STAT 240 Homework 3

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1) Show that a permutation test based on $\bar{X}$ and a permutation test based on t are equivalent when $m = n$

Note that the t statistic is defined by

$$t = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{Var(X)}{n} + \frac{Var(Y)}{m}}}$$

and that a permutation test based on t involves

1. Fill a box with the observed data.
2. Draw a simple random sample of size n and call if X, call the remaining elements in the box Y.
3. Compute t as you would if X and Y were your original data. Call this $t^{*(1)}$.
4. Repeat steps 1-3 L times to get $t^{*(2)}, t^{*(3)}, ..., t^{*(L)}$.
5. The distribution of the $t^{*(\ell)}$ approximates the true probability distribution of t under the strong null. In particular, a (left-tail) p-value can be computed as

$$\frac{1}{L}\#\{t^{*(\ell)} \leq t\}$$

We thus need to show that we can write

$$t^* = \frac{\bar{X}^* - \bar{Y}^*}{\sqrt{\frac{Var(X^*)}{n} + \frac{Var(Y^*)}{m}}}$$

in terms of $\bar{X}^*$ only.

We note, however, that if we simply write $A = \sum_{i=1}^n X_i + \sum_{i=1}^m Y_i = \sum_{i=1}^n X_i^* + \sum_{i=1}^m Y_i^*$ to the the sum of all observations (which can be considered our new population from which we are drawing), we have that

$$\bar{X}^* - \bar{Y}^* = \left(1 + \frac{n}{m}\right)\bar{X}^* - \frac{1}{m}A$$

Thus the RHS depends only on $\bar{X}^*$. Next, if we write $B = \sum_{i=1}^n X_i^2 + \sum_{i=1}^m Y_i^2 = \sum_{i=1}^n X_i^{*2} + \sum_{i=1}^m Y_i^{*2}$, then

$$
\begin{aligned}
\frac{Var(X^*)}{n}+\frac{Var(Y^*)}{m} &= \frac{1}{n}\left[\frac{1}{n}\sum_{i=1}^{n}X_i^{*2}-\frac{1}{n^2}\left(\sum_{i=1}^{n}X_i^*\right)^2\right]+\frac{1}{m}\left[\frac{1}{m}\sum_{i=1}^{m}Y_i^{*2}-\frac{1}{m^2}\left(\sum_{i=1}^{m}Y_i^*\right)^2\right]\\
&= \left[\frac{1}{n^2}\sum_{i=1}^{n}X_i^{*2}+\frac{1}{m^2}\sum_{i=1}^{m}Y_i^{*2}\right]-\frac{1}{n}\bar{X}^{*2}-\frac{1}{m^3}\left[A-n\bar{X}^*\right]^2\\
&= \frac{1}{n^2m^2}\left[m^2\sum_{i=1}^{n}X_i^{*2}+n^2\left(B-\sum_{i=1}^{n}X_i^{*2}\right)\right]-\frac{1}{n}\bar{X}^{*2}-\frac{1}{m^3}\left[A-n\bar{X}^*\right]^2
\end{aligned}
$$

How to show that the RHS depends only on $\bar{X}^{*2}$.??? Have I just gone about this all wrong???

2) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on $\bar{X}$ is smaller than the p-value of a permutation test based on t. Try to make the difference substantial.

3) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on t is smaller than the p-value of a permutation test based on $\bar{X}$. Try to make the difference substantial.

4) Construct a hypothetical dataset (with at least 3 data points in treatment and at least 3 in control) for which the p-value of a permutation test based on $\bar{X}$ is smaller than the p-value of a standard t test. Try to make the difference substantial.