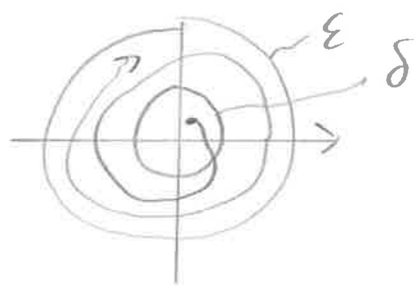
Stability of Nonlinear Systems (24.2)

Consider a nonlinear differential equation of the form $\dot{\underline{x}} = \underline{f}(\underline{x})$. When discussing stability of this system, we must discuss stability of one of its equilibrium points (i.e., $\bar{\underline{x}}$ such that $\underline{f}(\bar{\underline{x}}) = \underline{0}$).
If $\bar{\underline{x}} \neq \underline{0}$ is equilibrium pt, then we can "shift" coordinates such that $\bar{\underline{x}} = \underline{0}$ is equilibrium pt.

1) $\bar{\underline{x}} = \underline{0}$ is a <u>stable (Lyapunov stable)</u> equil. pt. of $\dot{\underline{x}} = \underline{f}(\underline{x})$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\underline{x}(t_0)\| < \delta \underset{\text{(implies)}}{\Longrightarrow} \|\underline{x}(t)\| < \varepsilon \quad \text{for all } t \geq t_0$$



2) $\bar{\underline{x}} = \underline{0}$ is <u>unstable</u> if not stable (Lyapunov stable).

3) $\bar{\underline{x}} = \underline{0}$ is <u>asymptotically stable</u> if it is stable (Lyapunov stable) and there exists $\delta > 0$ such that

$$\|\underline{x}(t_0)\| < \delta \underset{\text{(implies)}}{\Longrightarrow} \lim_{t \to \infty} \underline{x}(t) = \underline{0}$$

4) $\bar{\underline{x}} = \underline{0}$ is <u>globally asymptotically stable</u> if it is stable (Lyapunov stable) and

$$\lim_{t \to \infty} \underline{x}(t) = \underline{0} \quad \text{for all } \underline{x}(t_0)$$

<u>Summary</u>

Stable (Lyapunov Stable) (LS): Trajectories "stay around" the equilibrium pt. (EP).

Unstable: Trajectories leave vicinity of EP.

Asymptotically Stable (AS): Trajectories converge to EP for initial conditions "close enough" to EP.

Globally Asymptotically stable (GAS):
Trajectories converge to EP given any initial conditions.

Lyapunov Stability Analysis

Lyapunov's Indirect (First) Method

Let $\underline{x} = \underline{\bar{x}} + \delta \underline{x}$

$$\underline{\dot{x}} = \underline{f}(\underline{x}) \qquad \underline{f}(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{bmatrix}$$

$$\underbrace{\cancel{\dot{\underline{\bar{x}}}}}_{\underline{0}} + \delta \underline{\dot{x}} = \underline{f}(\underline{\bar{x}} + \delta \underline{x})$$

$$\delta \underline{\dot{x}} \simeq \underbrace{\underline{f}(\cancel{\underline{\bar{x}}})}_{\underline{0}} + \underbrace{\left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\bar{x}}}}_{\underline{A}} \delta \underline{x}$$

$$\delta \underline{\dot{x}} = \underline{A}\, \delta \underline{x} \quad , \quad \frac{\partial \underline{f}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Theorem:

$\underline{\bar{x}}$ is asymptotically stable if $Re\{\lambda_i(\underline{A})\} < 0$ for all $i = 1, \ldots, n$.

Cannot conclude global asymptotic stability using this approach.

Lyapunov's Direct (Second) Method (24.3)

Consider Euler's Equation, where derivative control has been implemented to de-tumble the spacecraft:

$$\underline{\underline{I}}_b^B \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times} \underline{\underline{I}}_b^B \underline{\omega}_b^{ba} = \underline{\tau}_b^{B,c}$$

$$\underline{\tau}_b^{Bc} = -K_d \underline{\omega}_b^{ba} \quad , \quad K_d > 0$$

For simplicity, drop subscripts and superscripts:

$$\underline{\underline{I}} \dot{\underline{\omega}} + \underline{\omega}^\times \underline{\underline{I}} \underline{\omega} = -K_d \underline{\omega}$$

Recall $\bar{\underline{\omega}} = \underline{0}$ is equil. pt. of Euler's Eq.

Define kinetic energy: $T = \frac{1}{2} \underline{\omega}^T \underline{\underline{I}} \underline{\omega} \geq 0$

Also notice that $T(\underline{\omega}=0) = 0$, $T \to \infty$ as $\|\underline{\omega}\| \to \infty$

Take derivative

$$\dot{T} = \frac{1}{2}\left(\dot{\underline{\omega}}^T \underline{\underline{I}} \underline{\omega} + \underline{\omega}^T \underline{\underline{I}} \dot{\underline{\omega}}\right)$$

$$= \underline{\omega}^T \underline{\underline{I}} \dot{\underline{\omega}} \quad \text{sub in EoM}$$

$$= \underline{\omega}^T\left(-K_d \underline{\omega} - \underline{\omega}^\times \underline{\underline{I}} \underline{\omega}\right)$$

$$= -K_d \underline{\omega}^T \underline{\omega} - \underbrace{\underline{\omega}^T \underline{\omega}^\times}_{=\underline{0}} \underline{\underline{I}} \underline{\omega}$$

$$\dot{T} = -K_d \underline{\omega}^T \underline{\omega} \leq 0 \quad , \text{ also } \dot{T} < 0 \text{ if } \underline{\omega} \neq \underline{0}$$

Can integrate to show

$$\int_0^t \dot{T}\, d\tau = -\int_0^t K_d\, \underline{\omega}^T \underline{\omega}\, d\tau$$

$$T(t) - T(0) = \underbrace{-K_d \int_0^t \underline{\omega}^T \underline{\omega}\, d\tau}_{\leq 0}$$

$T(t) \leq T(0)$ → energy decreases over time

If $\dot{T} \leq 0$, energy of system is not "blowing up" and must be at least stable.

If $\dot{T} < 0$, energy of system is decreasing and must be asymptotically stable.

This is the idea behind Lyapunov's direct method.

Theorem: Consider $\dot{\underline{x}} = \underline{f}(\underline{x})$ with equilibrium pt. $\bar{\underline{x}} = \underline{0}$. Let there exist function $V(\underline{x})$.

1) $\bar{\underline{x}} = \underline{0}$ is Lyapunov stable if

   i) $V(\underline{0}) = 0$

   ii) $V(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$

   iii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{x}$

2) $\bar{\underline{x}} = \underline{0}$ is globally asymptotically stable if

   i) $V(\underline{0}) = 0$

   ii) $V(\underline{x}) > 0$ for all $\underline{x} \neq 0$

   iii) $V(\underline{x}) \to \infty$ as $\|\underline{x}\| \to \infty$

   iv) $\dot{V}(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$

$V(\underline{x})$ is called a Lyapunov function.

In previous example, $T = \frac{1}{2} \underline{\omega}^T \underline{I}\, \underline{\omega}$ was a Lyapunov function that satisfied i) – iv) of 2).

Therefore $\underline{\bar{\omega}} = \underline{0}$ is globally asymptotically stable equilibrium pt. when D control is implemented (i.e., $\underline{\omega} \to \underline{0}$ as $t \to \infty$).

Nonlinear Attitude Control

Consider PD attitude control with the quaternion

$$\underline{\tau} = -K_p \underline{\varepsilon} - K_d \underline{\omega}. \qquad (\underline{\omega} \to 0,\ \underline{\varepsilon} \to 0 \text{ as } t \to \infty)$$

Recall Euler's Eqn. and quaternion attitude kinematics

$$\underline{I}\,\dot{\underline{\omega}} + \underline{\omega}^{\times} \underline{I}\,\underline{\omega} = \underline{\tau} = -K_p \underline{\varepsilon} - K_d \underline{\omega}$$

$$\dot{\underline{\varepsilon}} = \frac{1}{2}(\eta \underline{1} + \underline{\varepsilon}^{\times})\underline{\omega} \qquad \underline{x} = \begin{bmatrix} \underline{\omega} \\ \underline{\varepsilon} \\ \eta - 1 \end{bmatrix}$$

$$\dot{\eta} = -\frac{1}{2}\underline{\varepsilon}^T \underline{\omega}$$

Can show that $\underline{\omega} = 0$, $\underline{\varepsilon} = 0$, $\eta = 1$ is an equilibrium pt. : $\bar{\underline{x}} = \underline{0}$

Define $V = \frac{1}{2}\underline{\omega}^T \underline{I}\,\underline{\omega} + K_p\left(\underline{\varepsilon}^T\underline{\varepsilon} + (\eta - 1)^2\right)$

$V > 0$ for all $\underline{x}$ except $\bar{\underline{x}} = \underline{0}$

(Also, $V \to \infty$ as $\|\underline{x}\| \to \infty$)

$$\dot{V} = \underline{\omega}^T \underline{I}\,\dot{\underline{\omega}} + K_p\Big(\underbrace{\dot{\underline{\varepsilon}}^T\underline{\varepsilon} + \underline{\varepsilon}^T\dot{\underline{\varepsilon}}}_{2\underline{\varepsilon}^T\dot{\underline{\varepsilon}}} + 2(\eta - 1)\dot{\eta}\Big)$$

$$= \underline{\omega}^T \underline{I}\,\dot{\underline{\omega}} + \cancel{2}K_p\left(\underline{\varepsilon}^T \cancel{\tfrac{1}{2}}(\eta \underline{1} + \underline{\varepsilon}^{\times})\underline{\omega} + \cancel{\tfrac{1}{2}}(\eta - 1)(-\underline{\varepsilon}^T\underline{\omega})\right)$$

$$= \underline{\omega}^T\left(-K_p\underline{\varepsilon} - K_d\underline{\omega} - \underline{\omega}^{\times}\underline{I}\,\underline{\omega}\right) + K_p\Big(\cancel{\eta\underline{\varepsilon}^T\underline{\omega}} + \underbrace{\underline{\varepsilon}^T\underline{\varepsilon}^{\times}}_{=\underline{0}}\underline{\omega}$$

$$- \cancel{\eta\underline{\varepsilon}^T\underline{\omega}} + \underline{\varepsilon}^T\underline{\omega}\Big)$$

$$\dot{V} = -K_p \cancel{\underline{\omega}^T \underline{\varepsilon}} - K_d \underline{\omega}^T \underline{\omega} - \underbrace{\underline{\omega}^T \underline{\omega}^\times}_{=\underline{0}} \underline{I}\, \underline{\omega}$$

$$+ K_p \underbrace{\underline{\varepsilon}^T \underline{\omega}}_{= \cancel{\underline{\omega}^T \underline{\varepsilon}}}$$

$$\dot{V} = -K_d \underline{\omega}^T \underline{\omega} \leq 0$$

$\hookrightarrow$ Can conclude Lyapunov stability, but not asymptotic stability since $\dot{V} \leq 0$. We need one more trick (Theorem).

<u>LaSalle's Theorem</u> helps us prove asymptotic stability in situations like this, where $\dot{V} \leq 0$ rather than $\dot{V} < 0$.

The idea is to check what "invariant set" or states are allowed when $\dot{V} = 0$. See (24.4) for more technical details.

Back to our example:

$$\dot{V} = 0 \text{ only if } \underline{\omega} = \underline{0} \Rightarrow \dot{\underline{\omega}} = \underline{0}$$

Sub this into Euler's Eq.

$$\cancel{\underline{I}\, \underline{0}} + \cancel{\underline{0}^\times \underline{I}\, \underline{0}} = -K_p \underline{\varepsilon} - \cancel{K_d \underline{0}}$$

$$K_p \underline{\varepsilon} = \underline{0}$$

Since $K_p > 0$, $\underline{\varepsilon} = 0$

From $\underline{\varepsilon}^T \underline{\varepsilon} + \eta^2 = 1$, $\underline{\varepsilon} = 0 \Rightarrow \eta^2 = 1$

$\eta^2 = 1 \Rightarrow \eta = \pm 1$ which both correspond to same attitude.

Therefore $\dot{V} = 0$ only when $\underline{x} = \underline{\bar{x}} = \underline{0}$, therefore we can conclude that $\underline{\bar{x}} = \underline{0}$ is asymptotically stable (i.e., $\underset{\sim}{\omega} \to 0$, $\underline{\varepsilon} \to 0$, $\eta \to \pm 1$ as $t \to \infty$).

We can also track a non-zero attitude with PD attitude control law!

$$\underline{\tau} = -K_p \eta_{err} \underline{\varepsilon}_{err} - K_d \underline{\omega},$$

where
$$q_{err} = \begin{bmatrix} \underline{\varepsilon}_{err} \\ \eta_{err} \end{bmatrix} = \begin{bmatrix} \eta_d \underline{1} - \underline{\varepsilon}_d^{\times} & -\underline{\varepsilon}_d \\ \underline{\varepsilon}_d^T & \eta_d \end{bmatrix} \begin{bmatrix} \underline{\varepsilon} \\ \eta \end{bmatrix}$$

$q_d = \begin{bmatrix} \underline{\varepsilon}_d \\ \eta_d \end{bmatrix}$ is desired quaternion attitude

Note that q_{err} is in fact a quaternion.

Can also implement PID tracking:

$$\underline{\tau} = -K_p \eta_{err} \underline{\varepsilon}_{err} - K_d(\underline{\omega} - \underline{\omega}_d)$$

$$- K_i \int_0^t (\eta_{err} \underline{\varepsilon}_{err} + k(\underline{\omega} - \underline{\omega}_d)\, dt$$

where

$K_i > 0, \quad k > 0.$

Could also implement PD/PID attitude control using DCM:

Recall $(\underline{u}^{\times})^{\vee} = \underline{u}$

$(\cdot)^{\vee}$ is "uncross" operator

PD: $$\underline{\tau} = K_p\left(\underline{C}_{bd} - \underline{C}_{bd}^{T}\right)^{\vee} - K_d\,\underline{\omega}$$

where $\underline{C}_{bd} = \underline{C}_{ba}\underline{C}_{da}^{T}$, $\underline{C}_{da}$ is desired attitude

$$\underline{I}\,\dot{\underline{\omega}} + \underline{\omega}^{\times}\underline{I}\,\underline{\omega} = \underline{\tau} = K_p\left(\underline{C}_{bd} - \underline{C}_{bd}^{T}\right)^{\vee} - K_d\underline{\omega}$$

$$\dot{\underline{C}}_{ba} = -\underline{\omega}^{\times}\underline{C}_{ba}$$

$\dot{\underline{C}}_{da} = 0$ (assume $\underline{C}_{da}$ is constant)

$$\dot{\underline{C}}_{bd} = \dot{\underline{C}}_{ba}\underline{C}_{da}^{T} + \underline{C}_{ba}\cancelto{0}{\dot{\underline{C}}_{da}^{T}}$$

$$= -\underline{\omega}^{\times}\underline{C}_{ba}\underline{C}_{da}^{T}$$

$$= -\underline{\omega}^{\times}\underline{C}_{bd}$$

Can show that $\underline{C}_{bd} = \underline{1}$, $\underline{\omega} = \underline{0}$ is an equilibrium point.

Let $V = \frac{1}{2}\,\underline{\omega}^T \underline{I}\,\underline{\omega} + \frac{1}{2} K_p \,\text{trace}\left(\underline{1} - \underline{C}_{bd}\right)$

$V > 0$ for all $\underline{\omega} \neq 0$, $\underline{C}_{bd} \neq \underline{1}$

since $\text{trace}\left(\underline{1} - \underline{C}_{bd}\right) \geq 0$

$$\dot{V} = \underline{\omega}^T \underline{I}\,\dot{\underline{\omega}} - \frac{1}{2} K_p \,\text{trace}\left(\dot{\underline{C}}_{bd}\right)$$

$$= \underline{\omega}^T\left(K_p \left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee - K_d\,\underline{\omega} - \underline{\omega}^\times \underline{I}\,\underline{\omega}\right)$$

$$+ K_p \underbrace{\frac{1}{2}\,\text{trace}\left(\underline{\omega}^\times \underline{C}_{bd}\right)}_{= -\underline{\omega}^T\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee}$$

$$= -K_d\,\underline{\omega}^T\underline{\omega} - \underbrace{\underline{\omega}^T\underline{\omega}^\times}_{=\underline{0}} \underline{I}\,\underline{\omega} + \cancel{K_p\,\underline{\omega}^T\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee}$$

$$\cancel{- K_p\,\underline{\omega}^T\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee}$$

$$= -K_d\,\underline{\omega}^T\underline{\omega} \leq 0$$

LaSalle's Theorem:

$$\dot{V} = 0 \;\Rightarrow\; \underline{\omega} = 0 \;\Rightarrow\; \dot{\underline{\omega}} = 0$$

$$\underline{I}\,\underline{0} + \underline{0}^\times \underline{I}\,\underline{0} = K_p\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee - K_d\,\underline{0}$$

$$K_p\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee = \underline{0}$$

$$\underline{C}_{bd} - \underline{C}_{bd}^T = \underline{0} \;\Rightarrow\; \underline{C}_{bd} = \underline{C}_{bd}^T$$

$$\underline{C}_{bd}\,\underline{C}_{bd} = \underline{1} \Rightarrow \underline{C}_{bd} = \pm\underline{1}$$

For $\underline{C}_{bd}$ to be valid DCM, $\underline{C}_{bd} = +\underline{1}$

Therefore equilibrium pt. is asymptotically stable

$\underline{\omega} \to 0$, $\underline{C}_{bd} \to \underline{1}$ ($\underline{C}_{ba} \to \underline{C}_{da}$) as $t \to \infty$.

Can also track non-zero angular velocity and implement PID attitude controller:

$$\underline{\tau} = K_p\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee + K_i \int_0^t \left[\left(\underline{C}_{bd} - \underline{C}_{bd}^T\right)^\vee - K\left(\underline{\omega} - \underline{\omega}_d\right)\right] d\tau$$

$$+ K_d\left(\underline{\omega} - \underline{\omega}_d\right)$$

Tougher to prove stability.

Complementary Filter for Attitude Estimation

Recall that both attitude determination methods we discussed had disadvantages:

- Inertial Navigation: Unknown initial attitude and biased rate gyro measurement
- TRIAD: Only uses instantaneous measurements to determine attitude. No model.

We can combine both methods to obtain a better attitude estimate than either one individually.

Consider estimated attitude kinematics of the form:

$$\dot{\hat{q}} = \Gamma_b^{ba}(\hat{q})\left(\underline{\omega}_b^{ba,meas} + K_e \eta_{err} \underline{\varepsilon}_{err}\right)$$

where

$$q_{err} = \begin{bmatrix} \underline{\varepsilon}_{err} \\ \eta_{err} \end{bmatrix} = \begin{bmatrix} (\hat{\eta}\underline{1} - \hat{\underline{\varepsilon}}^{\times}) & -\hat{\underline{\varepsilon}} \\ \hat{\underline{\varepsilon}}^T & \hat{\eta} \end{bmatrix} \underbrace{\begin{bmatrix} \underline{\varepsilon} \\ \eta \end{bmatrix}}_{q:\text{ true quaternion}}, \quad K_e > 0$$

We don't know the true value of q when calculating q_{err}, so use TRIAD or another method to determine q^{meas}.

Therefore

$$q_{err} \simeq \begin{bmatrix} (\hat{\eta}\mathbf{1} - \hat{\underline{\varepsilon}}^{\times}) & -\hat{\underline{\varepsilon}} \\ \hat{\underline{\varepsilon}}^{T} & \hat{\eta} \end{bmatrix} q^{meas}$$

Notice how q^{meas} is obtained from vector measurements (e.g., Sun sensor, star tracker, etc.), $\underline{\omega}_b^{ba,meas}$ is from a rate gyro measurement and $\hat{q}$ is obtained by integrating a function of both of these types of measurement.

A lengthy proof using Lyapunov stability analysis can show that $\hat{q} \to q$ as $t \to \infty$ if sensors have no noise and rate gyro has no bias. With sensor noise and bias, typically $\hat{q}$ "approaches" q as $t \to \infty$, but $\hat{q} \not\to q$ as $t \to \infty$.

The effect of a constant rate gyro bias can be removed by using

$$\dot{\hat{q}} = \Gamma_b^{ba}(\hat{q})(\underline{\omega}_b^{ba,meas} - \hat{\underline{b}} + K_e \eta_{err} \varepsilon_{err})$$

$$\dot{\hat{\underline{b}}} = -K_b \eta_{err} \varepsilon_{err} \quad , \quad K_b > 0 .$$

$\hat{\underline{b}}$ is an estimate of the bias and can be thought of as an integrator that cancels out the effect of the constant "measurement disturbance."

The reason this is called a "complementary filter" is because the vector and rate gyro measurements complement each other. Vector measurements are typically accurate at low frequencies, but are noisy at high frequencies. A rate gyro measurement is typically inaccurate (biased) at low frequencies, but relatively accurate at high frequencies.

We can also implement a DCM-based Complementary Filter:

$$\dot{\underline{C}}_{ea} = -(\underline{\omega}_b^{ba,meas} - k_e \underline{e})^\times \underline{C}_{ea}$$

where

$$\underline{e} = +(\underline{C}_{be} - \underline{C}_{be}^T)^\vee, \quad \underline{C}_{be} = \underline{C}_{ba}^{meas} \underline{C}_{ea}^T,$$

$k_e > 0$, and $\underline{C}_{ba}^{meas}$ is obtained from TRIAD or another method.

Assume $\underline{\omega} = \underline{\omega}_b^{ba}$ is a perfect rate gyro measurement and $\underline{C}_{ba} = \underline{C}_{ba}^{meas}$ is a perfect estimate of $\underline{C}_{ba}$. We can prove asymptotic convergence of our attitude estimate (i.e., $\underline{C}_{ea} \to \underline{C}_{ba}$ as $t \to \infty$).

First, $\underline{C}_{be} = \underline{C}_{ba} \underline{C}_{ea}^T$ $\qquad \dot{\underline{C}}_{ea} = -(\underline{\omega} - k_e \underline{e})^\times \underline{C}_{ea}$

$\qquad \dot{\underline{C}}_{ba} = -\underline{\omega}^\times \underline{C}_{ba}$

$$\dot{\underline{C}}_{be} = \dot{\underline{C}}_{ba} \underline{C}_{ea}^T + \underline{C}_{ba} \dot{\underline{C}}_{ea}^T$$

$$= -\underline{\omega}^\times \underbrace{\underline{C}_{ba} \underline{C}_{ea}^T}_{\underline{C}_{be}} + \underbrace{\underline{C}_{ba} \underline{C}_{ea}^T}_{\underline{C}_{be}} (\underline{\omega} - k_e \underline{e})^\times$$

$$= -\underline{\omega}^\times \underline{C}_{be} + \underline{C}_{be} \underline{\omega}^\times - k_e \underline{C}_{be} \underline{e}^\times$$

Choose

$$V = \frac{1}{2} \operatorname{trace} (\underline{1} - \underline{C}_{be}) \geq 0$$

$$\dot{V} = \frac{1}{2} \operatorname{trace} (- \dot{\underline{C}}_{be})$$

$$= - \frac{1}{2} \operatorname{trace} (\dot{\underline{C}}_{be})$$

$$= - \frac{1}{2} \operatorname{trace} (- \underline{\omega}^{\times} \underline{C}_{be} + \underline{C}_{be} \underline{\omega}^{\times} - k_e \underline{C}_{be} \underline{e}^{\times})$$

The trace has the property that for $\underline{A}, \underline{B} \in \mathbb{R}^{n \times n}$

$$\operatorname{trace} (\underline{A} \, \underline{B}) = \operatorname{trace} (\underline{B} \, \underline{A})$$

So,

$$\dot{V} = - \frac{1}{2} \operatorname{trace} (- \cancel{\underline{\omega}^{\times} \underline{C}_{be}} + \cancel{\underline{\omega}^{\times} \underline{C}_{be}} - k_e \underline{C}_{be} \underline{e}^{\times})$$

$$= - \frac{1}{2} k_e \operatorname{trace} (\underline{C}_{be} \underline{e}^{\times})$$

Use property that $- \frac{1}{2} \operatorname{trace} (\underline{C}_{be} \underline{e}^{\times}) = \underline{e}^{T} (\underline{C}_{be} - \underline{C}_{be}^{T})^{\vee}$

$$\dot{V} = - k_e \underline{e}^{T} (\underline{C}_{be} - \underline{C}_{be}^{T})^{\vee}$$

Sub in $\underline{e} = + (\underline{C}_{be} - \underline{C}_{be}^{T})^{\vee}$ to get

$$\dot{V} = - k_e \left[(\underline{C}_{be} - \underline{C}_{be}^{T})^{\vee} \right]^{T} (\underline{C}_{be} - \underline{C}_{be}^{T})^{\vee}$$

Notice that

$$\dot{V} \leq 0 \quad \text{and} \quad \dot{V} < 0 \text{ as long as}$$

$$\underline{C}_{be} - \underline{C}_{be}^T \neq \underline{0} \iff \underline{C}_{ba} = \pm \underline{1}$$

$\underline{C}_{be} = -\underline{1}$ is not possible, so $\dot{V} < 0$ for all

$\underline{C}_{be} \neq +\underline{1}$.

Therefore, from Lyapunov's Direct Method we can conclude that $\underline{C}_{be} = +\underline{1}$ is an asymptotically stable equilibrium point (i.e., $\underline{C}_{ea} \rightarrow \underline{C}_{ba}$ as $t \rightarrow \infty$).

Can also account for constant bias.

$$\dot{\underline{C}}_{ea} = -\left(\underline{\omega}_b^{ba,\text{meas}} - \hat{\underline{b}} - k_e \underline{e}\right)^\times \underline{C}_{ea}$$

$$\dot{\hat{\underline{b}}} = -K_b \underline{e}$$

where $K_b > 0$, $\underline{e} = \left(\underline{C}_{be} - \underline{C}_{be}^T\right)^\vee$.