Spacecraft Attitude Determination (25.2)

Given the sensors available on a spacecraft, there are different approaches we can take to estimate the attitude of the spacecraft:

- Inertial navigation using rate gyro
  - uses Kinematics model ←
- Direct estimate of $\underline{C}_{ba}$ or q
  - TRIAD (25.2.4) ←
  - Davenport's q-Method (25.2.2)
  - QUEST (25.2.3)
- Combine model and measurements to predict and correct attitude estimate
  - Kalman Filtering (25.3)
  - Complementary Filter (End of course) ←

Inertial Navigation with Rate Gyros

Recall the attitude kinematics of the DCM and the quaternion:

$$\dot{\underline{C}}_{ba} = -\underline{\omega}_b^{ba^\times} \underline{C}_{ba} \quad , \quad \dot{q} = \underline{\Gamma}_b^{ba}(q)\, \underline{\omega}_b^{ba}$$

If we have a current estimate of our attitude ($\hat{\underline{C}}_{ba}$ or $\hat{q}$) and a measurement of our angular velocity ($\hat{\underline{\omega}}_b^{ba}$), where the "hat" signifies an estimate or a measurement, then we can determine how to propogate our attitude estimate forward in time:

$$\dot{\hat{\underline{C}}}_{ba} = -\hat{\underline{\omega}}_b^{ba^\times} \hat{\underline{C}}_{ba} \quad , \quad \dot{\hat{q}} = \underline{\Gamma}_b^{ba}(\hat{q})\, \hat{\underline{\omega}}_b^{ba}$$

Integrate these equations

$$\int_{t_1}^{t_2} \dot{\hat{\underline{C}}}_{ba}\, d\tau = -\int_{t_1}^{t_2} \hat{\underline{\omega}}_b^{ba^\times} \hat{\underline{C}}_{ba}\, d\tau$$

$$\Rightarrow \hat{\underline{C}}_{ba}(t_2) = \hat{\underline{C}}_{ba}(t_1) - \int_{t_1}^{t_2} \hat{\underline{\omega}}_b^{ba^\times} \hat{\underline{C}}_{ba}\, d\tau$$

$$\int_{t_1}^{t_2} \dot{\hat{q}}\, d\tau = \int_{t_1}^{t_2} \underline{\Gamma}_b^{ba}(\hat{q})\, \hat{\underline{\omega}}_b^{ba}\, d\tau$$

$$\Rightarrow \hat{q}(t_2) = \hat{q}(t_1) + \int_{t_1}^{t_2} \underline{\Gamma}_b^{ba}(\hat{q})\, \hat{\underline{\omega}}_b^{ba}\, d\tau$$

For small enough time step $\Delta T = t_2 - t_1$

$$\hat{\underline{C}}_{ba}(t_2) \approx \underline{C}_{ba}(t_1) - \Delta T \, \underline{\omega}_b^{ba^\times}(t_1) \, \underline{C}_{ba}(t_1)$$

$$\hat{q}(t_2) \approx \hat{q}(t_1) + \Delta T \, \underline{\Gamma}_b^{ba}(\hat{q}(t_1)) \, \hat{\underline{\omega}}_b^{ba}(t_1)$$

Therefore, if we know $\hat{\underline{C}}_{ba}(t_1)$ or $\hat{q}(t_1)$ (hopefully $\hat{\underline{C}}_{ba}(t_1) \approx \underline{C}_{ba}(t_1)$ or $\hat{q}(t_1) \approx q(t_1)$) and have a measurement of $\hat{\underline{\omega}}_b^{ba}(t_1)$, we can determine $\hat{\underline{C}}_{ba}(t_2)$ or $\hat{q}(t_2)$.

When performed recursively, this is known as inertial navigation or propogation of the attitude kinematics.

Is this a good approach to attitude determination?

If used alone, this will not give a good estimate of $\underline{C}_{ba}$ or q due to noisy and <u>biased</u> measurement of $\underline{\omega}_b^{ba}$.

<u>Example</u>: Consider estimation of q with biased measurement of $\underline{\omega} = \underline{\omega}_b^{ba}$.

Let $\underline{\Gamma}(q) = \underline{\Gamma}_b^{ba}(q)$

$$\underset{\substack{\uparrow \\ \text{measurement} \\ \text{of } \underline{\omega} \\ \text{(hat means "estimate")}}}{\hat{\underline{\omega}}} = \underline{\omega} + \underset{\substack{\uparrow \\ \text{measurement} \\ \text{bias}}}{\underline{b}}$$

True attitude kinematics:

$$q(t_2) \approx q(t_1) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{\omega}(t_1)$$

$$\begin{aligned} q(t_3) &\approx q(t_2) + \Delta T\, \underline{\Gamma}(q(t_2))\, \underline{\omega}(t_2) \\ &= q(t_1) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{\omega}(t_1) \\ &\quad + \Delta T\, \underline{\Gamma}(q(t_2))\, \underline{\omega}(t_2) \end{aligned}$$

Assume $q(t_1)$ is Known (not true in practice)

Estimated attitude Kinematics :

$$\hat{q}(t_2) \approx \hat{q}(t_1) + \Delta T\, \underline{\Gamma}(\hat{q}(t_1))\, \underline{\hat{\omega}}(t_1)$$

$$= q(t_1) + \Delta T\, \underline{\Gamma}(q(t_1)) \left(\underline{\omega}(t_1) + \underline{b}(t_1)\right)$$

$$= q(t_2) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{b}(t_1)$$

$$\hat{q}(t_3) \approx \hat{q}(t_2) + \Delta T\, \underline{\Gamma}(\hat{q}(t_2))\, \underline{\hat{\omega}}(t_2)$$

$$= q(t_2) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{b}(t_1)$$

$$+ \Delta T\, \underline{\Gamma}\left(q(t_2) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{b}(t_1)\right)\left(\underline{\omega}(t_2) + \underline{b}(t_2)\right)$$

$$= q(t_2) + \Delta T\, \underline{\Gamma}(q(t_1))\, \underline{b}(t_1)$$

$q(t_3)$ $$+ \Delta T\, \underline{\Gamma}(q(t_2))\left(\underline{\omega}(t_2) + \underline{b}(t_2)\right)$$

$$+ (\Delta T)^2\, \underline{\Gamma}\left(\underline{\Gamma}(q(t_1))\, \underline{b}(t_1)\right)\left(\underline{\omega}(t_2) + \underline{b}(t_2)\right)$$

$$= q(t_3) + \Delta T\left(\underline{\Gamma}(q(t_1))\, \underline{b}(t_1) + \underline{\Gamma}(q(t_2))\, \underline{b}(t_2)\right)$$

$$+ (\Delta T)^2\, \underline{\Gamma}\left(\underline{\Gamma}(q(t_1))\, \underline{b}(t_1)\right)\left(\underline{\omega}(t_2) + \underline{b}(t_2)\right)$$

Error between $q(t)$ and $\hat{q}(t)$ Keeps growing.

Remarks:

- Due to bias and noise and numerical integration error, $\hat{q}(t_k)$ may not be a valid quaternion (i.e., $\hat{q}^T(t_k)\hat{q}(t_k) \neq 1$) and $\hat{\underline{C}}_{ba}(t_k)$ may not be a valid DCM (i.e., $\hat{\underline{C}}_{ba}^T(t_k)\hat{\underline{C}}_{ba}(t_k) \neq \underline{1}$, $\det(\hat{\underline{C}}_{ba}(t_k)) \neq +1$)

  To correct this, we can normalize our estimate:

$$\hat{q}^{norm}(t_k) = \frac{\hat{q}(t_k)}{\sqrt{\hat{q}^T(t_k)\hat{q}(t_k)}}$$

$$\hat{\underline{C}}_{ba}^{norm}(t_k) = \left[\hat{\underline{C}}_{ba}^1(t_k) \quad \hat{\underline{C}}_{ba}^2(t_k) \quad \hat{\underline{C}}_{ba}^{1^\times}(t_k)\hat{\underline{C}}_{ba}^2(t_k)\right]$$

where $\hat{\underline{C}}_{ba}(t_k) = \left[\hat{\underline{C}}_{ba}^1(t_k) \quad \hat{\underline{C}}_{ba}^2(t_k) \quad \hat{\underline{C}}_{ba}^3(t_k)\right]$.

normalized columns

- Attitude estimation errors do not come from errors in attitude Kinematics model, they are due to noise and bias in measurement of $\underline{\omega}_b^{ba}$

Wahba's Problem and TRIAD (25.2)

Most sensors we discussed provide a measurement of some vector $\underrightarrow{s}^k$ in the spacecraft body frame (i.e., $\underrightarrow{s}^k = \underrightarrow{\mathcal{F}}_b^T \underline{s}_b^k$).

If we have some sort of model for $\underrightarrow{s}^k$ (e.g., model of Earth's magnetic field, catalog of stars), then we also know $\underrightarrow{s}^k$ resolved in inertial frame (i.e., $\underrightarrow{s}^k = \underrightarrow{\mathcal{F}}_a^T \underline{s}_a^k$).

For convenience, normalize the measurements:

$$\hat{\underline{s}}_b^k = \frac{\underline{s}_b^k}{\sqrt{\underline{s}_b^{kT}\underline{s}_b^k}}, \quad \hat{\underline{s}}_a^k = \frac{\underline{s}_a^k}{\sqrt{\underline{s}_a^{kT}\underline{s}_a^k}}$$

Recall that

$$\hat{\underline{s}}_b^k = \underline{C}_{ba}\hat{\underline{s}}_a^k$$

(1965)

<u>Wahba's Problem</u>: Given $\hat{\underline{s}}_b^k$, $\hat{\underline{s}}_a^k$, $k=1,\ldots,N$, determine $\underline{C}_{ba}$ subject to $\underline{C}_{ba}^T\underline{C}_{ba} = \underline{1}$, $\det(\underline{C}_{ba}) = +1$, such that $\hat{\underline{s}}_b^k = \underline{C}_{ba}\hat{\underline{s}}_a^k$, $k=1,\ldots,N$.

Mathematically we pose this as follows.

Let $\underline{r}^k = \hat{\underline{s}}_b^k - \underline{C}_{ba} \hat{\underline{s}}_a^k$

Solve for $\underline{C}_{ba}$ that minimizes

$$\frac{1}{2} \sum_{k=1}^{N} w_k \underline{r}^{k^T} \underline{r}^k \quad \text{such that } \underline{C}_{ba}^T \underline{C}_{ba} = \underline{1}$$

where $w_k = \frac{1}{\sigma_k^2}$ is a weight related to the accaracy of a given measurement.

Grace Wahba posed this problem in 1965, which led to a number of useful attitude determination methods, including Davenport's q-Method (1968) and QUEST (Shuster, 1978). These methods directly estimate the quaternion (see textbook).

We will look at a simpler method that directly estimates the DCM called TRIAD.

TRIAD Algorithm (25.2.4)

Consider two vectors ($\underrightarrow{s}^1$, $\underrightarrow{s}^2$) measured in $\mathcal{F}_b$ ($\underline{s}_b^1$, $\underline{s}_b^2$). These measurements could come from a sun sensor and a magnetometer, or 2 star tracker measurements, etc, as long as $\underrightarrow{s}^1$ and $\underrightarrow{s}^2$ are not parallel and $\underline{s}_a^1$ and $\underline{s}_a^2$ are known.

Normalize vectors and measurements:

$$\underrightarrow{\hat{s}}^1 = \frac{\underrightarrow{s}^1}{\|\underrightarrow{s}^1\|_2} = \underrightarrow{\mathcal{F}}_a^T \underline{\hat{s}}_a^1 = \underrightarrow{\mathcal{F}}_b^T \underline{\hat{s}}_b^1$$

$$\underrightarrow{\hat{s}}^2 = \frac{\underrightarrow{s}^2}{\|\underrightarrow{s}^2\|_2} = \underrightarrow{\mathcal{F}}_a^T \underline{\hat{s}}_a^2 = \underrightarrow{\mathcal{F}}_b^T \underline{\hat{s}}_b^1$$

Define

$$\underrightarrow{w}^1 = \underrightarrow{\hat{s}}^1 = \underrightarrow{\mathcal{F}}_a^T \underline{\hat{s}}_a^1 = \underrightarrow{\mathcal{F}}_b^T \underline{\hat{s}}_b^1$$

$$= \underrightarrow{\mathcal{F}}_a^T \underline{w}_a^1 = \underrightarrow{\mathcal{F}}_b^T \underline{w}_b^1$$

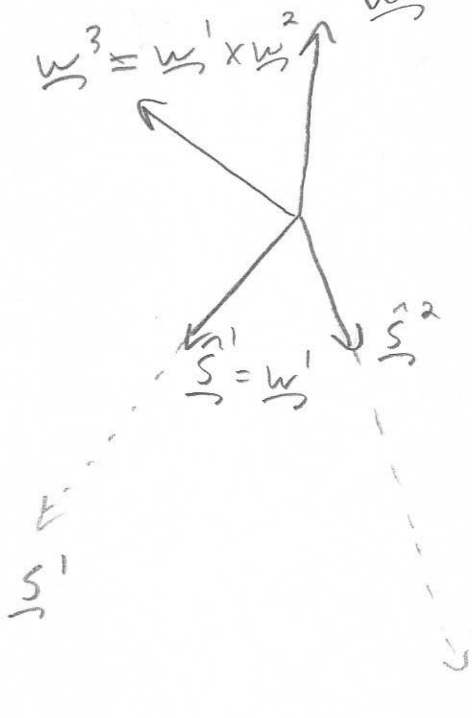
$$\underrightarrow{w}^2 = \frac{\underrightarrow{\hat{s}}^1 \times \underrightarrow{\hat{s}}^2}{\|\underrightarrow{s}^1 \times \underrightarrow{s}^2\|_2} = \underrightarrow{\mathcal{F}}_a^T \left(\frac{\underline{\hat{s}}_a^{1\times} \underline{\hat{s}}_a^2}{\|\underline{\hat{s}}_a^{1\times} \underline{\hat{s}}_a^2\|_2} \right) = \underrightarrow{\mathcal{F}}_b^T \left(\frac{\underline{\hat{s}}_b^{1\times} \underline{\hat{s}}_b^2}{\|\underline{\hat{s}}_b^{1\times} \underline{\hat{s}}_b^2\|_2} \right)$$

$$= \underrightarrow{\mathcal{F}}_a^T \underline{w}_a^2 = \underrightarrow{\mathcal{F}}_b^T \underline{w}_b^2$$

$$\vec{w}^3 = \vec{w}^1 \times \vec{w}^2 = \vec{\mathcal{F}}_a^T \underline{w}_a^{1\times} \underline{w}_a^2 = \vec{\mathcal{F}}_b^T \underline{w}_b^{1\times} \underline{w}_b^2$$

$$= \vec{\mathcal{F}}_a^T \underline{w}_a^3 = \vec{\mathcal{F}}_b^T \underline{w}_b^3$$

$$\vec{w}^2 = \frac{\hat{\vec{s}}^1 \times \hat{\vec{s}}^2}{\|\hat{\vec{s}}^1 \times \hat{\vec{s}}^2\|_2}$$



Let $\underrightarrow{\mathcal{F}}_w$ be intermediate frame defined by $\underrightarrow{w}^1, \underrightarrow{w}^2, \underrightarrow{w}^3$

$$\underrightarrow{\mathcal{F}}_w^T = \begin{bmatrix} \underrightarrow{w}^1 & \underrightarrow{w}^2 & \underrightarrow{w}^3 \end{bmatrix}$$

$$= \begin{bmatrix} \underrightarrow{\mathcal{F}}_a^T \underline{w}_a^1 & \underrightarrow{\mathcal{F}}_a^T \underline{w}_a^2 & \underrightarrow{\mathcal{F}}_a^T \underline{w}_a^3 \end{bmatrix}$$

$$= \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \underline{w}_a^1 & \underline{w}_a^2 & \underline{w}_a^3 \end{bmatrix}$$

Also,

$$\underrightarrow{\mathcal{F}}_w^T = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} \underline{w}_b^1 & \underline{w}_b^2 & \underline{w}_b^3 \end{bmatrix}$$

Recall

$$\underline{C}_{aw} = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_w^T$$

$$= \underbrace{\underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_a^T}_{\underline{1}} \begin{bmatrix} \underline{w}_a^1 & \underline{w}_a^2 & \underline{w}_a^3 \end{bmatrix}$$

$$\underline{C}_{aw} = \begin{bmatrix} \underline{w}_a^1 & \underline{w}_a^2 & \underline{w}_a^3 \end{bmatrix}$$

Similarly, $\underline{C}_{bw} = \begin{bmatrix} \underline{w}_b^1 & \underline{w}_b^2 & \underline{w}_b^3 \end{bmatrix}$

Then

$$\underline{C}_{ba} = \underline{C}_{bw} \underline{C}_{wa} = \underline{C}_{bw} \underline{C}_{aw}^T$$

$\Rightarrow \underline{C}_{ba}$ can be calculated given $\underline{s}_a^1, \underline{s}_a^2, \underline{s}_b^1, \underline{s}_b^2$

Remarks:
- In reality $\underline{s}_b^1, \underline{s}_b^2$ will be noisy measurements, $\underline{s}_a^1, \underline{s}_a^2$ rely on accuracy of model. Therefore $\hat{\underline{C}}_{ba} = \hat{\underline{C}}_{bw} \hat{\underline{C}}_{wa}$ is only an estimate, not the true $\underline{C}_{ba}$.

- $\hat{\underline{C}}_{ba} = \hat{\underline{C}}_{bw} \hat{\underline{C}}_{wa}$ will always be a valid DCM based on method we used to construct it.

- TRIAD can only use 2 vector measurements.
- TRIAD does not take into account estimates of $\hat{\underline{C}}_{ba}$ at previous instances in time, and does not consider accuracy or noise (σ_K) of measurements.