

STAT 240 Homework 3

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1) Show that a permutation test based on \bar{X} and a permutation test based on t are equivalent when $m = n$

Note that the t statistic is defined by

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{Var(X)}{n} + \frac{Var(Y)}{m}}}$$

and that a permutation test based on t involves

1. Fill a box with the observed data.
2. Draw a simple random sample of size n and call it X , call the remaining elements in the box Y .
3. Compute t as you would if X and Y were your original data. Call this $t^{*(1)}$.
4. Repeat steps 1-3 L times to get $t^{*(2)}, t^{*(3)}, \dots, t^{*(L)}$.
5. The distribution of the $t^{*(\ell)}$ approximates the true probability distribution of t under the strong null. In particular, a (left-tail) p -value can be computed as

$$\frac{1}{L} \# \{t^{*(\ell)} \leq t\}$$

We thus need to show that we can write

$$t^* = \frac{\bar{X}^* - \bar{Y}^*}{\sqrt{\frac{Var(X^*)}{n} + \frac{Var(Y^*)}{n}}}$$

in terms of \bar{X}^* only.

We note, however, that if we simply write $A = \sum_{i=1}^n X_i + \sum_{i=1}^n Y_i = \sum_{i=1}^n X_i^* + \sum_{i=1}^n Y_i^*$ to be the sum of all observations (which can be considered our new population from which we are drawing), we have that

$$\bar{X}^* - \bar{Y}^* = \left(1 + \frac{n}{n}\right) \bar{X}^* - \frac{1}{n} A = 2\bar{X}^* - \frac{1}{n} A$$

Thus the RHS depends only on \bar{X}^* . Next, if we write $B = \sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2 = \sum_{i=1}^n X_i^{*2} + \sum_{i=1}^n Y_i^{*2}$, then

$$\begin{aligned}
\frac{Var(X^*)}{n} + \frac{Var(Y^*)}{n} &= \frac{1}{n} \left[\frac{1}{n} \sum_{i=1}^n X_i^{*2} - \frac{1}{n^2} \left(\sum_{i=1}^n X_i^* \right)^2 \right] + \frac{1}{n} \left[\frac{1}{n} \sum_{i=1}^n Y_i^{*2} - \frac{1}{n^2} \left(\sum_{i=1}^n Y_i^* \right)^2 \right] \\
&= \left[\frac{1}{n^2} \sum_{i=1}^n X_i^{*2} + \frac{1}{n^2} \sum_{i=1}^n Y_i^{*2} \right] - \frac{1}{n} \bar{X}^{*2} - \frac{1}{n^3} [A - n\bar{X}^*]^2 \\
&= \frac{1}{n^4} \left[n^2 \sum_{i=1}^n X_i^{*2} + n^2 \left(B - \sum_{i=1}^n X_i^{*2} \right) \right] - \frac{1}{n} \bar{X}^{*2} - \frac{1}{n^3} [A - n\bar{X}^*]^2 \\
&= \frac{B}{n^2} - \frac{1}{n} \bar{X}^{*2} - \frac{1}{n^3} [A - n\bar{X}^*]^2
\end{aligned}$$

which depends only on \bar{X}^* . Thus we have shown that we can write t^* in terms of \bar{X}^* only, implying that t and \bar{X} are equivalent test statistics for a permutation test when $m = n$.

2) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p -value of a permutation test based on \bar{X} is smaller than the p -value of a permutation test based on t . Try to make the difference substantial.

Let the treatment group consist of the observations

$$X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 100, 1000\}$$

and the control group consist of the integers from -10 to 10 . Suppose that in the treatment group, 100 and 1000 are highly unusual observations. If we were to exclude these, the two groups would have the same means. We would like our test to be robust to these outliers and tell us that the difference in means between the treatment and control groups is not significantly different from 0 under the strong null. We will compare the strong null to the alternative hypothesis that the difference in means is greater than 0.

The observed difference in means is $\bar{X} - \bar{Y} = 84.6154$ and the observed t -statistic is 1.1034. Using 10000 simulations, the permutation test p -values are

$$P(\bar{X} - \bar{Y} \geq 84.6154) = 0.0162$$

$$P(t \geq 1.1034) = 0.1492$$

The p -value of the permutation test based on \bar{X} is smaller than the p -value of the permutation test based on t because of the extreme outliers in the treatment group.

3) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p -value of a permutation test based on t is smaller than the p -value of a permutation test based on \bar{X} . Try to make the difference substantial.

4) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p -value of a permutation test based on \bar{X} is smaller than the p -value of a standard t test. Try to make the difference substantial.