Second Moment of Mass as a Tensor

You may have noticed that we always resolve the second moment of mass in $\mathcal{F}_b$ (i.e., $\underline{J}_b^{Bc}$), We have not discussed a "physical vector" version of the second moment of mass.

Much like $\underline{r}_b$ is in fact the physical vector $\underrightarrow{r}$ resolved in $\mathcal{F}_b$, $\underline{J}_b^{Bc}$ is the <u>tensor</u> $\underrightarrow{J}^{Bc}$ resolved in $\mathcal{F}_b$.

<u>Def</u>: A second-order tensor $\underrightarrow{I}$ is a linear operator such that for all physical vectors $\underrightarrow{u}$,

$\underrightarrow{I} \cdot \underrightarrow{u}$ is also a physical vector.

We can write a secon-order tensor in vectrix notation as

$$\underrightarrow{I} = \underrightarrow{\mathcal{F}}_b^T \underline{I}_b \underrightarrow{\mathcal{F}}_b$$

Example: $\underrightarrow{I} = \underrightarrow{\mathcal{F}}_b^T \underline{I}_b \underrightarrow{\mathcal{F}}_b$, $\underrightarrow{u} = \underrightarrow{\mathcal{F}}_b^T \underline{u}_b$

$$\underrightarrow{I} \cdot \underrightarrow{u} = (\underrightarrow{\mathcal{F}}_b^T \underline{I}_b \underrightarrow{\mathcal{F}}_b) \cdot \underrightarrow{\mathcal{F}}_b^T \underline{u}_b$$

$$= \underrightarrow{\mathcal{F}}_b^T \underline{I}_b \underline{u}_b$$

Therefore, the second moment of mass can be written as a second moment of mass tensor as

$$\underrightarrow{J}^{Bz} = \underrightarrow{\mathcal{F}}_b^{T}\, \underline{J}_b^{Bz}\, \underrightarrow{\mathcal{F}}_b$$

Recall Euler's equation resolved in $\mathcal{F}_b$:

$$\underline{J}_b^{Bc}\, \underline{\dot{\omega}}_b^{ba} + \underline{\omega}_b^{ba^\times}\, \underline{J}_b^{Bc}\, \underline{\omega}_b^{ba} = \underline{m}_b^{Bc}$$

This can be rewritten in vector form as

$$\underrightarrow{J}^{Bc} \cdot \underrightarrow{\omega}^{ba^{\cdot b}} + \underrightarrow{\omega}^{ba} \times \left(\underrightarrow{J}^{Bc} \cdot \underrightarrow{\omega}^{ba} \right) = \underrightarrow{m}^{Bc}$$

Just like for physical vectors, we can use the DCM to relate a tensor resolved in different frames.

Consider

$$\begin{aligned} \underrightarrow{I} &= \underrightarrow{\mathcal{F}}_a^T \, \underline{I}_a \, \underrightarrow{\mathcal{F}}_a \\ &= \underrightarrow{\mathcal{F}}_b^T \, \underline{I}_b \, \underrightarrow{\mathcal{F}}_b \qquad (*) \end{aligned}$$

Recall that $\underrightarrow{\mathcal{F}}_a^T = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ba}$, $\underrightarrow{\mathcal{F}}_a = \underline{C}_{ba}^T \underrightarrow{\mathcal{F}}_a$

$$\left(\underrightarrow{r} = \underrightarrow{\mathcal{F}}_a^T \underline{r}_a = \underrightarrow{\mathcal{F}}_b^T \underline{C}_{ba} \underline{r}_a \right)$$

Therefore,

$$\begin{aligned} \underrightarrow{I} &= \underrightarrow{\mathcal{F}}_a^T \, \underline{I}_a \, \underrightarrow{\mathcal{F}}_a \\ &= \underrightarrow{\mathcal{F}}_b^T \, \underline{C}_{ba} \, \underline{I}_a \, \underline{C}_{ba}^T \, \underrightarrow{\mathcal{F}}_b \qquad (**) \end{aligned}$$

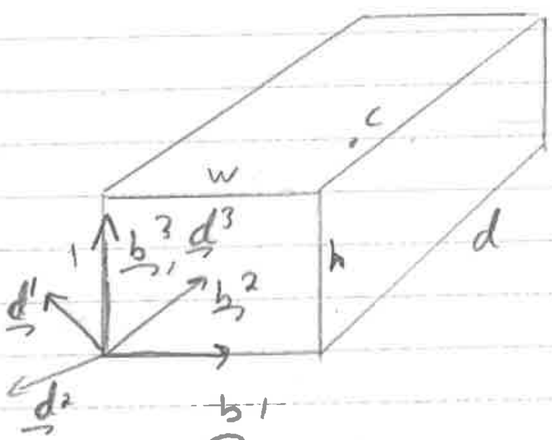
Comparing $(*)$ and $(**)$, we see that

$$\underline{I}_b = \underline{C}_{ba} \, \underline{I}_a \, \underline{C}_{ba}^T = \underline{C}_{ba} \, \underline{I}_a \, \underline{C}_{ab}$$

We can therefore resolve the second moment of mass tensor in any frame we want and use the DCM to go between frames.

$$\underline{J}_b^{Bz} = \underline{C}_{ba}\,\underline{J}_a^{Bz}\,\underline{C}_{ba}^T$$

Example: Consider a rectangular prism



$$\mathcal{F}_b \xrightarrow{\underline{C}_3(\theta)} \mathcal{F}_d$$

$$\underline{C}_{db} = \underline{C}_3(\theta)$$

$$\underline{J}_b^{Bc} = \begin{bmatrix} \frac{1}{12}m(h^2+d^2) & 0 & 0 \\ 0 & \frac{1}{12}m(h^2+w^2) & 0 \\ 0 & 0 & \frac{1}{12}m(w^2+d^2) \end{bmatrix}$$

Let $J_{b11}^{Bc} = \frac{1}{12}m(h^2+d^2)$
$J_{b22}^{Bc} = \frac{1}{12}m(h^2+w^2)$
$J_{b33}^{Bc} = \frac{1}{12}m(w^2+d^2)$

$$\underline{J}_d^{Bc} = \underline{C}_{db}\,\underline{J}_b^{Bc}\,\underline{C}_{db}^T$$

$$= \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_{b11}^{Bc} & 0 & 0 \\ 0 & J_{b22}^{Bc} & 0 \\ 0 & 0 & J_{b33}^{Bc} \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta J_{b11}^{BC} & -s_\theta J_{b11}^{BC} & 0 \\ s_\theta J_{b22}^{BC} & c_\theta J_{b22}^{BC} & 0 \\ 0 & 0 & J_{b33}^{BC} \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta^2 J_{b11}^{BC} + s_\theta^2 J_{b22}^{BC} & -s_\theta c_\theta J_{b11}^{BC} + s_\theta c_\theta J_{b22}^{BC} & 0 \\ -s_\theta c_\theta J_{b11}^{BC} + s_\theta c_\theta J_{b22}^{BC} & s_\theta^2 J_{b11}^{BC} + c_\theta^2 J_{b22}^{BC} & 0 \\ 0 & 0 & J_{b33}^{BC} \end{bmatrix}$$

Notice that $\underline{J}_d^{BC}$ is no longer diagonal.

Also notice that if $\theta = 0°$, then $c_\theta = 1$, $s_\theta = 0$, and we revert to $\underline{J}_b^{BC}$.

Remark: In some textbooks, when the second moment of mass is calculated relative to the center of mass, then the notation $\underline{I}_b^{B} = \underline{I}_b^{Bc}$ is used, or in tensor form $\underline{\underline{I}}^{B} = \underline{\underline{I}}^{Bc}$.