(Forbes 4.1-4.3)

Systems of Multiple Particles (Continued)

Recall for a system $\mathcal{B}$ composed of ℓ particles:

Mass of $\mathcal{B}$: $m_{\mathcal{B}} = \sum_{i=1}^{\ell} m_i$

Center of mass of $\mathcal{B}$ relative to z: $\underset{\rightharpoonup}{r}^{cz} = \frac{1}{m_{\mathcal{B}}} \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{y_i z}$

Translational momentum of $\mathcal{B}$ relative to w wrt $\mathcal{F}_a$:

$$\underset{\rightharpoonup}{p}^{\mathcal{B}w/a} = \sum_{i=1}^{\ell} \underset{\rightharpoonup}{p}^{y_i w/a} = m_{\mathcal{B}} \underset{\rightharpoonup}{v}^{cw/a}$$

Force applied to $\mathcal{B}$: $\underset{\rightharpoonup}{f}^{\mathcal{B}} = \sum_{i=1}^{\ell} \underset{\rightharpoonup}{f}^{y_i}$

N2L: $\underset{\rightharpoonup}{f}^{\mathcal{B}} = \underset{\rightharpoonup}{p}^{\mathcal{B}w/a \, \cdot a} = m_{\mathcal{B}} \underset{\rightharpoonup}{v}^{cw/a \, \cdot a}$

where w is an unforced particle and $\mathcal{F}_a$ is an inertial frame.

We will now consider rotations of the system of particles.

Def: Consider point z, frame $\mathcal{F}_a$, and body $\mathcal{B}$ composed of particles $y_1, y_2, \ldots, y_\ell$. The <u>angular momentum</u> of $\mathcal{B}$ relative to z with respect to $\mathcal{F}_a$ is

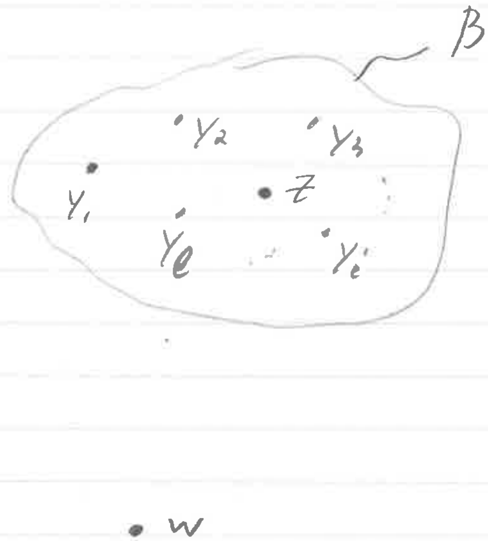
$$\underset{\rightharpoonup}{h}^{\mathcal{B}z/a} = \sum_{i=1}^{\ell} \underset{\rightharpoonup}{h}^{y_i z/a} = \sum_{i=1}^{\ell} \underset{\rightharpoonup}{r}^{y_i z} \times \underset{\rightharpoonup}{p}^{y_i z/a}$$

$$= \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{y_i z} \times \underset{\rightharpoonup}{v}^{y_i z/a}$$

We would like to derive a version of N2L for the rotational motion of body B.

First, consider N2L for an individual particle

$$m_i \underset{\rightharpoondown}{v}^{y_i w/a \cdot a} = \underset{\rightharpoondown}{f}^{y_i} + \sum_{j=1}^{\ell} \underset{\rightharpoondown}{f}^{y_i y_j} \quad (1)$$



Consider $\underset{\rightharpoondown}{r}^{y_i w} = \underset{\rightharpoondown}{r}^{y_i z} + \underset{\rightharpoondown}{r}^{z w}$

Take derivatives wrt $\mathcal{F}_a$ to get

$$\underset{\rightharpoondown}{v}^{y_i w/a \cdot a} = \underset{\rightharpoondown}{v}^{y_i z/a \cdot a} + \underset{\rightharpoondown}{v}^{z w/a \cdot a} \quad (2)$$

Substitute (2) into (1) to get

$$m_i \left(\underset{\rightharpoondown}{v}^{y_i z/a \cdot a} + \underset{\rightharpoondown}{v}^{z w/a \cdot a} \right) = \underset{\rightharpoondown}{f}^{y_i} + \sum_{j=1}^{\ell} \underset{\rightharpoondown}{f}^{y_i y_j}$$

or

$$(3) \quad m_i \underset{\rightharpoondown}{v}^{y_i z/a \cdot a} = \underset{\rightharpoondown}{f}^{y_i} + \sum_{j=1}^{\ell} \underset{\rightharpoondown}{f}^{y_i y_j} - m_i \underset{\rightharpoondown}{v}^{z w/a \cdot a}$$

Take derivative of $\vec{h}^{Bz/a}$ wrt $\mathcal{F}_a$

$$\vec{h}^{Bz/a \cdot a} = \left(\sum_{i=1}^{\ell} m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,y_i z/a}\right)^{\cdot a}$$

$$= \sum_{i=1}^{\ell}\left(m_i \vec{r}^{\,y_i z \cdot a} \times \vec{v}^{\,y_i z/a} + m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,y_i z/a \cdot a}\right)$$

$$= \sum_{i=1}^{\ell}\left(\cancelto{\vec{0}}{m_i \vec{v}^{\,y_i z/a} \times \vec{v}^{\,y_i z/a}} + m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,y_i z/a \cdot a}\right)$$

$$= \sum_{i=1}^{\ell} m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,y_i z/a \cdot a} \qquad (4)$$

Substitute (3) into (4) to get

$$\vec{h}^{Bz/a \cdot a} = \sum_{i=1}^{\ell} \vec{r}^{\,y_i z} \times \left(\vec{f}^{\,y_i} + \sum_{j=1}^{\ell} \vec{f}^{\,y_i y_j} - m_i \vec{v}^{\,zw/a \cdot a}\right)$$

$$= \sum_{i=1}^{\ell}\left(\vec{r}^{\,y_i z} \times \vec{f}^{\,y_i} + \sum_{j=1}^{\ell} \vec{r}^{\,y_i z} \times \vec{f}^{\,y_i y_j} - m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,zw/a \cdot a}\right)$$

$$= \underbrace{\sum_{i=1}^{\ell} \vec{r}^{\,y_i z} \times \vec{f}^{\,y_i}}_{\vec{m}^{Bz}} + \underbrace{\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} \vec{r}^{\,y_i z} \times \vec{f}^{\,y_i y_j}}_{\vec{0} \text{ since } \vec{f}^{\,y_i y_j} = -\vec{f}^{\,y_j y_i}}$$

$$- \sum_{i=1}^{\ell} m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,zw/a \cdot a}$$

$$= \vec{m}^{Bz} - \sum_{i=1}^{\ell} m_i \vec{r}^{\,y_i z} \times \vec{v}^{\,zw/a \cdot a}$$

Def: The <u>first moment of mass</u> of $\mathcal{B}$ relative to z is

$$\underset{\rightharpoonup}{c}^{\mathcal{B}z} = \sum_{i=1}^{\ell} m_i \underset{\rightharpoonup}{r}^{\gamma_i z} = m_{\mathcal{B}} \underset{\rightharpoonup}{r}^{cz}$$

Therefore

$$\underset{\rightharpoonup}{h}^{\mathcal{B}z/a \cdot a} = \underset{\rightharpoonup}{m}^{\mathcal{B}z} - \underset{\rightharpoonup}{c}^{\mathcal{B}z} \times \underset{\rightharpoonup}{v}^{zw/a \cdot a}$$

or

$$\boxed{\underset{\rightharpoonup}{h}^{\mathcal{B}z/a \cdot a} + \underset{\rightharpoonup}{c}^{\mathcal{B}z} \times \underset{\rightharpoonup}{v}^{zw/a \cdot a} = \underset{\rightharpoonup}{m}^{\mathcal{B}z}}$$

This is Newton's 2nd Law for rotation of a discrete body of particles.

If we choose z to be the location of the center of mass, then

$$\underset{\rightharpoonup}{c}^{\mathcal{B}z} = m_{\mathcal{B}} \cancel{\underset{\rightharpoonup}{r}^{cz}}^{\,\underset{\rightharpoonup}{0}} = \underset{\rightharpoonup}{0}$$ and N2LR becomes

$$\boxed{\underset{\rightharpoonup}{h}^{\mathcal{B}c/a \cdot a} = \underset{\rightharpoonup}{m}^{\mathcal{B}c}}$$

Notice that the unforced particle w does not show up in Newton's 2nd Law for rotation of a body when $\underset{\rightharpoonup}{c}^{\mathcal{B}z} = \underset{\rightharpoonup}{0}$!

Kinetic and Potential Energy of a System of Particles (Forbes 4.3.4)

Def: Consider point w, frame $\mathcal{F}_a$, and body $\mathcal{B}$ composed of particles $y_1, y_2, \dots, y_\ell$. The **Kinetic energy** of body $\mathcal{B}$ relative to w wrt $\mathcal{F}_a$ is

$$T_{\mathcal{B}w/a} = \sum_{i=1}^{\ell} T_{y_i w/a} = \frac{1}{2}\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i w/a} \cdot \underset{\rightharpoondown}{v}^{y_i w/a}$$

Consider $\underset{\rightharpoondown}{r}^{y_i w} = \underset{\rightharpoondown}{r}^{y_i z} + \underset{\rightharpoondown}{r}^{zw}$

$$\underset{\rightharpoondown}{v}^{y_i w/a} = \underset{\rightharpoondown}{v}^{y_i z/a} + \underset{\rightharpoondown}{v}^{zw/a}$$

This gives

$$T_{\mathcal{B}w/a} = \frac{1}{2}\sum_{i=1}^{\ell} m_i \left(\underset{\rightharpoondown}{v}^{y_i z/a} + \underset{\rightharpoondown}{v}^{zw/a}\right) \cdot \left(\underset{\rightharpoondown}{v}^{y_i z/a} + \underset{\rightharpoondown}{v}^{zw/a}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{\ell} m_i \left(\underset{\rightharpoondown}{v}^{y_i z/a} \cdot \underset{\rightharpoondown}{v}^{y_i z/a} + 2\underset{\rightharpoondown}{v}^{y_i z/a} \cdot \underset{\rightharpoondown}{v}^{zw/a} + \underset{\rightharpoondown}{v}^{zw/a} \cdot \underset{\rightharpoondown}{v}^{zw/a}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i z/a} \cdot \underset{\rightharpoondown}{v}^{y_i z/a} + \underbrace{\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}}_{\underset{\rightharpoondown}{c}^{\mathcal{B}z}}{}^{y_i z/a} \cdot \underset{\rightharpoondown}{v}^{zw/a}$$

$$+ \frac{1}{2} m_{\mathcal{B}} \underset{\rightharpoondown}{v}^{zw/a} \cdot \underset{\rightharpoondown}{v}^{zw/a}$$

$$= \frac{1}{2}\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i z/a} \cdot \underset{\rightharpoondown}{v}^{y_i z/a} + \underset{\rightharpoondown}{c}^{\mathcal{B}z} \cdot \underset{\rightharpoondown}{v}^{zw/a}$$

$$+ \frac{1}{2} m_{\mathcal{B}} \underset{\rightharpoondown}{v}^{zw/a} \cdot \underset{\rightharpoondown}{v}^{zw/a}$$

If $z = c$, then $\underset{\rightharpoondown}{c}^{Bz} = \underset{\rightharpoondown}{0}$ and

$$T_{Bz/a} = \frac{1}{2} \sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{\gamma_i c/a} \cdot \underset{\rightharpoondown}{v}^{\gamma_i c/a} + \frac{1}{2} m_B \underset{\rightharpoondown}{v}^{cw/a} \cdot \underset{\rightharpoondown}{v}^{cw/a}$$

Def: Consider point w and body B composed of particles $\gamma_1, \gamma_2, \ldots, \gamma_\ell$. The <u>gravitational potential</u> of B relative to w is

$$V_{Bw} = \sum_{i=1}^{\ell} V_{\gamma_i w}(\underset{\rightharpoondown}{r}^{\gamma_i w}) = -\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{\gamma_i w}$$

$$= -\underset{\rightharpoondown}{g} \cdot \left(\sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{r}^{\gamma_i w} \right)$$

$$= -\underset{\rightharpoondown}{g} \cdot (m_B \underset{\rightharpoondown}{r}^{cw})$$

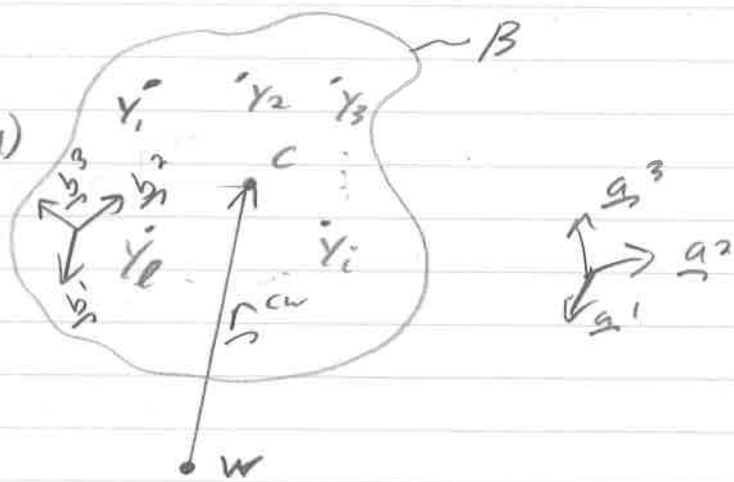
$$= -m_B \underset{\rightharpoondown}{g} \cdot \underset{\rightharpoondown}{r}^{cw}$$

Dynamics of a Discrete Rigid Body (DRB)

<u>Def</u>: A <u>discrete rigid body</u> (DRB) is a finite collection of particles such that the distance between each pair of particles is constant.

$\mathcal{F}_a$: inertial frame (e.g., fixed to ground)

$\mathcal{F}_b$: body frame fixed to DRB



If B is a DRB, then $\vec{r}^{\,r_i r_j \cdot b} = \vec{0}$, $\vec{r}^{\,r_i c \cdot b} = \vec{0}$

Let's consider N2L, where we have $\vec{v}^{\,cw/a}$ and $\vec{f}^{\,B}$ resolved in $\mathcal{F}_a$

$$\text{N2L} \rightarrow \quad \vec{f}^{\,B} = m_B \, \vec{v}^{\,cw/a \cdot a}$$

$$\underset{\rightarrow}{\mathcal{F}}_a^{T} \underline{f}_a^{B} = m_B \, \underset{\rightarrow}{\mathcal{F}}_a^{T} \dot{\underline{v}}_a^{cw/a}$$

Therefore

$$\boxed{m_B \, \dot{\underline{v}}_a^{cw/a} = \underline{f}_a^{B}}$$

This is the equation of motion that represents the translation of the DRB.

Let's consider N2L for rotation about $z = c$.

$$\underrightarrow{h}^{Bc/a \cdot a} = \underrightarrow{m}^{Bc}$$

Recall that $\underrightarrow{h}^{Bc/a} = \sum_{i=1}^{l} m_i \underrightarrow{r}^{r_i c} \times \underrightarrow{v}^{r_i c/a}$

and $\underrightarrow{v}^{r_i c/a} = \underrightarrow{r}^{r_i c \cdot a} = \overbrace{\cancel{\underrightarrow{r}^{r_i c \cdot b}}}^{\to \underrightarrow{0}\ (\text{DRB})} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}^{r_i c}$

so,

$$\underrightarrow{h}^{Bc/a} = \sum_{i=1}^{l} m_i \underrightarrow{r}^{r_i c} \times \left(\underrightarrow{\omega}^{ba} \times \underrightarrow{r}^{r_i c} \right)$$

Use fact that $\underrightarrow{u} \times \underrightarrow{v} = -\underrightarrow{v} \times \underrightarrow{u}$ to get

$$\underrightarrow{h}^{Bc/a} = -\sum_{i=1}^{l} m_i \underrightarrow{r}^{r_i c} \times \left(\underrightarrow{r}^{r_i c} \times \underrightarrow{\omega}^{ba} \right)$$

Let's resolve this in $\mathcal{F}_b$:

$$\underrightarrow{h}^{Bc/a} = \underrightarrow{\mathcal{F}}_b^{T} \left(-\sum_{i=1}^{l} m_i \underline{r}_b^{r_i c^\times} \underline{r}_b^{r_i c^\times} \underline{\omega}_b^{ba} \right)$$

Take derivative wrt $\mathcal{F}_a$:

$$\begin{aligned}
\underrightarrow{h}^{Bc/a \cdot a} &= \underrightarrow{h}^{Bc/a \cdot b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{h}^{Bc/a} \\
&= \underrightarrow{\mathcal{F}}_b^{T} \left(\dot{\underline{h}}_b^{Bc/a} + \underline{\omega}_b^{ba^\times} \underline{h}_b^{Bc/a} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{T} \Bigg(-\sum_{i=1}^{l} m_i \underline{r}_b^{r_i c^\times} \underline{r}_b^{r_i c^\times} \dot{\underline{\omega}}_b^{ba} \\
&\qquad + \underline{\omega}_b^{ba^\times} \left(-\sum_{i=1}^{l} m_i \underline{r}_b^{r_i c^\times} \underline{r}_b^{r_i c^\times} \underline{\omega}_b^{ba} \right) \Bigg)
\end{aligned}$$

Def: Consider point z and body $\mathcal{B}$ composed of particles $y_1, y_2, \ldots, y_\ell$. The <u>second moment of mass</u> of body $\mathcal{B}$ relative to z resolved in $\underrightarrow{\mathcal{F}}_b$ is defined as

$$\underline{J}_b^{\mathcal{B}z} = -\sum_{i=1}^{\ell} m_i \, \underline{r}_b^{y_i z^\times} \, \underline{r}_b^{y_i z^\times}$$

Therefore,

$$\underrightarrow{h}^{\mathcal{B}c/a \cdot a} = \underrightarrow{\mathcal{F}}_b^T \left(\underline{J}_b^{\mathcal{B}c} \, \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba} \right)$$

Applying N2LR gives

$$\underrightarrow{\mathcal{F}}_b^T \left(\underline{J}_b^{\mathcal{B}c} \, \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba} \right) = \underrightarrow{\mathcal{F}}_b^T \underline{m}_b^{\mathcal{B}c}$$

or

$$\boxed{\underline{J}_b^{\mathcal{B}c} \, \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba} = \underline{m}_b^{\mathcal{B}c}}$$

This is the equation of motion that describes the rotational motion of body $\mathcal{B}$ about point c, the center of mass of the DRB. (<u>Euler's Equation</u>)

Note that $\underline{J}_b^{\mathcal{B}c}$ is sometimes referred to as the <u>moment of inertia</u> of $\mathcal{B}$ relative to c resolved in frame $\underrightarrow{\mathcal{F}}_b$

This is not the same "moment of inertia" you most likely saw in statics. In statics you dealt with the "second moment of <u>area</u>", not the second moment of <u>mass</u>.

If we return to our definition for $\underset{\rightarrow}{h}^{Bc/a}$, we realize that

$$\underset{\rightarrow}{h}^{Bc/a} = \underset{\rightarrow}{\mathcal{F}}_b^T \left(-\sum_{i=1}^{\ell} m_i \underline{r}_b^{v_i c \times} \underline{r}_b^{v_i c \times}\right) \underline{\omega}_b^{ba}$$

$$= \underset{\rightarrow}{\mathcal{F}}_b^T \underline{J}_b^{Bc} \underline{\omega}_b^{ba},$$

or

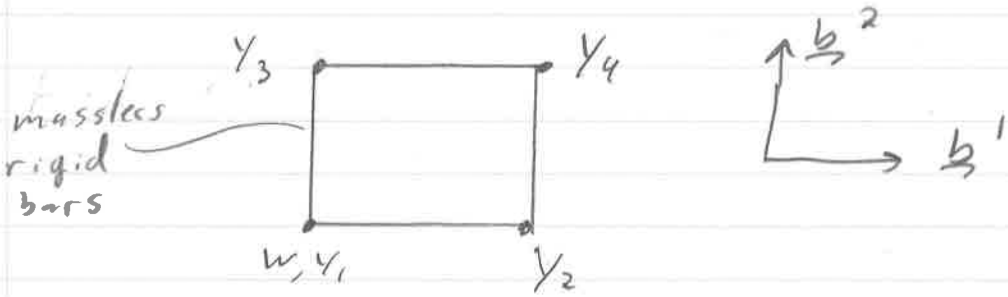
$$\underline{h}_b^{Bc/a} = \underline{J}_b^{Bc} \underline{\omega}_b^{ba},$$

where $\underline{J}_b^{Bc}$ is constant for a DRB.

Example: Consider a DRB composed of 4 particles of mass $m_i = m$ whose positions at a given instant in time are

$$\underrightarrow{r}^{y_1 w} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} m, \quad \underrightarrow{r}^{y_2 w} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} m$$

$$\underrightarrow{r}^{y_3 w} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} m, \quad \underrightarrow{r}^{y_4} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} m$$



Find $\underline{J}_b^{Bc}$.

Solution: $$m_B = \sum_{i=1}^{4} m_i = \sum_{i=1}^{4} m = 4m$$

$$\underrightarrow{r}^{cw} = \frac{1}{m_B} \sum_{i=1}^{4} m_i \underrightarrow{r}^{y_i w}$$

$$= \frac{\cancel{m}}{4\cancel{m}} \left(\underrightarrow{r}^{y_1 w} + \underrightarrow{r}^{y_2 w} + \underrightarrow{r}^{y_3 w} + \underrightarrow{r}^{y_4 w} \right)$$

$$= \frac{1}{4} \underrightarrow{\mathcal{F}}_b^T \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) m$$

$$= \frac{1}{4} \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} m = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} m$$

$$\underline{r}^{y_i c} = \underline{r}^{y_i w} - \underline{r}^{cw}$$

$$\underline{r}^{y_1 c} = \underline{\mathcal{F}}_b^T \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \end{bmatrix} m, \quad \underline{r}^{y_2 c} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} m$$

$$\underline{r}^{y_3 c} = \underline{\mathcal{F}}_b^T \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} m, \quad \underline{r}^{y_4 c} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} m$$

$$\underline{J}_b^{Bc} = - \sum_{i=1}^{4} m_i \underline{r}_b^{y_i c^\times} \underline{r}_b^{y_i c^\times}$$

$$= - m \left(\underline{r}_b^{y_1 c^\times} \underline{r}_b^{y_1 c^\times} + \underline{r}_b^{y_2 c^\times} \underline{r}_b^{y_2 c^\times} + \underline{r}_b^{y_3 c^\times} \underline{r}_b^{y_4 c^\times} + \underline{r}_b^{y_4 c^\times} \underline{r}_b^{y_4 c^\times} \right)$$

$$= - m \left(\begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 0 & -1/2 \\ 1/2 & +1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 0 & -1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \right.$$

$$+ \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

$$\left. + \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} \right) m^2$$

$$= -m\left(\begin{bmatrix} -1/4 & 1/4 & 0 \\ 1/4 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} + \begin{bmatrix} -1/4 & -1/4 & 0 \\ -1/4 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} -1/4 & -1/4 & 0 \\ -1/4 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} + \begin{bmatrix} -1/4 & 1/4 & 0 \\ 1/4 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}\right) m^2$$

$$= -m \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} m^2 = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} m^2$$

Say $m = 1\,kg$, then

$$\underline{J}_b^{Bc} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} kg \cdot m^2$$

3 Steps to Success for a DRB (rotations about c)

1) Kinematics
   i) Frames and DCMs
   ii) Angular Velocity
   iii) Positions
      - position of center of mass
      - positions of particles relative to center of mass
   iv) velocity
      - velocity of center of mass
   v) Acceleration
      - acceleration of center of mass

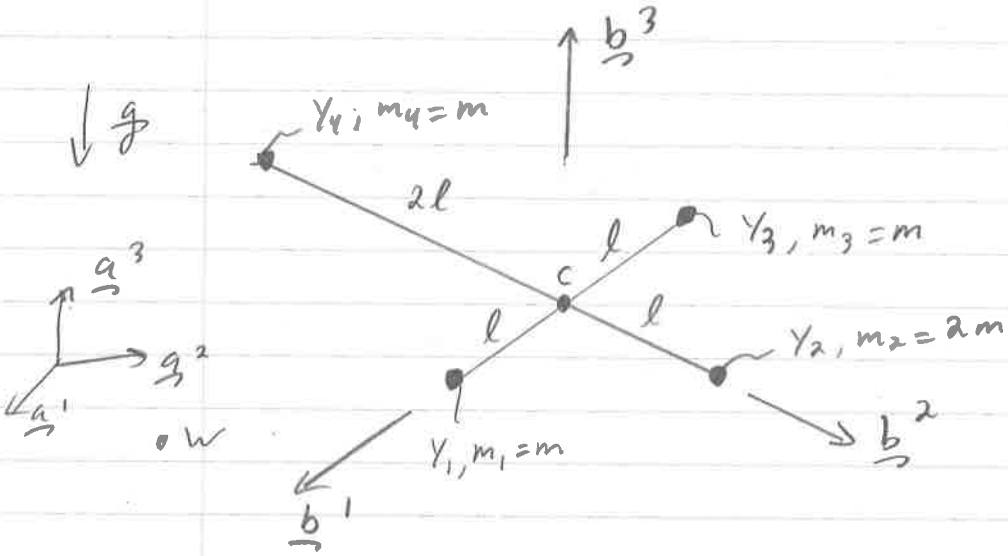
2) FBD of DRB
   - forces ($\underrightarrow{f}^{B}$)
   - moments ($\underrightarrow{m}^{Bc}$)

3) Newton's Equations
   i) angular momentum ($\underrightarrow{h}^{Bc/a} = \underrightarrow{\mathcal{F}}_b^{T} \underline{J}_b^{Bc} \underline{\omega}_b^{ba}$)
   ii) N2L ($\underrightarrow{f}^{B} = m_B \underrightarrow{a}^{cw/a/a}$)
   iii) N2LR ($\underrightarrow{h}^{Bc/a \cdot a} = \underrightarrow{m}^{Bc}$) or $\underrightarrow{h}$
      - Can use form of N2LR resolved in $\mathcal{F}_b$:

$$\underline{J}_b^{Bc} \dot{\underline{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times} \underline{J}_b^{Bc} \underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$$

(Euler's Equation)

Example: Consider a simplified DRB model of an aircraft constrained to pitching motion



$$\mathcal{F}_a \xrightarrow{\underline{C}_1(\theta)} \mathcal{F}_b \qquad \underline{C}_{ba} = \underline{C}_1(\theta)$$

Aerodynamic forces

$$\underset{\rightharpoondown}{f}^{y_1 a} = \underset{\rightharpoondown}{f}^{y_3 a} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -D^w/2 \\ L^w/2 \end{bmatrix}, \quad \underset{\rightharpoondown}{f}^{y_4 a} = \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -D^e \\ L^e \end{bmatrix}$$

$$\underset{\rightharpoondown}{r}^{cw} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$$

1) Kinematics

i) $\mathcal{F}_a \xrightarrow{\underline{C}_1(\theta)} \mathcal{F}_b \qquad \underline{C}_{ba} = \underline{C}_1(\theta)$

ii) $\underline{\omega}^{ba} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

iii) $\underrightarrow{r}^{cw} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$

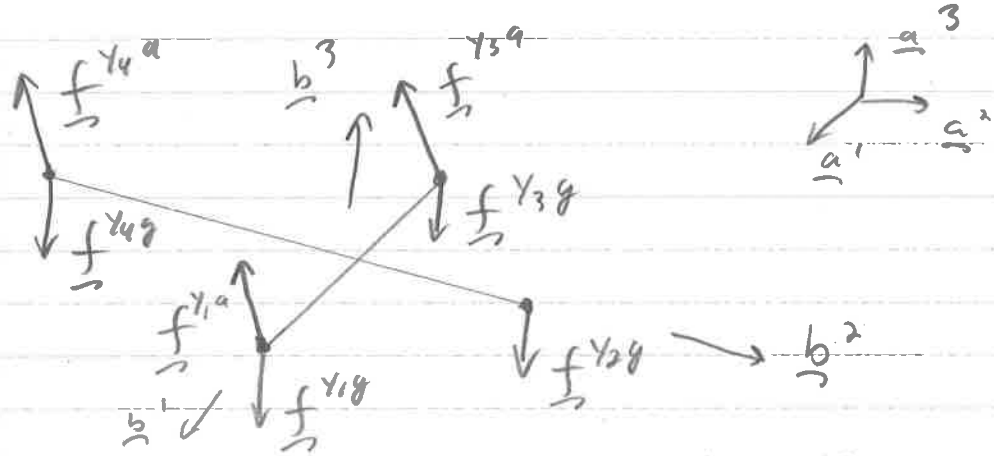
$\underrightarrow{r}^{y_1 c} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}, \quad \underrightarrow{r}^{y_2 c} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}$

$\underrightarrow{r}^{y_3 c} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} -l \\ 0 \\ 0 \end{bmatrix}, \quad \underrightarrow{r}^{y_4 c} = \underrightarrow{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -2l \\ 0 \end{bmatrix}$

iv) $\underrightarrow{v}^{cw/a} = \underrightarrow{r}^{cw^{\cdot a}} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix}$

v) $\underrightarrow{a}^{cw/a/a} = \underrightarrow{v}^{cw/a^{\cdot a}} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{bmatrix}$

2) FBD



i) $$\underline{f}^{Y_1 g} = \underline{f}^{Y_3 g} = \underline{f}^{Y_4 g} = \underline{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -mg\, s_\theta \\ -mg\, c_\theta \end{bmatrix}$$

$$\underline{f}^{Y_2 g} = \underline{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ 0 \\ -2mg \end{bmatrix} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -2mg\, s_\theta \\ -2mg\, c_\theta \end{bmatrix}$$

$$\underline{f}^{Y_1 a} = \underline{f}^{Y_3 a} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -D^w/2 \\ L^w/2 \end{bmatrix} =$$

$$\underline{f}^{Y_4 a} = \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ -D^e \\ L^e \end{bmatrix}$$

$$\underline{f}^B = \sum_{i=1}^{4} \underline{f}^{Y_i g} + \underline{f}^{Y_1 a} + \underline{f}^{Y_3 a} + \underline{f}^{Y_4 a}$$

$$= \underline{\mathcal{F}}_a^T \left(\begin{bmatrix} 0 \\ 0 \\ -5mg \end{bmatrix} + \underline{C}_1^T(\theta) \begin{bmatrix} 0 \\ -(D^w + D^e) \\ L^w + L^e \end{bmatrix} \right)$$

$$= \underline{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ -(D^w + D^e)\, c_\theta - (L^w + L^e)\, s_\theta \\ -(D^w + D^e)\, s_\theta + (L^w + L^e)\, c_\theta - 5mg \end{bmatrix}$$

ii)

$$\underset{\rightharpoondown}{m}^{Bc} = \sum_{i=1}^{4} \underset{\rightharpoondown}{r}^{y_i c} \times \underset{\rightharpoondown}{f}^{y_i}$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \Big(\underline{r}_b^{y_1 c \times} \left(\underline{f}_b^{y_1 a} + \underline{f}_b^{y_1 g}\right) + \underline{r}_b^{y_2 c \times} \underline{f}_b^{y_2 g}$$

$$+ \underline{r}_b^{y_3 c \times} \left(\underline{f}_b^{y_3 a} + \underline{f}_b^{y_3 g}\right) + \underline{r}_b^{y_4 c \times} \left(\underline{f}_b^{y_4 a} + \underline{f}_b^{y_4 g}\right) \Big)$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l \\ 0 & l & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -D^w/2 - mg\,s_\theta \\ L^w/2 - mg\,c_\theta \end{bmatrix} + \begin{bmatrix} 0 & 0 & l \\ 0 & 0 & 0 \\ -l & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2mg\,s_\theta \\ -2mg\,c_\theta \end{bmatrix} \right.$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & l \\ 0 & -l & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -D^w/2 - mg\,s_\theta \\ L^w/2 - mg\,c_\theta \end{bmatrix}$$

$$\left. + \begin{bmatrix} 0 & 0 & -2l \\ 0 & 0 & 0 \\ +2l & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -D^e - mg\,s_\theta \\ L^e - mg\,c_\theta \end{bmatrix} \right)$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \left(\begin{bmatrix} 0 \\ -l/2\, L^w + mgl\,c_\theta \\ -l/2\, D^w - mgl\,s_\theta \end{bmatrix} + \begin{bmatrix} -2mgl\,c_\theta \\ 0 \\ 0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 \\ l/2\, L^w - mgl\,c_\theta \\ l/2\, D^w + mgl\,s_\theta \end{bmatrix} + \begin{bmatrix} -2lL^e + 2mgl\,c_\theta \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \underset{\rightharpoondown}{\mathcal{F}}_b^T \begin{bmatrix} -2lL^e \\ 0 \\ 0 \end{bmatrix}$$

3)

i) $\underset{\sim}{h}^{Bc/a} = \underset{\rightarrow}{\mathcal{F}}_b^T \, \underline{J}_b^{Bc} \, \underline{\omega}_b^{ba}$

$$\underline{J}_b^{Bc} = -\sum_{i=1}^{4} m_i \underline{r}_b^{v_ic\times} \underline{r}_b^{v_ic\times}$$

$$= -m\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\ell \\ 0 & \ell & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\ell \\ 0 & \ell & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell \\ 0 & -\ell & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell \\ 0 & -\ell & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2\ell \\ 0 & 0 & 0 \\ -2\ell & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2\ell \\ 0 & 0 & 0 \\ -2\ell & 0 & 0 \end{bmatrix}\right)$$

$$= -m\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\ell^2 & 0 \\ 0 & 0 & -\ell^2 \end{bmatrix} + 2\begin{bmatrix} -\ell^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\ell^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\ell^2 & 0 \\ 0 & 0 & -\ell^2 \end{bmatrix} + \begin{bmatrix} -4\ell^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4\ell^2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 6m\ell^2 & 0 & 0 \\ 0 & 2m\ell^2 & 0 \\ 0 & 0 & 8m\ell^2 \end{bmatrix}$$

$$\underline{h}_b^{Bc/a} = \underline{J}_b^{Bc}\,\underline{\omega}_b^{ba}$$

$$= \begin{bmatrix} 6ml^2 & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 8ml^2 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6ml^2\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

ii) N2L

$$\underrightarrow{f}^{B} = m_B\,\underrightarrow{a}^{cw/a/a}$$

$$\underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} 0 \\ -(D^w+D^e)c_\theta - (L^w+L^e)s_\theta \\ -(D^w+D^e)s_\theta + (L^w+L^e)c_\theta - 5mg \end{bmatrix}$$

$$= \underrightarrow{\mathcal{F}}_a^T\, 5m \begin{bmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{bmatrix}$$

Translational EoMs:

$$5m\ddot{x}_a = 0 \qquad (\ddot{x}_a = 0)$$

$$5m\ddot{y}_a + (D^w+D^e)c_\theta + (L^w+L^e)s_\theta = 0$$

$$5m\ddot{z}_a + (D^w+D^e)s_\theta - (L^w+L^e)c_\theta + 5mg = 0$$

N2LR

$$\underset{\sim}{h}^{Bc/c'\,a} = \underset{\sim}{m}^{Bc}$$

Use form resolved in $\mathcal{F}_b$ for DRB

$$\underline{J}_b^{Bc}\,\underline{\dot{\omega}}_b^{ba} + \underline{\omega}_b^{ba\times}\,\underline{J}_b^{Bc}\,\underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$$

$$\begin{bmatrix} 6ml^2 & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 8ml^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} 6ml^2 & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 8ml^2 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2lL^e \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6ml^2\ddot{\theta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} 6ml^2\dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2lL^e \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6ml^2\ddot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2lL^e \\ 0 \\ 0 \end{bmatrix}$$

Rotational EoM: $3\cancel{6}ml^2\ddot{\theta} + \cancel{2}lL^e = 0$

or $3ml\ddot{\theta} + L^e = 0$

Kinetic and Potential Energy of a DRB

Def: Consider point w, frame $\mathcal{F}_a$, and DRB $\mathcal{B}$ composed of particles, $y_1, y_2, \ldots, y_\ell$. The <u>Kinetic energy</u> of $\mathcal{B}$ relative to w wrt $\mathcal{F}_a$ is

$$T_{\mathcal{B}w/a} = \sum_{i=1}^{\ell} T_{y_i w/a} = \frac{1}{2} \sum_{i=1}^{\ell} m_i \underset{\rightharpoondown}{v}^{y_i w/a} \cdot \underset{\rightharpoondown}{v}^{y_i w/a}$$

Consider $\underset{\rightharpoondown}{r}^{y_i w} = \underset{\rightharpoondown}{r}^{y_i c} + \underset{\rightharpoondown}{r}^{cw}$

$$\underset{\rightharpoondown}{v}^{y_i w/a} = \underset{\rightharpoondown}{r}^{y_i w \cdot a} = \underset{\rightharpoondown}{r}^{y_i c \cdot a} + \underset{\rightharpoondown}{r}^{cw \cdot a}$$

$$= \underbrace{\cancel{\underset{\rightharpoondown}{r}^{y_i c \cdot b}}}_{\underset{\rightharpoondown}{0}} + \underset{\rightharpoondown}{\omega}^{ba} \times \underset{\rightharpoondown}{r}^{y_i c} + \underset{\rightharpoondown}{v}^{cw/a}$$

$$= \underset{\rightharpoondown}{v}^{cw/a} + \underset{\rightharpoondown}{\omega}^{ba} \times \underset{\rightharpoondown}{r}^{y_i c}$$

$$= \underset{\rightharpoondown}{v}^{cw/a} - \underset{\rightharpoondown}{r}^{y_i c} \times \underset{\rightharpoondown}{\omega}^{ba}$$

$$T_{\mathcal{B}w/a} = \frac{1}{2}\sum_{i=1}^{\ell} m_i \left(\underrightarrow{v}^{cw/a} - \underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right) \cdot \left(\underrightarrow{v}^{cw/a} - \underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right)$$

$$= \frac{1}{2}\sum_{i=1}^{\ell} m_i \Big(\underrightarrow{v}^{cw/a} \cdot \underrightarrow{v}^{cw/a} - \underrightarrow{v}^{cw/a} \cdot \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right)$$

$$- \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right) \cdot \underrightarrow{v}^{cw/a} + \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right) \cdot \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right) \Big)$$

$$= \frac{1}{2} m_{\mathcal{B}}\, \underrightarrow{v}^{cw/a} \cdot \underrightarrow{v}^{cw/a} - \frac{1}{2} \underrightarrow{v}^{cw/a} \cdot \Big(\underbrace{\sum_{i=1}^{\ell} m_i \underrightarrow{r}^{v_i c}}_{\underrightarrow{c}^{\mathcal{B}c} = \underrightarrow{0}} \times \underrightarrow{\omega}^{ba} \Big)$$

$$- \frac{1}{2} \Big(\underbrace{\sum_{i=1}^{\ell} m_i \underrightarrow{r}^{v_i c}}_{\underrightarrow{c}^{\mathcal{B}c} = \underrightarrow{0}} \times \underrightarrow{\omega}^{ba} \Big) \cdot \underrightarrow{v}^{cw/a} + \frac{1}{2} \sum_{i=1}^{\ell} \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right) \cdot \left(\underrightarrow{r}^{v_i c} \times \underrightarrow{\omega}^{ba} \right)$$

$$= \frac{1}{2} m_{\mathcal{B}}\, \underline{v}_a^{cw/a\,T} \underline{v}_a^{cw/a} + \frac{1}{2} \sum_{i=1}^{\ell} \left(\underline{r}_b^{v_i c \times} \underline{\omega}_b^{ba} \right)^T \left(\underline{r}_b^{v_i c \times} \underline{\omega}_b^{ba} \right)$$

$$= \frac{1}{2} m_{\mathcal{B}}\, \underline{v}_a^{cw/a\,T} \underline{v}_a^{cw/a} - \frac{1}{2} \sum_{i=1}^{\ell} \underline{\omega}_b^{ba\,T} \underline{r}_b^{v_i c \times} \underline{r}_b^{v_i c \times} \underline{\omega}_b^{ba}$$

$$= \frac{1}{2} m_{\mathcal{B}}\, \underline{v}_a^{cw/a\,T} \underline{v}_a^{cw/a} + \frac{1}{2} \underline{\omega}_b^{ba\,T} \underbrace{\Big(- \sum_{i=1}^{\ell} \underline{r}_b^{v_i c \times} \underline{r}_b^{v_i c \times} \Big)}_{\underline{J}_b^{\mathcal{B}c}} \underline{\omega}_b^{ba}$$

$$= \frac{1}{2} m_{\mathcal{B}}\, \underline{v}_a^{cw/a\,T} \underline{v}_a^{cw/a} + \frac{1}{2} \underline{\omega}_b^{ba\,T} \underline{J}_b^{\mathcal{B}c} \underline{\omega}_b^{ba}$$

$$= \frac{1}{2} \underbrace{\begin{bmatrix} \underline{v}_a^{cw/a\,T} & \underline{\omega}_b^{ba\,T} \end{bmatrix}}_{\underline{\nu}^{\mathcal{B}}} \underbrace{\begin{bmatrix} m_{\mathcal{B}} \underline{1} & 0 \\ 0 & \underline{J}_b^{\mathcal{B}c} \end{bmatrix}}_{\underline{M}^{\mathcal{B}c}} \underbrace{\begin{bmatrix} \underline{v}_a^{cw/a} \\ \underline{\omega}_b^{ba} \end{bmatrix}}_{\underline{\nu}^{\mathcal{B}}}$$

$$= \frac{1}{2} \underline{\nu}^{\mathcal{B}} \underline{M}^{\mathcal{B}c} \underline{\nu}^{\mathcal{B}}$$

Def: Consider point w and DRB $\mathcal{B}$ composed of particles $y_1, y_2, \ldots, y_\ell$. The <u>gravitational potential</u> of $\mathcal{B}$ relative to w is

$$V_{\mathcal{B}w} = \sum_{i=1}^{\ell} V_{y_i w} = -\sum_{i=1}^{\ell} m_i \underset{\sim}{g} \cdot \underset{\sim}{r}^{y_i w}$$

$$= -\underset{\sim}{g} \cdot \left(\sum_{i=1}^{\ell} m_i \underset{\sim}{r}^{y_i w} \right)$$

$$= -\underset{\sim}{g} \cdot \left(m_{\mathcal{B}} \underset{\sim}{r}^{cw} \right)$$

$$= -m_{\mathcal{B}} \underset{\sim}{g} \cdot \underset{\sim}{r}^{cw}$$