STAT 240 Homework 3

Rebecca Barter, Andrew Do, and Kellie Ottoboni

March 13, 2015

1) Show that a permutation test based on $\bar{X}$ and a permutation test based on t are equivalent when $m = n$

Note that the t statistic is defined by

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{Var(X)}{n} + \frac{Var(Y)}{m}}}$$

and that a permutation test based on t involves

1. Fill a box with the observed data.
2. Draw a simple random sample of size n and call if X, call the remaining elements in the box Y.
3. Compute t as you would if X and Y were your original data. Call this $t^{*(1)}$.
4. Repeat steps 1-3 L times to get $t^{*(2)}, t^{*(3)}, ..., t^{*(L)}$.
5. The distribution of the $t^{*(\ell)}$ approximates the true probability distribution of t under the strong null. In particular, a (left-tail) p-value can be computed as

$$\frac{1}{L}\#\{t^{*(\ell)} \leq t\}$$

We thus need to show that we can write

$$t^* = \frac{\bar{X}^* - \bar{Y}^*}{\sqrt{\frac{Var(X^*)}{n} + \frac{Var(Y^*)}{n}}}$$

in terms of $\bar{X}^*$ only.

We note, however, that if we simply write $A = \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i = \sum_{i=1}^n X_i^* + \sum_{i=1}^n Y_i^*$ to the the sum of all observations (which can be considered our new population from which we are drawing), we have that

$$\bar{X}^* - \bar{Y}^* = \left(1 + \frac{n}{n}\right)\bar{X}^* - \frac{1}{n}A = 2\bar{X}^* - \frac{1}{n}A$$

Thus the RHS depends only on $\bar{X}^*$. Next, if we write $B = \sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2 = \sum_{i=1}^n X_i^{*2} + \sum_{i=1}^n Y_i^{*2}$, then

$$
\begin{aligned}
\frac{Var(X^*)}{n}+\frac{Var(Y^*)}{n} &= \frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n}X_i^{*2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}X_i^*\right)^2\right]+\frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n}Y_i^{*2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}Y_i^*\right)^2\right] \\
&= \left[\frac{1}{n^2}\sum_{i=1}^{n}X_i^{*2}+\frac{1}{n^2}\sum_{i=1}^{n}Y_i^{*2}\right]-\frac{1}{n}\bar{X}^{*2}-\frac{1}{n^3}\left[A-n\bar{X}^*\right]^2 \\
&= \frac{1}{n^4}\left[n^2\sum_{i=1}^{n}X_i^{*2}+n^2\left(B-\sum_{i=1}^{n}X_i^{*2}\right)\right]-\frac{1}{n}\bar{X}^{*2}-\frac{1}{n^3}\left[A-n\bar{X}^*\right]^2 \\
&= \frac{B}{n^2}-\frac{1}{n}\bar{X}^{*2}-\frac{1}{n^3}\left[A-n\bar{X}^*\right]^2
\end{aligned}
$$

which depends only on $\bar{X}^*$. Thus we have shown that we can write t^* in terms of $\bar{X}^*$ only, implying that t and $\bar{X}$ are equivalent test statistics for a permutation test when $m = n$.

2) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on $\bar{X}$ is smaller than the p-value of a permutation test based on t. Try to make the difference substantial.

Suppose that the treatment group X contains only 5 observations, drawn from a Gaussian distribution with mean 0 and standard deviation 20. In this example, the generated sample is

$$X = \{10.9934, -16.8321, 0.6600, 10.4830, -34.5521\}$$

Let the control group Y contains 100 standard normal observations. A situation like this might occur if a treatment is very expensive to administer and has variable effects.

Suppose we want to test the strong null hypothesis using a two-sided test. The observed difference in means is $\bar{X} - \bar{Y} = -5.7513$ and the observed t-statistic is -0.6560. Using 10000 simulations, the permutation test p-values for the two-sided test are

$$
\begin{aligned}
P(|\bar{X}-\bar{Y}| \geq 5.7513) &= 0.05 \\
P(|t| \geq 0.6560) &= 0.559
\end{aligned}
$$

The p-value of the permutation test based on $\bar{X}$ is smaller than the p-value of the permutation test based on t because of the extreme noise and small sample size in the treatment group.

3) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on t is smaller than the p-value of a permutation test based on $\bar{X}$. Try to make the difference substantial.

4) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on $\bar{X}$ is smaller than the p-value of a standard t test. Try to make the difference substantial.