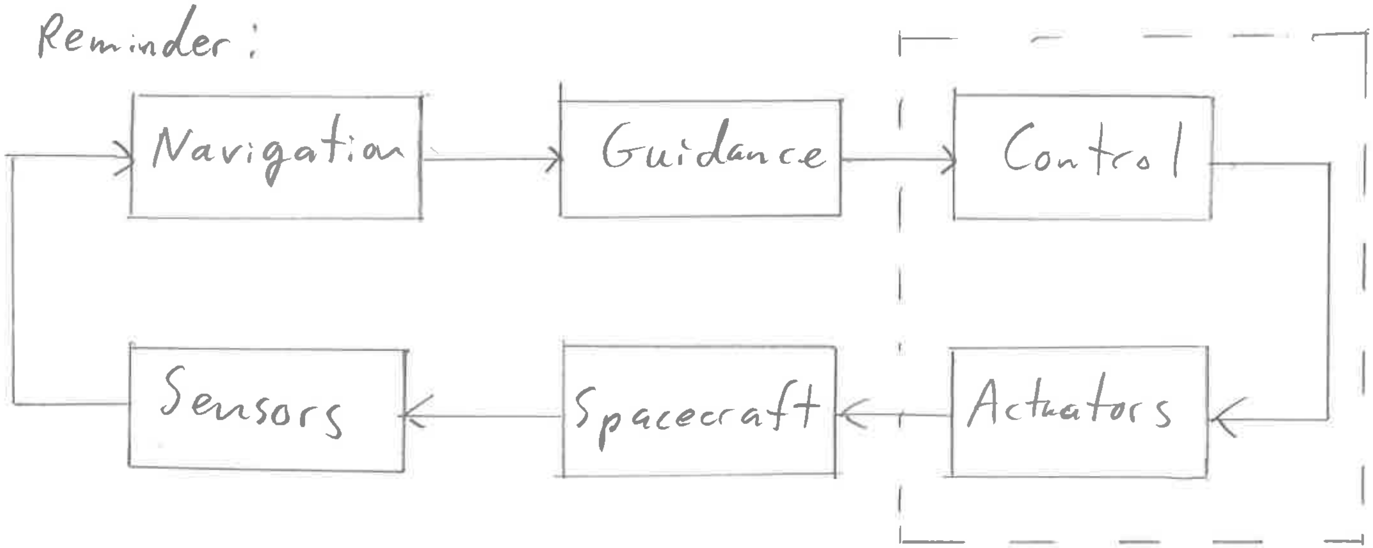
Active Spacecraft Attitude Control (Ch. 17-23)

Passive attitude stabilization methods are useful, but typically cannot achieve accurate attitude control (e.g. $\theta \sim \pm 20^\circ$). Active control (feedback control) is required in many applications.

Reminder:



Navigation: Where am I?

Guidance: Where do I want to go?

Control: How do I get there?

Let $\underline{I}_b^B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$, $\underline{\omega}_b^{ba} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$\underline{\tau}_b^{Bc} = \begin{bmatrix} \tau_x^c + \tau_x^d \\ \tau_y^c + \tau_y^d \\ \tau_z^c + \tau_z^d \end{bmatrix}$ τ_i^c: control torques

τ_i^d: disturbance torques

Euler's Eqn: $I_x \dot{\omega}_x + (I_z - I_y)\omega_y \omega_z = \tau_x^c + \tau_x^d$

$$I_y \dot{\omega}_y + (I_x - I_z)\omega_x \omega_z = \tau_y^c + \tau_y^d$$

$$I_z \dot{\omega}_z + (I_y - I_x)\omega_x \omega_y = \tau_z^c + \tau_z^d$$

Assume small $\omega_x, \omega_y, \omega_z$, then $\omega_i \omega_j \approx 0$

$$I_x \dot{\omega}_x = \tau_x^c + \tau_x^d$$
$$I_y \dot{\omega}_y = \tau_y^c + \tau_y^d$$
$$I_z \dot{\omega}_z = \tau_z^c + \tau_z^d$$

Assume $\underline{C}_{ba}$ is parameterized by a 3-2-1 rotation sequence ψ, θ, ϕ

Then $\underline{\omega}_b^{ba} \approx \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \Rightarrow \dot{\underline{\omega}}_b^{ba} \approx \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix}$

Then

$$I_x \ddot{\phi} = \tau_x^c + \tau_x^d \quad ①$$

$$I_y \ddot{\theta} = \tau_y^c + \tau_y^d \quad ②$$

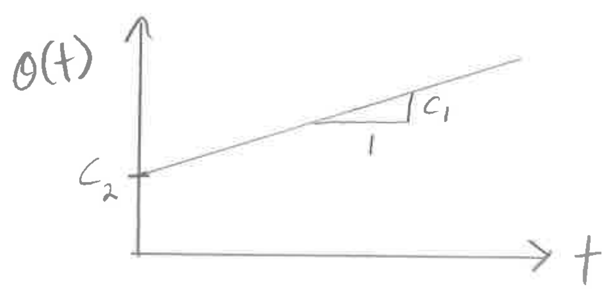
$$I_z \ddot{\psi} = \tau_z^c + \tau_z^d \quad ③$$

All 3 equations have same form. Let's examine generic version given by

$$I \ddot{\theta} = \tau^c + \tau^d$$

If $\tau^c = \tau^d = 0$ (no control torque or disturbance torque), then

$$I \ddot{\theta} = 0 \implies \dot{\theta}(t) = c_1 \implies \theta(t) = c_1 t + c_2$$



If $\theta(0) = \theta_0$
$\dot{\theta}(0) = \dot{\theta}_0$

$$\theta(t) = \dot{\theta}_0 t + \theta_0$$

If $\dot{\theta}_0 \neq 0$, then $\theta(t) \to \infty$ as $t \to \infty$ and rotation is unstable.

If $\dot{\theta}_0 = 0$, then $\theta(t) = \theta_0$ for all t and rotation is stable.

Since there is an initial condition for which $\theta(t) \to \infty$ as $t \to \infty$, we say the system is <u>unstable</u>.

How do we get $\theta(t) \to 0$ as $t \to \infty$?

Or what if we want $\theta(t) \to \theta_d$ as $t \to \infty$?

Use Feedback control!

Option 1: Proportional "p" Control

Let $\tau^c = K_p(\theta_d - \theta)$

If $\theta_d = 0$, Then $\tau^c = -K_p \theta$

where $K_p > 0$ is known as <u>proportional gain</u>

Assume $\theta_d = 0$, $\tau^d = 0$, then

$$I\ddot{\theta} = -K_p \theta \quad \text{or} \quad I\ddot{\theta} + K_p \theta = 0$$

This is a 2nd order linear ODE. Solution has the form $\theta(t) = Ae^{\lambda t}$. Sub into ODE

$$\dot{\theta}(t) = \lambda A e^{\lambda t}$$
$$\ddot{\theta}(t) = \lambda^2 A e^{\lambda t}$$

$$I(\lambda^2 A e^{\lambda t}) + K_p A e^{\lambda t} = 0$$
$$(I\lambda^2 + K_p) A e^{\lambda t} = 0$$

recall $K_p > 0$

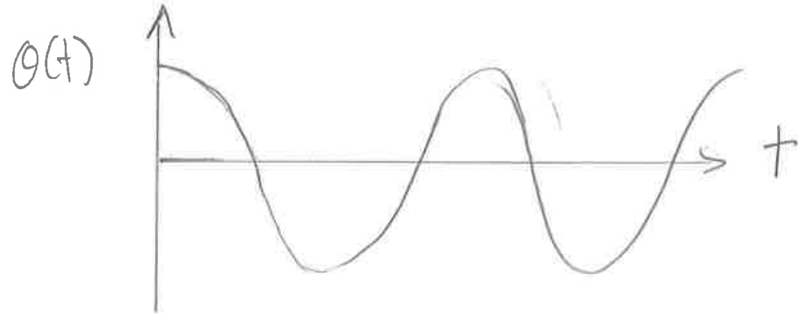
Therefore $I\lambda^2 + K_p = 0 \Rightarrow \lambda^2 = -\frac{K_p}{I}$

$$\lambda = \pm\sqrt{-\frac{K_p}{I}} = \pm\sqrt{\frac{K_p}{I}}\, i$$

$$\theta(t) = A_1 e^{\sqrt{\frac{K_p}{I}}\, i t} + A_2 e^{-\sqrt{\frac{K_p}{I}}\, i t}$$

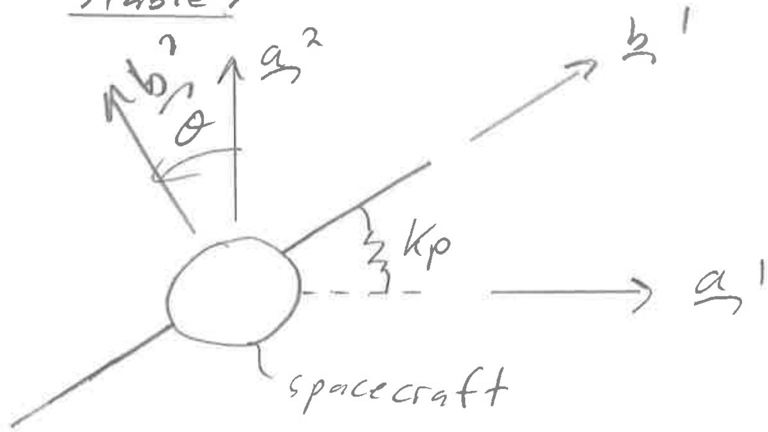
$\downarrow$ trig identities

$$= C_1 \cos\left(\sqrt{\tfrac{K_p}{I}}\, t\right) + C_2 \sin\left(\sqrt{\tfrac{K_p}{I}}\, t\right)$$



$\theta(t) \not\to 0$ as $t \to \infty$, but $\theta(t)$ remains bounded and system is <u>stable</u>.

Interpretation:



P control is like adding a spring with stiffness K_p to spacecraft.

Option 2: Proportional - Derivative "PD" Control

Let $\tau^c = K_p(\theta_d - \theta) + K_d(\dot{\theta}_d - \dot{\theta})$

where $\dot{\theta}_d$ is desired value of $\dot{\theta}$ and $K_d > 0$ is known as <u>derivative gain</u>.

If $\theta_d = 0$, $\dot{\theta}_d = 0$, then

$$\tau^c = -K_p \theta - K_d \dot{\theta}$$

Assume $\theta_d = 0$, $\dot{\theta}_d = 0$, $\tau^d = 0$, then

$$I\ddot{\theta} = -K_p\theta - K_d\dot{\theta} \quad \text{or} \quad I\ddot{\theta} + K_d\dot{\theta} + K_p\theta = 0$$

Try $\theta(t) = Ae^{\lambda t}$

$$I(\lambda^2 Ae^{\lambda t}) + K_d(\lambda Ae^{\lambda t}) + K_p(Ae^{\lambda t}) = 0$$

$$(I\lambda^2 + K_d\lambda + K_p)Ae^{\lambda t} = 0$$

$$I\lambda^2 + K_d\lambda + K_p = 0$$

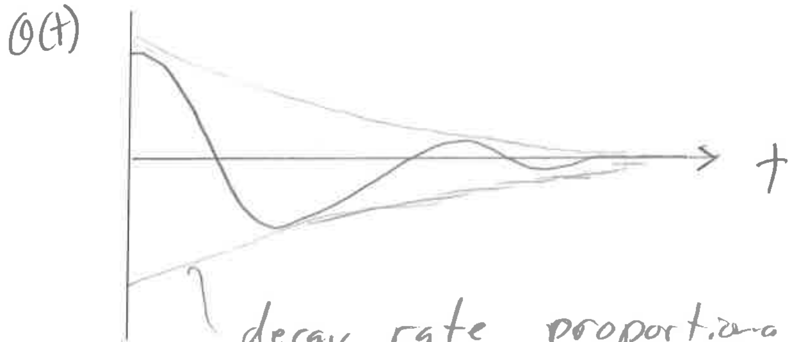
$$\lambda = \frac{-K_d}{2I} \pm \frac{1}{2I}\sqrt{K_d^2 - 4IK_p}$$

$\rightarrow$ If $K_d^2 < 4IK_p$, then $\frac{1}{2I}\sqrt{K_d^2 - 4IK_p} = \omega i$

$$\theta(t) = A_1 e^{\left(-\frac{K_d}{2I} + \omega i\right)t} + A_2 e^{\left(-\frac{K_d}{2I} - \omega i\right)t}$$

$$= e^{-\frac{K_d}{2I}t}\left(C_1\cos(\omega t) + C_2\sin(\omega t)\right)$$

Since $\frac{K_d}{2I} > 0$, then $\theta(t) \to 0$ as $t \to \infty$



decay rate proportional to $\frac{K_d}{2I}$

$\uparrow K_d$ and $\theta(t) \to 0$ faster

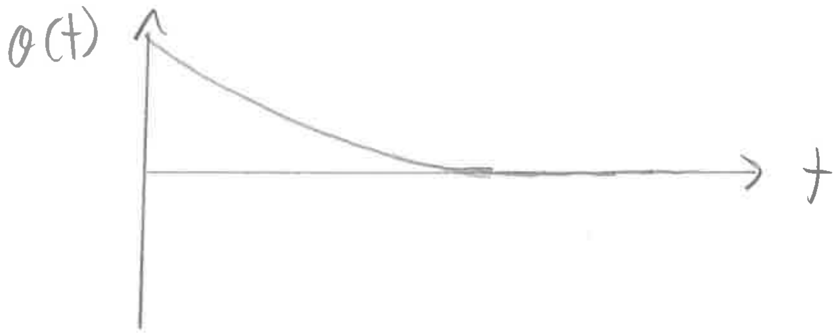
This is known as an underdamped response.

→ If $K_d^2 > 4IK_p$, then $\frac{1}{2I}\sqrt{K_d^2 - 4IK_p} > 0$

and $\frac{1}{2I}\sqrt{K_d^2 - 4IK_p} < \frac{K_d}{2I}$

$$\theta(t) = c_1 e^{\left(-\frac{K_d}{2I} + \frac{1}{2I}\sqrt{K_d^2 - 4IK_p}\right)t} + c_2 e^{\left(-\frac{K_d}{2I} - \frac{1}{2I}\sqrt{K_d^2 - 4IK_p}\right)t}$$

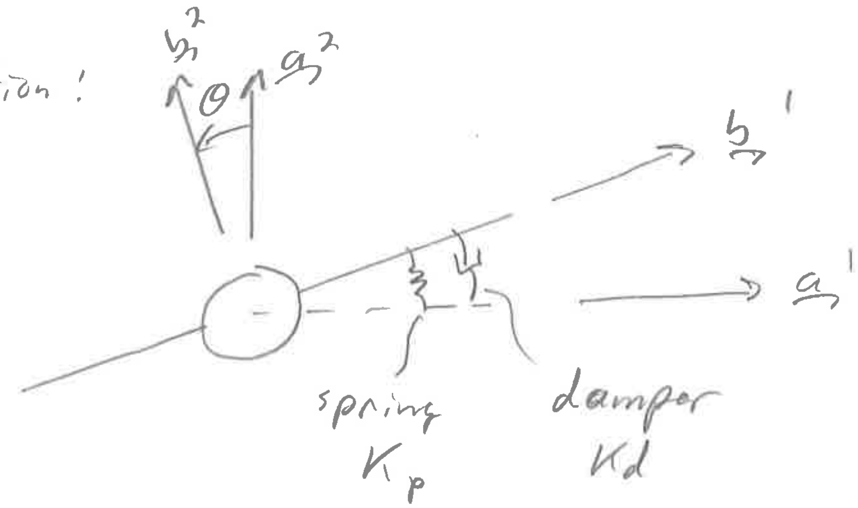
$\theta(t) \to 0$ as $t \to \infty$



This is known as an overdamped response.

Notice that in both cases $\theta(t) \to 0$ regardless of value of I. Therefore, this PD control law works even if we do not know the spacecraft inertia!

Interpretation:



PD control is like adding a spring of stiffness K_p and a damper with coefficient K_d to spacecraft.

If $\theta_d \neq 0$, then define $\tilde{\theta} = \theta - \theta_d$

Assume $\dot{\theta}_d = 0$, $\dot{\tilde{\theta}} = \dot{\theta} - \dot{\theta}_d = \dot{\theta}$

$$\ddot{\tilde{\theta}} = \ddot{\theta} - \ddot{\theta}_d = \ddot{\theta}$$

If $\tau^d = 0$, then $\tau^c = K_p \underbrace{(\theta_d - \theta)}_{-\tilde{\theta}} - K_d \underbrace{\dot{\theta}}_{\dot{\tilde{\theta}}}$

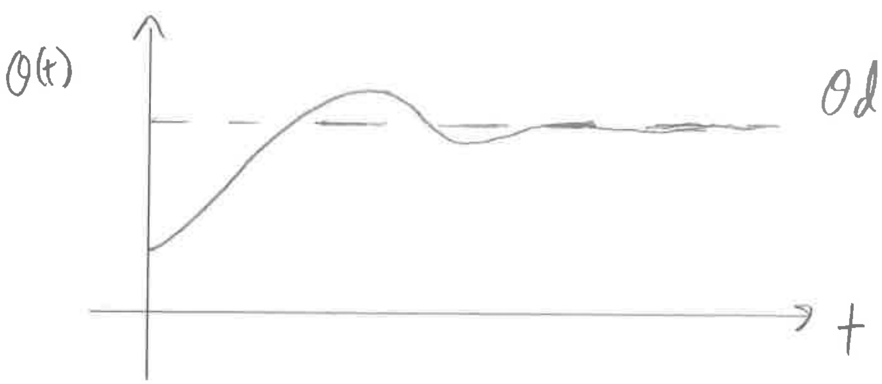
$$I \ddot{\tilde{\theta}} + K_d \dot{\tilde{\theta}} + K_p \tilde{\theta} = 0$$

From previous analysis,

$$\tilde{\theta}(t) = \theta(t) - \theta_d(t) \longrightarrow 0 \quad \text{as } t \to \infty$$

which implies $\theta(t) \to \theta_d(t)$ as $t \to \infty$.

Therefore, we can track a desired attitude with PD control.



What if $\tau^d \neq 0$?

$$I\ddot{\tilde{\theta}} + K_d\dot{\tilde{\theta}} + K_p\tilde{\theta} = \tau^d$$

Need to find particular solution for $\tilde{\theta}(t)$

Assume τ^d is a constant. Try $\tilde{\theta}_p(t) = A$

Sub into EoM: $\qquad \dot{\tilde{\theta}}_p(t) = \ddot{\tilde{\theta}}_p(t) = 0$

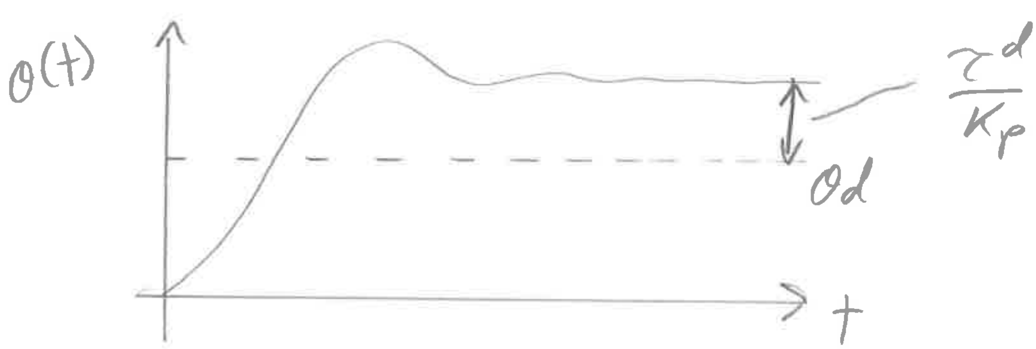
$$K_p A = \tau^d$$

$$A = \frac{\tau^d}{K_p}$$

Assume $K_d^2 < 4IK_p$ (could show other case as well)

$$\tilde{\theta}(t) = e^{-\frac{K_d}{I}t}\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right) + \frac{\tau^d}{K_p}$$

As $t \to \infty$, $e^{-\frac{K_d}{I}t} \to 0$ and $\tilde{\theta}(t) \to \frac{\tau^d}{K_p}$



If $K_p \to \infty$ (not possible in practice with real sensors and actuators), then $\theta(t) \to \theta_d$, but otherwise there is a steady-state error of $-\frac{\tau^d}{K_p}$

Option 3: Proportional-Integral-Derivative "PID" Control

Let $\tau^c = K_p(\theta_d - \theta) + K_i \int_0^t (\theta_d - \theta)\, d\tau + K_d(\dot{\theta}_d - \dot{\theta})$

where $K_i > 0$ is the <u>integral gain</u>.

or $\tau^c = -K_p \tilde{\theta} - K_i \int_0^t \tilde{\theta}\, d\tau - K_d \dot{\tilde{\theta}}$

Sub into EoM:

$$I \ddot{\tilde{\theta}} + K_d \dot{\tilde{\theta}} + K_p \tilde{\theta} + K_i \int_0^t \tilde{\theta}\, d\tau = \tau^d$$

Take derivative of EoM:

$$I \tilde{\theta}^{(3)} + K_d \ddot{\tilde{\theta}} + K_p \dot{\tilde{\theta}} + K_i \tilde{\theta} = \cancel{\dot{\tau}^d} \rightarrow 0$$

Assume τ^d is constant.

Try $\tilde{\theta}(t) = A e^{\lambda t}$, $\dot{\tilde{\theta}}(t) = \lambda A e^{\lambda t}$; $\ddot{\tilde{\theta}}(t) = \lambda^2 A e^{\lambda t}$

$\tilde{\theta}^{(3)}(t) = \lambda^3 A e^{\lambda t}$

$$(I\lambda^3 + K_d \lambda^2 + K_p \lambda + K_i) A e^{\lambda t} = 0$$

$$I\lambda^3 + K_d \lambda^2 + K_p \lambda + K_i = 0 \quad (*)$$

Recall that $Re(\lambda_i) < 0$ for all λ_i for asymptotic stability ($\tilde{\theta}(t) \to 0$ as $t \to \infty$).

($\dot{\tilde{\theta}}(t) \to 0$ as $t \to \infty$)

We could evaluate solutions λ_i to (*), or there is a useful tool called <u>Routh's Stability Criterion</u> that states $Re(\lambda_i) < 0$ if and only if

$$I > 0 \quad \checkmark$$

$$K_d > 0 \quad \checkmark$$

$$\frac{K_p K_d - I K_i}{K_d} = K_p - I\frac{K_i}{K_d} > 0$$

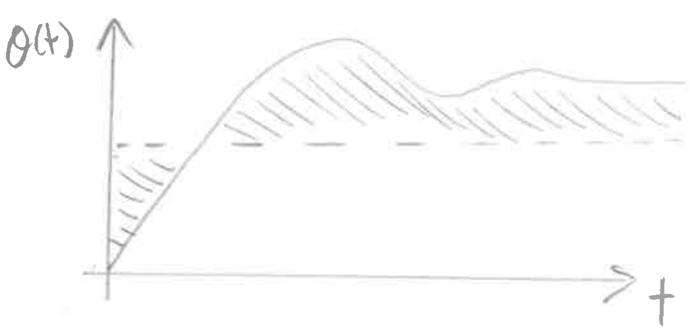
$$K_i > 0 \quad \checkmark$$

Control gains must satisfy $K_p - I\frac{K_i}{K_d} > 0$ for asymptotic stability. Notice that this depends on I, so stability depends on knowledge of spacecraft inertia!

What happens to τ^d?

We showed $\tilde{\theta}(t) \to 0$ and $\dot{\tilde{\theta}}(t) \to 0$ in the presence of a constant τ^d.

Think of integral control as an input that is proportional to accumulated error:



As $\int_0^t \tilde{\theta}(\tau)d\tau$ grows effect of $-K_i\int_0^t \tilde{\theta}\, d\tau$ in control law grows to the point where it cancels out effect of τ^d. $\uparrow K_i$ increases rate at which this occurs, but must also satisfy $K_p - I\frac{K_i}{K_d} > 0$.

PID Attitude Control with Reaction Wheels

Consider spacecraft equipped with 3 reaction wheels whose spin axes are $\underline{b}^1, \underline{b}^2, \underline{b}^3$. On HW6 derived EoMs:

$$\underline{I}_b^B \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times}\left(\underline{I}_b^B \underline{\omega}_b^{ba} + \underline{h}_b^{wheels}\right) = \underline{\tau}_b^{B,wheels} + \underline{\tau}_b^{B,d}$$

$$\dot{\underline{h}}_b^{wheels} = -\underline{\tau}_b^{B,wheels}$$

Let $\underline{I}_b^B = diag\{I_x, I_y, I_z\}$, $\underline{\omega}_b^{ba} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$$\underline{h}_b^{wheels} = \begin{bmatrix} I_s \dot{\gamma}_1 \\ I_s \dot{\gamma}_2 \\ I_s \dot{\gamma}_3 \end{bmatrix}, \quad \underline{\tau}_b^{B,wheels} = \begin{bmatrix} \tau_x^{wheels} \\ \tau_y^{wheels} \\ \tau_z^{wheels} \end{bmatrix}, \quad \underline{\tau}_b^{B,d} = \begin{bmatrix} \tau_x^d \\ \tau_y^d \\ \tau_z^d \end{bmatrix}$$

Sub into EoMs and linearize about $\omega_x = \omega_y = \omega_z = 0$
$\dot{\gamma}_1 = \dot{\gamma}_2 = \dot{\gamma}_3 = 0$

$$I_x \dot{\omega}_x = \tau_x^{wheels} + \tau_x^d$$

$$I_y \dot{\omega}_y = \tau_y^{wheels} + \tau_y^d$$

$$I_z \dot{\omega}_z = \tau_z^{wheels} + \tau_z^d$$

$$I_s \ddot{\gamma}_1 = -\tau_x^{wheels}$$

$$I_s \ddot{\gamma}_2 = -\tau_y^{wheels}$$

$$I_s \ddot{\gamma}_3 = -\tau_z^{wheels}$$

Look at one axis:

$$I_y \dot{\omega}_y = \tau_y^{wheels} + \tau_y^{d}$$

$$I_s \ddot{\gamma}_2 = -\tau_y^{wheels}$$

Let $\omega_y = \dot{\theta}$, drop "y", "2" subscripts

$$I \ddot{\theta} = \tau^{wheel} + \tau^{d}$$

$$I_s \ddot{\gamma} = -\tau^{wheel}$$

assume constant disturbance torque

Say we want $\theta \to \theta_d$, $\dot{\theta} \to \dot{\theta}_d$.

Define

$$\tau^{wheel} = -K_p \tilde{\theta} - K_i \int_0^t \tilde{\theta}\, d\tau - K_d \dot{\tilde{\theta}},$$

where $\tilde{\theta} = \theta - \theta_d$, $\dot{\tilde{\theta}} = \dot{\theta} - \dot{\theta}_d$, $\ddot{\tilde{\theta}} = \ddot{\theta} - \cancel{\ddot{\theta}_d}$ assume zero

This gives

① $$I \ddot{\tilde{\theta}} + K_d \dot{\tilde{\theta}} + K_p \tilde{\theta} + K_i \int_0^t \tilde{\theta}\, d\tau = \tau^d$$

② $$I_s \ddot{\gamma} = K_p \tilde{\theta} + K_i \int_0^T \tilde{\theta}\, d\tau + K_d \dot{\tilde{\theta}}$$

From before we know $\tilde{\theta} \to 0$, $\dot{\tilde{\theta}} \to 0$ if

$I > 0$, $K_d > 0$, $K_i > 0$, $K_p - I \frac{K_i}{K_d} > 0$

what about ② ?

Since $\tilde{\theta} \to 0$, $\dot{\tilde{\theta}} \to 0$, then

$$I_s \ddot{\gamma} \to K_i \int_0^\infty \tilde{\theta} \, d\tau = \tau^d \quad \text{as } t \to \infty$$

So $\ddot{\gamma} \to \dfrac{\tau^d}{I_s}$ as $t \to \infty$

Integrate

$$\lim_{t \to \infty} \dot{\gamma} = \lim_{t \to \infty} \frac{\tau^d}{I_s} t = \infty$$

Wheels keep spinning up to cancel out constant disturbance force. This is why we need to perform momentum management.

Practical Issues with Feedback Control

(26.3 - 26.4)

We already discussed limitations related to physical actuators. In addition, we need to consider the following when implementing feedback control:

- Digitized Control

Our control law (possibly PID control) will most likely determine a continuous control torque $\tau^c(t)$ that we would like to implement. There are two issues with this:

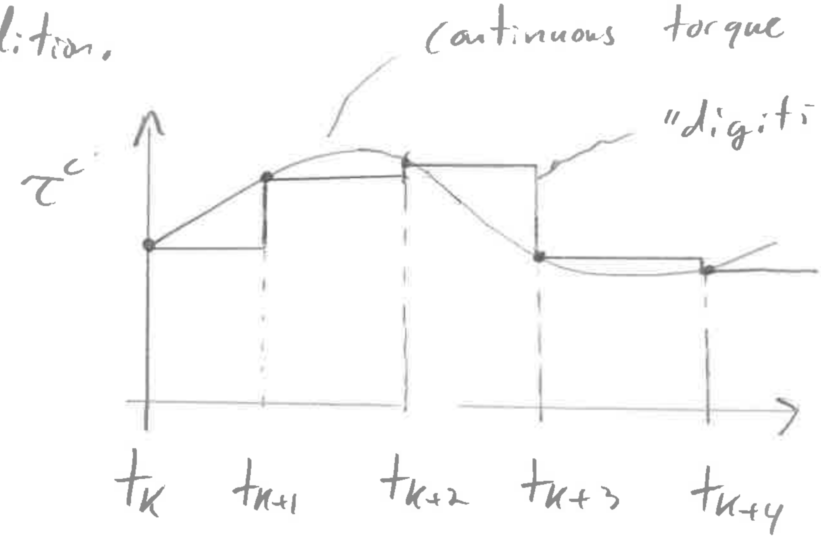
1) We can only take measurements from sensors at discrete instances in time (e.g. $\underline{\omega}_b^{ba}(t_k)$, $\underline{\omega}_b^{ba}(t_{k+1}), \ldots$)

2) We can only apply control torques over discrete intervals in time.

This is due to onboard digital computer taking a finite amount of time to process information and perform calculations.

We can "digitize" the control torque using what is called a <u>zero-order hold</u> (ZOH) condition.

Ex



The smaller the time interval, the better we can approximate continuous control torque.

Even though we can mathematically show stability with continuous control torque, the "digitized" control torque could render the system unstable if $\Delta T = t_{k+1} - t_k$ is too large!

- <u>Unmodeled Dynamics</u>

Real spacecraft are not true rigid bodies, and typically have flexible components and fuel that can slosh inside the fuel tank. Controllers that do not take these effect into consideration can render a spacecraft unstable! This motivates the need for robust control techniques and/or better modeling techniques.