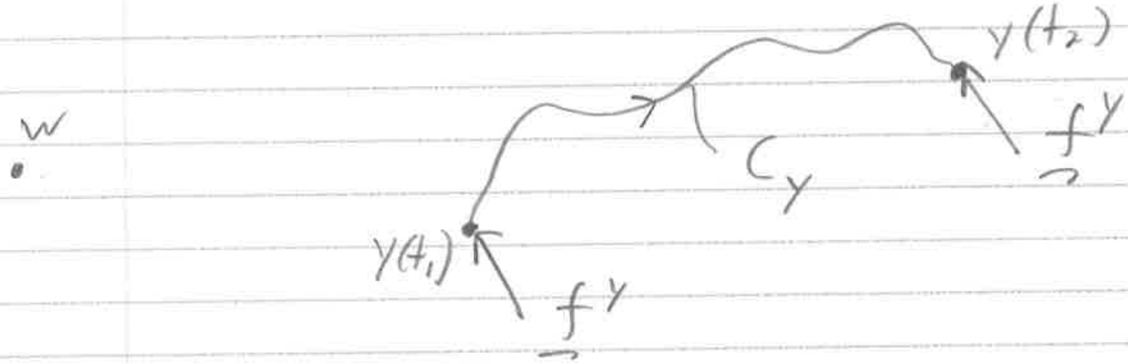
Energy of a Particle

Def: Consider frame $\mathcal{F}_a$, point w, and particle y of mass m. The <u>kinetic energy</u> of y relative to w with respect to $\mathcal{F}_a$ is

$$T_{yw/a} = \frac{1}{2} m \underset{\rightharpoonup}{v}^{yw/a} \cdot \underset{\rightharpoonup}{v}^{yw/a}$$
$$= \frac{1}{2} m \, \underline{v}_i^{yw/a\,T} \, \underline{v}_i^{yw/a}, \text{ where } \mathcal{F}_i \text{ is any frame.}$$
$$= \frac{1}{2} m \, \| \underset{\rightharpoonup}{v}^{yw/a} \|_2^2$$

Def: Consider particle y, unforced particle w, and force $\underset{\rightharpoonup}{f}^{y}$ applied to particle y as y moves along path C_y.



The <u>work done on y</u> relative to w by $\underset{\rightharpoonup}{f}^{y}$ as particle y moves along C_y is

$$W_{yw}(\underset{\rightharpoonup}{f}^{y}, C_y) = \int_{C_y} \underset{\rightharpoonup}{f}^{y} \cdot d\underset{\rightharpoonup}{r}^{yw}$$

Recall N2L, where

$$\underset{\rightharpoondown}{f}^{Y} = \underset{\rightharpoondown}{p}^{yw/a \cdot a}$$

If m is constant, then

$$\underset{\rightharpoondown}{f}^{Y} = m\,\underset{\rightharpoondown}{a}^{yw/a/a} = m\,\underset{\rightharpoondown}{v}^{yw/a \cdot a} = m\,\underset{\rightharpoondown}{r}^{yw \cdot a \cdot a}$$

Using this in the expression for work gives

$$W_{yw}(\underset{\rightharpoondown}{f}^{Y}, C_Y) = \int_{C_Y} m\,\underset{\rightharpoondown}{v}^{yw/a \cdot a} \cdot d\underset{\rightharpoondown}{r}^{yw}$$

We can simplify this further, start with

$$= \frac{1}{2} m \frac{d}{dt}\left(\underline{V}_a^{yw/a^T} \underline{V}_a^{yw/a}\right) dt$$

$$= \frac{1}{2} m \left(\dot{\underline{V}}_a^{yw/a^T} \underline{V}_a^{yw/a} + \underline{V}_a^{yw/a^T} \dot{\underline{V}}_a^{yw/a}\right) dt$$

$$= \frac{1}{2} m \left(\dot{\underline{V}}_a^{yw/a^T} \underline{V}_a^{yw/a} + \dot{\underline{V}}_a^{yw/a^T} \underline{V}_a^{yw/a}\right) dt$$

$$= m\,\dot{\underline{V}}_a^{yw/a^T} \underline{V}_a^{yw/a}\, dt \qquad (**)$$

Recall $\underset{\rightharpoondown}{v}^{yw/a} = \underset{\rightharpoondown}{\mathcal{F}}_a^{T} \underline{V}_a^{yw/a}$ and

$\underset{\rightharpoondown}{r}^{yw \cdot a} = \underset{\rightharpoondown}{\mathcal{F}}_a^{T} \dot{\underline{r}}_a^{yw}$, therefore

$$\underline{V}_a^{yw/a} = \dot{\underline{r}}_a^{yw} = \frac{d}{dt}\left(\underline{r}_a^{yw}\right)$$

$$\underline{v}_a^{yw/a}\, dt = \frac{d}{dt}\left(\underline{r}_a^{yw}\right) dt = d\,\underline{r}_a^{yw}$$

Returning to (**)

$$\frac{1}{2} m \frac{d}{dt}\left(\underline{v}_a^{yw/a^T} \underline{v}_a^{yw/a}\right) dt$$

$$= m\, \dot{\underline{v}}_a^{yw/a^T}\, d\,\underline{r}_a^{yw}$$

$$= m\, \vec{v}^{yw/a^{\circ a}} \cdot d\vec{r}^{yw}$$

Going back to our expression for work, we get

$$W_{yw}\left(\vec{f}^{Y}, C_Y\right) = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt}\left(\underline{v}_a^{yw/a^T} \underline{v}_a^{yw/a}\right) dt$$

$$= \frac{1}{2} m\, \underline{v}_a^{yw/a^T}(t_2)\, \underline{v}_a^{yw/a}(t_2)$$

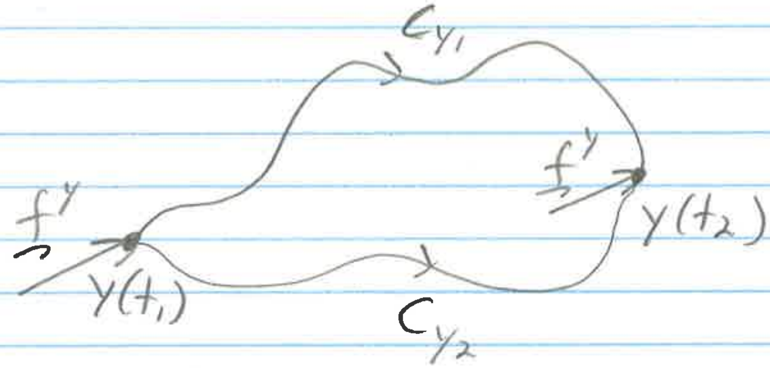
$$- \frac{1}{2} m\, \underline{v}_a^{yw/a^T}(t_1)\, \underline{v}_a^{yw/a}(t_1)$$

$$= \frac{1}{2} m\, \vec{v}^{yw/a}(t_2) \cdot \vec{v}^{yw/a}(t_2)$$

$$- \frac{1}{2} m\, \vec{v}^{yw/a}(t_1) \cdot \vec{v}^{yw/a}(t_1)$$

$$= T_{yw/a}(t_2) - T_{yw/a}(t_1)$$

Def: Consider unforced particles y and w, and force $\underset{\rightharpoondown}{f}^{y}$ applied to particle y. C_{y_1} and C_{y_2} are two different paths with the same beginning and end position



If $$\int_{C_{y_1}} \underset{\rightharpoondown}{f}^{y} \cdot d\underset{\rightharpoondown}{r}^{yw} = \int_{C_{y_2}} \underset{\rightharpoondown}{f}^{y} \cdot d\underset{\rightharpoondown}{r}^{yw}$$ holds

for any paths C_{y_1} and C_{y_2}, then $\underset{\rightharpoondown}{f}^{y}$ is a conservative force.

If $\underset{\rightharpoondown}{f}^{y}$ is conservative, then we can find a scalar potential function $V_{yw}(\underset{\rightharpoondown}{r}^{yw})$ such that

$$dV_{yw} = -\underset{\rightharpoondown}{f}^{y} \cdot d\underset{\rightharpoondown}{r}^{yw}$$

Accordingly, then

$$W_{yw}(\underset{\rightharpoondown}{f}^{y}) = \int_{C_y} \underset{\rightharpoondown}{f}^{y} \cdot d\underset{\rightharpoondown}{r}^{yw} = -\int_{V_{yw}(t_1)}^{V_{yw}(t_2)} dV_{yw} = V_{yw}(t_1) - V_{yw}(t_2)$$

where V_{yw} is the potential energy of y rel. to w

We already derived

$$W_{yw}(\underline{f}^{y}, C_y) = T_{yw}(t_2) - T_{yw}(t_1),$$

which combined with the previous result gives

$$T_{yw/a}(t_1) + V_{yw}(t_1) = T_{yw/a}(t_2) + V_{yw}(t_2)$$

Def: the <u>total energy</u> of y relative to w wrt $\mathcal{F}_a$ is

$$E_{yw/a} = T_{yw/a} + V_{yw}$$

The total energy of particle y relative to an unforced particle w wrt an inertial frame $\mathcal{F}_a$ is conserved if the forces acting on y are conservative. That is,

$$\boxed{E_{yw/a}(t_1) = E_{yw/a}(t_2)}$$

Or alternatively,

$$\boxed{\dot{E}_{yw/a} = 0}$$

This is known as conservation of energy.

Potential Energy of Conservative Forces

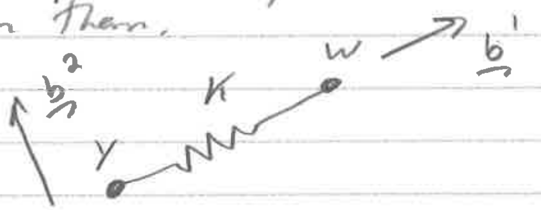
We have talked about 4 difference kinds of forces:

1) linear spring force (conservative)
2) linear viscous force (non-conservative)
3) gravitational force (conservative)
4) reaction force (non-conservative, but usually N/A)

Since linear spring forces and gravitational forces are conservative, we can write out a potential function for each.

Potential Energy of a Linear Spring

Consider particles y and w with a spring connected between them.



d: unstretched length of spring

ℓ: stretched length of spring

$x_s = \ell - d$

The force acting on y due to the spring is

$$\underrightarrow{f}^{y} = -K x_s \frac{\underrightarrow{r}^{yw}}{\|\underrightarrow{r}^{yw}\|_2}$$

$$\underrightarrow{r}^{yw} = \underrightarrow{\mathcal{I}}_b{}^{1} \begin{bmatrix} -\ell \\ 0 \\ 0 \end{bmatrix}, \quad \|\underrightarrow{r}^{yw}\|_2 = \ell \quad \text{so} \quad \frac{\underrightarrow{r}^{yw}}{\|\underrightarrow{r}^{yw}\|_2} = -\underrightarrow{b}^1$$

and $\underset{\rightharpoondown}{f}^{y} = +K x_s \underset{\rightharpoondown}{b}'$

An infinitessimally small change in the length of the spring is given by

$$d\underset{\rightharpoondown}{r}^{yw} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} -dl \\ 0 \\ 0 \end{bmatrix} = -dl\, \underset{\rightharpoondown}{b}'$$

But $l = x_b + d$, so $dl = dx_s + \cancel{d(d)}^{\,0}$

$$d\underset{\rightharpoondown}{r}^{yw} = -dx_s \underset{\rightharpoondown}{b}'$$

$$W_{yw}(\underset{\rightharpoondown}{f}^{y}) = \int_{x_s(t_1)}^{x_s(t_2)} -K x_s \cancel{\underset{\rightharpoondown}{b}' \cdot \underset{\rightharpoondown}{b}'}^{\,1} dx_s$$

$$= \int_{x_s(t_1)}^{x_s(t_2)} -K x_b \, dx_b$$

$$= -\frac{K}{2}\left(x_s^2(t_2) - x_s^2(t_1)\right)$$

$$= \frac{1}{2} K x_s^2(t_1) - \frac{1}{2} K x_s^2(t_2)$$

Since $\underset{\rightharpoondown}{f}^{y}$ is a conservative force, we know

$$W_{yw}(\underset{\rightharpoondown}{f}^{y}) = V_{yw}(t_1) - V_{yw}(t_2), \quad \text{so}$$

by inspection

$$\boxed{V_{yw}(t) = \frac{1}{2} K x_s^2(t)}$$

Gravitational Potential Energy

Consider particles x and y with masses m_x and m_y

If we assume the motion of y is much smaller than $\|\underset{\rightharpoonup}{r}^{yx}\|$, then the gravitational force acting on y due to x is

$$\underset{\rightharpoonup}{f}^{y} = m_y \underset{\rightharpoonup}{g}$$

where

$$\underset{\rightharpoonup}{g} = -\frac{G m_x}{\|\underset{\rightharpoonup}{r}^{yx}\|_2^3}\underset{\rightharpoonup}{r}^{yx} \quad \text{is constant.}$$

The work done on particle y relative to w due to $\underset{\rightharpoonup}{f}^{y}$ is

$$W_{yw}(\underset{\rightharpoonup}{f}^{y}) = \int_{\underset{\rightharpoonup}{r}^{yw}(t_1)}^{\underset{\rightharpoonup}{r}^{yw}(t_2)} \underset{\rightharpoonup}{f}^{y}\cdot d\underset{\rightharpoonup}{r}^{yw}$$

$$= \int_{\underset{\rightharpoonup}{r}^{yw}(t_1)}^{\underset{\rightharpoonup}{r}^{yw}(t_2)} m_y \underset{\rightharpoonup}{g}\cdot d\underset{\rightharpoonup}{r}^{yw}$$

Since $\underset{\rightharpoonup}{g}$ is constant,

$$W_{yw}(\underset{\rightharpoonup}{f}^{y}) = m_y \underset{\rightharpoonup}{g}\cdot \int_{\underset{\rightharpoonup}{r}^{yw}(t_1)}^{\underset{\rightharpoonup}{r}^{yw}(t_2)} d\underset{\rightharpoonup}{r}^{yw}$$

$$= m_y \underset{\rightharpoonup}{g}\cdot\left(\underset{\rightharpoonup}{r}^{yw}(t_2) - \underset{\rightharpoonup}{r}^{yw}(t_1)\right)$$

$$W_{yw}(\underset{\rightharpoondown}{f}^{Y}) = m_y \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{yw}(t_2) - m_y \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{yw}(t_1)$$

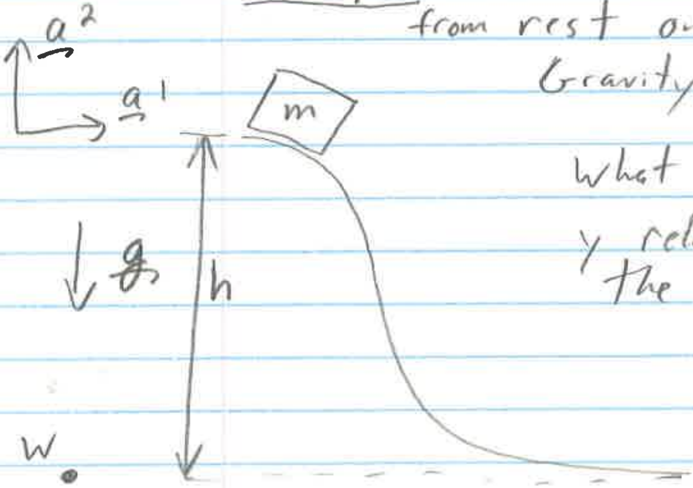
Since $\underset{\rightharpoondown}{f}^{Y}$ is a conservative force, we know

$$W_{yw}(\underset{\rightharpoondown}{f}^{Y}) = V_{yw}(t_1) - V_{yw}(t_2)$$

by inspection,

$$\boxed{V_{yw}(t) = - m_y \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{yw}(t)}$$

Example: Consider block (particle) y of mass m released from rest on a frictionless slide of height h. Gravity is acting in the downwards direction.



What is the magnitude of the velocity of y relative to w wrt $\mathcal{F}_a$ when y reaches the bottom of the ramp?

$$T_{yw/a} = \frac{1}{2} m \, \underline{v}^{yw/a} \cdot \underline{v}^{yw/a} = \frac{1}{2} m \, \| \underline{v}^{yw/a} \|_2^2$$

$$V_{yw} = -m \, \underline{g} \cdot \underline{r}^{yw}$$

1) Kinematics

i) Frames and DCMs: N/A

ii) Angular Velocity: N/A

iii) Position

$$\underline{r}^{yw} = \underline{\mathcal{F}}_a^{T} \begin{bmatrix} x_a \\ y_a \\ 0 \end{bmatrix}$$

iv) Velocity

$$\underline{v}^{yw/a} = \underline{r}^{yw^{\cdot a}} = \underline{\mathcal{F}}_a^{T} \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ 0 \end{bmatrix}$$

2)

$$T_{yw/a} = \frac{1}{2} m \underset{\rightharpoondown}{V}^{yw/a} \cdot \underset{\rightharpoondown}{V}^{yw/a}$$

$$= \frac{1}{2} m \begin{bmatrix} \dot{x}_a & \dot{y}_a & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m \left(\dot{x}_a^{2} + \dot{y}_a^{2} \right)$$

$$= \frac{1}{2} m \, \| \underset{\rightharpoondown}{V}^{yw/a} \|_2^2$$

$$\underset{\rightharpoondown}{g} = \underset{\rightharpoondown}{I}_a^{T} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$V_{yw} = -m \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{yw}$$

$$= -m \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \cdot \begin{bmatrix} x_a \\ y_a \\ 0 \end{bmatrix}$$

$$= -m g y_a$$

$$E_{yw/a} = T_{yw/a} + V_{yw}$$

$$= \frac{1}{2} m \| \underset{\rightharpoondown}{V}^{yw/a} \|_2^2 + m g y_a$$

3) $E_{yw/a}(t_1) = E_{yw/a}(t_2)$

$$\frac{1}{2} m \| \underset{\rightharpoondown}{V}^{yw/a}(t_1) \|_2^2 + m g y_a(t_1)$$

$$= \frac{1}{2} m \| \underset{\rightharpoondown}{V}^{yw/a}(t_2) \|_2^2 + m g y_a(t_2)$$

$t_1: \|\underset{\rightharpoondown}{v}^{yw/a}(t_1)\|_2 = 0$

$y_a(t_1) = h$

$t_2: \|\underset{\rightharpoondown}{v}^{yw/a}(t_2)\|_2$: unknown

$y_a(t_2) = 0$

$$\cancel{m} g h = \frac{1}{2} \cancel{m} \|\underset{\rightharpoondown}{v}^{yw/a}(t_2)\|_2^2$$

$$\|\underset{\rightharpoondown}{v}^{yw/a}(t_2)\|_2 = \sqrt{2gh}$$

Notice that we did not need to consider reaction forces to get this answer.

However, we did not obtain the equations of motion that govern the behavior of y, only the velocity of y at a particular instance in time.

Power Due to a Force on a Particle

Def: Consider particles y and w and force $\underset{\rightharpoondown}{f}^y$ applied to particle y as y moves along path C_y.

The *power* of the force $\underset{\rightharpoondown}{f}_y$ acting on y relative to w wrt $\mathcal{F}_a$ is

$$P_{yw/a}(\underset{\rightharpoondown}{f}^y) = \frac{d}{dt}\left(W_{yw}(\underset{\rightharpoondown}{f}^y, C_y)\right)$$

$$= \underset{\rightharpoondown}{f}^y \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

We can relate $P_{yw/a}$ to Kinetic energy as follows

$$T_{yw/a} = \frac{1}{2} m \, \underset{\rightharpoondown}{v}^{yw/a} \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

Take the time derivative of $T_{yw/a}$

$$\frac{d}{dt}\left(T_{yw/a}\right) = \frac{1}{2} m \frac{d}{dt}\left(\underset{\rightharpoondown}{v}^{yw/a} \cdot \underset{\rightharpoondown}{v}^{yw/a}\right)$$

$$= \frac{1}{2} m \left(\underset{\rightharpoondown}{v}^{yw/a \cdot a} \cdot \underset{\rightharpoondown}{v}^{yw/a} + \underset{\rightharpoondown}{v}^{yw/a} \cdot \underset{\rightharpoondown}{v}^{yw/a \cdot a}\right)$$

$$= m \, \underset{\rightharpoondown}{v}^{yw/a \cdot a} \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

$$= \underbrace{m \, \underset{\rightharpoondown}{a}^{yw/a/a}}_{\underset{\rightharpoondown}{f}^y \text{ from N2L}} \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

$$\boxed{\frac{d}{dt}\left(T_{\gamma w/a}\right) = \underset{\rightarrow}{f}^{\gamma} \cdot \underset{\rightarrow}{v}^{\gamma w/a} = P_{\gamma w/a}\left(\underset{\rightarrow}{f}^{\gamma}\right)}$$

This is known as the work-energy theorem for a particle. $\mathcal{F}_a$ must be an inertial frame, w must be an unforced particle.

This theorem is valid for any force (conservative or non conservative) acting on particle γ.

This theorem gives us an alternative approach to deriving the equations of motion of a particle.

We do not need to consider reaction forces that are not in the direction of motion of γ, since they will disappear when we take the dot product with $\underset{\rightarrow}{v}^{\gamma w/a}$

Only valid for single degree-of-freedom systems.

This approach is a stepping stone towards the Lagrangian approach to dynamics. You will see this in more advanced dynamics courses, such as "Intermediate Dynamics"

"4 Alternative Steps to Success in Dynamics"

1) Kinematics
   i) Frames and DCMs
   ii) Angular Velocity
   iii) Position
   iv) Velocity

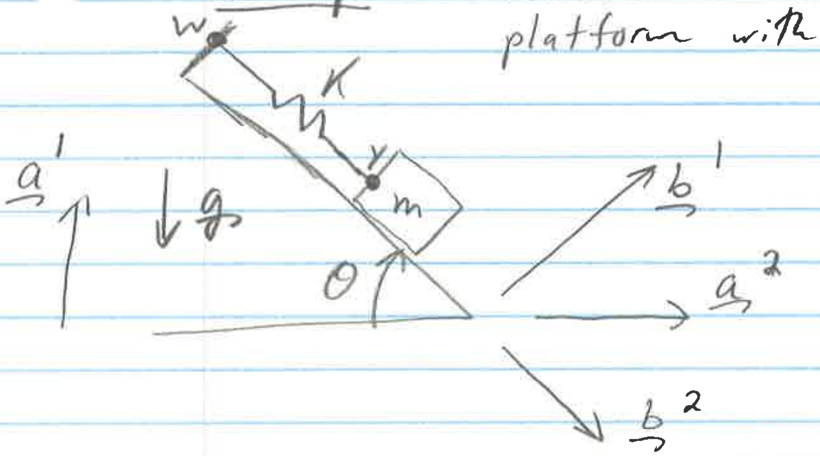
2) Kinetic Energy

3) Forces

4) Work-Energy Theorem

Only valid for single DOF systems!

Example: Consider a mass y on a frictionless inclined platform with a spring



θ: constant
l: unstretched length of spring
x_s: extension of spring

Let's use the "4 alternative steps to success"

1) Kinematics

i) Frames and DCMs

$$\underset{\rightharpoondown}{\mathcal{F}}_a \xrightarrow{\underline{C}_3(\theta)} \underset{\rightharpoondown}{\mathcal{F}}_b \qquad \underline{C}_{ba} = \underline{C}_3(\theta)$$

ii) Angular Velocity

$$\underset{\rightharpoondown}{\omega}^{ba} = \underset{\rightharpoondown}{0}$$

iii) Position

$$\underset{\rightharpoondown}{r}^{yw} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ l + x_s \\ 0 \end{bmatrix}$$

iv) Velocity

$$\underset{\rightharpoondown}{v}^{yw/a} = \underset{\rightharpoondown}{r}^{yw \cdot a} = \underset{\rightharpoondown}{r}^{yw \cdot b} + \cancelto{0}{\underset{\rightharpoondown}{\omega}^{ba}} \times \underset{\rightharpoondown}{r}^{yw}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ \dot{x}_s \\ 0 \end{bmatrix}$$

2) Energy

Kinetic Energy

$$T_{y w/a} = \frac{1}{2} m \underrightarrow{V}^{y w/a} \cdot \underrightarrow{V}^{y w/a}$$

$$= \frac{1}{2} m \underline{V}_b^{y w/a^T} \underline{V}_b^{y w/a}$$

$$= \frac{1}{2} m \begin{bmatrix} 0 & \dot{x}_s & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{x}_s \\ 0 \end{bmatrix} = \frac{1}{2} m \dot{x}_s^2$$

3) Forces

$$\underrightarrow{f}^{yg} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} -mg \\ 0 \\ 0 \end{bmatrix} = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ba} \begin{bmatrix} -mg \\ 0 \\ 0 \end{bmatrix} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} -mg C_\theta \\ mg S_\theta \\ 0 \end{bmatrix}$$

$$\underrightarrow{f}^{ys} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -K x_s \\ 0 \end{bmatrix}$$

4) Work-Energy Theorem

$$\frac{d}{dt}\left(T_{y w/a}\right) = m \dot{x}_s \ddot{x}_s$$

$$\left(\underrightarrow{f}^{yg} + \underrightarrow{f}^{ys}\right) \cdot \underrightarrow{V}^{y w/a} = \left(\underline{f}_b^{yg} + \underline{f}_b^{ys}\right)^T \underline{V}_b^{y w/a}$$

$$= \begin{bmatrix} -mg C_\theta & mg S_\theta - K x_s & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{x}_s \\ 0 \end{bmatrix}$$

$$= (mg\, s_\theta - K x_s)\, \dot{x}_s$$

$$\frac{d}{dt}\left(T_{y w/a} \right) = \left(\underset{\rightharpoondown}{f}^{yg} + \underset{\rightharpoondown}{f}^{ys} \right) \cdot \underset{\rightharpoondown}{v}^{yw/a}$$

$$m \dot{x}_s \ddot{x}_s = (mg\, s_\theta - K x_s)\, \dot{x}_s$$

$$\dot{x}_s (m \ddot{x}_s + K x_s - mg\, s_\theta) = 0$$

This equation must hold for all time, so $\dot{x}_s = 0$ is not an option if the mass is to move. Therefore, we have

$$m \ddot{x}_s + K x_s - mg\, s_\theta = 0.$$

What if we used the original "3 steps to success"?

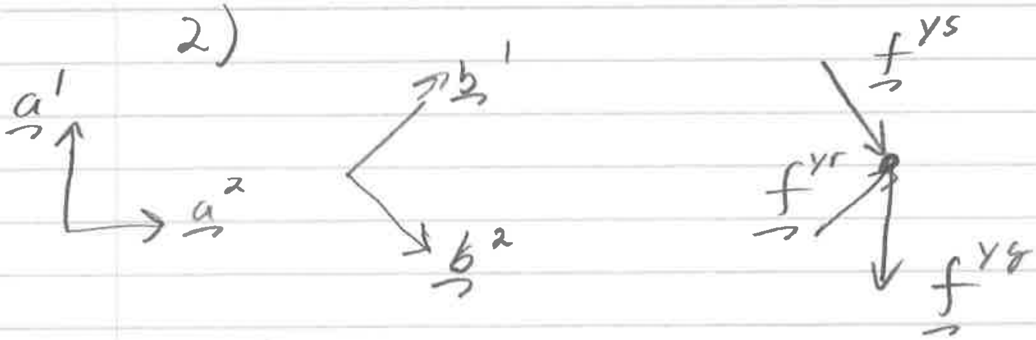
1) i) to iv) already done

v) Acceleration

$$\underset{\rightharpoondown}{a}^{yw/a/a} = \underset{\rightharpoondown}{v}^{yw/a\,\cdot a} = \underset{\rightharpoondown}{v}^{yw/a\,\cdot b} + \overbrace{\cancel{\underset{\rightharpoondown}{\omega}^{ba}}}^{0} \times \underset{\rightharpoondown}{v}^{yw/a}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ \ddot{x}_s \\ 0 \end{bmatrix}$$

2)



$$\vec{f}^{ys} = \vec{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -Kx_s \\ 0 \end{bmatrix}, \quad \vec{f}^{yg} = \vec{\mathcal{F}}_b^T \begin{bmatrix} -mg\,c_\theta \\ mg\,s_\theta \\ 0 \end{bmatrix}$$

$$\vec{f}^{yr} = \vec{\mathcal{F}}_b^T \begin{bmatrix} f_{b1}^{yr} \\ 0 \\ 0 \end{bmatrix}$$

3) N2L

$$\vec{f}^{ys} + \vec{f}^{yg} + \vec{f}^{yr} = m\,\vec{a}^{yw/a/a}$$

$$\begin{bmatrix} -mg\,c_\theta + f_{b1}^{yr} \\ -Kx_s + mg\,s_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ m\ddot{x}_s \\ 0 \end{bmatrix}$$

(1) $\quad -Kx_s + mg\,s_\theta = m\ddot{x}_s$

(2) $\quad -mg\,c_\theta + f_{b1}^{yr} = 0$

(1) $\Rightarrow \quad m\ddot{x}_s + Kx_s - mg\,s_\theta = 0$

↳ same as with the alternative method.

(2) $\Rightarrow \quad f_{b1}^{yr} = mg\,c_\theta$