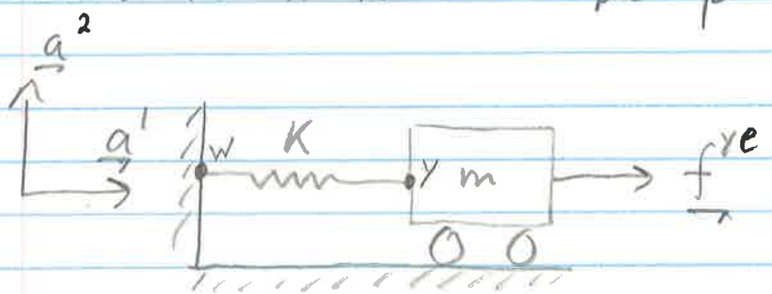
Vibrations

Let's return to a simple particle dynamics problem:



, $\| \underrightarrow{f}^{ye} \|_2 = f$

Let $\underrightarrow{r}^{yw} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} l+q \\ 0 \\ 0 \end{bmatrix}$, where l: constant

q: time varying elongation of spring

Find the equation of motion

1) Kinematics

i) Frames and DCMs : N/A

ii) Angular Velocity : N/A

iii) Position

$$\underrightarrow{r}^{yw} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} l+q \\ 0 \\ 0 \end{bmatrix}$$

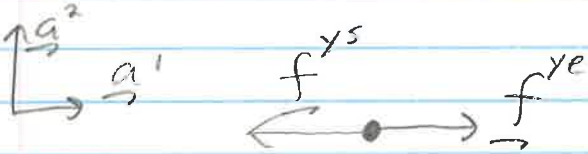
iv) Velocity

$$\underrightarrow{v}^{yw/a} = \underrightarrow{r}^{yw \cdot a} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \dot{q} \\ 0 \\ 0 \end{bmatrix}$$

v) Acceleration

$$\underrightarrow{a}^{yw/a/a} = \underrightarrow{v}^{yw/a \cdot a} = \underrightarrow{\mathcal{F}}_a^T \begin{bmatrix} \ddot{q} \\ 0 \\ 0 \end{bmatrix}$$

2) FBD



$$\vec{f}^{ys} = \vec{\mathcal{F}}_a^{T} \begin{bmatrix} -kq \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}^{ye} = \vec{\mathcal{F}}_a^{T} \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

3) N2L

$$\vec{f}^{ys} + \vec{f}^{ye} = m\,\vec{a}^{yw/a/a}$$

$$\vec{\mathcal{F}}_a^{T} \left(\begin{bmatrix} -kq \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} \right) = \vec{\mathcal{F}}_a^{T} \begin{bmatrix} m\ddot{q} \\ 0 \\ 0 \end{bmatrix}$$

$$-kq + f = m\ddot{q}$$

EoM: $m\ddot{q} + kq = f$

This is a second order linear ordinary differential equation that we can solve analytically.

First, we will assume $f=0$, giving

$$m\ddot{q} + Kq = 0 \qquad (*)$$

A general solution to a linear ODE is $q(t) = ae^{\lambda t}$

$$\dot{q}(t) = \lambda a e^{\lambda t}, \quad \ddot{q}(t) = \lambda^2 a e^{\lambda t}$$

Sub into $(*)$

$$m(\lambda^2 a e^{\lambda t}) + K(ae^{\lambda t}) = 0$$

$$(m\lambda^2 + K)ae^{\lambda t} = 0$$

$ae^{\lambda t} = 0$ give trivial solution, so we consider

$$\lambda^2 m + K = 0$$

$$\lambda^2 = -\frac{K}{m}$$

$$\lambda = \pm\sqrt{\frac{K}{m}}\, i$$, where i is imaginary

Therefore the solution to the ODE is

$$q(t) = a_1 e^{i\sqrt{K/m}\,t} + a_2 e^{-i\sqrt{K/m}\,t}$$

We can rewrite this using the trig identity

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$$

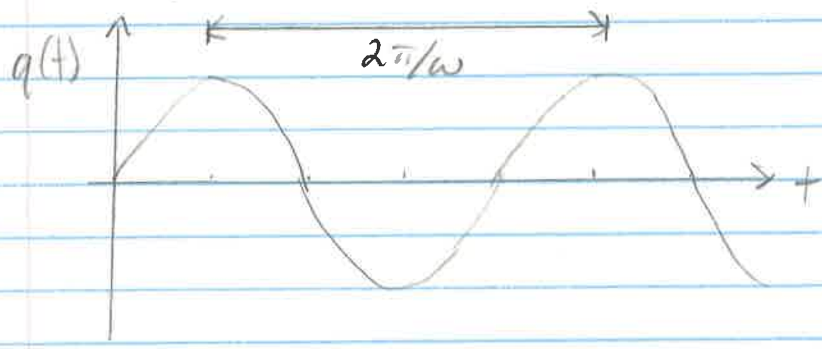
This gives

$$q(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t\right) + B\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

where $A = a_1 + a_2$

$B = (a_1 - a_2)i$

The response of $q(t)$ will look like



The frequency of oscillation is $\omega = \sqrt{\frac{K}{m}}$

The magnitude and phase of oscillation is determined by A and B (initial conditions)

$\omega = \sqrt{\frac{K}{m}}$ is known as the natural frequency of the system. This is the frequency of oscillation when no external forces act on the system.

We also sometimes write the EoM as

$$\ddot{q} + \omega^2 q = 0 \quad , \text{ where } \omega = \sqrt{\frac{K}{m}}$$

What if the force acting on the mass is non-zero?

Let's consider $f(t) = f_p \cos(\omega_p t)$

The EoM is now

$$m\ddot{q} + Kq = f_p \cos(\omega_p t) \qquad (**)$$

Remember from your differential equations course that

$$q(t) = q_h(t) + q_p(t),$$

where $q_h(t)$ is the homogeneous solution where the right hand side of the ODE is zero. We already solved for this and found

$$q_h(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t\right) + B\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

$q_p(t)$ is the particular solution that has the form

$$q_p(t) = C\cos(\omega_p t)$$

Same form as forcing function

$$\dot{q}_p(t) = -\omega_p C\sin(\omega_p t), \quad \ddot{q}_p(t) = -\omega_p^2 C\cos(\omega_p t)$$

Sub into $(**)$

$$\left(-m\omega_p^2 + K\right)C\cos(\omega_p t) = f_p\cos(\omega_p t)$$

Rearranging gives

$$\left[(-m\,\omega_p^2+K)C - f_p\right]\cos(\omega_p t) = 0$$

$\cos(\omega_p t) \neq 0$, so

$$(-m\,\omega_p^2+K)C - f_p = 0$$

$$C(-m\,\omega_p^2+K) = f_p$$

$$C = \frac{f_p}{K-m\omega_p^2}\cdot\frac{1/m}{1/m}$$

$$= \frac{f_p/m}{\frac{K}{m}-\omega_p^2} = \frac{f_p/m}{\omega^2-\omega_p^2}$$

So, we have

$$q_p(t) = \frac{f_p/m}{\omega^2-\omega_p^2}\cos(\omega_p t)$$

And the total solution is

$$q(t) = q_h(t) + q_p(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t\right) + B\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

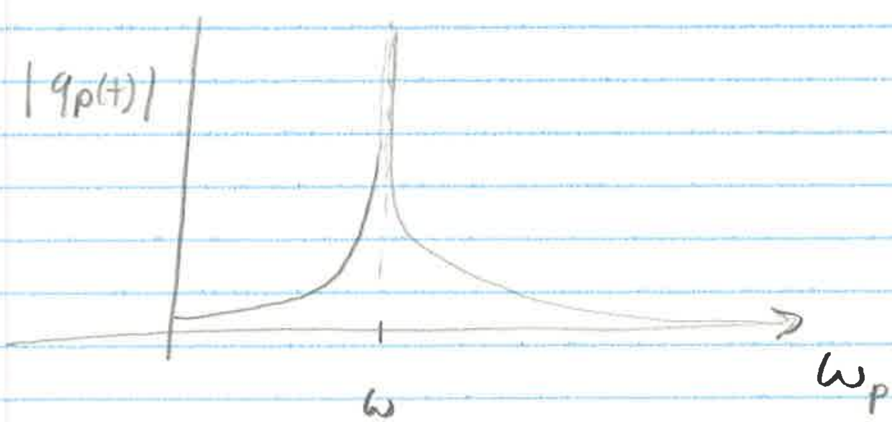
$$+ \frac{f_p/m}{\omega^2-\omega_p^2}\cos(\omega_p t)$$

Based on the solution for $q_p(t)$, what happens if $\omega_p \approx \omega$ (i.e., the forcing frequency approaches the natural frequency)?

$$\lim_{\omega_p \to \omega} q_p(t) = \lim_{\omega_p \to \omega} \frac{f_p/m}{\omega^2 - \omega_p^2} \cos(\omega_p t) \longrightarrow \infty$$

This is known as resonance, and is why the natural frequency of the system is sometimes referred to as the resonant frequency.

$$|q_p(t)| = \left| \frac{f_p/m}{\omega^2 - \omega_p^2} \right|$$



Dynamics of a System of Particles

(4.3 Forbes)

Newton's 3rd Law (N3L)

Consider two particles, y and x, connected by a massless link, a spring, or a damper. Then,

$$\underline{f}^{yx} = -\underline{f}^{xy}$$

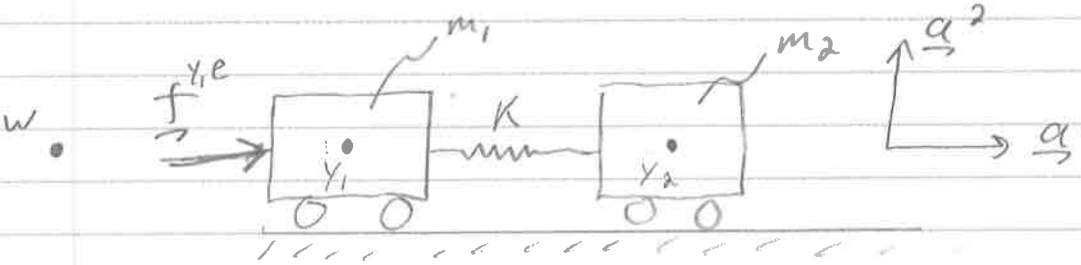
That is, the force acting on y due to x is equal and opposite to the force acting on x due to y.

N3L is useful when working with a system of particles.

Multi Degree of Freedom Vibrations

The concepts we discussed for single degree of freedom vibrations apply to multi degree of freedom vibrations.

Let's consider two masses connected by a spring:



$$\underline{r}^{Y_1 W} = \underline{\mathcal{F}}_a^T \begin{bmatrix} q_1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{r}^{Y_2 W} = \underline{\mathcal{F}}_a^T \begin{bmatrix} q_2 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{f}^{Y_1 e} = \underline{\mathcal{F}}_a^T \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

For simplicity we assume unstretched length of spring is zero.

Derive EoMs using 3 steps to success

1) Kinematics
   i) Frames and DCMs : N/A
   ii) Angular Velocity : N/A
   iii) Positions

$$\underline{r}^{Y_1 W} = \underline{\mathcal{F}}_a^T \begin{bmatrix} q_1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{r}^{Y_2 W} = \underline{\mathcal{F}}_a^T \begin{bmatrix} q_2 \\ 0 \\ 0 \end{bmatrix}$$

   iv) Velocities

$$\underline{v}^{Y_1 W/a} = \underline{r}^{Y_1 W \cdot a} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \dot{q}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{v}^{Y_2 W/a} = \underline{r}^{Y_1 W \cdot a} = \underline{\mathcal{F}}_a^T \begin{bmatrix} \dot{q}_2 \\ 0 \\ 0 \end{bmatrix}$$

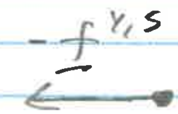
v) Accelerations

$$\vec{a}^{\,y_1 w/a/a} = \vec{v}^{\,y_1 w/a \cdot a} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} \ddot{q}_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{a}^{\,y_2 w/a/a} = \vec{v}^{\,y_2 w/a \cdot a} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} \ddot{q}_2 \\ 0 \\ 0 \end{bmatrix}$$

2) FBDs

Particl y_1 | Particle y_2



$$\vec{f}^{\,y_1 e} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}, \quad \vec{f}^{\,y_1 s} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} -Kq_1 + Kq_2 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} -K(q_1 - q_2) \\ 0 \\ 0 \end{bmatrix}$$

3) N2L

Particle y_1 : $\vec{f}^{\,y_1 e} + \vec{f}^{\,y_1 s} = m_1 \vec{a}^{\,y_1 w/a/a}$

$$\vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} f - K(q_1 - q_2) \\ \\ \end{bmatrix} = \vec{\mathcal{J}}_a^{\,T} \begin{bmatrix} m_1 \ddot{q}_1 \\ 0 \\ 0 \end{bmatrix}$$

$$m_1 \ddot{q}_1 + K(q_1 - q_2) = f$$

Particle y_2

$$-\underset{\rightharpoondown}{f}^{y_1 s} = m_2 \underset{\rightharpoondown}{a}^{y_2 w/a \, to \, a}$$

$$\underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} -K(q_2 - q_1) \\ 0 \\ 0 \end{bmatrix} = \underset{\rightharpoondown}{\mathcal{F}}_a^T \begin{bmatrix} m_2 \ddot{q}_2 \\ 0 \\ 0 \end{bmatrix}$$

$$m_2 \ddot{q}_2 + K(q_2 - q_1) = 0$$

Writing the EoMs together gives

$$m_1 \ddot{q}_1 + K(q_1 - q_2) = f \qquad (*)$$

$$m_2 \ddot{q}_2 + K(q_2 - q_1) = 0 \qquad (**)$$

Or in matrix form as

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\underline{M}} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}}_{\ddot{\underline{q}}} + \underbrace{\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}}_{\underline{K}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\underline{q}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{\underline{b}}} f$$

$$\underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} = \hat{\underline{b}} f$$

This is a second order matrix ordinary differential equation that is equivalent to $(*)$ and $(**)$.

Notice that $\underline{M} = \underline{M}^T$ and $\underline{K} = \underline{K}^T$.
$\underline{M}$ must also be positive definite (all eigenvalues strictly positive)

$\underline{K}$ must be positive semi definite (all eigenvalues zero or positive).

Let's solve this ODE first with $f=0$

$$\underline{M}\,\ddot{\underline{q}} + \underline{K}\,\underline{q} = \underline{0}$$

General solution of the form $\underline{q}(t) = \underline{a}\,e^{i\omega t}$

$$\dot{\underline{q}}(t) = i\omega\,\underline{a}\,e^{i\omega t}, \quad \ddot{\underline{q}}(t) = -\omega^2\,\underline{a}\,e^{i\omega t}$$

Sub into EoM to give

$$\underline{M}(-\omega^2\underline{a}\,e^{i\omega t}) + \underline{K}(\underline{a}\,e^{i\omega t}) = \underline{0}$$

$$(-\omega^2\underline{M} + \underline{K})\,\underline{a}\,e^{i\omega t} = \underline{0}$$

$e^{i\omega t} \neq 0$, so $(-\omega^2\underline{M} + \underline{K})\,\underline{a} = \underline{0}$

$\underline{M}$ is invertible, so we can rewrite as

always true because $\underline{M}$ is positive definite

$$(-\omega^2\underline{1} + \underline{M}^{-1}\underline{K})\,\underline{a} = \underline{0}$$

or $$\underline{M}^{-1}\underline{K}\,\underline{a} = \omega^2\,\underline{a}$$

Is this equation familiar? $(\underline{A}\,\underline{v} = \lambda\,\underline{v})$

ω^2 is an eigenvalue of $\underline{M}^{-1}\underline{K}$!

In our case,

$$\underline{M}^{-1}\underline{K} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} k/m_1 & -k/m_1 \\ -k/m_2 & k/m_2 \end{bmatrix}$$

Solve for eigenvalues of $\underline{M}^{-1}\underline{K}$

$$\det(\omega^2 \underline{1} - \underline{M}^{-1}\underline{K}) = 0$$

$$\det\left(\begin{bmatrix} \omega^2 - K/m_1 & K/m_1 \\ K/m_2 & \omega^2 - K/m_2 \end{bmatrix}\right) = 0$$

$$(\omega^2 - K/m_1)(\omega^2 - K/m_2) - (K/m_1)(K/m_2) = 0$$

$$\omega^4 - \left(\frac{1}{m_1} + \frac{1}{m_2}\right) K \omega^2 + \cancel{\frac{K^2}{m_1 m_2}} - \cancel{\frac{K^2}{m_1 m_2}} = 0$$

$$\left(\omega^2 - \frac{K(m_1 + m_2)}{m_1 m_2}\right)\omega^2 = 0$$

Solutions: $\omega^2 = \frac{K(m_1+m_2)}{m_1 m_2}$ or $\omega^2 = 0$

Natural frequencies are

$$\omega_1 = \sqrt{\frac{K(m_1+m_2)}{m_1 m_2}} \quad \text{and} \quad \omega_2 = 0$$

Check eigenvectors

$$\omega_1 : \quad (\omega_1^2 \underline{1} - \underline{M}^{-1}\underline{K})\underline{a}_1 = 0$$

$$\begin{bmatrix} \frac{K(m_1+m_2)}{m_1 m_2} - \frac{K}{m_1} & \frac{K}{m_1} \\ \frac{K}{m_2} & \frac{K(m_1+m_2)}{m_1 m_2} - \frac{K}{m_2} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{K}{m_2} & \frac{K}{m_1} \\ \frac{K}{m_2} & \frac{K}{m_1} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{K}{m_2} a_{11} + \frac{K}{m_1} a_{12} = 0$$

$$m_1 a_{11} + m_2 a_{12} = 0$$

$$a_{11} = -\frac{m_2}{m_1} a_{12} \Rightarrow \underline{a}_1 = \begin{bmatrix} -m_2 \\ m_1 \end{bmatrix}$$

$\omega_2 = 0$

$$(\omega_2^2 \underline{1} - \underline{M}^{-1} \underline{K}) \underline{a}_2 = \underline{0}$$

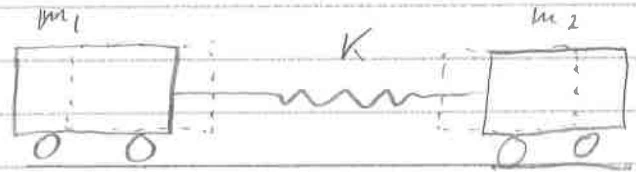
$$\begin{bmatrix} -\frac{K}{m_1} & \frac{K}{m_1} \\ \frac{K}{m_2} & -\frac{K}{m_2} \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-a_{21} + a_{22} = 0 \Rightarrow \underline{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{a}_1$ and $\underline{a}_2$ are the mode shapes associated with ω_1 and ω_2 that describe the motion caused by vibrating at those frequencies.

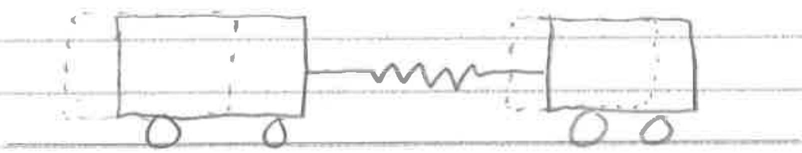
In this case

$\omega_1^2 = \dfrac{K(m_1+m_2)}{m_1 m_2}$ and $\underline{a}_1 = \begin{bmatrix} -m_2 \\ m_1 \end{bmatrix}$ refer to



an elongation in the spring where the center of mass of the system doesn't move.

$\omega_2^2 = 0$ and $\underline{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ refers to

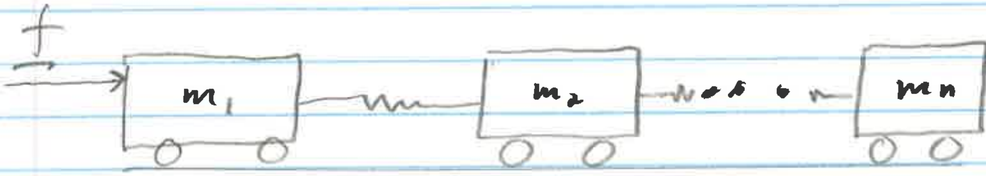


Both carts move the same amount and the spring is not stretched or compressed. This is known as a rigid body mode. The center of mass of the system does move in this case.

The total motion of the system is a combination of both the rigid body mode and the first mode of oscillation

$$q(t) = \underbrace{\underline{c}_1 \cos(\omega_1 t) + \underline{c}_2 \sin(\omega_1 t)}_{\text{harmonic motion (oscillations)}} + \underbrace{\underline{c}_3 + \underline{c}_4 t}_{\text{rigid body motion}}$$

The same procedure applies to multi-degree-of-freedom vibrations. Consider n carts



The EoMs will have the form

$$\underline{M}\,\ddot{q} + \underline{K}\,q = \hat{\underline{b}}\,f$$

where $\underline{M}, \underline{K} \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^{n \times 1}$.

The natural frequencies of vibration are the eigenvalues of

$$\underline{M}^{-1}\underline{K}$$

Notes

- Natural frequencies are system properties and do not depend on external forces.
- External forces are set to zero when solving for natural frequencies. This becomes an eigenvalue problem.