underset{\rightarrow}{f}^{y_1 y_2} = -\underset{\rightarrow}{f}^{y_2 y_1} \qquad (N3L)$$

Therefore, in the absence of external forces acting on y_1 and y_2, the sum of the forces acting on the system is zero.

Even if there are external forces acting on the system, we can often ignore them, since the magnitude of the forces during impact are much greater than most external forces

Since there are no external forces acting on the system, translational momentum must be conserved, that is,

$$\underset{\rightharpoonup}{p}^{sw/a}(t_1) = \underset{\rightharpoonup}{p}^{sw/a}(t_2)$$

$$\underset{\rightharpoonup}{p}^{\gamma_1 w/a}(t_1) + \underset{\rightharpoonup}{p}^{\gamma_2 w/a}(t_1) = \underset{\rightharpoonup}{p}^{\gamma_1 w/a}(t_2) + \underset{\rightharpoonup}{p}^{\gamma_2 w/a}(t_2)$$

or

$$m_1 \underset{\rightharpoonup}{v}^{\gamma_1 w/a}(t_1) + m_2 \underset{\rightharpoonup}{v}^{\gamma_2 w/a}(t_1) = m_1 \underset{\rightharpoonup}{v}^{\gamma_1 w/a}(t_2) + m_2 \underset{\rightharpoonup}{v}^{\gamma_2 w/a}(t_2)$$

where t_1 and t_2 are any two times during the collision, but we will often be interested in

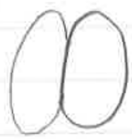
t_1: right before collision.

t_2: right after collision.

To summarize, translational momentum will be conserved throughout a collision in the absence of external forces acting on the system of particles.

We will model particle collisions as two steps:

1) Compression: Starts when particles begin to touch and ends when particles reach maximum deformation

$\Rightarrow$ In between  (maximum deformation) particles will momentarily have same velocity.

2) Restitution: Starts when particles have reached maximum deformation and ends when particles are no longer touching.

Since we will assume particle collisions occur over very short intervals, we will use impulses to describe the collision forces acting on Y_1 and Y_2

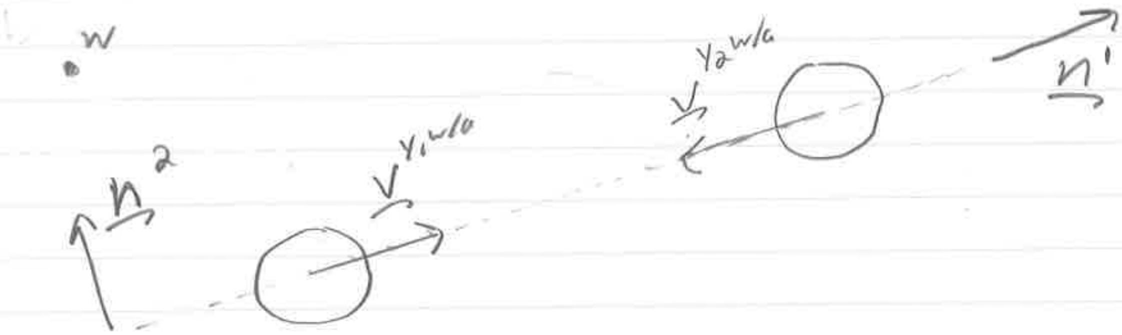
$\underset{\rightharpoonup}{\hat{f}}^{Y_1 Y_2 C}$: impulse on Y_1 due to Y_2 during compression

$\underset{\rightharpoonup}{\hat{f}}^{Y_1 Y_2 r}$: " " " " " " during restitution

$\underset{\rightharpoonup}{\hat{f}}^{Y_2 Y_1 C} = -\underset{\rightharpoonup}{\hat{f}}^{Y_1 Y_2 C}$: impulse on Y_2 due to Y_1 during compression

$\underset{\rightharpoonup}{\hat{f}}^{Y_2 Y_1 r} = -\underset{\rightharpoonup}{\hat{f}}^{Y_1 Y_2 r}$: " " " " " " during restitution

Direct Impact: For now, assume that $\underline{v}^{Y_1 w/a}$, $\underline{v}^{Y_2 w/a}$, and $\underline{r}^{Y_1 Y_2}$ are parallel or one of $\underline{v}^{Y_1 w/a}$ or $\underline{v}^{Y_2 w/a}$ is zero. This is called a direct impact.



$$\underline{v}^{Y_1 w/a} = \underline{\mathcal{F}}_n^T \begin{bmatrix} v_{n_1}^{Y_1 w/a} \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}^{Y_2 w/a} = \underline{\mathcal{F}}_n^T \begin{bmatrix} v_{n_1}^{Y_2 w/a} \\ 0 \\ 0 \end{bmatrix}$$

All motion and forces (impulses) during impact will be in the $\underline{n}^1$ direction. Therefore,

$$\underline{\hat{f}}^{Y_1 Y_2 c} = \underline{\mathcal{F}}_n^T \begin{bmatrix} \hat{f}_{n_1}^{Y_1 Y_2 c} \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\hat{f}}^{Y_1 Y_2 r} = \underline{\mathcal{F}}_n^T \begin{bmatrix} \hat{f}_{n_1}^{Y_1 Y_2 r} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\hat{f}}^{Y_2 Y_1 c} = \underline{\mathcal{F}}_n^T \begin{bmatrix} \hat{f}_{n_1}^{Y_2 Y_1 c} \\ 0 \\ 0 \end{bmatrix}, \quad \underline{\hat{f}}^{Y_2 Y_1 r} = \underline{\mathcal{F}}_n^T \begin{bmatrix} \hat{f}_{n_1}^{Y_2 Y_1 r} \\ 0 \\ 0 \end{bmatrix}$$

Def: The coefficient of restitution (COR) of the impact is defined as

$$e = \frac{\hat{f}_{n_1}^{\,\gamma_1 \gamma_2 r}}{\hat{f}_{n_1}^{\,\gamma_1 \gamma_2 c}} = \frac{\hat{f}_{n_1}^{\,\gamma_2 \gamma_1 r}}{\hat{f}^{\,\gamma_2 \gamma_1 c}}$$

The value of e for an impact will depend on many factors, including the material properties of the particles, the initial velocities of the particles, etc.

The COR can only take on values $0 \leq e \leq 1$

If $e = 1$, the impact is perfectly elastic and kinetic energy is conserved throughout the impact.

If $e = 0$, the impact is perfectly plastic and kinetic energy will decrease throughout the impact.

All real impacts have $0 < e < 1$.

Define

compression { t_1 : time right before impact
t_i : time between compression and restitution
restitution { t_2 : time right after impact

From the definition of a translational impulse we know that

$$\underset{\rightharpoonup}{\hat{f}}^{\,y_1 y_2 c} = m_1 \underset{\rightharpoonup}{v}^{y_1 w/a}(t_i) - m_1 \underset{\rightharpoonup}{v}^{y_1 w/a}(t_1)$$

Since all physical vectors are in $\underset{\rightharpoonup}{n}_1$ direction

$$\hat{f}_{n_1}^{\,y_1 y_2 c} = m_1 v_{n_1}^{y_1 w/a}(t_i) - m_1 v_{n_1}^{y_1 w/a}(t_1)$$

Also, $$\hat{f}_{n_1}^{\,y_1 y_2 r} = m_1 v_{n_1}^{y_1 w/a}(t_2) - m_1 v_{n_1}^{y_1 w/a}(t_i)$$

Therefore

$$e = \frac{\cancel{m_1}\left(v_{n_1}^{y_1 w/a}(t_2) - v_{n_1}^{y_1 w/a}(t_i)\right)}{\cancel{m_1}\left(v_{n_1}^{y_1 w/a}(t_i) - v_{n_1}^{y_1 w/a}(t_1)\right)} \quad (*)$$

We can do the same for particle y_2 to get

$$e = \frac{v_{n_1}^{y_2 w/a}(t_2) - v_{n_1}^{y_2 w/a}(t_i)}{v_{n_1}^{y_2 w/a}(t_i) - v_{n_1}^{y_2 w/a}(t_1)} \quad (**)$$

At t_i particles y_1 and y_2 will have the same velocity, therefore

$$V_{n_1}^{y_1 w/a}(t_i) = V_{n_1}^{y_2 w/a}(t_i) \equiv V_{n_1}^{y w/a}(t_i)$$

(*) becomes

$$e = \frac{V_{n_1}^{y_1 w/a}(t_2) - V_{n_1}^{y w/a}(t_i)}{V_{n_1}^{y w/a}(t_i) - V_{n_1}^{y_1 w/a}(t_1)}$$

or $$e\left(V_{n_1}^{y w/a}(t_i) - V_{n_1}^{y_1 w/a}(t_1)\right) = V_{n_1}^{y_1 w/a}(t_2) - V_{n_1}^{y w/a}(t_i)$$

$$(1+e)\, V_{n_1}^{y w/a}(t_i) = V_{n_1}^{y_1 w/a}(t_2) + e\, V_{n_1}^{y_1 w/a}(t_1)$$

$$V_{n_1}^{y w/a}(t_i) = \frac{V_{n_1}^{y_1 w/a}(t_2) + e\, V_{n_1}^{y_1 w/a}(t_1)}{1+e}$$

Same procedure on (**) to get

$$V_{n_1}^{y w/a}(t_i) = \frac{V_{n_1}^{y_2 w/a}(t_2) + e\, V_{n_1}^{y_2 w/a}(t_1)}{1+e}$$

Equating both expressions gives

$$V_{n_1}^{y_1 w/a}(t_2) + e V_{n_1}^{y_1 w/a}(t_1) = V_{n_1}^{y_2 w/a}(t_2) - e\, V_{n_1}^{y_2 w/a}(t_1)$$

$$e\left(V_{n_1}^{y_1 w/a}(t_1) - V_{n_1}^{y_2 w/a}(t_1)\right) = V_{n_1}^{y_2 w/a}(t_2) - V_{n_1}^{y_1 w/a}(t_2)$$

$$e = \frac{V_{n_1}^{Y_2 w/a}(t_2) - V_{n_1}^{Y_1 w/a}(t_2)}{V_{n_1}^{Y_1 w/a}(t_1) - V_{n_1}^{Y_2 w/a}(t_1)}$$

If $e = 1$: $V_{n_1}^{Y_2 w/a}(t_1) - V_{n_1}^{Y_1 w/a}(t_1) = V_{n_1}^{Y_1 w/a}(t_2) - V_{n_1}^{Y_2 w/a}(t_2)$

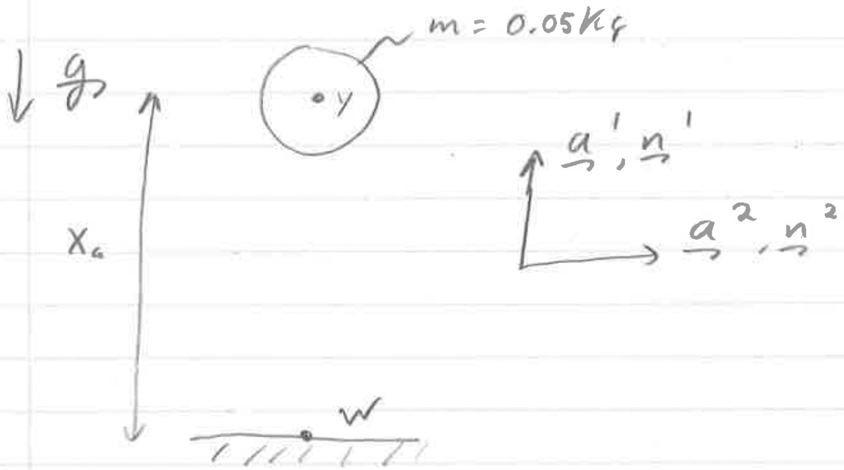
No loss of energy

If $e = 0$: $V_{n_1}^{Y_2 w/a}(t_2) = V_{n_1}^{Y_1 w/a}(t_2)$

particles stick together following impact

Example: A tennis ball of mass $m = 0.05\,Kg$ is dropped from a height of 1 m. The COR of impact between the ground and the ball is $e = 0.7$. Gravity is acting in the negative $\underline{a}^1$ direction.

Neglecting drag, what height will the tennis ball rebound to?



Solution: Separate problem into 3 stages (before impact, impact, after impact)

Before impact:

What do we need to solve for? $\underline{v}^{yw/a}$ right before impact.

Let's use conservation of energy.

1) Kinematics: i) $\underline{\mathcal{F}}_a = \underline{\mathcal{F}}_n$, ii) $\underline{\omega}^{na} = \underline{0}$

iii) Position $\underline{r}^{yw} = \underline{\mathcal{F}}_a^T \begin{bmatrix} x_a \\ 0 \\ 0 \end{bmatrix}$

iV) Velocity

$$\underset{\rightharpoondown}{v}^{yw/a} = \dot{\underset{\rightharpoondown}{r}}^{yw/a} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} \dot{x}_a \\ 0 \\ 0 \end{bmatrix}$$

2) Energy

$$T_{yw/a} = \frac{1}{2} m \underset{\rightharpoondown}{v}^{yw/a} \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

$$= \frac{1}{2} m \dot{x}_a^2$$

$$V_{yw/a} = -m \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{yw}, \quad \underset{\rightharpoondown}{g} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}$$

$$= m g x_a$$

3) Conservation of Energy

t_1 : ball released from rest
t_2 : ball just before impact

$$E_{yw/a}(t_1) = E_{yw/a}(t_2)$$

$$T_{yw/a}(t_1) + V_{yw}(t_1) = T_{yw/a}(t_2) + V_{yw}(t_2)$$

$$\frac{1}{2} m \underbrace{\cancel{\dot{x}_a^2}}_{0}(t_1) + \cancel{m} g x_a(t_1) = \frac{1}{2} \cancel{m} \dot{x}_a^2(t_2) + m g \underbrace{\cancel{x_a}}_{0}(t_2)$$

$$\dot{x}_a^2(t_2) = 2 g x_a(t_1)$$

$$\dot{x}_a(t_2) = \pm\sqrt{2 g x_a(t_1)} = \pm\sqrt{2(9.81 m/s^2)(1 m)}$$

$$\simeq \pm 4.43 \, m/s$$

$$\dot{x}_a(t_2) = -4.43 \, m/s \quad \text{(downwards)}$$

During Impact:

Consider the ground to be a massive particle that has no velocity relative to w.

t_2: just before impact
t_3: just after impact

$$e = \frac{v_{n_1}^{\gamma_2 w/a}(t_3) - v_{n_1}^{\gamma_1 w/a}(t_3)}{v_{n_1}^{\gamma_1 w/a}(t_2) - v_{n_1}^{\gamma_2 w/a}(t_2)}$$

$\gamma_1 : \gamma \Rightarrow v_{n_1}^{\gamma_1 w/a} = \dot{x}_a$

γ_2 : ground $\Rightarrow v_{n_1}^{\gamma_2 w/a} = 0$

$$e = \frac{-\dot{x}_a(t_3)}{\dot{x}_a(t_2)} \Rightarrow 0.7 = \frac{-\dot{x}_a(t_3)}{-4.43 \text{ m/s}}$$

$$\dot{x}_a(t_3) = 3.10 \text{ m/s}$$

After impact:

Kinematics and Energy already done

3) Conservation of Energy

t_3 : just after impact
t_4 : ball reaches maximum height

$$E_{yw/a}(t_3) = E_{yw/a}(t_4)$$

$$T_{yw/a}(t_3) + V_{yw}(t_3) = T_{yw/a}(t_4) + V_{yw}(t_4)$$

$$\frac{1}{2}\cancel{m}\,\dot{x}_a^2(t_3) + mg\,\underbrace{x_a(t_3)}_{0} = \frac{1}{2}m\,\underbrace{\dot{x}_a^2(t_4)}_{0} + \cancel{m}g\,x_a(t_4)$$

$$x_a(t_4) = \frac{1}{2g}\,\dot{x}_a^2(t_3)$$

$$\approx \frac{1}{2(9.81\,m/s^2)}\left(3.10\,m/s\right)^2$$

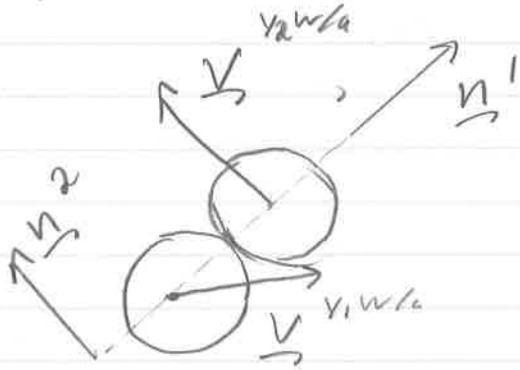
$$\approx 0.49\ m$$

Ball will rebound to 0.49 m

Notice that $\frac{0.49\ m}{1\ m} = 0.49 = 0.7^2 = e^2$!

It turns out that e^2 is proportional to the amount of kinetic energy remaining following impact.

Oblique Impact: In general, $\underset{\sim}{v}^{Y_1 w/a}$ and $\underset{\sim}{v}^{Y_2 w/a}$ will not be parallel, and therefore the forces (impulses) due to impact will not be parallel to these physical vectors either. This is known as oblique impact.



In this case, the coefficient of restitution is

$$e = \frac{\hat{\underset{\sim}{f}}^{Y_1 Y_2 r} \cdot \underset{\sim}{n}^1}{\hat{\underset{\sim}{f}}^{Y_1 Y_2 c} \cdot \underset{\sim}{n}^1} = \frac{\hat{\underset{\sim}{f}}^{Y_2 Y_1 r} \cdot \underset{\sim}{n}^1}{\hat{\underset{\sim}{f}}^{Y_2 Y_1 c} \cdot \underset{\sim}{n}^1}$$

A similar procedure to that used for direct impacts yields

$$e = \frac{\underset{\sim}{v}^{Y_2 w/a}(t_2) \cdot \underset{\sim}{n}^1 - \underset{\sim}{v}^{Y_1 w/a}(t_2) \cdot \underset{\sim}{n}^1}{\underset{\sim}{v}^{Y_1 w/a}(t_1) \cdot \underset{\sim}{n}^1 - \underset{\sim}{v}^{Y_2 w/a}(t_1) \cdot \underset{\sim}{n}^1}$$

$$= \frac{\left(\underset{\sim}{v}^{Y_2 w/a}(t_2) - \underset{\sim}{v}^{Y_1 w/a}(t_2)\right) \cdot \underset{\sim}{n}^1}{\left(\underset{\sim}{v}^{Y_1 w/a}(t_1) - \underset{\sim}{v}^{Y_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^1}$$

If we resolve velocities in $\underset{\sim}{\mathcal{F}}_n$, this becomes

$$e = \frac{V_{n_1}^{Y_2 w/a}(t_2) - V_{n_1}^{Y_1 w/a}(t_2)}{V_{n_1}^{Y_1 w/a}(t_1) - V_{n_1}^{Y_2 w/a}(t_1)}$$

where

$$\underset{\sim}{V}^{Y_1 w/a} = \underset{\sim}{\mathcal{F}}_n^T \begin{bmatrix} V_{n_1}^{Y_1 w/a} \\ V_{n_2}^{Y_1 w/a} \\ V_{n_3}^{Y_1 w/a} \end{bmatrix}$$

$$\underset{\sim}{V}^{Y_2 w/a} = \underset{\sim}{\mathcal{F}}_n^T \begin{bmatrix} V_{n_1}^{Y_2 w/a} \\ V_{n_2}^{Y_2 w/a} \\ V_{n_3}^{Y_2 w/a} \end{bmatrix}$$

Define $\underset{\sim}{\hat{f}}^{Y_1 Y_2} = \underset{\sim}{\hat{f}}^{Y_1 Y_2 c} + \underset{\sim}{\hat{f}}^{Y_1 Y_2 r} = m_1 \underset{\sim}{V}^{Y_1 w/a}(t_2) - m_1 \underset{\sim}{V}^{Y_1 w/a}(t_1)$

$\underset{\sim}{\hat{f}}^{Y_1 Y_2}$ is in the direction of $\underset{\sim}{n}^1$, therefore

$$\underset{\sim}{\hat{f}}^{Y_1 Y_2} \cdot \underset{\sim}{n}^2 = 0 \quad \text{and} \quad \underset{\sim}{\hat{f}}^{Y_1 Y_2} \cdot \underset{\sim}{n}^3 = 0$$

or

$$\left(\underset{\sim}{V}^{Y_1 w/a}(t_2) - \underset{\sim}{V}^{Y_1 w/a}(t_1)\right) \cdot \underset{\sim}{n}^2 = 0$$

$$\left(\underset{\sim}{V}^{Y_1 w/a}(t_2) - \underset{\sim}{V}^{Y_1 w/a}(t_1)\right) \cdot \underset{\sim}{n}^3 = 0$$

The same applies to particle Y_2

$$\left(\underset{\sim}{V}^{Y_2 w/a}(t_2) - \underset{\sim}{V}^{Y_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^2 = 0$$

$$\left(\underset{\sim}{V}^{Y_2 w/a}(t_2) - \underset{\sim}{V}^{Y_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^3 = 0$$

When solving problems that involve an impact, break up the problem into 3 phases:

1) Before Impact
2) The Impact
3) After Impact

Before and after the impact will be solved using techniques we have previously learned:

N2L (3 steps to success)
Work - Energy Theorem (4 steps to success)
Conservation of Energy
Vibrations

The steps to solving for "the impact" will depend on the problem and the information given. The following is a guideline to help you:

1) Kinematics (If not done in previous stage)

2) Impact Relationships

i)

$$e = \frac{\left(\underline{v}^{Y_2 w/a}(t_2) - \underline{v}^{Y_1 w/a}(t_2)\right) \cdot \underline{n}'}{\left(\underline{v}^{Y_1 w/a}(t_1) - \underline{v}^{Y_2 w/a}(t_1)\right) \cdot \underline{n}'}$$

(coefficient of restitution)

$$= \frac{v_{n_1}^{Y_2 w/a}(t_2) - v_{n_1}^{Y_1 w/a}(t_2)}{v_{n_1}^{Y_1 w/a}(t_1) - v_{n_1}^{Y_2 w/a}(t_1)}$$

ii) Conservation of translational momentum

$$m_1 \underset{\sim}{v}^{\gamma_1 w/a}(t_1) + m_2 \underset{\sim}{v}^{\gamma_2 w/a}(t_1) = m_1 \underset{\sim}{v}^{\gamma_1 w/a}(t_2) + m_2 \underset{\sim}{v}^{\gamma_2 w/a}(t_2)$$

Usually this is only necessary in the $\underset{\sim}{n}^1$ direction

$$\left(m_1 \underset{\sim}{v}^{\gamma_1 w/a}(t_1) + m_2 \underset{\sim}{v}^{\gamma_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^1 = \left(m_1 \underset{\sim}{v}^{\gamma_1 w/a}(t_2) + m_2 \underset{\sim}{v}^{\gamma_2 w/a}(t_2)\right) \cdot \underset{\sim}{n}^1$$

or

$$m_1 v_{n1}^{\gamma_1 w/a}(t_1) + m_2 v_{n1}^{\gamma_2 w/a}(t_1) = m_1 v_{n1}^{\gamma_1 w/a}(t_2) + m_2 v_{n1}^{\gamma_2 w/a}(t_2)$$

iii) Conservation of translation momentum in $\underset{\sim}{n}^2$ and $\underset{\sim}{n}^3$ directions for each particle

$$\left(\underset{\sim}{v}^{\gamma_1 w/a}(t_2) - \underset{\sim}{v}^{\gamma_1 w/a}(t_1)\right) \cdot \underset{\sim}{n}^2 = 0$$
$$\left(\underset{\sim}{v}^{\gamma_1 w/a}(t_2) - \underset{\sim}{v}^{\gamma_1 w/a}(t_1)\right) \cdot \underset{\sim}{n}^3 = 0$$

$$\left(\underset{\sim}{v}^{\gamma_2 w/a}(t_2) - \underset{\sim}{v}^{\gamma_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^2 = 0$$
$$\left(\underset{\sim}{v}^{\gamma_2 w/a}(t_2) - \underset{\sim}{v}^{\gamma_2 w/a}(t_1)\right) \cdot \underset{\sim}{n}^3 = 0$$

If using iii), then only consider ii) in $\underset{\sim}{n}^1$ direction. Other directions will be redundant, since conservation of momentum of each particle in $\underset{\sim}{n}^2, \underset{\sim}{n}^3$ directions implies conservation of total system momentum in $\underset{\sim}{n}^2, \underset{\sim}{n}^3$ directions.

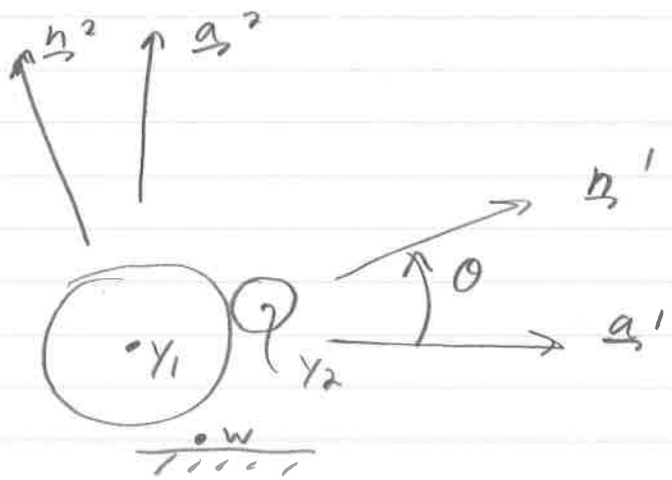
particle Y_1

<u>Example</u>: Consider the impact of a driver on a golf ball. The golf ball is initially at rest and just before impact the velocity of the driver is

particle Y_2

$$\underset{\rightharpoonup}{v}^{Y_1 w/a} = \underset{\rightharpoonup}{\mathcal{F}}_a^T \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} \text{ m/s}$$ and the COR is $e = 0.83$.

The driver has a mass of $m_1 = 0.2$ kg and the golf ball has a mass of $m_2 = 0.05$ kg. The driver impacts the golf ball obliquely with an angle of $\theta = 10°$.



What are the velocities of Y_1 and Y_2 following the impact?

1) Kinematics

i) $\underset{\rightharpoonup}{\mathcal{F}}_a \xrightarrow{C_3(\theta)} \underset{\rightharpoonup}{\mathcal{F}}_n \qquad C_{na} = C_3(\theta)$

ii) $\underset{\rightharpoonup}{\omega}^{na} = \underset{\rightharpoonup}{0}$

iii) $\underline{r}^{y_1 w} = \underline{\mathcal{I}}_n^T \begin{bmatrix} x_{n1} \\ y_{n1} \\ 0 \end{bmatrix}$ $\underline{r}^{y_2 w} = \underline{\mathcal{I}}_n^T \begin{bmatrix} x_{n2} \\ y_{n2} \\ 0 \end{bmatrix}$

iv) $$\underline{v}^{y_1 w/a} = \underline{r}^{y_1 w \cdot a} = \underline{r}^{y_1 w \cdot n} + \cancel{\underline{\omega}^{na}}^{\,\underline{0}} \times \underline{r}^{y_1 w}$$
$$= \underline{\mathcal{I}}_n^T \begin{bmatrix} \dot{x}_{n1} \\ \dot{y}_{n1} \\ 0 \end{bmatrix}$$

$$\underline{v}^{y_2 w/a} = \underline{r}^{y_2 w \cdot a} = \underline{r}^{y_2 w \cdot n} + \cancel{\underline{\omega}^{na}}^{\,\underline{0}} \times \underline{r}^{y_2 w}$$
$$= \underline{\mathcal{I}}_n^T \begin{bmatrix} \dot{x}_{n2} \\ \dot{y}_{n2} \\ 0 \end{bmatrix}$$

2) Impact

t_1 : just before impact

t_2 : just following impact

COR:

① $$e = \frac{v_{n1}^{y_2 w/a}(t_2) - v_{n1}^{y_1 w/a}(t_2)}{v_{n1}^{y_1 w/a}(t_1) - v_{n1}^{y_2 w/a}(t_1)}$$

Conservation of momentum in $\underline{n}'$ direction of system

② $$m_1 v_{n1}^{y_1 w/a}(t_1) + m_2 v_{n1}^{y_2 w/a}(t_1) = m_1 v_{n1}^{y_1 w/a}(t_2) + m_2 v_{n1}^{y_2 w/a}(t_2)$$

Conservation of momentum in $\underset{\rightharpoondown}{n}^2$ direction of each particle

(3) $V_{n2}^{Y_1 w/a}(t_2) - V_{n2}^{Y_1 w/a}(t_1) = 0$

(4) $V_{n2}^{Y_2 w/a}(t_2) - V_{n2}^{Y_2 w/a}(t_1) = 0$

Need $\underset{\rightharpoondown}{V}^{Y_1 w/a}(t_1)$ in $\mathcal{F}_n$

$$\underset{\rightharpoondown}{V}^{Y_1 w/a}(t_1) = \underset{\rightharpoondown}{\mathcal{F}}_n^T \underline{C}_{na} \underline{V}_a^{Y_1 w/a}(t_1)$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_n^T \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} m/s$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_n^T \begin{bmatrix} 50 c_\theta \\ -50 s_\theta \\ 0 \end{bmatrix} m/s$$

Substitute in Known Values

(1) $\Rightarrow$ $$0.83 = \frac{\dot{x}_{n2}(t_2) - \dot{x}_{n1}(t_2)}{50 c_\theta \, m/s - (0 \, m/s)}$$

$$-\dot{x}_{n1}(t_2) + \dot{x}_{n2}(t_2) = 41.5 c_\theta \, m/s$$

(2) $\Rightarrow$ $(0.2\,Kg)\,\dot{x}_{n1}(t_2) + (0.05\,Kg)\,\dot{x}_{n2}(t_2)$

$$= (0.2\,Kg)(50 c_\theta \, m/s) + (0.05\,Kg)(0\,m/s)$$

$$(0.2\,Kg)\,\dot{x}_{n1}(t_2) + (0.05\,Kg)\,\dot{x}_{n2}(t_2) = 10 c_\theta \, Kg\,m/s$$

③ $\Rightarrow \quad \dot{y}_{n1}(t_2) = -50 s_\theta \text{ m/s} = -8.68 \text{ m/s}$

④ $\Rightarrow \quad \dot{y}_{n2}(t_2) = 0 \text{ m/s}$

Combine ① and ②:

$$-\dot{x}_{n1}(t_2) + \dot{x}_{n2}(t_2) = 41.5 c_\theta \text{ m/s}$$

$$(0.2\,\text{kg})\dot{x}_{n1}(t_2) + (0.05\,\text{kg})\dot{x}_{n2}(t_2) = 10 c_\theta \text{ kg m/s}$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0.2\,\text{kg} & 0.05\,\text{kg} \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} \dot{x}_{n1}(t_2) \\ \dot{x}_{n2}(t_2) \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 41.5 c_\theta \text{ m/s} \\ 10 c_\theta \text{ kg m/s} \end{bmatrix}}_{\underline{b}}$$

$\underline{x} = \underline{A}^{-1}\underline{b}$ gives

$$\dot{x}_{n1}(t_2) = 31.22 \text{ m/s}$$

$$\dot{x}_{n2}(t_2) = 72.09 \text{ m/s}$$

Therefore $\underset{\sim}{v}^{Y_1 w/a}(t_2) = \underset{\sim}{\mathcal{F}}_n^T \begin{bmatrix} 31.22 \\ -8.68 \\ 0 \end{bmatrix} \text{m/s}$

$$\underset{\sim}{v}^{Y_2 w/a}(t_2) = \underset{\sim}{\mathcal{F}}_n^T \begin{bmatrix} 72.09 \\ 0 \\ 0 \end{bmatrix} \text{m/s}$$

Or resolved in $\mathcal{F}_a$

$$\underline{v}^{y_1 w/a}(t_2) = \underrightarrow{\mathcal{F}}_a^T \, \underline{C}_{na}^T \begin{bmatrix} 31.22 \\ -8.68 \\ 0 \end{bmatrix} \text{ m/s}$$

$$= \underrightarrow{\mathcal{F}}_n^T \begin{bmatrix} 32.25 \\ -3.13 \\ 0 \end{bmatrix} \text{ m/s}$$

$$\underline{v}^{y_2 w/a}(t_2) = \underrightarrow{\mathcal{F}}_a^T \, \underline{C}_{na}^T \begin{bmatrix} 72.09 \\ 0 \\ 0 \end{bmatrix} \text{ m/s}$$

$$= \underrightarrow{\mathcal{F}}_n^T \begin{bmatrix} 70.99 \\ 12.52 \\ 0 \end{bmatrix} \text{ m/s}$$