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Threshold Dynamics of Stochastic SIS Epidemic Models with Logistic Recruitment Rate

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This study explores an SIS epidemic model that integrates logistic growth and a saturated incidence rate among susceptible individuals. The primary objective is to examine the long-term behavior of the stochastic system. For this purpose, a key parameter, λ, is introduced to differentiate the system dynamics into two cases. In scenarios where $\lambda < 0$, the model predicts the eventual extinction of the disease. In contrast, for $\lambda > 0$, we show an ergodic stationary distribution. Numerical simulations validate our theoretical conditions for extinction, persistence, and the existence of a stationary distribution.

Keywords: SIS epidemic model; logistic population growth; disease extinction; stability in distribution; ergodic properties.

~~Mathematics Subject Classification 2020:~~

1. Introduction

During the early 20th century, epidemiology underwent a significant transformation, driven by the groundbreaking contributions of eminent scientists such as Anderson Gray McKendrick and Janet Leigh. Their pioneering work introduced the concept of mathematical modeling, which has since evolved into an indispensable tool in the field. This mathematical modeling has profoundly impacted the management of outbreaks and epidemics, playing a pivotal role in guiding evidence-based public health

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interventions. Epidemiology has evolved significantly, owing to notable physicians such as Quinto Tiberio Angelerio, who demonstrated remarkable proficiency in managing the plague outbreak in Alghero, Sardinia, in 1582. However, the actual emergence of modern epidemiology as a formal scientific discipline occurred during the 19th century. Often referred to as the "father of modern epidemiology," John Snow made a meaningful breakthrough when he meticulously traced a devastating cholera outbreak in London to water contamination from the Broad Street pump. This groundbreaking investigation is the pivotal moment that laid the cornerstone for contemporary epidemiology, shaping it into the scientific field we recognize today. Epidemiology is a scientific discipline investigating epidemics, diseases, and various health-related conditions, including those unrelated to diseases. Its roots can be traced to ancient Greece, notably through the influential work of Hippocrates of Kos, a renowned physician who made notable contributions by distinguishing between epidemic and endemic diseases. Epidemiology, in its broader scope, also encompasses the study of diseases affecting plants, domestic animals, and livestock. An epidemic is characterized by a significant and abnormal surge in the occurrence of a specific disease within a population, typically manifesting rapidly. The intricate disease transmission process is influenced by many factors, encompassing both the characteristics of the infectious agent and the complex dynamics within the host population. Regarding the infectious agent, its inherent characteristics, such as its mode of transmission (e.g. respiratory droplets, direct contact), the duration of infectivity, and its responsiveness to medical interventions like treatments and vaccines, are crucial factors that determine its ability to spread among individuals. Equally important are the host population elements that influence the dynamics of an epidemic. Factors such as social interactions, demographics (e.g. age, gender), cultural practices, geographic distribution, and economic conditions are pivotal in determining a population's susceptibility and resilience in the face of the disease. Across the pages of recorded history, human civilization has wrestled with recurrent epidemics and pandemics. These outbreaks of infectious diseases have inflicted significant human suffering, societal upheaval, and economic turbulence. Given the formidable nature of these challenges, the precise prediction of outbreak progression becomes paramount to effectively mitigating their adverse impacts. Epidemiologic modeling is at the heart of this pursuit, a fundamental tool for comprehending the intricacies of disease transmission dynamics and formulating informed strategies for containment and prevention. Using dynamical systems to model infectious diseases is crucial in examining both theoretical and quantitative facets of epidemic spread. Following the foundational contributions of McKendrick and Kermack,[1] various compartmental models have emerged, employing ordinary, fractional, and stochastic differential equations (SDEs). Every model presented in existing literature has its unique characteristics and limitations, and it is essential to recognize that no model is flawless, regardless of the number of variables or parameters included. Additionally, there tends to be an inverse relationship between model complexity and the depth of mathematical analysis. Commonly, researchers focus on either

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SIS (Susceptible–Infectious–Susceptible) or SIR (Susceptible–Infectious–Recovered) models, where S denotes susceptible individuals, I represents infectious persons, and R refers to those who have recovered. It is also possible to incorporate additional compartments, such as vaccinated individuals or those exposed to the disease. In this paper, our attention is specifically directed toward SIS epidemic models, which are effectively employed in the study of diseases such as sexually transmitted infections and bacterial diseases.[2–15] These models aptly encapsulate the progression of diseases where lasting immunity is not conferred, resulting in individuals reverting to a susceptible state post-infection. While certain researchers[16,17] have explored analytical or explicit series solutions for two-dimensional SIS differential systems, other mathematical methodologies such as bifurcation theory, stability theory, or the Lyapunov method are more commonly utilized. This preference arises because most two-dimensional differential equations, particularly non-linear ones, are seldom solvable via integration. Additionally, the pursuit of explicit solutions is often less prioritized by authors, as these alternative mathematical tools yield valuable insights into aspects such as the asymptotic behavior of the solutions, which can be more informative for understanding system dynamics. Epidemic models aim to address various queries, including determining the number of new infections that arise when infectious individuals interact with a susceptible group. The resolution to this query typically hinges on two key assumptions. The first assumption posits that each individual in a population is equally likely to come into contact with any other individual at any given moment, a concept referred to as homogeneous mixing. The second assumption is the mass-action principle, which suggests that a certain proportion of these contacts will result in infection at any given time.[18,19] The concepts above can be mathematically articulated as follows: Let $S(t)$ and $I(t)$ represent the number of susceptible and infectious individuals at time t, respectively. Then, the term $\beta S(t)I(t)$ signifies the new portion of susceptible individuals who become infectious at time t. In this study, we retain the assumption that transmission occurs almost instantaneously. However, the second assumption is modifiable by adopting a different incidence rate function. Rather than the conventional $\beta S(t)I(t)$, alternative models suggest using $\beta g(S, I)$, where $g(S, I)$ is a function that is not bilinear, introducing a variation in the interaction dynamics between susceptible and infectious individuals. In our study, we have opted for a Holling-type functional response function, defined as $g(S, I) = SI/(1 + \alpha S)$. This formulation was initially introduced into epidemic modeling by Macdonald[20] and later adopted by May *et al.*[21] Notably, when the number of susceptible individuals is considerably large, the incidence rate function $\beta g(S, I)$ tends to approximate $\beta I/\alpha$, with β representing the contact parameter between susceptible and infected individuals. This characteristic is why $g(S, I)$ is a saturated function concerning the susceptibles. Commonly in the literature, the population growth rate is assumed to be constant, leading to exponential demographic changes. However, this assumption usually holds only over short time frames. In practical scenarios, the notion of fluctuating total populations is more applicable, especially in the context of diseases with high mortality rates

or those that persist over a considerable duration.[22–28] Additionally, population growth is naturally constrained by environmental factors, such as the availability of resources. A notable approach for encapsulating this concept is the logistic model, initially proposed by Verhulst in Ref. 29. This model illustrates exponential growth within a population, eventually tempered by the environmental carrying capacity, thus providing a more realistic representation of population dynamics under such constraints. Numerous researchers have explored and developed epidemic models incorporating logistic growth equations.[14,30–33] Building upon this foundation, we present a deterministic SIS epidemic model that includes logistic growth, structured as follows:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = rS\left(1-\dfrac{S}{K}\right) - \mu S - \dfrac{\beta SI}{1+\alpha S} + \gamma I, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = -(\mu+\gamma)I + \dfrac{\beta SI}{1+\alpha S}. \end{cases} \tag{1.1}$$

At a given moment, $S(t)$ denotes the quantity of susceptible individuals, and $I(t)$ indicates the number of currently infected persons. The parameters in the model, all positive, have specific meanings: r is the intrinsic growth rate of the susceptible population, and K symbolizes the carrying capacity. Additionally, β signifies the rate of disease transmission, α represents the half-saturation constant, and γ refers to the recovery rate of infected individuals. The parameter μ reflects the natural mortality rate within the population. We operate under the assumption that $r > \mu$, as the scenario where $\mu \geq r$ lacks biological relevance. Concurrently, stochastic variations are crucial in industrial production and ecosystem dynamics. Consequently, many researchers have integrated the concept of random noise, often represented as stochastic white noise, into deterministic epidemic models. In certain studies, it is posited that every individual in the population is concurrently influenced by various random environmental factors.[3,4,34] In a recent study,[5] researchers introduced an SIS epidemic model that incorporates multiple independent Brownian motions. They identified a stochastic threshold crucial for determining whether the disease persists or becomes extinct within the host population. This threshold is the Lyapunov exponent of $I(t)$, particularly when $I(t)$ remains small over an extended period. Inspired by these findings, our proposed model considers stochastic perturbations as environmental noise impacting individuals directly. These perturbations are assumed to be proportional to the variables in the system, and the model can be expressed as follows:

$$\begin{cases} \mathrm{d}S = \left[rS\left(1-\dfrac{S}{K}\right) - \mu S - \dfrac{\beta SI}{1+\alpha S} + \gamma I\right]\mathrm{d}t + \sigma_1 S\mathrm{d}B_1, \\ \mathrm{d}I = \left[-(\mu+\gamma)I + \dfrac{\beta SI}{1+\alpha S}\right]\mathrm{d}t + \sigma_2 I\mathrm{d}B_2. \end{cases} \tag{1.2}$$

In our model, we incorporate independent Brownian motions, represented by $B_i(t)$ for $i = 1, 2$, each associated with their respective intensities of white noise, denoted

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as σ_i, and all are positive. The organization of this paper follows a systematic approach. Starting with Sec. 2, we present essential results necessary for the subsequent sections. Section 3 is devoted to deriving the required conditions for the extinction of the disease. In Sec. 4, we establish the criteria needed to ensure the system attains an ergodic stationary distribution. Following this, Sec. 5 comprises a series of numerical simulations to corroborate the accuracy of our analytical results. The paper concludes with a summary of our findings and a discussion of potential future research directions.

2. Preliminary Results

We define $(\Omega, \mathcal{T}, \{\mathcal{T}_t\}_{t\geq 0}, \mathbb{P})$ as a complete probability space, complete with a filtration $\{\mathcal{T}_t\}_{t\geq 0}$ that meets the usual conditions. The notations $\mathbb{E}_{(s,i)}$ and $\mathbb{P}_{(s,i)}$ are used to denote the expected value and probability, starting from $S(0) = s$, $I(0) = i$. In this paper, we adopt the following notations:

$$\mathbb{R}^{2,o}_+ = \{(s,i) : s > 0, i > 0\}, \quad \text{for the strictly positive quadrant.}$$
$$\mathbb{R}^{2}_+ = \{(s,i) : s \geq 0, i \geq 0\}, \quad \text{for the non-negative quadrant.}$$
$$\mathbb{R}^{2,*}_+ = \{(s,i) : s \geq 0, i > 0\}, \quad \text{for the semi-positive quadrant.}$$

Our research focuses on examining a d-dimensional Itô process, characterized by the subsequent SDE:

$$\mathrm{d}X(t) = f(X)\mathrm{d}t + g(X)\mathrm{d}B(t) \quad \text{for each} \quad t \geq t_0. \tag{2.1}$$

Here, $B(t)$ represents a d-dimensional white noise and the starting value $X(0) \in \mathbb{R}^d$. In virtue of the Itô formula, the stochastic equation verified by $V(X)$, with V being a function that is continuously twice differentiable and defined on $\mathbb{R}^d \times \mathbb{R}^+$ is

$$\mathrm{d}V(X) = \mathcal{L}V(X)\mathrm{d}t + \nabla V(X)g(X)\mathrm{d}B(t),$$

where $\nabla V = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$ is gradient of V and $\mathcal{L}$ is the differential operator associated to system (2.1) defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g^T(X)\cdot g(X))]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

An appropriate beginning would be introducing a theorem that establishes the existence, uniqueness of solutions, and bounded nature.

Theorem 2.1. (i) *For any given initial value $X(0) \in \mathbb{R}^2_+$, a unique and positive solution X to the SDE as stated in Eq.* (1.2) *can be assured for all $t \geq 0$, such*

that

$$\mathbb{P}_{(s,i)}\left\{X(t) \in \mathbb{R}_+^2\right\} = 1.$$

Furthermore

$$\mathbb{P}_{(s,0)}\left[I(t) = 0; \textit{ for any } t \geq 0\right] = 1,$$

then

$$\mathbb{P}_{(s,i)}\left[(S(t), I(t)) \in \mathbb{R}_+^{(2,o)}; \textit{ for all } \ t > 0\right] = 1, \quad i > 0.$$

(ii) *For each $\theta > 0$ and sufficiently small, it can be established that there are values θ_1, θ_2, both exceeding zero, for which the following conditions hold*:

$$\mathbb{E}_{(s,i)}\left[S(t) + I(t) + 1\right]^{\theta+1} \leq \left[s + i + 1\right]^{\theta+1} \exp\left(-\theta_2 t\right) + \frac{\theta_1}{\theta_2}. \tag{2.2}$$

Moreover, for all $M, T, \epsilon > 0$, there exists $H_{M,T,\epsilon} > 0$ such that

$$\mathbb{P}_{(s,i)}\left[\sup_{t\in[0,T]}\left\{S_t + I_t\right\} \leq H_{M,\epsilon,T}\right] \geq 1 - \epsilon \ \textit{ for } \ (s,i) \in [0,M]^2. \tag{2.3}$$

Proof. The demonstration of solution positivity is based on methods akin to those found in Refs. 14 and 35, Theorem 2.1. However, to maintain conciseness, these details are not included here. Considering (1.2), it becomes evident that $I(t)$ conforms to the following criteria:

$$I(t) = i \exp\left[\int_0^t \frac{\beta S(u)}{1 + \alpha S(u)} \mathrm{d}u - (\mu + \gamma)t + \sigma_2 B_2\right]. \tag{2.4}$$

Furthermore, deriving from Eq. (2.4), when setting $i = 0$, it leads to the following result:

$$\mathbb{P}_{(s,0)}\left[I(t) = 0; \text{ for each } t \geq 0\right] = 1.$$

For $i > 0$ and $s = 0$, consider $X(t)$ to represent the positive solution to the SDE, which is depicted in the following manner:

$$\mathrm{d}X = \left[rX\left(1 - \frac{X}{K}\right) - \mu X - \frac{\beta X I}{1 + \alpha X}\right]\mathrm{d}t + \sigma_1 X \mathrm{d}B_1(t).$$

Applying the comparison theorem as delineated in Ref. 36, it can be established that for each $t \geq 0$:

$$S(t) > X(t) \ \text{ a.s.},$$

which implies

$$\mathbb{P}_{(0,i)}\left(S_t > 0; \text{ for each } t > 0\right) = 1.$$

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We now transition to the discussion on bounds. To begin, let us introduce and define the subsequent function:

$$G(S, I) = (1 + S + I)^{1+\theta},$$

wherein θ represents a positive real number that satisfies

$$\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2) > 0.$$

Applying the operator $\mathcal{L}$ on G, gives

$$\begin{aligned}
\mathcal{L}G(S, I) &= (\theta + 1)\,(S + I + 1)^{\theta} \left[rS\left(1 - \frac{S}{K}\right) - \mu(S + I)\right] \\
&\quad + \frac{\theta}{2}(\theta + 1)(S + I + 1)^{\theta-1}\left(\sigma_1^2 S^2 + \sigma_2^2 I^2\right) \\
&\leq (\theta + 1)\,(S + I + 1)^{\theta} \left[\sup_{S\in(0,\infty)} \left\{ -\frac{r}{K}S^2 + rS \right\} + \mu - \mu(S + I + 1)\right] \\
&\quad + \frac{\theta}{2}(1 + \theta)(S + I + 1)^{1+\theta}\left(\sigma_1^2 \vee \sigma_2^2\right) \\
&\leq (\theta + 1)\,(S + I + 1)^{\theta} \left[\frac{r}{4}K + \mu - \left(\mu - \left(\sigma_1^2 \vee \sigma_2^2\right)\frac{\theta}{2}\right)(S + I + 1)\right].
\end{aligned}$$

Consequently, following the selection of θ, it can be established that there are positive real numbers $\theta_1, \theta_2 > 0$ such that

$$\mathcal{L}G(S, I) \leq \theta_1 - \theta_2 G(S, I). \tag{2.5}$$

For values of n that are sufficiently large, let us establish the stopping time τ_n in the following manner:

$$\tau_n = \inf\{t \geq 0 : n \leq G(S_t, I_t)\}.$$

Using the Itô formula in conjunction with Eq. (2.5), we get

$$\begin{aligned}
&\mathbb{E}_{(s,i)}\left[\exp\left(\theta_2(t \wedge \tau_n)\right)G(S(t \wedge \tau_n), I(t \wedge \tau_n))\right] \\
&\quad \leq G(s, i) + \mathbb{E}_{(s,i)}\left(\int_0^{t\wedge\tau_n} \theta_1 \exp\left(\theta_2 u\right)\mathrm{d}u\right) \\
&\quad \leq G(s, i) + \frac{\theta_1}{\theta_2}\exp\left(\theta_2 t\right). \qquad (2.6)
\end{aligned}$$

Using the Fatou lemma and letting $n \to \infty$, we get

$$\mathbb{E}_{(s,i)}\left[\exp\left(\theta_2 t\right)G(S(t), I(t))\right] \leq G(s, i) + \frac{\theta_1}{\theta_2}\exp\left(\theta_2 t\right).$$

Thus

$$\mathbb{E}_{(s,i)}[1 + S(t) + I(t)]^{1+\theta} \leq (s + i + 1)^{1+\theta}\exp\left(-\theta_2 t\right) + \frac{\theta_1}{\theta_2}.$$

As a result, we achieve the necessary inequality as specified in (2.2) of part (ii).

Conversely, based on Eq. (2.6), it is deduced that for any given $\overline{T} > 0$, the following holds:

$$\begin{aligned} n^{\theta+1}\mathbb{P}_{(s,i)}\left\{\tau_n \leq \overline{T}\right\} &= \mathbb{E}_{(s,i)}\left[G\left(S\left(\tau_n\right), I\left(\tau_n\right)\right)\mathbb{I}_{\left\{\tau_n \leq \overline{T}\right\}}\right] \\ &\leq \mathbb{E}_{(s,i)}\left[\exp\left(\theta_2\tau_n\right)G\left(S\left(\tau_n\right), I\left(\tau_n\right)\right)\mathbb{I}_{\left\{\tau_n \leq \overline{T}\right\}}\right] \\ &\leq G(s,i) + \frac{\theta_1}{\theta_2}\exp\left(\theta_2\overline{T}\right). \end{aligned}$$

Consequently, we establish the requisite inequality (2.3) of part (ii). This concludes the proof of Theorem 2.1. □

We now establish the threshold that governs the system model dynamics as outlined in Eq. (1.2). Moving forward, we examine the specific equation obtained by assigning $I(t) = 0$ in the initial equation of model (1.2), and we refer to $\widehat{S}(t)$ as the solution to this equation. Hence,

$$\mathrm{d}\widehat{S}(t) = \left[r\widehat{S}(t)\left(1 - \frac{\widehat{S}(t)}{K}\right) - \mu\widehat{S}(t)\right]\mathrm{d}t + \sigma_1\widehat{S}(t)\mathrm{d}B_1. \tag{2.7}$$

Beginning with the initial value $\widehat{S}(0)$, it is proven in Ref. 33 that the process $\widehat{S}(t)$ manifests a singular stationary distribution, denoted as $\mathbf{\Theta}_0$, under the condition $\frac{\sigma_1^2}{2} < r - \mu$. The density of this distribution is articulated through the utilization of the Fokker–Planck equation, as follows:

$$f^*(x) = \mathcal{H}x^{-2+\frac{2(r-\mu)}{\sigma_1^2}}\exp\left(\frac{-2rx}{K\sigma_1^2}\right), \quad x \in (0,\infty),$$

where

$$\mathcal{H} = \left[\sigma_1^{-2}\left(\frac{K\sigma_1^2}{2r}\right)^{-1+2\frac{(r-\mu)}{\sigma_1^2}}\Gamma\left(-1 + 2\frac{(r-\mu)}{\sigma_1^2}\right)\right]^{-1}.$$

By ergodicity of $\widehat{S}(t)$, one obtains

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \widehat{S}(x)\,\mathrm{d}x = \int_0^\infty s f^*(s)\,\mathrm{d}s < \infty \quad \text{a.s.}$$

In the subsequent analysis, we aim to show how the long-term dynamics of $I(t)$ are influenced by

$$\lambda = \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_0(\mathrm{d}s) - \mu - \gamma - \frac{\sigma_2^2}{2}.$$

Let us examine the modified version of Eq. (2.7), which is presented as follows:

$$\mathrm{d}\bar{S}^{(\vartheta_0)}(t) = \left[r\bar{S}^{(\vartheta_0)}\left(1 - \frac{\bar{S}^{(\vartheta_0)}}{K}\right) - \mu\bar{S}^{(\vartheta_0)} + \gamma\vartheta_0\right]\mathrm{d}t + \sigma_1\bar{S}^{(\vartheta_0)}\mathrm{d}B_1(t). \tag{2.8}$$

Similar to Eq. (2.7). If $r - \mu > \frac{\sigma_1^2}{2}$, then the function $\bar{S}^{(\vartheta_0)}(t)$ admits an ergodic invariant measure, which is denoted by $\mathbf{\Theta}_{\vartheta_0}$. Thus, we have

$$\int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_{\vartheta_0}(\mathrm{d}s) < \infty.$$

Subsequently, we introduce a crucial lemma that elucidates the connection between Eqs. (2.7) and (2.8).

Lemma 2.1. *Let* $\vartheta_0 \in [0,1]$, *we have*

$$\lim_{\vartheta_0 \to 0}\left|\int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_{\vartheta_0}(\mathrm{d}s) - \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_0(\mathrm{d}s)\right| = 0.$$

Proof. It is easy to verify that

$$\lim_{t\to\infty}\sup\mathbb{E}\left(\bar{S}^{(\vartheta_0)}(t)\right)^{1+q} < \infty \text{ for } q > 0.$$

Consequently, the process $(\bar{S}^{(\vartheta_0)}, t \geq 0)$ exhibits uniform integrability. This characteristic implies that the family $\{\mathbf{\Theta}_{\vartheta_0}, \vartheta_0 \in [0,1]\}$ demonstrates tightness over the interval $(0,\infty)$. Consider an integer n such that ϑ_n, a sequence within $[0,1]$, approaches 0 as n tends to infinity. From this, it follows that $\{\mathbf{\Theta}_{\vartheta_n}, n \geq 0\}$ is also tight. Therefore, one can extract a subsequence that converges weakly to a probability measure $\bar{\mathbf{\Theta}}$ over the interval $(0,\infty)$). For ease of notation, this subsequence will continue to be referred to as $\{\mathbf{\Theta}_{\vartheta_n}, n \geq 0\}$. Thus, we have

$$\lim_{n\to\infty}\int_0^\infty g(s)\mathbf{\Theta}_{\vartheta_n}(\mathrm{d}s) = \int_0^\infty g(s)\bar{\mathbf{\Theta}}(\mathrm{d}s).$$

For every smooth function, g possesses compact support. Furthermore, applying the dominated convergence theorem leads to the conclusion that

$$\begin{aligned}\int_0^\infty \mathcal{L}^{\vartheta_n}g(s)\mathbf{\Theta}_{\vartheta_n}(\mathrm{d}s) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\mathcal{L}^{\vartheta_n}g\left(\bar{S}^{(\vartheta_n)}(u)\right)\mathrm{d}u \\ &= \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[g\left(\bar{S}^{(\vartheta_n)}(t)\right) - g\left(\bar{S}^{(\vartheta_n)}(0)\right)\right] \\ &= 0,\end{aligned}$$

where $\bar{S}^{(\vartheta_n)}(t)$ denotes the invariant solution corresponding to Eq. (2.8), with the substitution of ϑ_n for ϑ_0. It is important to observe that

$$\mathcal{L}^{\vartheta_n}g(s) = \mathcal{L}^0 g(s) + \gamma\vartheta_n,$$

which implies that

$$\int_0^\infty \mathcal{L}^0 g(s)\mathbf{\Theta}_{\vartheta_n}(\mathrm{d}s) + \gamma\vartheta_n = 0. \tag{2.9}$$

By Eq. (2.9) and considering the weak convergence of $\mathbf{\Theta}_{\vartheta_n}$ toward $\bar{\mathbf{\Theta}}$, we obtain the following conclusion:

$$\int_0^\infty \mathcal{L}^0 g(s)\bar{\mathbf{\Theta}}(\mathrm{d}s) = 0.$$

This indicates that $\bar{\mathbf{\Theta}}$ serves as an invariant measure for Eq. (2.7). Given its unique nature, it follows that $\bar{\mathbf{\Theta}}$ is equivalent to $\mathbf{\Theta}_0$. Consequently, $\mathbf{\Theta}_{\vartheta_0}$ exhibits weak convergence toward $\mathbf{\Theta}_0$ as ϑ_0 approaches zero. □

3. Patterns of Extinction Dynamics

In this section, the assumption remains that $(r-\mu) > \frac{\sigma_1^2}{2}$. The ensuing finding confirms that $\lambda < 0$ represents the rate at which the count of infectious individuals approaches zero.

Lemma 3.1. *For any given values of ϵ and $M > 0$, there is a corresponding $\vartheta_1 > 0$ such that*

$$\mathbb{P}_{(s,i)}\left(\lim_{t\to\infty} I(t) = 0\right) \geq 1-\epsilon \text{ for each } (s,i) \in [0,M]\times[0,\vartheta_1].$$

Proof. Assuming that ϑ_0 adheres to the conditions outlined in Lemma 2.1, it then follows that

$$\left|\int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_{\vartheta_0}(\mathrm{d}s) - \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_0(\mathrm{d}s)\right| \leq \frac{|\lambda|}{4}.$$

Using the ergodicity properties of $\bar{S}^{\vartheta_0}(t)$, it can be established that a certain $T_1 > 0$ exists, ensuring that for every $\epsilon > 0$, the probability $\mathbb{P}(\Omega_1) \geq 1-\frac{\epsilon}{4}$, wherein

$$\Omega_1 = \left\{w : \frac{1}{t}\int_0^t \frac{\beta\bar{S}^{\vartheta_0}(u)}{1+\alpha\bar{S}^{\vartheta_0}(u)}\,\mathrm{d}u \leq \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_{\vartheta_0}(\mathrm{d}s) + \frac{|\lambda|}{4},\ t \geq T_1\right\}. \tag{3.1}$$

On the other hand, by the strong law of large numbers applicable to local martingale (see, e.g. Ref. 37 for more information and the references cited therein), we derive the following conclusion:

$$\lim_{t\to\infty}\frac{1}{t}B_2(t) = 0, \quad \text{a.s.} \tag{3.2}$$

Therefore, there is a T_2 such that the probability $\mathbb{P}(\Omega_2) \geq 1-\frac{\epsilon}{4}$, wherein

$$\Omega_2 = \left\{w : \frac{1}{t}\left|\sigma_2 B_2(t)\right| \leq \frac{|\lambda|}{4}, \text{ for all } t \geq T_2\right\}. \tag{3.3}$$

Define $T = \max\{T_1, T_2\}$. Additionally, we choose a sufficiently large $C > \frac{\beta}{\alpha}T$ to ensure that the probability $\mathbb{P}(\Omega_3) \geq 1 - \frac{\epsilon}{4}$, where

$$\Omega_3 = \left\{ w : |\sigma_2 B_2(t)| \leq C - \frac{\beta}{\alpha}T \text{ for all } t \in [0, T] \right\}. \tag{3.4}$$

By analyzing the I-equation in conjunction with Eq. (3.4), it leads us to the following result:

$$\begin{aligned} I(t) &= i \exp\left\{ \int_0^t \frac{\beta S(u)}{1 + \alpha S(u)} \mathrm{d}u - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t) \right\} \\ &\leq i \exp\left\{ \frac{\beta}{\alpha} t + \sigma_2 B_2(t) \right\} \\ &\leq \vartheta_1 \exp(C). \end{aligned} \tag{3.5}$$

Assume $\vartheta_1 \in [0, \vartheta_0 \exp(-C)]$. Under the condition that $I(0) = i \leq \vartheta_1$, it can be deduced that

$$I(t) < \vartheta_0 \quad \text{for all } t \in [0, T], \quad w \in \Omega_3.$$

Subsequently, let us establish the stopping time as follows:

$$\bar{\nu} = \inf\{t \geq 0 : \vartheta_0 \leq I(t)\}.$$

At this juncture, it can be shown that $\bar{\nu} = \infty$. For any $\omega \in \Omega_3$, it follows that $\bar{\nu} > T$. Consequently, considering $S(t) \leq \bar{S}^{\vartheta_0}$, it results in the following for $t \leq \bar{\nu}$:

$$\begin{aligned} I(t) &= i \exp\left\{ \int_0^t \frac{\beta S(u)}{1 + \alpha S(u)} \mathrm{d}u - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t) \right\} \\ &\leq i \exp\left\{ \int_0^t \frac{\beta \bar{S}^{\vartheta_0}}{1 + \alpha \bar{S}^{\vartheta_0}} \mathrm{d}u - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t) \right\}. \end{aligned} \tag{3.6}$$

Using Eqs. (3.1), (3.3), and (3.5) into Eq. (3.6), it follows that for any $\omega \in (\cap_{j=1}^3 \Omega_j)$, and for every $t \in [T, \bar{\nu})$, we deduce the following:

$$\begin{aligned} I(t) &\leq i \exp\left\{ t \int_0^\infty \frac{\beta s}{1 + \alpha s} \mathbf{\Theta}_{\vartheta_0}(\mathrm{d}s) + \frac{|\lambda|}{4} t - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) t + \frac{|\lambda|}{4} t \right\} \\ &\leq i \exp\left\{ \left(\int_0^\infty \frac{\beta s}{1 + \alpha s} \mathbf{\Theta}_0(\mathrm{d}s) + \frac{|\lambda|}{4} \right) t + \frac{|\lambda|}{4} t - \left(\mu + \gamma + \frac{\sigma_2^2}{2} \right) t + \frac{|\lambda|}{4} t \right\} \\ &\leq i \exp\left\{ t\lambda + \frac{|\lambda|}{4} t + \frac{|\lambda|}{4} t + \frac{|\lambda|}{4} t \right\} \end{aligned}$$

$$\leq i \exp\left\{t\lambda - \frac{3\lambda}{4}t\right\}$$

$$= i \exp\left\{\frac{\lambda}{4}t\right\} < i < \vartheta_0.$$

As a result, we must have $\bar{\nu} = \infty$ for $\omega \in (\cap_{j=1}^{3}\Omega_j)$, when $I(0) \leq \vartheta_1$. Under these conditions, we can establish that for all $t \geq T$, and $\omega \in (\cap_{j=1}^{3}\Omega_j)$, with $I(0) = i \leq \vartheta_1$, the following holds:

$$I(t) \leq i \exp\left\{\frac{\lambda}{4}t\right\}.$$

Consequently, this leads to the conclusion that $\lim_{t\to\infty} I(t) = 0$. □

Lemma 3.2. *For every ϵ and $M > 0$, there exists $\vartheta_1 > 0$ such that*

$$\mathbb{P}_{(s,i)}\left(\lim_{t\to\infty}\frac{1}{t}\ln(I(t)) = \lambda < 0\right) \geq 1 - \epsilon, \quad \textit{for each } (s,i) \in [0,M] \times [0,\vartheta_1].$$

Proof. Given that $I(t) \leq \vartheta_0$ holds for all $t \geq 0$ and for the majority of $\omega \in (\cap_{j=1}^{3}\Omega_j)$, one can deduce that $S(t) \leq \bar{S}^{\vartheta_0}$. Moreover, for almost all $\omega \in (\cap_{j=1}^{3}\Omega_j)$ and considering the ergodic properties of $S^{\bar{\vartheta}_0}(u)$, it is established that for a certain small $q > 0$:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (S(u))^{1+q}\mathrm{d}u \leq \lim_{t\to\infty}\frac{1}{t}\int_0^t (S^{\bar{\vartheta}_0}(u))^{1+q}\mathrm{d}u$$

$$\leq \int_0^\infty s^{1+q}\boldsymbol{\Theta}_{\vartheta_0}(\mathrm{d}s) < \infty. \tag{3.7}$$

Applying Lemma 3.1 and considering the uniform integrability specified in Eq. (3.7), the assortment of random occupation measures is presented in the following manner:

$$\bar{\mathcal{O}}^t(\cdot) = \frac{1}{t}\int_0^t \mathbb{I}_{\{S(u),I(u)\in(\cdot)\}}\mathrm{d}u.$$

For the majority of $\omega \in (\cap_{j=1}^{3}\Omega_j)$, the set is notably tight. Moreover, one can easily prove that the process $X(t)$ possesses a distinctive invariant probability measure, $(\boldsymbol{\Theta}_0 \times \delta)$, over the interval $[0,\infty] \times \{0\}$, with δ representing the Dirac measure concentrated at 0. Furthermore, it has been shown (refer to Refs. 38, 39) that as t approaches infinity, any weak limit of $\bar{\mathcal{O}}^t$ a.s. becomes an invariant probability measure, predominantly supported on $[0,\infty] \times \{0\}$. Consequently, $\bar{\mathcal{O}}^t(\cdot)$ exhibits weak convergence toward $(\boldsymbol{\Theta}_0 \times \delta)$ a.s. in $(\cap_{j=1}^{3}\Omega_j)$ when t tends to infinity. Following this line of weak convergence (as discussed in Ref. 38), it is concluded that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\beta S(u)}{1+\alpha S(u)}\mathrm{d}u = \int_0^\infty \frac{\beta s}{1+\alpha s}\boldsymbol{\Theta}_0(\mathrm{d}s). \tag{3.8}$$

Injecting Eq. (3.8) into (3.5), for almost every $\omega \in (\cap_{j=1}^{3}\Omega_j)$, and for each pair $(s,i) \in [0,M] \times [0,\vartheta_1]$, yields

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\ln(I(t)) &= \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_0(\mathrm{d}s) - \mu - \gamma - \frac{\sigma_2^2}{2} \\ &= \lambda < 0.\end{aligned}$$

Thus, given that the probability $\mathbb{P}(\omega \in \cap_{j=1}^{3}\Omega_j) > 1-\epsilon$, we obtain the desired assertion. The proof is complete. □

Theorem 3.1. *If the threshold is* $\lambda < 0$, *then*

$$\mathbb{P}_{(s,i)}\left(\lim_{t\to\infty} \frac{1}{t}\ln(I(t)) = \lambda < 0\right) = 1. \tag{3.9}$$

Proof. Consider any $\epsilon > 0$ as given. By Lemma 3.2, $X(t)$ exhibits transience within $\mathbb{R}^{(2,o)}$, leading to the absence of an invariant probability measure for $X(t)$ in $\mathbb{R}^{(2,o)}$. Consequently, $(\mathbf{\Theta}_0 \times \delta)$ stands as the sole invariant probability measure for $X(t)$ in $\mathbb{R}^2_+$. We then select M to be sufficiently large to satisfy the subsequent condition:

$$1 - \frac{\varepsilon}{2} < \mathbf{\Theta}_0 \quad (\{s \in (0,M)\}).$$

By Eq. (2.2), the process $X(t)$ exhibits tightness. Then, we can conclude that the occupation measure is characterized as follows:

$$\bar{\mathrm{O}}^t_{(s,i)}(\cdot) = \frac{1}{t}\int_0^t \mathbb{P}_{(s,i)}\left[X(u) \in (\cdot)\right]\mathrm{d}u,$$

is tightness within the state space $\mathbb{R}^2_+$. Additionally, the process $X(t)$ is characterized by an invariant measure, identified as the weak limit of $\bar{\mathrm{O}}^t_{(s,i)}$ with the progression of t toward infinity. Therefore, it is evident that $\bar{\mathrm{O}}^t_{(s,i)}$ converges to $(\mathbf{\Theta}_0 \times \delta)$ as t approaches infinity. Consequently, there exists $\bar{T} > 0$ and for any given $\vartheta > 0$ such that

$$\bar{\mathrm{O}}^{\bar{T}}_{(s,i)}\left[(0,H) \times (0,\vartheta)\right] > 1 - \varepsilon.$$

In other words, we have

$$\frac{1}{\bar{T}}\int_0^{\bar{T}} \mathbb{P}_{(s,i)}\left\{(S_t, I_t) \in (0,M) \times (0,\vartheta)\right\}\mathrm{d}t > 1 - \varepsilon.$$

Therefore, upon establishing the following setting:

$$\widehat{\nu} = \inf\left\{t \geq 0 : (S_t, I_t) \in (0,M) \times (0,\vartheta)\right\},$$

hence

$$\mathbb{P}_{(s,i)}\{\widehat{\nu} \leq \bar{T}\} > 1 - \varepsilon.$$

Using Lemma 3.2 in conjunction with the strong Markov property leads to the subsequent conclusion:

$$\mathbb{P}_{(s,i)}\left(\lim_{t\to\infty}\frac{1}{t}\ln(I(t))=\lambda<0\right)\geq 1-\epsilon \text{ for each } (s,i)\in\mathbb{R}_+^{(2,*)}.$$

Consequently, as ϵ approaches zero, we get the necessary outcome outlined in Eq. (3.9). The proof is complete. □

4. Ergodic Stationary Distribution

In this part of the study, we validate the existence of an ergodic stationary distribution for our epidemic model (1.2), underscoring the ongoing and significant impact of disease on the host population.

Theorem 4.1. *Assuming $\lambda > 0$ and $\mu > \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$, the system defined by Eq. (1.2) not only has a unique stationary distribution, denoted as $\upsilon^*(\cdot)$, but is also ergodic. Furthermore,*

(i) *For every function $f : \mathbb{R}_+^{(2,o)} \mapsto \mathbb{R}$ that is integrable with respect to υ^*, it follows that*

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(\tau))\mathrm{d}\tau=\int_{\mathbb{R}_+^{(2,o)}} f(x)\upsilon^*(\mathrm{d}x)\right\}=1.$$

(ii) *For each $(s,i)\in\mathbb{R}_+^{(2,*)}$, we have*

$$\lim_{t\to\infty}\|\mathbb{P}[t,(s,i),(\cdot)]-\upsilon^*(\cdot)\|=0,$$

where $\mathbb{P}[t,(s,i),(\cdot)]$ is the transition probability.

Proof. (i) Considering the framework of system (1.2), let us define

$$V_1=-\ln(I)+c_1[\ln(\widehat{S})-\ln(S)].$$

Applying the Itô formula, we get

$$\begin{aligned}\mathcal{L}V_1&=\left(\mu+\gamma+\frac{\sigma_2^2}{2}\right)-\frac{\beta S}{1+\alpha S}+c_1\left[r\left(1-\frac{\widehat{S}}{K}\right)-r\left(1-\frac{S}{K}\right)+\frac{\beta I}{1+\alpha S}-\gamma\frac{I}{S}\right]\\&\leq-\lambda+\int_0^\infty\frac{\beta s}{1+\alpha s}\mathbf{\Theta_0}(\mathrm{d}s)-\frac{\beta\widehat{S}}{1+\alpha\widehat{S}}+\frac{\beta\widehat{S}}{1+\alpha\widehat{S}}-\frac{c_1r}{K}\widehat{S}+\frac{c_1r}{K}S+\frac{c_1\beta I}{1+\alpha S}\\&\quad-c_1\gamma\frac{I}{S}\\&\leq-\lambda+\int_0^\infty\frac{\beta s}{1+\alpha s}\mathbf{\Theta_0}(\mathrm{d}s)-\frac{\beta\widehat{S}}{1+\alpha\widehat{S}}+\left(\beta-\frac{c_1r}{K}\right)\widehat{S}+\frac{c_1r}{K}S+c_1\beta I-c_1\gamma\frac{I}{S}.\end{aligned}$$

Choosing $c_1 = \frac{\beta K}{r}$, then

$$\mathcal{L}V_1 \leq -\lambda + \int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta_0}\,(\mathrm{d}s) - \frac{\beta\widehat{S}}{1+\alpha\widehat{S}} - c_1\gamma\frac{I}{S} + \frac{c_1 r}{K}S + c_1\beta I. \tag{4.1}$$

Moreover, we have

$$\begin{aligned}
&\mathcal{L}\left(\frac{1}{m+2}(S+I)^{m+2}\right)\\
&= (S+I)^{m+1}\left[rS\left(1-\frac{S}{K}\right) - \mu S - \mu I\right] + \frac{m+1}{2}(S+I)^m\left(\sigma_1^2S^2 + \sigma_2^2I^2\right)\\
&\leq rS(S+I)^{m+1} - \frac{r}{K}S^{m+3} - \mu I^{m+2} + \frac{m+1}{2}(S+I)^{m+2}\left(\sigma_1^2\vee\sigma_2^2\right)\\
&\leq -\frac{r}{2K}S^{m+3} - m\mu I^{m+2} - \frac{r}{2K}S^{m+3} - \mu(1-m)I^{m+2} + rS(S+I)^{m+1}\\
&\quad + \frac{m+1}{2}(S+I)^{m+2}\left(\sigma_1^2\vee\sigma_2^2\right)\\
&\leq -\frac{r}{2K}S^{m+3} - m\mu I^{m+2} + B,
\end{aligned} \tag{4.2}$$

where m is a constant satisfying

$$0 < m < \frac{2\mu - \sigma_1^2\vee\sigma_2^2}{2\mu + \sigma_1^2\vee\sigma_2^2}$$

and

$$\begin{aligned}
B = \sup_{X\in\mathbb{R}_+^{(2,o)}}\Big\{ &-\frac{r}{2K}S^{m+3} - \mu(1-m)I^{m+2}\\
&+ \frac{1}{2}(m+1)(S+I)^{m+2}(\sigma_1^2\vee\sigma_2^2) + rS(S+I)^{m+1}\Big\}.
\end{aligned}$$

Let $\tilde{V}\in\mathcal{C}^2(\mathbb{R}_+^{(2,o)};\mathbb{R})$, we define

$$\tilde{V}(S,I) = \overline{M}V_1(S,I,R) + \frac{1}{m+2}(S+I)^{m+2},$$

where $\overline{M} > 0$ is chosen to be sufficiently large to satisfy the subsequent condition:

$$-\overline{M}\lambda + E \leq -2, \tag{4.3}$$

where

$$E = \sup_{S\in(0,\infty)}\left\{-\frac{r}{4K}S^{m+3} + \frac{rc_1M}{K}S + B\right\}.$$

Showing that $\tilde{V}(S,I)$ attains a minimum at the point (S^*,I^*) is straightforward. Subsequently, we establish a non-negative $\mathcal{C}^2$-class function $W(S,I)$ in the following manner:

$$W(S,I) = \tilde{V}(S,I) - \tilde{V}(S^*,I^*).$$

Using Eqs. (4.1) and (4.2), we derive the following result:

$$\begin{aligned}\mathcal{L}W &\leq -\overline{M}\lambda + \frac{rc_1\overline{M}}{K}S + c_1\overline{M}\beta I - \frac{r}{2K}S^{m+3} - \mu m I^{m+2} + B - c_1\gamma\overline{M}\frac{I}{S} \\ &\quad + \overline{M}\bigg(\int_0^\infty \frac{\beta s}{1+\alpha s}\Theta_0(\mathrm{d}s) - \frac{\beta\widehat{S}}{1+\alpha\widehat{S}}\bigg), \\ &= F(S,I) + \overline{M}\left(\int_0^\infty \frac{\beta s}{1+\alpha s}\boldsymbol{\Theta_0}(\mathrm{d}s) - \frac{\beta\widehat{S}}{1+\alpha\widehat{S}}\right),\end{aligned}$$

where

$$F(S,I) = -\overline{M}\lambda + \frac{rc_1\overline{M}}{K}S + c_1\overline{M}\beta I - \frac{r}{2K}S^{m+3} - \mu m I^{m+2} + B - c_1\gamma\overline{M}\frac{I}{S}.$$

Moreover, by applying the specific expression of $F(S,I)$, it leads us to the following:

Case 1. Under the conditions where either $S \rightsquigarrow 0^+$, $S \rightsquigarrow \infty$, or $I \rightsquigarrow \infty$, it follows that

$$\begin{aligned}F(S,I) &= -\overline{M}\lambda + \frac{rc_1\overline{M}}{K}S + c_1\overline{M}\beta I - \frac{r}{2K}S^{m+3} - \mu m I^{m+2} + B - c_1\gamma\overline{M}\frac{I}{S} \\ &\leq -\infty.\end{aligned}$$

Case 2. If $I \rightsquigarrow 0^+$, then it implies that

$$\begin{aligned}F(S,I) &= -\overline{M}\lambda + \frac{rc_1\overline{M}}{K}S + c_1\overline{M}\beta I - \frac{r}{2K}S^{m+3} - \mu m I^{m+2} + B - c_1\gamma\overline{M}\frac{I}{S} \\ &\leq -\overline{M}\lambda + c_1\overline{M}\beta I + E \rightsquigarrow -\overline{M}\lambda + E \\ &\leq -2.\end{aligned}$$

This conclusion is derived from Eq. (4.3). Consequently, with a sufficiently small $\epsilon > 0$, it is possible to deduce that

$$F(X) \leq -1 \text{ for each } X = (S,I) \in (\mathbb{R}_+^{(2,o)} \backslash \mathcal{D}), \tag{4.4}$$

where

$$\mathcal{D} = \left\{(S,I) \in \mathbb{R}_+^{(2,o)} : \ \epsilon < S < \frac{1}{\epsilon}, \, \epsilon < I < \frac{1}{\epsilon}\right\}.$$

Furthermore, there exists a constant $\bar{C} > 0$ such that

$$F(X) \leq \bar{C} \text{ for all } X \in \mathbb{R}_+^{(2,o)}. \tag{4.5}$$

Therefore, it results in the following conclusion:

$$\begin{aligned}-\mathbb{E}\left[W(X_0)\right] &\leq -\mathbb{E}\left[W(X_0)\right] + \mathbb{E}\left[W(X(t))\right] \\ &= \int_0^t \mathbb{E}\left[\mathcal{L}W(X(\tau))\right] \mathrm{d}\tau \\ &\leq \int_0^t \mathbb{E}\left[F(X(\tau))\right] \mathrm{d}\tau + \overline{M}\mathbb{E}\Big[\int_0^t \int_0^\infty \frac{\beta s}{1+\alpha s}\Theta_0\left(\mathrm{d}s\right)\mathrm{d}\tau \\ &\quad - \int_0^t \frac{\beta \widehat{S}(\tau)}{1+\alpha\widehat{S}(\tau)}\,\mathrm{d}\tau\Big].\end{aligned}$$

Consequently, by applying Eqs. (4.4), (4.5) and considering the ergodic nature of $\widehat{S}$, we get the following result:

$$\begin{aligned}0 &\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[F\left(X(\tau)\right)\right]\mathrm{d}\tau \\ &= \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left[\mathbb{E}\left(F(X(\tau))\mathbb{I}_{\{X(\tau)\in\mathcal{D}\}}\right) + \mathbb{E}\left(F(X(\tau))\mathbb{I}_{\{X(\tau)\in\mathcal{D}^c\}}\right)\right]\mathrm{d}\tau \\ &\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left[\bar{C}\mathbb{P}(X(\tau)\in\mathcal{D}) - \mathbb{P}\left(X(\tau)\in\mathcal{D}^c\right)\right]\mathrm{d}\tau \\ &= -1 + \left(\bar{C}+1\right)\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}(X(\tau)\in\mathcal{D})\mathrm{d}\tau.\end{aligned}$$

This indicates that for every initial condition $X_0 = (s, i)$ within space $\mathbb{R}_+^{(2,o)}$,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}(\tau,(s,i),\mathcal{D})\mathrm{d}\tau \geq \frac{1}{\bar{C}+1}. \tag{4.6}$$

Given the invariance of the space $\mathbb{R}_+^{(2,o)}$ as outlined in Eq. (1.2), it is appropriate to analyze $X(t)$ within this state space. Moreover, the invariance of $\mathbb{R}_+^{(2,o)}$, coupled with the implications of inequality (4.6), indicates the existence of a stable probability measure υ^* on $\mathbb{R}_+^{(2,o)}$ (refer to Ref. 40 for more information and the references cited therein). Furthermore, the fact that $B_i(t)$ for $i = 1, 2$ are independent suggests that $\mathbb{R}_+^{(2,o)}$ serves as the support for the measure υ^*, as detailed in Lemma 4.1 of Khasminskii.[41] In conclusion, with the insights provided in Refs. 42–44, we can substantiate assertions (i) and (ii), thereby completing the proof. □

5. Numerical Simulations

In this section, we will conduct a range of numerical simulations using the Euler-Maruyama method, as further elaborated in the referenced work (refer to Ref. 45 for further information and the references cited therein). These simulations aim

to represent and validate our research conclusions visually. We will focus on the discretized version of Eq. (1.2) as follows:

$$\begin{cases} S_{k+1} = S_k + \left[rS_k \left(1 - \dfrac{S_k}{K}\right) - \mu S_k - \dfrac{\beta S_k I_k}{1+\alpha S_k} + \gamma I_k \right] \Delta t + \sigma_1 S_k \sqrt{\Delta t}\tau_{(1,k)}, \\ I_{k+1} = I_k + \left[\dfrac{\beta S_k I_k}{1+\alpha S_k} - (\mu+\gamma) I_k \right] \Delta t + \sigma_2 I_k \sqrt{\Delta t}\tau_{(2,k)}. \end{cases}$$

In this formulation, $\tau_{(i,k)}$ for $i = 1, 2$ are independent Gaussian random variables, each adhering to a standard normal distribution, symbolized as $\mathcal{N}(0, 1)$. These simulations aim to offer an extensive visual depiction of the real-world applications of our key findings. Additionally, we consistently have

$$\mathbb{E}_{\mathbf{\Theta}_0}(\widehat{S}) = \int_0^\infty s\mathbf{\Theta}_0(\mathrm{d}s),$$

where $\mathbf{\Theta}_0$ exhibits absolute continuity compared to the Lebesgue measure over the $(0, \infty)$ range. Moreover, this can be further elaborated as follows:

$$\lambda = \mathbb{E}_{\mathbf{\Theta}_0}(f(\widehat{S})) - \mu - \gamma - \frac{\sigma_2^2}{2}.$$

Here, $f(\widehat{S}) = \beta\widehat{S}/(1+\alpha\widehat{S})$. Using the Monte-Carlo method enables the approximation of the expected value $\mathbb{E}_{\mathbf{\Theta}_0}(f(\widehat{S}))$ at a specific time t. With this methodology, the following examples are presented:

Example 1. In system (1.2), we choose $K = 1.5$, $r = 1.0$, $\beta = 0.3$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.2$, $\sigma_1 = 0.4$, $\sigma_2 = 0.45$, $(S_0, I_0) = (0.8, 0.6)$. These values make the assumption in Theorem 3.1 satisfied. Namely,

$$r - \mu = 0.8 > \frac{\sigma_1^2}{2} \quad \text{and} \quad \lambda \approx -0.13 < 0.$$

Consequently, the condition for extinction outlined in Theorem 3.1 is met. This outcome is validated through computer simulations illustrated in Fig. 1.

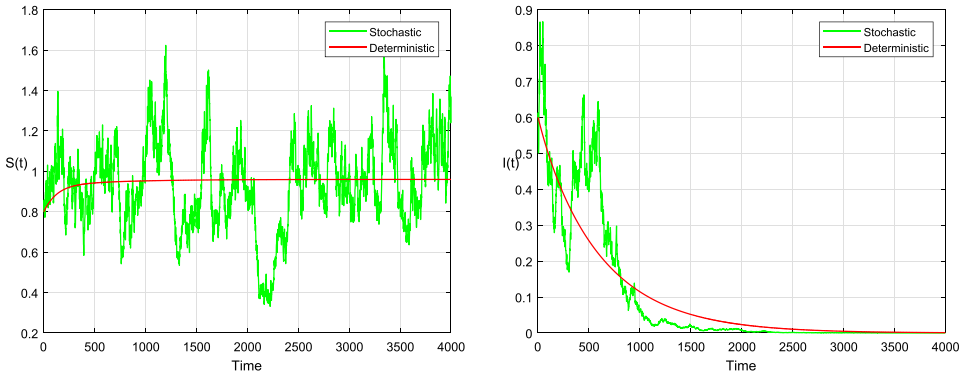

Fig. 1. Dynamics of the solution $(S(t), I(t))$ with initial condition $(S_0, I_0) = (0.8, 0.6)$, indicating that the disease dies out when $\lambda < 0$.

Example 2. Choosing $K = 1.5$, $r = 1.0$, $\beta = 0.6$, $\mu = 0.2$, $\gamma = 0.3$, $\alpha = 0.2$, $\sigma_1 = 0.1$, $\sigma_2 = 0.15$. By direct calculation, we have

$$\mu = 0.2 > \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2) \quad \text{and} \quad \lambda \approx 0.22 > 0.$$

According to Theorem 4.1, it is established that a unique ergodic stationary distribution, represented as $\upsilon^*(\cdot)$, exists for the system detailed in Eq. (1.2). This is demonstrated in Fig. 3. Furthermore, the persistence of the solution for the stochastic model (1.2) is evident in its mean behavior, as depicted in Fig. 2.

The numerical simulations presented in this section exhibit strong concordance with the theoretical predictions. In Example 1, where $\lambda < 0$, the conditions for disease extinction are satisfied, and the results shown in Fig. 1 confirm that the epidemic eventually vanishes. In contrast, in Example 2, where $\lambda > 0$, the persistence condition is met, and the simulation results illustrated in Figs. 2 and 3

AQ: Figures 2 and 3 are out of sequence in the text citation. Please check.

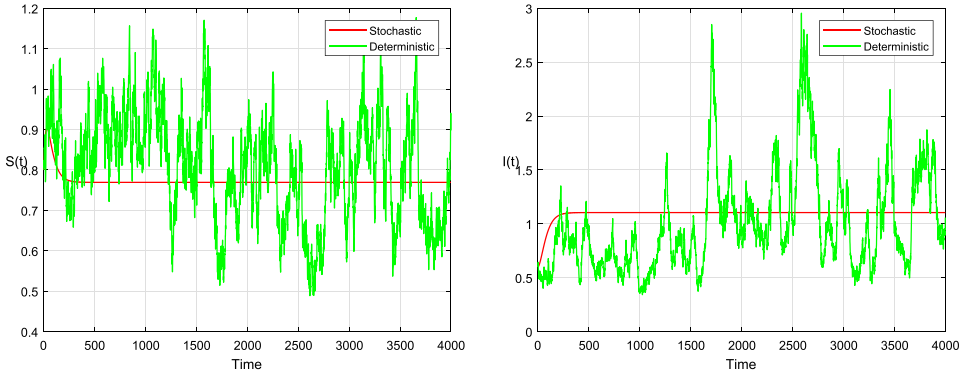

Fig. 2. Dynamics of the solution $(S(t), I(t))$ with initial condition $(S_0, I_0) = (0.8, 0.6)$, indicating that the disease persists when $\lambda > 0$.

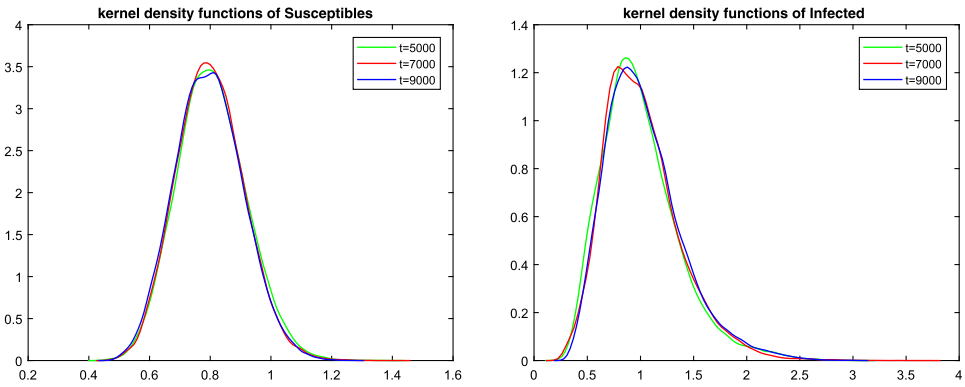

Fig. 3. Kernel density functions of $S(t)$ and $I(t)$ for the SDE described in (1.2).

indicate that the system converges asymptotically to a unique ergodic stationary distribution. These numerical findings provide strong empirical support for the theoretical analysis, reinforcing the validity of the extinction and persistence conditions established in Theorems 3.1 and 4.1.

6. Concluding Remarks

Our study thoroughly explored a stochastic SIS epidemic model, integrating logistic growth and saturation incidence elements. Our comprehensive research has yielded a significant understanding of the disease transmission dynamics. We have delineated crucial criteria that govern whether the disease persists or is eradicated within this sophisticated model. Our findings indicate that a negative value of λ leads to disease extinction, while a positive value implies persistence. To concisely represent this pivotal discovery, we introduce the parameter $\mathcal{R}_0^s$, which is defined in the following manner:

$$\mathcal{R}_0^s := \left(\mu+\gamma+\frac{\sigma_2^2}{2}\right)^{-1}\int_0^\infty \frac{\beta s}{1+\alpha s}\mathbf{\Theta}_0\left(\mathrm{d}s\right) = \lambda\left(\mu+\gamma+\frac{\sigma_2^2}{2}\right)^{-1}+1.$$

Our research has established that $\mathcal{R}_0^s$ acts as the critical reproduction number in the stochastic system (1.2), providing a coherent and meaningful interpretation. When $\mathcal{R}_0^s$ falls below one ($\mathcal{R}_0^s < 1$), it indicates a trend toward disease extinction, whereas a value exceeding one ($\mathcal{R}_0^s > 1$) denotes the likelihood of disease persistence. This parameter, $\mathcal{R}_0^s$, is essential for analyzing and comprehending the key processes that dictate the disease trajectory within the studied framework.

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Appendix A

Proof of Theorem 2.1. Note that the coefficients in system (1.2) exhibit local Lipschitz continuity. Hence, given any initial value $X(0) \in \mathbb{R}_+^2$, there exists a single, unique local solution for $t \in [0, \nu_e]$ almost surely, where ν_e represents the explosion time (for additional details, refer to Ref. 37). Let $m_0 \geq 1$ be such that S_0 and I_0 all lie within the interval $[\frac{1}{m_0}, m_0]$. We define a stopping time as per Ref. 37 for each

integer $m \geq m_0$

$$\nu_m = \inf\left\{0 \leq t < \nu_e : \frac{1}{m} \geq \min\{S_t, I_t\} \text{ or } m \leq \max\{S_t, I_t\}\right\}.$$

Evidently, ν_m is increasing as $m \to \infty$. Denote $\nu_\infty = \lim_{m\to\infty} \nu_m$, from which we deduce $\nu_\infty \leq \nu_e$ a.s. We need to prove that $\nu_\infty = \infty$ a.s. Suppose instead that

$$\mathbb{P}\{\nu_\infty < \infty\} > 0.$$

Then, there exists $T > 0$ and $\kappa \in (0, 1)$ such that

$$\mathbb{P}\{\nu_\infty \leq T\} \geq \kappa.$$

Thus, there exists $m_1 \geq m_0$, where $\mathbb{P}\{\nu_m \leq T\} \geq \kappa$. There is an integer $m \geq m_1$, denote $\Omega_m = \{\nu_m \leq T\}$, then

$$\mathbb{P}(\Omega_m) \geq \kappa. \tag{6.7}$$

Now, consider the Lyapunov function $\hat{V} \in C^2(\mathbb{R}^2_+; \mathbb{R}_+)$ given by

$$\hat{V}(X) = \left[S - a\ln\left(\frac{S}{a}\right) - a\right] + [I - \ln(I) - 1],$$

where a is a positive constant to be determined later. Applying Itô's formula, we obtain

$$\mathrm{d}\hat{V}(X) = \mathcal{L}\hat{V}\mathrm{d}t + \sigma_1(S-a)\mathrm{d}B_1 + \sigma_2(I-1)\mathrm{d}B_2,$$

where

$$\mathcal{L}\hat{V} \leq \frac{-rS^2}{K} + \frac{(K+a)r}{K}S + \frac{a\beta I}{1+\alpha S} + a\mu - \mu I + \gamma + 2\mu + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2}.$$

Choosing $a = \frac{\mu}{\beta}$, we obtain

$$\mathcal{L}\hat{V} \leq \widehat{K},$$

where $\widehat{K}$ is a positive constant. Consequently,

$$\mathrm{d}\hat{V}(X) \leq \widehat{K}\mathrm{d}t + \sigma_1(S-a)\mathrm{d}B_1 + \sigma_2(I-1)\mathrm{d}B_2. \tag{6.8}$$

Integrating inequality (6.8) over $(0, \nu_m \wedge T)$ and taking expectations, we get

$$\mathbb{E}\hat{V}(X(\nu_m \wedge T)) \leq \hat{V}(X(0)) + \widehat{K}T.$$

For each $\omega \in \Omega_m$, either $S(\nu_m, \omega)$ or $I(\nu_m, \omega)$ equals either m or $\frac{1}{m}$. Hence

$$\hat{V}(X(\nu_m, \omega)) \geq \kappa\bigg(\left[m - a - a\ln\frac{m}{a}\right] \wedge \left[\frac{1}{m} - a + a\ln(am)\right] \wedge [m - 1 - \ln(m)],$$
$$\wedge \left[\frac{1}{m} - 1 + \ln(m)\right]\bigg).$$

By letting $m \to \infty$, we get

$$\infty > \hat{V}(X(0)) + \widehat{K}T = \infty.$$

This is a contradiction. Hence

$$\nu_\infty = \infty \quad \text{a.s.}$$

The proof is completed. □

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