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Stochastic Analysis of COVID-19 Epidemics Under Quarantine Measures

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To enhance our understanding of the complex dynamics of the COVID-19 epidemic, we investigated a stochastic SIQS epidemic model that includes a compartment specifically for populations under quarantine. This study aims to examine the impact of environmental noise on the asymptotic properties of a stochastic variant of the classical SIQS epidemic model. Our model incorporates white noise, colored noise, and Lévy noise. Under certain parameter conditions, we demonstrate that the disease-free equilibrium state is globally asymptotically stable in probability. Furthermore, we establish the conditions under which the disease persists, considering the intensities of various noises, the model parameters, and the stationary distribution of the Markov chain. A key finding of our work is that our conditions are sufficient and nearly necessary to predict the extinction or persistence of the epidemic. Numerical simulations support the results presented.

Keywords: Stochastic SIQS model; Lévy jumps; Extinction; Exponentially stability; Persistence.

1. Introduction

The COVID-19 pandemic, caused by the novel SARS-CoV-2 virus, has emerged as one of the most significant global health crises of the 21st century. First detected in Wuhan, China, in December 2019, the virus has since spread to 212 countries and territories, resulting in over 800 million infections and more than 7 million deaths as of September 2024. The World Health Organization (WHO) declared COVID-19 a Public Health Emergency of International Concern on January 30, 2020, and subsequently classified it as a global pandemic. The virus rapid and extensive spread has placed immense strain on healthcare systems worldwide, with hospitals reporting over 70% capacity utilization for ICU beds [1]. Daily cumulative cases have fluctuated significantly, with a peak of approximately 800,000 new cases reported in a single day during the pandemic peak periods. In response, nations have implemented various mitigation strategies, including quarantine, isolation, travel bans, and lockdowns. Despite these efforts, the ongoing lack of effective treatments and vaccines continues to strain healthcare systems, particularly in economically disadvantaged regions [2]. According to the International Monetary Fund, the broader socio-economic impacts of the pandemic are evident, with an estimated global economic contraction of 3% in 2020 and widespread job losses. These challenges underscore the need for advanced methodologies in disease detection and forecasting [3]. Addressing these needs through innovative research is crucial for developing effective prevention strategies and informing comprehensive public health and economic policies for future outbreaks [3]. The SIQS epidemiological model is a compartmental approach used to analyze the dynamics of infectious disease spread by categorizing the population into three key groups: The Susceptible (S) group consists of individuals who have not yet been infected but are vulnerable to contracting the disease. The Infectious (I) group includes individuals who are currently carrying the disease and capable of transmitting it to others. Finally, the Quarantined (Q) group encompasses individuals who have been confirmed as infected and placed in isolation to prevent further spread of the infection.The transitions between these compartments are modeled using differential equations that describe the rates at which individuals move from one state to another, capturing the epidemic's progression. This enhanced model incorporates additional compartments for asymptomatic, quarantined, and isolated individuals, which are crucial for accurately reflecting the diverse stages of the disease and the various interventions implemented to curb its spread [4,5,6,7,8,9]. A simple SIQS epidemic model is described by the following differential system:

$$\begin{cases} dS \ = [\Lambda \ - \mu \ S - \beta \ SI \ + \gamma \ I + \epsilon \ Q]\, dt, \\ dI \ = [-(\mu \ + d + \lambda \ + \gamma \ ) \ I \ + \beta \ SI \ ]\, dt, \\ dQ \ = [-(\mu \ + d \ + \epsilon \ )Q + \lambda \ I]\, dt. \end{cases} \tag{1.1}$$

This model integrates multiple parameters to depict disease dynamics accurately. The parameter Λ represents the rate at which new individuals enter the susceptible population through births and immigration. The natural death rate for all individuals is denoted by μ. The parameter β quantifies the average number of interactions per susceptible individual that result in infection outside quarantine. The recovery rate of infected individuals, who return to the susceptible population, is described by γ. The parameter ϵ represents the rate at which quarantined individuals recover and rejoin the susceptible population. Additionally, λ indicates the rate at which infected individuals are moved to quarantine. Finally, δ denotes the constant death rate due to the disease for both infected and quarantined individuals. This study analyzes a model to gain insights into various factors influencing disease transmission. These factors include the effects of quarantine measures, identification of infected individuals, vaccination efforts, fluctuations in transmission rates, and the evolution of the disease over time. Hethcote et al. laid the groundwork for infectious disease modeling with their SIQS and SIQR models [10]. The inherent stochasticity of real-world events necessitates the development of stochastic models. To account for this, several studies have explored parameter perturbation techniques to construct stochastic epidemic models [11,12,13,14,15,16,17,18,19,20,21,22]. Zhang and Huo [23] further explored these models by comparing the long-term behaviors of a stochastic SIQS model with a deterministic counterpart, accounting for non-linear effects. They also introduced a stochastic SIQS model and investigated the conditions under which diseases persist or become extinct [24]. We further develop this model by incorporating random fluctuations in the transmission rate using white noise. A specific noise term is added, with σ controlling the intensity of the randomness. To enhance our approach, we have incorporated Lévy perturbations, allowing us to capture the inherent variability in disease spread better. This modification facilitates the simulation of more realistic scenarios, including predicting rare events such as widespread outbreaks or the emergence of new variants. Additionally, it enables the assessment of intervention measures like lockdowns and vaccination campaigns, the estimation of potential risks for informed decision-making, and the improvement of model accuracy by accounting for observed data variability. For those interested in stochastic models influenced by Lévy noise [25,26,27,28,29]. The basic SIQS epidemic model, incorporating both white noise and Lévy noise, can be described by the following differential equations

$$\begin{cases} dS &= [\Lambda \ - \mu \ S - \beta \ SI + \gamma \ I + \epsilon \ Q] \, dt - \sigma \ SIdB_t - dK_t, \\ dI &= [-(\mu \ + d + \lambda \ + \gamma \ I + \beta \ SI] \, dt + \sigma \ SIdB_t + dK_t, \\ dQ &= [-(\mu \ + d + \epsilon \ )Q + \lambda \ I] \, dt. \end{cases} \tag{1.2}$$

Consider the integral expression

$$K_t = \int_0^t \int_{\mathbb{Y}} \mathcal{G} \ (\ y \ ) S(s-) I(s-) \breve{\mathcal{N}}(ds \ , \ dy),$$

where $\breve{\mathcal{N}}(ds, dy) = \mathcal{N}(ds, dy) - \pi(dy)ds$ denotes the compensated Poisson measure. Here, $\mathcal{N}(ds, dy)$ is a Poisson counting measure that is independent of $B(t)$, ds represents the Lebesgue measure and π is a Lévy measure defined on a measurable subset $\mathbb{Y}$ of $[0, \infty)$. The function $\mathcal{G}(\cdot)$ is continuously differentiable. It characterizes the impact of random jumps in the population, satisfying $|\mathcal{G}(y)| \leq \dfrac{\overset{\vee}{\mu}}{\hat{\Lambda}}$ for all $y \in \mathbb{Y}$. The terms $S(s-)$ and $I(s-)$ denote the left-hand limits of the functions $S(s)$ and $I(s)$, respectively. Thereafter, we use the notations

$$\overset{\vee}{\Lambda} = \min_{s \in \mathcal{E}} \Lambda_{\upsilon(s)}, \quad \hat{\Lambda} = \max_{s \in \mathcal{E}} \Lambda_{\upsilon(s)}, \quad \hat{d} = \max_{s \in \mathcal{E}} \Lambda_{\upsilon(s)}, \quad \overset{\vee}{\mu} := \min_{s \in \mathcal{E}} \mu_{\upsilon(s)} \quad \text{and} \quad \hat{\mu} = \max_{s \in \mathcal{E}} \mu_{\upsilon(s)}.$$

We will refer to these limits as $S(s)$ and $I(s)$ for simplicity. In recent work (See for instance [30,31,32,33,34]) several authors have explored a different form of environmental disturbance known as color noise or telegraph noise. This type of noise represents transitions between two or more environmental states. These states vary due to factors like nutrition, climate or sociocultural influences, impacting the speed of disease transmission either accelerating or decelerating it. So, incorporating regime switching into the SIQS model for studying COVID-19 offers a multifaceted advantage by enabling dynamic intervention modeling, accounting for seasonal variations, capturing behavioral shifts, adapting to new variants, analyzing population heterogeneity, assessing crisis responses and guiding policy decisions. A finite-state Markov chain effectively models this regime-switching mechanism. Let $\upsilon(t)$ be defined in a finite state space $\mathcal{E} = \{1, 2, ..., m\}$ with the generator $\Upsilon = (\chi_{uv})_{1 \leq u,v \leq m}$ given, for $\delta > 0$, by

$$\mathbb{P}\left(\upsilon(t+\delta) = v | \upsilon(t) = u\right) = \begin{cases} \chi_{uv}\delta + o(\delta), & \text{if } \ u \neq v, \\ 1 + \chi_{uu}\delta + o(\delta), & \text{if } \ u = v. \end{cases} \tag{1.3}$$

Here, χ_{uv} is the transition rate from u to v while

$$\chi_{uu} = - \sum_{u \neq v} \chi_{uv}. \tag{1.4}$$

Our stochastic SIQS model under regime switching, jump perturbation and white noise is described by

$$\begin{cases} dS = \left[\Lambda_{\upsilon(t)} - \mu_{\upsilon(t)}S - \beta_{\upsilon(t)}SI + \gamma_{\upsilon(t)}I + \epsilon_{\upsilon(t)}Q\right]dt - \sigma_{\upsilon(t)}SIdB_t - dK_t, \\ dI = \left[-(\mu_{\upsilon(t)} + d_{\upsilon(t)} + \lambda_{\upsilon(t)} + \gamma_{\upsilon(t)})I + \beta_{\upsilon(t)}SI\right]dt + \sigma_{\upsilon(t)}SIdB_t + dK_t, \\ dQ = \left[-(\mu_{\upsilon(t)} + d_{\upsilon(t)} + \epsilon_{\upsilon(t)})Q + \lambda_{\upsilon(t)}I\right]dt. \end{cases} \tag{1.5}$$

2. Global existence and positivity

To begin with, we define the subset $\Delta = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3_+ \mid \dfrac{\overset{\vee}{\Lambda}}{\hat{\mu} + \hat{d}} \le x_1 + x_2 + x_3 \le \dfrac{\hat{\Lambda}}{\overset{\vee}{\mu}}\right\}$.

Lemma 2.1. (Lipster [35]) Consider the local martingale N_t, $t \ge t_0$, which vanishes at time 0. Define $\varphi_{N_t} := \displaystyle\int_{t_0}^{t} \frac{d < N_s, N_s >}{(1+s)^2}$, where $< N_t, N_t >$ is Meyer angle bracket process. Then,

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{N_t}{t} = 0\right) = 1, \quad \text{provided that} \qquad \mathbb{P}\left(\lim_{t\to\infty} \varphi_{N_t} < \infty\right) = 1.$$

Theorem 2.1. If for some constants $\varrho_j < 1$, $j \in \mathcal{E}$ we have

$$|\mathcal{G}(y, j)| \le \varrho_j, \quad \text{for every} \quad (y, j) \in \mathbb{Y} \times \mathcal{E}. \tag{2.6}$$

Then, for every $(S_0, I_0, Q_0) \in \Delta$, there is a unique solution $(S, I, Q) \in \Delta$ to model (1.5).

Proof. Let $(S_0, I_0, Q_0) \in \Delta$ and $t \ge 0$. We can easily show that the total population $N = S + I + Q$ in system (1.5) verifies for all $0 \le s \le t$,

$$dN(s) = \left(\Lambda_{\upsilon(s)} - \mu_{\upsilon(s)}N(s) - d_{\upsilon(s)}(I(s) + Q(s))\right)ds, \tag{2.7}$$

then

$$dN(s) \ge (\overset{\vee}{\Lambda} - (\hat{\mu} + \hat{d})N(s))ds \qquad \text{a.s.},$$

which implies by integration

$$N(t) - \frac{\overset{\vee}{\Lambda}}{\hat{\mu} + \hat{d}} \ge \left(N_0 - \frac{\overset{\vee}{\Lambda}}{\hat{\mu} + \hat{d}}\right) e^{-(\hat{\mu} + \hat{d})t} \qquad \text{for all } s \in (0, t) \qquad \text{a.s.},$$

so we have

$$N(s) \ge \frac{\overset{\vee}{\Lambda}}{\hat{\mu} + \hat{d}} \qquad \text{for all } s \in (0, t) \qquad \text{a.s.}. \tag{2.8}$$

Beside, using the positivity of I and Q, we drive from (2.7) that

$$dN(t) \leq \left(\hat{\Lambda} - \overset{\vee}{\mu} N(t)\right) dt.$$

Integrating the above inequality we get

$$N(s) \leq \frac{\hat{\Lambda}}{\overset{\vee}{\mu}} \quad \text{for all } s \in (0, t) \quad \text{a.s..} \tag{2.9}$$

As the coefficients of the equations are locally Lipschliz-continuous, for all (S_0, I_0, Q_0) in Δ, there is a unique maximal local solution (S_t, I_t, Q_t) for $t \in [0, \tau_e)$, with τ_e represents the timing of the explosion, taking $\varepsilon, \varepsilon_0 > 0$ where $S_0, I_0, Q_0 > \varepsilon_0$. For $\varepsilon \leq \varepsilon_0$, assuming the stopping time

$$\tau_\varepsilon = \inf\{t \in [0, \tau_e), \quad S_t \leq \varepsilon \ \text{ or } \ I_t \leq \varepsilon \ \text{ or } \ Q_t \leq \varepsilon\}. \tag{2.10}$$

Let us define the function for $(S, I, Q, j) \in \Delta \times \mathcal{E}$

$$W_1(S, I, Q, j) = \ln\left(\frac{1}{S}\right) + \ln\left(\frac{1}{I}\right) + \ln\left(\frac{1}{Q}\right).$$

Using Itô formula, we drive for every $t \geq 0$ and $s \in [0, t \wedge \tau_\varepsilon]$

$$\begin{aligned} dW_1(S, I, Q, j) = &\left(3\mu_\upsilon + 2d_r + \lambda_r + \gamma_\upsilon + \epsilon_\upsilon - \frac{\Lambda_\upsilon}{S} + \beta_\upsilon I - \frac{\gamma_\upsilon I}{S} - \frac{\epsilon_\upsilon Q}{S} + \frac{\sigma_\upsilon^2 I^2}{2}\right. \\ &\left. - \beta_\upsilon S + \frac{\sigma_\upsilon^2 S^2}{2} - \frac{\lambda_\upsilon I}{Q}\right) ds + \int_{\mathbb{Y}} \left(-\left(\ln\left(1 - \mathcal{G}(y,\upsilon)I\right) - \mathcal{G}(y,\upsilon)I\right)\right. \\ &\left. + \left(-\ln\left(1 + \mathcal{G}(y,\upsilon)S\right) + \mathcal{G}(y,\upsilon)S\right)\right) \pi(dy)dt + \sigma_\upsilon(I - S)dB \\ &- \int_{\mathbb{Y}} \left(\ln\left(1 - \mathcal{G}(y,\upsilon)I\right) + \ln\left(1 + \mathcal{G}(y,\upsilon)S\right)\right) \breve{\mathcal{N}}(dt, dy). \end{aligned}$$

By (2.6) and (2.9), we get

$$\begin{aligned} dW_1(S, I, Q, j) \leq &\left(3\mu_\upsilon + 2d_\upsilon + \lambda_\upsilon + \gamma_\upsilon + \epsilon_\upsilon + \beta_\upsilon I + \frac{\sigma_\upsilon^2 I^2}{2} + \frac{\sigma_\upsilon^2 S^2}{2}\right. \\ &\left. - 2\ln\left(1 - \varrho_{\upsilon(s)} \frac{\hat{\Lambda}}{\overset{\vee}{\mu}}\right) \pi(\mathbb{Y})\right) dt + \sigma_\upsilon(I - S)dB \\ &- \int_{\mathbb{Y}} \left(\ln\left(1 - \mathcal{G}(y,\upsilon)I\right) + \ln\left(1 + \mathcal{G}(y,\upsilon)S\right)\right) \breve{\mathcal{N}}(dt, dy). \end{aligned}$$

Combining it with (2.9), we get

$$W_1(X, j) \leq W_1(S_0, I_0, Q_0, j) + \max_{j \in \mathcal{E}}\{k_j\}s + \Lambda_t^0 - \Lambda_t^1,$$

so

$$W_1(X, j) \leq \ln\left(\frac{1}{S_0}\right) + \ln\left(\frac{1}{I_0}\right) + \ln\left(\frac{1}{Q_0}\right) + \max_{j \in \mathcal{E}}\{k_j\}s + \Lambda_t^0 - \Lambda_t^1,$$

where

$$k_j = 3\mu_j + 2d_j + \lambda_j + \gamma_j + \epsilon_j + \beta_j I + \frac{\sigma_j^2 I^2}{2} + \frac{\sigma_\upsilon^2 S^2}{2} - 2\ln\left(1 - \varrho_j \left(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}}\right)\right) \pi(\mathbb{Y}),$$

and

$$\Lambda_s^0 = \int_0^s \sigma_\upsilon (I - S)\, dB(u),$$
$$\Lambda_s^1 = \int_0^s \int_{\mathbb{Y}} (\ln(1 - \mathcal{G}(y,\upsilon)I) + \ln(1 + \mathcal{G}(y,\upsilon)S))\breve{\mathcal{N}}(dt, dy).$$

Hence, Λ_s^0 and Λ_s^1 are real-valued continuous martingales having the following Meyer angle bracket process

$$\langle \Lambda_s^0, \Lambda_s^0 \rangle = \int_0^s \sigma_\upsilon^2 (I-S)^2\, ds \le 2 \max_{j\in\mathcal{E}} \{\sigma_j^2\} s,$$

and

$$\begin{aligned} < \Lambda_s^1, \Lambda_s^1 > &= \int_0^s \int_{\mathbb{Y}} (\ln(1 - \mathcal{G}(y,\upsilon)I) + \ln(1 + \mathcal{G}(y,\upsilon)S))^2 \pi(dy)ds, \\ &\le \int_0^s \int_{\mathbb{Y}} (\ln(1 - \mathcal{G}(y,\upsilon)I))^2 + (\ln(1 + \mathcal{G}(y,\upsilon)S))^2 \pi(dy)ds, \qquad (2.11) \\ &\le 2\pi(\mathbb{Y}) \times \max_{j\in\mathcal{E}} \left\{ \max\left\{ \left(\ln\left(1 - \varrho_j \left(\frac{\hat{\Lambda}}{\underset{\vee}{\mu}}\right)\right)\right)^2, \left(ln\left(1 + \varrho_j \left(\frac{\hat{\Lambda}}{\underset{\vee}{\mu}}\right)\right)\right)^2 \right\} \right\} s, \end{aligned}$$

which yields $E(\Lambda_s^0) = E(\Lambda_s^1) = 0$ and then

$$E\left(W_1(X(s), j)\right) \le 3\ln\left(\frac{1}{\varepsilon_0}\right) + \max_{j\in\mathcal{E}}\{k_j\}s, \text{ for all } s \in [0, t\wedge\tau_\varepsilon].$$

Using (2.9) we affirm that

$$\begin{aligned} E\left(W_1\left(X\left(t\wedge\tau_\varepsilon\right), j\right) 1_{t\le t\wedge\tau_\varepsilon}\right) \le E\left(W_1\left(X\left(t\wedge\tau_\varepsilon\right), j\right)\right) &\le \max_{j\in\mathcal{E}}\{k_j\} t\wedge\tau_\varepsilon \\ &\le 3\ln\left(\frac{1}{\varepsilon_0}\right) + \max_{j\in\mathcal{E}}\{k_j\}t. \qquad (2.12) \end{aligned}$$

Note that $S\left(t\wedge\tau_\varepsilon\right) \le \varepsilon$ or $I\left(t\wedge\tau_\varepsilon\right) \le \varepsilon$ or $Q\left(t\wedge\tau_\varepsilon\right) \le \varepsilon$, therefore $W_1\left(X\left(t\wedge\tau_\varepsilon\right), j\right) \ge -\ln(\varepsilon)$.
Combining it with (2.12) and the fact that $\tau_\varepsilon \le \tau_e$, we obtain

$$\mathbb{P}\left(\tau_e \le t\right) \le \mathbb{P}\left(\tau_\varepsilon \le t\right) \le \frac{3\ln\left(\frac{1}{\varepsilon_0}\right) + \max_{j\in\mathcal{E}}\{k_j\}t}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

Let $\varepsilon \longrightarrow 0$, which leads to $\mathbb{P}\left(\tau_e \le t\right) = 0$ for all $t \ge 0$. Consequently, $\tau_e = \infty \quad a.s$ □

3. Persistence

As a preliminary step, we define the following sets

$$\Pi^+ = \left\{ j \in \mathcal{E} \mid C_j^2 \le 0 \right\}, \quad \Pi^- = \left\{ j \in \mathcal{E} \mid C_j^2 > 0 \right\},$$

and

$$(\Pi^+)' = \left\{ j \ \in \mathcal{E} \ \mid \bar{C}_j^2 \le 0 \right\}, \quad (\Pi^-)' = \left\{ j \in \mathcal{E} \ \mid \bar{C}_j^2 > 0 \right\},$$

where

$$C_j^2 = -\beta_j + \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right),$$

and

$$\bar{C}_j^2 = -\beta_j + \left(\sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right),$$

and let, for all $S \in \left(0, \frac{\hat{\Lambda}}{\check{\mu}} \right)$ and $j \in \mathcal{E}$,

$$\begin{aligned}
\daleth_j(S) &= -\frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) S^2 + \beta_j S - (\mu_j + d_j + \lambda_j + \gamma_j), \\
\daleth_j'(S) &= -\frac{1}{2} \left(\sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) S^2 + \beta_j S - (\mu_j + d_j + \lambda_j + \gamma_j),
\end{aligned}$$

$$\begin{aligned}
C_j^3 = \daleth_j \left(\frac{\hat{\Lambda}}{\check{\mu}} \right) &= \left(\mathcal{R}_j^1 - 1 \right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right)^2 \right), && (3.13) \\
\mathcal{R}_j^1 &= \beta_j \left(\frac{\hat{\Lambda}}{\check{\mu}} \right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right)^2 \right)^{-1}, \\
\bar{C}_j^3 = \daleth_j' \left(\frac{\hat{\Lambda}}{\check{\mu}} \right) &= \left(\mathcal{R}_j^2 - 1 \right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right)^2 \right), \\
\mathcal{R}_j^2 &= \beta_j \left(\frac{\hat{\Lambda}}{\check{\mu}} \right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}} \right)^2 \right)^{-1}.
\end{aligned}$$

Theorem 3.1. For every $(S_0, I_0, R_0) \in \Delta$, if for any $j \in \Pi^-$

$$C_j^3 > 0 \quad \text{and} \quad \sum_{j \in \Pi^+} \pi_j C_j^3 > 0, \tag{3.14}$$

then, the solution of SDE (1.5) obeys

(a) $\limsup_{t \to \infty} S(t) \ge \ \nu_1$ a.s.,

(b) $\liminf_{t \to \infty} I(t) \le \dfrac{\max_{j \in \mathcal{E}} (\mu_j + d_j + \epsilon_j)}{\max_{j \in \mathcal{E}} (\mu_j + d_j + \epsilon_j) + \min_{j \in \mathcal{E}} \lambda_j} \left(\frac{\hat{\Lambda}}{\check{\mu}} - \nu_1 \right)$ a.s.,

(c) $\liminf_{t\to\infty} Q(t) \leq \dfrac{\max_{j\in\mathcal{E}} \lambda_j}{\min_{j\in\mathcal{E}}(\mu_j + d_j + \epsilon_j) + \max_{j\in\mathcal{E}} \lambda_j}(\dfrac{\hat{\Lambda}}{\overset{\vee}{\mu}} - \nu_1)$ a.s.,

with $\nu_1 = \min \left\{\eta, \min_{j\in\Pi^-}\{\eta_j\}\right\}$ and $\nu_2 = \max\left\{\eta, \max_{j\in\Pi^-}\{\eta_j\}\right\}$ where, for $j \in \Pi^-$, η_j and η are respectively the unique positives roots of

$$\daleth_j(x) = 0 \quad \text{and} \quad \sum_{j\in\Pi^+} \pi_j \daleth_j(x) = 0 \quad \text{on} \quad \left(0, \frac{\hat{\Lambda}}{\overset{\vee}{\mu}}\right).$$

If $0 \leq \mathcal{G}(y, j) < 1$ for every $(y, j) \in \mathbb{Y} \times \mathcal{E}$ and for any $j \in (\Pi^-)'$

$$\bar{C}_j^3 > 0 \text{ and } \sum_{j\in(\Pi^+)'} \pi_j \bar{C}_j^3 > 0, \tag{3.15}$$

then, the solution obeys

(a') $\liminf_{t\to\infty} S(t) \leq \nu_2'$ a.s,

(b') $\limsup_{t\to\infty} I(t) \geq \dfrac{\min_{j\in\mathcal{E}}(\mu_j + d_j + \epsilon_j)}{\min_{j\in\mathcal{E}}(\mu_j + d_j + \epsilon_j) + \max_{j\in\mathcal{E}} \lambda_j}(\dfrac{\hat{\Lambda}}{\overset{\vee}{\mu}} - \nu_2')$ a.s.,

(c') $\limsup_{t\to\infty} Q(t) \geq \dfrac{\min_{j\in\mathcal{E}} \lambda_j}{\max_{j\in\mathcal{E}}(\mu_j + \gamma_j) + \min_{j\in\mathcal{E}} \lambda_j}(\dfrac{\hat{\Lambda}}{\overset{\vee}{\mu}} - \nu_2')$ a.s.,

with $\nu_1' = \min\left\{\eta', \min_{j\in(\Pi^-)'}\{\eta_j'\}\right\}$ and $\nu_2' = \max\left\{\eta', \max_{j\in(\Pi^-)'}\{\eta_j'\}\right\}$, where, for $j \in (\Pi^-)'$, η_j' and η' are respectively the unique positives roots of

$$\daleth_j'(x) = 0 \quad \text{and} \quad \sum_{j\in(\Pi^+)'} \pi_j \daleth_j'(x) = 0 \quad \text{on} \quad \left(0, \frac{\hat{\Lambda}}{\overset{\vee}{\mu}}\right).$$

Proof. (a) By the Itô formula, we get from (1.5)

$$\begin{aligned} \ln(I_t) = \ln(I_0) &+ \int_0^t \left(-\frac{1}{2}\sigma_j^2 S^2 + \beta_j S - (\mu_j + d_j + \lambda_j + \gamma_j)\right) ds \\ &+ \int_0^t \int_{\mathbb{Y}} \left(\ln\left(1 + \mathcal{G}(y, \upsilon(s)S)\right) - \mathcal{G}(y, \upsilon(s)S)\right) \pi(dy) \\ &+ \int_0^t \sigma_{\upsilon(s)} S(s) dB(s) + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \mathcal{G}(y, \upsilon(s)S)\right) \breve{\mathcal{N}}(ds, dy). \end{aligned} \tag{3.16}$$

By using the inequality

$$\ln(1 + v) - v < -\frac{1}{4}v^2, \quad 0 < v \leq 1, \tag{3.17}$$

we get

$$\begin{aligned} \ln(I_t) \leq \ln(I_0) &+ \int_0^t \daleth_{\upsilon(s)}(S(s)) ds + \int_0^t \sigma_{\upsilon(s)} S(s) dB(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \mathcal{G}(y, \upsilon(s)S)\right) \breve{\mathcal{N}}(ds, dy). \end{aligned} \tag{3.18}$$

From (3.14), we have for any $j \in \Pi^-$,

$$\gimel_j(0) = -(\mu_j + d_j + \lambda_j + \gamma_j) < 0 \text{ and } \gimel_j\left(\frac{\hat{\Lambda}}{\check{\mu}}\right) = C_j > 0. \tag{3.19}$$

Thus, for every $j \in \Pi^-$, the equation $\gimel_j(x) = 0$ has a unique root $\eta_j \in \left(0, \frac{\hat{\Lambda}}{\check{\mu}}\right)$. Besides, $\gimel_j(x)$ is increasing on $(0, \eta_j)$ and for any $\varepsilon > 0$ sufficiently small, for all $j \in \Pi^-$ and $0 < x \leq \eta_j - \varepsilon$, we have

$$\gimel_j(x) \leq \gimel_j(\eta_j - \varepsilon) < 0. \tag{3.20}$$

In the same way, using (3.14), the equation $\sum_{j\in\Pi^+} \pi_j \gimel_j(x) = 0$ admits a unique root $\sum_{j\in\Pi^+} \pi_j \gimel_j(x) = 0.$

It is easy to show that, for any $\varepsilon > 0$ sufficiently small and x such that $0 < x \leq \eta - \varepsilon$, then we have

$$\sum_{j\in\Pi^+} \pi_j \gimel_j(x) \leq \sum_{j\in\Pi^+} \pi_j \gimel_j(\eta - \varepsilon) < 0. \tag{3.21}$$

Now, we will prove assertion (a). If it is not true, then there is a $\varepsilon > 0$ small enough that

$$\mathbb{P}\left(\limsup_{t\to\infty} S_t \leq \min\left\{\eta, \min_{j\in\Pi^-}\{\eta_j\}\right\} - 2\varepsilon\right) > 0.$$

Let us put

$$\Omega_1 = \left\{\limsup_{t\to\infty} S_t \leq \min\left\{\eta, \min_{j\in\Pi^-}\{\eta_j\}\right\} - 2\varepsilon\right\}. \tag{3.22}$$

Thus, for any $\omega \in \Omega_1$, there is a $\imath(\omega) > 0$ that

$$S_t \leq \min\left\{\eta, \min_{j\in\Pi^-}\{\eta_j\}\right\} - \varepsilon < \frac{\hat{\Lambda}}{\check{\mu}} \qquad \text{for all} \quad t \geq \imath(\omega), \tag{3.23}$$

that is, for any $s \geq \imath(\omega)$ such that $\upsilon(s) \in \Pi^-$, we get

$$0 \leq S(s) \leq \eta_{\upsilon(s)} - \varepsilon. \tag{3.24}$$

Form (3.20) and (3.24) we have, for any $s \geq \imath(\omega)$ such that $\upsilon(s) \in \Pi^-$

$$\gimel_{\upsilon(s)}(S(s)) \leq \gimel_{\upsilon(s)}(\eta_{\upsilon(s)} - \varepsilon) < 0. \tag{3.25}$$

Furthermore, for all $j \in \Pi^+$, the function $\gimel_j$ is increasing on $\left(0, \frac{\hat{\Lambda}}{\check{\mu}}\right)$. This means, by (3.23), that

$$\gimel_{\upsilon(s)}(S(s)) \leq \gimel_{\upsilon(s)}(\eta - \varepsilon) \qquad \text{for all} \quad s \geq \imath(\omega), \quad \upsilon(s) \in \Pi^+. \tag{3.26}$$

In addition, in the light of Lemma (2.1), we can easily show as (3.14) and (3.18), that there is a $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that for every $\omega \in \Omega_2$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma_{\upsilon(s)} S(s)dB(s) = 0,$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_{\mathbb{Y}} \ln\left(1+\mathcal{G}(y,\upsilon(s)S)\right)\breve{\mathcal{N}}(ds,dy) = 0. \tag{3.27}$$

Now, fix any $\omega \in \Omega_1 \cap \Omega_2$. We can deduce from (3.18), (3.25) and (3.26) that, for $t \geq \imath(\omega)$

$$\begin{aligned}\ln(I_t) \leq \ln(I_0) &+ \int_0^{\imath(\omega)} \daleth_{\upsilon(s)}(S(s))ds + \int_{\imath(\omega)}^t \mathbf{1}_{\{\upsilon(s)\in\Pi^+\}}\daleth_{\upsilon(s)}(\eta-\varepsilon)ds \\ &+ \int_{\imath(\omega)}^t \mathbf{1}_{\{\upsilon(s)\in\Pi^-\}}\daleth_{\upsilon(s)}(\eta_{\upsilon(s)}-\varepsilon)ds + \int_0^t \sigma_{\upsilon(s)}S(s)dB(s) \\ &+ \int_0^t \int_{\mathbb{Y}} \ln\left(1+\mathcal{G}(y,\upsilon(s)S)\right)\breve{\mathcal{N}}(ds,dy).\end{aligned} \tag{3.28}$$

From the ergodic theory of the Markov chain, we can derive

$$\limsup_{t\to\infty} \frac{1}{t}\int_{\imath(\omega)}^t \mathbf{1}_{\{\upsilon(s)\in\Pi^+\}}\daleth_{\upsilon(s)}(\eta-\varepsilon)ds = \sum_{j\in\Pi^+}\pi_j\daleth_j(\eta-\varepsilon), \tag{3.29}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_{\imath(\omega)}^t \mathbf{1}_{\{\upsilon(s)\in\Pi^-\}}\daleth_{\upsilon(s)}(\eta_{\upsilon(s)}-\varepsilon)ds = \sum_{j\in\Pi^-}\pi_j\daleth_j(\eta_j-\varepsilon). \tag{3.30}$$

From (3.20), (3.21), (3.27), (3.28), (3.29) and (3.30), we have

$$\limsup_{t\to\infty} \frac{1}{t}\ln(I_t) \leq \sum_{j\in\Pi^+}\pi_j\daleth_j(\eta-\varepsilon) + \sum_{j\in\Pi^-}\pi_j\daleth_j(\eta_j-\varepsilon) < 0. \tag{3.31}$$

Hence

$$\lim_{t\to\infty} I_t = 0. \tag{3.32}$$

Besides, the third equation of (1.5) gives

$$dQ_t + (\min_{j\in\mathcal{E}}(\mu_j+d_j+\epsilon_j))Q_t dt \leq (\max_{j\in\mathcal{E}}\lambda_j)I_t dt.$$

By integration, we find

$$\begin{aligned}Q_t \leq\, & Q_0\exp\{-\min_{j\in\mathcal{E}}(\mu_j+d_j+\epsilon_j)t\} \\ &+ (\max_{j\in\mathcal{E}}\lambda_j)\int_0^t I(t-s)\exp\{-\min_{j\in\mathcal{E}}(\mu_j+d_j+\epsilon_j)s\}ds,\end{aligned} \tag{3.33}$$

from which and Fatou's lemma, one can easily get

$$\limsup_{t\to\infty} Q_t \leq \frac{\max_{j\in\mathcal{E}}\lambda_j}{\min_{j\in\mathcal{E}}(\mu_j+d_j+\epsilon_j)}\limsup_{t\to\infty} I_t, \tag{3.34}$$

which in addition with (3.32) gives $\lim_{t\to\infty} Q_t = 0$ and then $\lim_{t\to\infty} S_t = \frac{\hat{\Lambda}}{\overset{\vee}{\mu}}$. But this contradicts (3.23). Consequently, (a) is shown.
(a') Similarly, from (3.16) and the inequality

$$-\frac{y^2}{2} \leq \ln(1+y) - y, \quad y \geq 0,$$

we get, using inequalities (3.17)

$$\begin{aligned} \ln(I_t) \geq \ln(I_0) + \int_0^t \beth'_{\upsilon(s)}(S(s))ds + \int_0^t \sigma_{\upsilon(s)} S(s) dB(s) \\ + \int_0^t \int_{\mathbb{Y}} \ln\left(1 + \mathcal{G}(y, \upsilon(s)S)\right) \breve{\mathcal{N}}(ds, dy). \end{aligned} \tag{3.35}$$

If (a') were not true, we can then find an $\varepsilon' > 0$ sufficiently small such that $\mathbb{P}(\Omega_3) > 0$, where

$$\Omega_3 = \left\{ \liminf_{t\to\infty} S_t \geq \max\left\{\eta', \max_{j\in(\Pi^-)'}\{\eta'_j\}\right\} + 2\varepsilon' \right\}.$$

Thus, for any $\omega \in \Omega_3$, there is a $\imath'(\omega) > 0$ such that

$$S_t \geq \max\left\{\eta', \max_{j\in(\Pi^-)'}\{\eta'_j\}\right\} + \varepsilon' \qquad \text{for all} \quad t \geq \imath'(\omega), \tag{3.36}$$

as in (3.25) and (3.26), by choosing $\varepsilon' > 0$ sufficiently small, it is easy to verify that

$$\beth'_{\upsilon(s)}(S(s)) \geq \beth'_{\upsilon(s)}(\eta'_{\upsilon(s)} + \varepsilon') > 0 \qquad \text{for all} \quad s \geq \imath'(\omega), \quad \upsilon(s) \in (\Pi^-)', \tag{3.37}$$

$$\sum_{j\in(\Pi^+)'} \pi_j \beth'_j(\eta' + \varepsilon') > 0, \tag{3.38}$$

and

$$\beth'_{\upsilon(s)}(S(s)) \geq \beth'_{\upsilon(s)}(\eta' + \varepsilon') \qquad \text{for all} \quad s \geq \imath'(\omega), \quad \upsilon(s) \in (\Pi^+)'. \tag{3.39}$$

By (3.35), (3.37)-(3.39) and similarly to (3.28), (3.29) and (3.30) we get

$$\limsup_{t\to\infty} \frac{1}{t} \ln(I_t) \geq \sum_{j\in(\Pi^+)'} \pi_j \beth'_j(\eta' + \varepsilon') + \sum_{j\in(\Pi^-)'} \pi_j \beth'_j(\eta'_j + \varepsilon') > 0.$$

Whence $\lim_{t\to\infty} I_t = \infty$. So, this contradicts $I_t < \frac{\hat{\Lambda}}{\overset{\vee}{\mu}}$. Thus, the proof of (a') is closed.
(b) By (a) and (2.9), we have

$$\liminf_{t\to\infty} I_t + \liminf_{t\to\infty} Q_t \leq \frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - \nu_1 \quad a.s. \tag{3.40}$$

By the third equation of (1.5), we have

$$dQ_t + (\max_{j\in\mathcal{E}}(\mu_j + d_j + \epsilon_j))Q_t dt \geq (\min_{j\in\mathcal{E}} \lambda_j) I_t dt,$$

which from (3.33) and (3.34), we have

$$\liminf_{t\to\infty} I_t \leq \frac{\max\limits_{j\in\mathcal{E}}(\mu_j + d_j + \epsilon_j)}{\min\limits_{j\in\mathcal{E}} \lambda_j} \liminf_{t\to\infty} Q_t. \tag{3.41}$$

Combining (3.40) and (3.41) we obtain (b).
Similarly (b') is easily obtained by (3.34), (a') and (2.8).
(c) - (c') they follow directly from (2.8), (2.9), (a), (a'), (b) and (b'). □

We shall present a more detailed account of theorem 3.1 by examining its implications in the following two specific cases.

Corollary 3.1. If for any $j \in \mathcal{E}$,

$$\beta_j \geq \sigma_j^2 + \frac{1}{2}\int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \text{ and } \sum_{j\in\mathcal{E}} \pi_j C_j^3 > 0,$$

then the $(a)-(c)$ assumptions of Theorem 3.1 are valid with $\nu_1 = \nu_2 = \eta$. Moreover , for all $j \in \mathcal{E}$,

$$\beta_j < \sigma_j^2 + \frac{1}{2}\int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \text{ and } C_j^3 > 0,$$

then, $(a)-(c)$ hold with $\nu_1 = \min\limits_{j\in\mathcal{E}}\{\eta_j\}$ and $\nu_2 = \max\limits_{j\in\mathcal{E}}\{\eta_j\}$.
If, $0 \leq \mathcal{G}(y,j) < 1$ for every $(y,j) \in \mathbb{Y}\times\mathcal{E}$ and for any $j \in \mathcal{E}$,

$$\beta_j \geq \sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \text{ and } \sum_{j\in\mathcal{E}} \pi_j \bar{C}_j^3 > 0,$$

so the $(a')-(c')$ assumptions of Theorem 3.1 are valid with $\nu_1' = \nu_2' = \eta'$. Otherwise, if for all $j \in \mathcal{E}$,

$$\beta_j < \sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \text{ and } \bar{C}_j^3 > 0,$$

then, $(a')-(c')$ hold with $\nu_1' = \min\limits_{j\in\mathcal{E}}\{\eta_j'\}$ and $\nu_2' = \max\limits_{j\in\mathcal{E}}\{\eta_j'\}$.

4. Extinction

Referring to Khasminskii et al. [36] and Yuan & Mao [37], we encounter the following lemma, which provides sufficient conditions for asymptotical stability in probability in terms of Lyapunov functions.

Lemma 4.1. Assume that there are functions $W \in \mathcal{C}^2\left(\mathbb{R}^3\times\mathbb{S};\mathbb{R}^+\right)$ and $w \in \left(\mathbb{R}^3;\mathbb{R}^+\right)$ vanishes only at E_0 such that

$$\mathcal{L}W(x,j) \leq -w(x) \quad \text{for all } (x,j) \in \mathbb{R}^3\times\mathbb{S}, \tag{4.42}$$

and

$$\lim_{|x|\to\infty} \inf_{j\in\mathbb{S}} W(x,j) = \infty. \tag{4.43}$$

Then, the equilibrium state E_0 of system (1.5) is globally asymptotically stable in probability.

To begin, we recall the following sets

$$\Pi^+ = \left\{ j \in \mathcal{E} \mid C_j^2 \le 0 \right\} \quad \text{and} \quad \Pi^- = \left\{ j \in \mathcal{E} \mid C_j^2 > 0 \right\},$$

and the following quantities

$$\begin{aligned}
\mathcal{R}_j^1 &= \beta_j \left(\frac{\hat{\Lambda}}{\check{\mu}}\right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^2 \right)^{-1}, \\
C_j^1 &= \frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right), \\
C_j^2 &= -\beta_j + \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}}\right), \\
C_j^3 &= \gimel_j \left(\frac{\hat{\Lambda}}{\check{\mu}}\right) = \left(\mathcal{R}_j^1 - 1\right) \left(\mu_j + d_j + \lambda_j + \gamma_j + \frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^2 \right)
\end{aligned}$$

and

$$\gimel_j(S) = -\frac{1}{2} \left(\sigma_j^2 + \frac{1}{2} \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \right) S^2 + \beta_j S - (\mu_j + d_j + \lambda_j + \gamma_j), \tag{4.44}$$

Theorem 4.1. For every $(S_0, I_0, Q_0) \in \Delta$. If (2.6) holds and for any $j \in \Pi^-$

$$(C_j^2)^2 + 4C_j^1 C_j^3 < 0 \text{ and } \sum_{j \in \Pi^+} \pi_j C_j^3 < 0, \tag{4.45}$$

then, the disease-free E_0 of system (1.5) is globally asymptotically stable in probability.

Proof. Let $(S_0, I_0, Q_0) \in \Delta$, $S \in \left(0, \frac{\hat{\Lambda}}{\check{\mu}}\right)$ and $j \in \mathcal{E}$

$$W_2(S, I, R, j) = \omega_1 \left(\frac{\hat{\Lambda}}{\check{\mu}} - S\right)^2 + (\ell + a_j) I^{\frac{1}{\ell}} + \omega_2 Q^2, \quad j \in \mathbb{S}, \tag{4.46}$$

with ω_1, ℓ, ω_2, and a_j denote real positive constants, which will be determined subsequently. We get

$$
\begin{aligned}
\mathcal{L}W_2 = & -2\omega_1\mu_j(\frac{\Lambda_j}{\mu_j} - S)(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S) + 2\omega_1\beta_j SI(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S) - 2\omega_1\gamma_j I(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S) - 2\omega_1\epsilon_j Q(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S), \\
& + \omega_1\sigma_j^2 S^2 I^2 - \frac{1}{\ell}(\ell + a_j)(\mu_j + d_j + \lambda_j + \gamma_j) I^{\frac{1}{\ell}} + \frac{1}{\ell}(\ell + a_j)\beta_j S I^{\frac{1}{\ell}}, \\
& + \frac{1}{2\ell}\left(\frac{1}{\ell} - 1\right)(\ell + a_j)\sigma_j^2 S^2 I^{\frac{1}{\ell}} - 2\omega_2(\mu_j + d_j + \epsilon_j)Q^2 + 2\omega_2\lambda_j IQ + I^{\frac{1}{\ell}} \sum_{k\neq j, k\in\mathbb{S}} \theta_{jk}(a_k - a_j), \\
& + \omega_1 \int_{\mathbb{Y}} \Big((1 - S + \mathcal{G}(y,j)SI)^2 - (1-S)^2 - 2\mathcal{G}(y,j)SI(1-S) \Big) \pi(dy), \\
& + (\ell + a_j) \int_{\mathbb{Y}} \left((I + \mathcal{G}(y,j)SI)^{\frac{1}{\ell}} - I^{\frac{1}{\ell}} - \frac{1}{\ell}\mathcal{G}(y,j)SI^{\frac{1}{\ell}} \right) \pi(dy).
\end{aligned}
$$

Since $S, I \in (0, \frac{\hat{\Lambda}}{\overset{\vee}{\mu}})$, $I \leq \frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S$ and $Q \leq \frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S$, we have, for all $\ell \geq 1$

$$
\begin{aligned}
\mathcal{L}V \leq & -2\omega_1\mu_j(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S)^2 + 2\omega_1\beta_j(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}})^2 I + \omega_1\sigma_j^2(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}})^2 I^2 - \frac{1}{\ell}(\ell + a_j)(\mu_j + d_j + \lambda_j + \gamma_j) I^{\frac{1}{\ell}}, \\
& + \frac{1}{\ell}(\ell + a_j)\beta_j S I^{\frac{1}{\ell}} + \frac{1}{2\ell}\left(\frac{1}{\ell} - 1\right)(\ell + a_j)\sigma_j^2 S^2 I^{\frac{1}{\ell}} - 2\omega_2(\mu_j + d_j + \epsilon_j)Q^2, \\
& + 2(\omega_2\lambda_j - \omega_1(\gamma_j + \epsilon_j))IQ + I^{\frac{1}{\ell}} \sum_{k\neq j, k\in\mathbb{S}} \theta_{jk}(a_k - a_j) + \omega_1 S^2 I^2 \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy), \\
& + \frac{\ell + a_j}{\ell} I^{\frac{1}{\ell}} \int_{\mathbb{Y}} \left(\ell\,(1 + \mathcal{G}(y,j)S)^{\frac{1}{\ell}} - \ell - \mathcal{G}(y,j)S \right) \pi(dy).
\end{aligned}
\tag{4.47}
$$

Using (2.6) one can have

$$
|\ell\,(1 + \mathcal{G}(y,j)S)^{\frac{1}{\ell}} - \ell| \leq \left(\ell\,(1 + \varrho_j)^{\frac{1}{\ell}} - \ell \right) - \left(\ell\,(1 - \varrho_j)^{\frac{1}{\ell}} - \ell \right),
$$

and

$$
\lim_{\ell\to\infty} \left(\left(\ell\,(1 + \varrho_j)^{\frac{1}{\ell}} - \ell \right) - \left(\ell\,(1 - \varrho_j)^{\frac{1}{\ell}} - \ell \right) \right) = \ln(1 - \varrho_j^2).
$$

Combined with the Lebesgue-dominated convergence theorem, we obtain

$$
\begin{aligned}
&\int_{\mathbb{Y}} \left(\ell \left(1+\mathcal{G}(y,j)S\right)^{\frac{1}{\ell}} - \ell - \mathcal{G}(y,j)S \right) \pi(dy), \\
&= \int_{\mathbb{Y}} \left(\ln\left(1+\mathcal{G}(y,j)S\right) - \mathcal{G}(y,j)S\right) \pi(dy) + \mathcal{O}\left(\frac{1}{\ell}\right).
\end{aligned} \tag{4.48}
$$

From (4.47), (4.48) and (3.17), we have

$$
\begin{aligned}
\mathcal{L}V_1 \leq & -2\omega_1\mu_j \left(\frac{\hat{\Lambda}}{\check{\mu}} - S\right)^2 - 2\omega_2(\mu_j + d_j + \epsilon_j)Q^2 + 2\left(\omega_2\lambda_j - \omega_1(\gamma_j+\epsilon_j)\right) IQ + 2\omega_1\beta_j \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^2 I \\
& + \omega_1\sigma_j^2 \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^2 I^2 - \frac{1}{\ell}(\ell + a_j)(\mu_j + d_j + \lambda_j + \gamma_j) I^{\frac{1}{\ell}} + \frac{1}{\ell}(\ell + a_j)\beta_j S I^{\frac{1}{\ell}} \\
& + \frac{1}{2\ell}\left(\frac{1}{\ell} - 1\right)(\ell + a_j)\sigma_j^2 S^2 I^{\frac{1}{\ell}} + I^{\frac{1}{\ell}} \sum_{k\neq j, k\in\mathbb{S}} \theta_{jk}(a_k - a_j) + \omega_1 S^2 I^2 \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) \\
& + \frac{1}{\ell}(\ell + a_j) I^{\frac{1}{\ell}} \left(-\frac{1}{4}S^2 \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy) + \mathcal{O}\left(\frac{1}{\ell}\right)\right).
\end{aligned}
$$

For ω_2 such that $\omega_2 < \min\limits_{j\in\mathbb{S}} \left\{\dfrac{\omega_1(\gamma_j+\epsilon_j)}{\lambda_j}\right\}$ and ℓ sufficiently large, and by using $I^j \leq \left(\dfrac{\hat{\Lambda}}{\check{\mu}}\right)^{j-\frac{1}{\ell}} I^{\frac{1}{\ell}}$ for $j \geq \dfrac{1}{\ell}$, we get

$$
\begin{aligned}
\mathcal{L}V_1 \leq & -2\omega_1\mu_j \left(\frac{\hat{\Lambda}}{\check{\mu}} - S\right)^2 - 2\omega_2(\mu_j + d_j + \epsilon_j)Q^2 + \frac{1}{\ell}(\ell + a_j) I^{\frac{1}{\ell}} \left(\frac{\ell\omega_1}{\ell + a_j} \left(2\beta_j \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{3-\frac{1}{\ell}} \right.\right. \\
& \left.+ \left(\sigma_j^2 + \int_{\mathbb{Y}} \mathcal{G}^2(y,j)\pi(dy)\right) \left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{4-\frac{1}{\ell}} \right) + \frac{1}{2\ell}\sigma_j^2 + \daleth_j(S) + \mathcal{O}\left(\frac{1}{\ell}\right) \\
& \left. + \frac{\ell}{\ell + a_j} \sum_{k\neq j, k\in\mathbb{S}} \theta_{jk}\left(a_k - a_j\right) \right).
\end{aligned}
$$

Subsequently, the functions $\daleth_j$ is increasing on the interval $\left(0, \dfrac{\hat{\Lambda}}{\check{\mu}}\right)$. This implies that $\gimel_j(S) \leq \Phi_j$, thus

$$
\begin{aligned}
\mathcal{L}V_1 \leq & -2\omega_1\mu_j\left(\frac{\hat{\Lambda}}{\check{\mu}}-S\right)^2-2\omega_2(\mu_j+d_j+\epsilon_j)Q^2 \qquad (4.49)\\
& +\frac{1}{\ell}(\ell+a_j)I^{\frac{1}{\ell}}\left(\frac{\ell\omega_1}{\ell+a_j}\left(2\beta_j\left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{3-\frac{1}{\ell}}+\left(\sigma_j^2+\int_{\mathbb{Y}}\mathcal{G}^2(y,j)\pi(dy)\right)\left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{4-\frac{1}{\ell}}\right)\right.\\
& \left.+\frac{1}{2\ell}\sigma_j^2-\frac{a_j}{\ell+a_j}\sum_{k\neq j,k\in\mathbb{S}}\theta_{jk}(a_k-a_j)+C_j+\mathcal{O}\left(\frac{1}{\ell}\right)+\sum_{k\neq j,k\in\mathbb{S}}\theta_{jk}(a_k-a_j)\right). \qquad (4.50)
\end{aligned}
$$

Given $\Phi_j = A_j(1)$ (3.13), the irreducibility of the generating matrix Θ ensures the existence of a solution $\Phi = (\Phi_1, \ldots, \Phi_m)^T$ to the system [38] for $\Lambda = (a_1, \ldots, a_m)^T$. The column vector with all entries equal to 1 is denoted as $\mathbf{e} = (1\,1\,\ldots\,1)$.

$$
\Theta\Lambda = -\Phi + \left(\sum_{j=1}^{m}\pi_j\Phi_j\right)\mathbf{e}, \qquad (4.51)
$$

substituting (4.51) into (4.49), we derive

$$
\begin{aligned}
\mathcal{L}V_1 \leq & -2\omega_1\mu_j\left(\frac{\hat{\Lambda}}{\check{\mu}}-S\right)^2-2\omega_2(\mu_j+d_j+\epsilon_j)Q^2+\frac{1}{\ell}(\ell+a_j)I^{\frac{1}{\ell}}\left[\frac{\ell\omega_1}{\ell+a_j}\left(2\beta_j\left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{3-\frac{1}{\ell}}\right.\right.\\
& \left.\left.+\left(\sigma_j^2+\int_{\mathbb{Y}}\mathcal{G}^2(y,j)\pi(dy)\right)\left(\frac{\hat{\Lambda}}{\check{\mu}}\right)^{4-\frac{1}{\ell}}\right)\right]+\frac{1}{2\ell}\sigma_j^2+\frac{a_j}{\ell+a_j}\left(\Phi_j-\sum_{j=1}^{m}\pi_j\Phi_j\right)\\
& +\sum_{j=1}^{m}\pi_j\Phi_j+\mathcal{O}\left(\frac{1}{\ell}\right)-K_j^1\left(\frac{\hat{\Lambda}}{\check{\mu}}-S\right)^2-K_j^2R^2-K_j^3Q^{\frac{1}{\ell}}. \qquad (4.52)
\end{aligned}
$$

By (4.52), we can select a sufficiently large value for ℓ such that

$$
\max_{j\in\mathbb{S}}\left\{\frac{1}{2\ell}\sigma_j^2+\frac{a_j}{\ell+a_j}\left(\Phi_j-\sum_{j=1}^{m}\pi_j\Phi_j\right)+\sum_{j=1}^{m}\pi_j\Phi_j+\mathcal{O}\left(\frac{1}{\ell}\right)\right\}<0,
$$

we can choose ω_1 such that

$$0 < \omega_1 < -\frac{\ell + \min_{j\in\mathbb{S}}\{a_j\}}{\ell\left(\max_{j\in\mathbb{S}}\left\{2\beta_j(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}})^{3-\dfrac{1}{\ell}} + \left(\sigma_j^2 + \int_{\mathbb{Y}}\mathcal{G}^2(y,j)\pi(dy)\right)(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}})^{4-\dfrac{1}{\ell}}\right\}\right)}$$
$$\times \max_{j\in\mathbb{S}}\left\{\frac{1}{2\ell}\sigma_j^2 + \frac{a_j}{\ell + a_j}\left(\Phi_j - \sum_{j=1}^{m}\pi_j\Phi_j\right) + \sum_{j=1}^{m}\pi_j\Phi_j + \mathcal{O}\left(\frac{1}{\ell}\right)\right\}.$$

This implies that the coefficients of $(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} - S)^2$, I^2, and Q^2 in equation (4.52) are all negative. We conclude the proof by referring to Lemma 4.1. □

Theorem 4.2. For every $(S_0, I_0, Q_0) \in \Delta$ and (2.6) holds. Assume further that

$$\sup_{0<S<\dfrac{\Lambda}{\mu}} \int_{\mathbb{Y}} \left(\ln(1 + \mathcal{G}(y,j)S)\right)^2 \pi(dy) < \infty, \tag{4.53}$$

then

$$\limsup_{t\to\infty} \frac{1}{t}\ln(I(t)) \le \sum_{j=1}^{m}\pi_j\Phi_j.$$

Moreover, if $\sum_{j=1}^{m}\pi_j\Phi_j < 0$, then the disease in (1.5) is extensive.

Proof. By Itô formula, we obtain

$$\begin{aligned}\ln(I(t)) = \ln(I_0) &+ \int_0^t \left(-\frac{1}{2}\sigma_{\upsilon(s)}^2 S^2 + \beta_{\upsilon(s)}S - (\mu_{\upsilon(s)} + d_{\upsilon(s)} + \lambda_{\upsilon(s)} + \gamma_{\upsilon(s)})\right) ds \\ &+ \int_0^t \int_{\mathbb{Y}} \left(\ln(1 + \mathcal{G}(y,\upsilon(s))S(s)) - \mathcal{G}(y,\upsilon(s))S(s)\right)\pi(dy)ds \\ &+ \int_0^t \sigma_{\upsilon(s)}S(s)\,dB_s + \int_0^t\int_{\mathbb{Y}} \ln(1 + \mathcal{G}(y,\upsilon(s))S(s))\breve{\mathcal{N}}(ds,dy).\end{aligned}$$

By (3.17), we find

$$\begin{aligned}\ln(I(t)) \le \ln(I_0) &+ \int_0^t \left[-\frac{1}{2}\left(\sigma_{\upsilon(s)}^2 + \frac{\overset{\vee}{\mu}}{\hat{\Lambda} + \overset{\vee}{\mu}}\int_{\mathbb{Y}}\mathcal{G}^2(y,\upsilon(s))\pi(dy)\right)S^2(s)\right. \\ &\left.+\beta_{\upsilon(s)}S(s) - (\mu_{\upsilon(s)} + d_{\upsilon(s)} + \lambda_{\upsilon(s)} + \gamma_{\upsilon(s)})\right] ds \\ &+ \int_0^t \sigma_{\upsilon(s)}S(s)\,dB_s + \int_0^t\int_{\mathbb{Y}} \ln\left(1 + \mathcal{G}(y,\upsilon(s))S(s)\right)\breve{\mathcal{N}}(ds,dy) \\ &\triangleq \ln(I_0) + \int_0^t \daleth_{\upsilon(s)}(S(s))\,ds + M_t^0 + M_t^1,\end{aligned} \tag{4.54}$$

where $M_t^0 = \int_0^t \sigma_{\upsilon(s)} S(s) dB_s$ and $M_t^1 = \int_0^t \int_{\mathbb{Y}} \ln(1 + \mathcal{G}(y, \upsilon(s)) S(s)) \breve{\mathcal{N}}(ds, dy)$ are real-valued local martingales, verifies

$$< M_t^0, M_t^0 >= \int_0^t \sigma^2_{\upsilon(s)} S^2(s) ds \leq \max_{j \in \mathcal{E}} \{\sigma_j^2\} \left(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} \right)^2 t,$$

$$\begin{aligned} < M_t^1, M_t^1 > &= \int_0^t \int_{\mathbb{Y}} \left(\ln(1 + \mathcal{G}(y, \upsilon(s)) S(s))\right)^2 \pi(dy) ds \\ &\leq \left[\pi(\mathbb{Y}) \times \max_{j \in \mathcal{E}} \left(\ln \left(1 + \varrho_j \left(\frac{\hat{\Lambda}}{\overset{\vee}{\mu}} \right) \right) \right)^2 \right] t, \end{aligned}$$

then, we have from (4.53) and the Lemma 2.1

$$\limsup_{t \to \infty} \frac{M_t^0}{t} = 0 \quad , \quad \limsup_{t \to \infty} \frac{M_t^1}{t} = 0 \quad a.s.$$

It follows from (4.54) that

$$\limsup_{t \to \infty} \frac{1}{t} \ln(I(t)) \leq \limsup_{t \to \infty} \frac{1}{t} \int_0^t (\Phi_j) \, ds + \limsup_{t \to \infty} \frac{M_t^0}{t} + \limsup_{t \to \infty} \frac{M_t^1}{t}.$$

By the ergodic theory of the Markov chain, we have

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t (\Phi_j) \, ds = \sum_{j=1}^m \pi_j \Phi_j. \tag{4.55}$$

Therefore, if $\sum_{j=1}^m \pi_j \Phi_j < 0$, then system (1.5) is globally exponentially stable. □

5. Numerical simulations

In this section, we explore the practical implications of our theoretical findings through numerical simulations. Specifically, we examine a right-continuous Markov chain, denoted as $(r(t))_{t \geq 0}$, which models the dynamics of a disease outbreak with two possible states. The dynamics of this Markov chain are governed by the following generator matrix

$$\Theta = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

To investigate the chain's behavior over time, we utilize the Euler-Maruyama method with a step size of $\Delta = 0.0001$. This numerical approach involves calculating the one-step transition probability matrix P, which represents the probabilities of transitioning between states at each discrete time step. The calculated transition probability matrix P and its associated stationary distribution π are given by

$$P = \begin{pmatrix} 0.9999 \; 0.0001 \\ 0.0002 \; 0.9998 \end{pmatrix}, \quad \pi = (0.6667, 0.3333). \tag{5.56}$$

Example 5.1. To illustrate the persistence property as outlined in Theorem 3.1, it is essential to carefully select specific parameter values that ensure the quantity $\sum_{j \in \Pi^+} \pi_j C_j^3$ remains positive. We set $\Lambda = (0.08, 0.06)$, $\mu = (0.045, 0.02)$, $d = (0.009, 0.012)$, $\beta = (0.39, 0.42)$, $\gamma = (0.009, 0.013)$, $\lambda = (0.06, 0.045)$, $\epsilon = (0.022, 0.018)$, $\sigma = (0.19, 0.33)$, and $J(z) = (0.3, 0.18)$, for $z \in \mathbb{R}^+$ and $j \in \{1, 2\}$. Hence $\sum_{j \in \Pi^+} \pi_j C_j^3 = 0.7503$ ¿ 0.

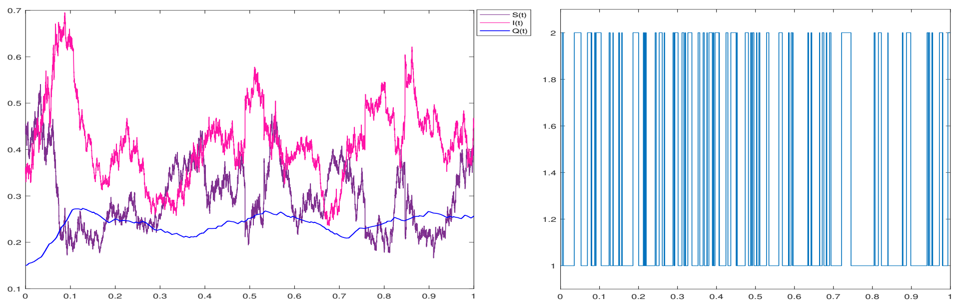

Fig. 1. Trajectory of the state variables $(S(t), I(t), Q(t))$ for the model described by equation (1.5) and its associated Markov chain $r(t)$, using the parameter values provided in Example 5.1.

In an alternative scenario, we retain the previously specified parameters but introduce a significant change: $\beta = (0.8, 0.82)$. Under this modification, we observe $\bar{C}^2 \approx (-0.4748, -0.3254)$ and $\bar{C}^3 \approx (2.0960, 1.9145)$, where $\sum_{j \in \Pi^+} \pi_j \bar{C}_j^3 = 2.0355$, which remains positive. This indicates, as per Theorem 3.1, that the infection is likely to persist within the population over an extended period.

Example 5.2. To illustrate the extinction property as described in Theorem 4.1, careful selection of specific parameter values is crucial to ensure that the expression $\sum_{j \in \Pi^+} \pi_j C_j^3 = -0.0063$ remains less than zero. For the study of the extinction scenario, the following parameter values were used: $\Lambda = (0.12, 0.098)$, $\mu = (0.099, 0.096)$, $d = (0.07, 0.005)$, $\beta = (0.3, 0.18)$, $\gamma = (0.03, 0.011)$, $\lambda = (0.04, 0.028)$, $\epsilon = (0.038, 0.042)$, $\sigma = (0.27, 0.22)$, and $J(z) = (0.4, 0.3)$, where $z \in \mathbb{R}^+$ and $j \in \{1, 2\}$. .

In the alternative scenario, we retain the parameter values outlined in the preceding paragraph but with a significant modification: $J(z) = (0.75, 0.6)$. As a result, we observe $C^2 \approx (0.0852, 0.0579)$ and $C^3 \approx (-0.0882, -0.0784)$, where C^3 is less than zero. According to Theorem 4.1, the infection may eventually become extinct within the population over an extended time horizon.

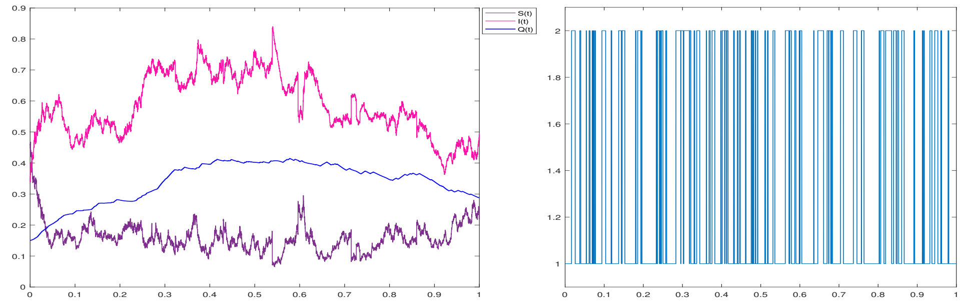

Fig. 2. Trajectory of the state variables $(S(t), I(t), Q(t))$ for the model described by equation (1.5) and its associated Markov chain $r(t)$, using the parameter values provided in Example 5.1.

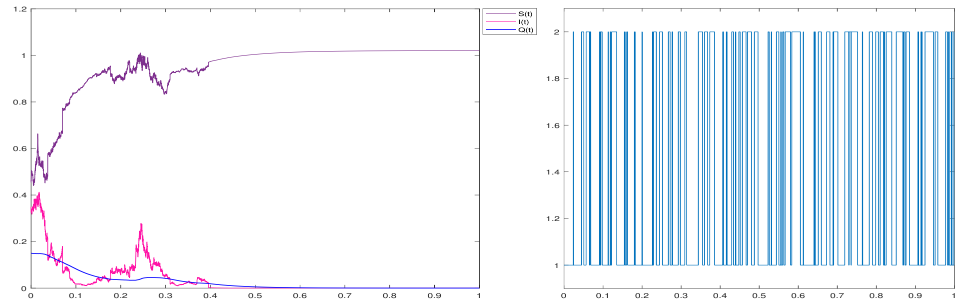

Fig. 3. Trajectory of the State Variables $(S(t), I(t), Q(t))$ for Model (1.5) and the Corresponding Markov Chain $r(t)$, Parameterized by Example 5.2.

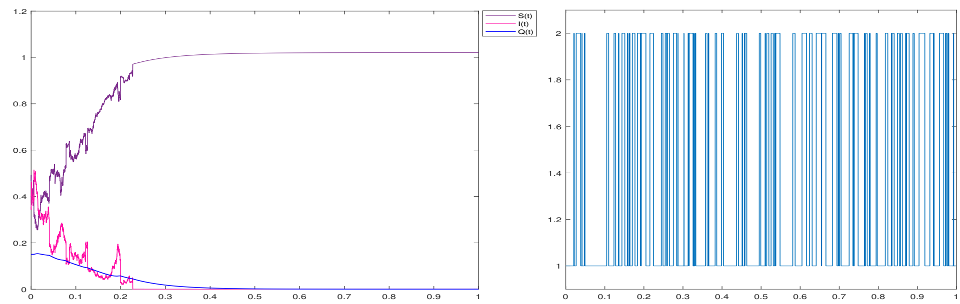

Fig. 4. Trajectories of $(S(t), I(t), Q(t))$ for Model (1.5) and the Associated Markov Chain $r(t)$ Based on Parameter Values from Example 5.2.

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