Influence of stochastic jumps on the dynamics of SIRS systems

Systems of stochastic differential equations

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Epidemiological Modeling

- Fundamental tool for understanding infectious disease dynamics
- Historical evolution: from SIR model (Kermack & McKendrick, 1927) to modern stochastic approaches
- Crucial importance of random perturbations in modeling

Limitations of Existing Approaches

- Deterministic models: unable to capture random fluctuations
- Classical stochastic models: underestimation of extreme events
- Traditional SIRS models: absence of Lévy jumps
- Major challenge: modeling the impact of sudden environmental shocks

Deterministic Differential Equations

$$\begin{cases} dS_t = [\rho(1 - S_t) + \eta R_t - \alpha S_t I_t]dt \\ dI_t = [\alpha S_t I_t - (\rho + \lambda) I_t]dt \\ dR_t = [\lambda I_t - (\rho + \eta) R_t]dt \end{cases}$$

State Variables

- S_t: Proportion of susceptible individuals in the population
- I_t: Proportion of infectious individuals in the population
- R_t: Proportion of recovered (immune) individuals in the population

Deterministic Parameters $(\alpha, \eta, \lambda, \rho) \in (0, 1)^4$

- ρ: Mortality/birth rate (constant population on average)
- α: Transmission rate (contacts $S \to I$)
- λ: Recovery rate ($I \to R$)
- η: Immunity loss rate ($R \to S$)

Stochastic Differential Equations (SDEs)

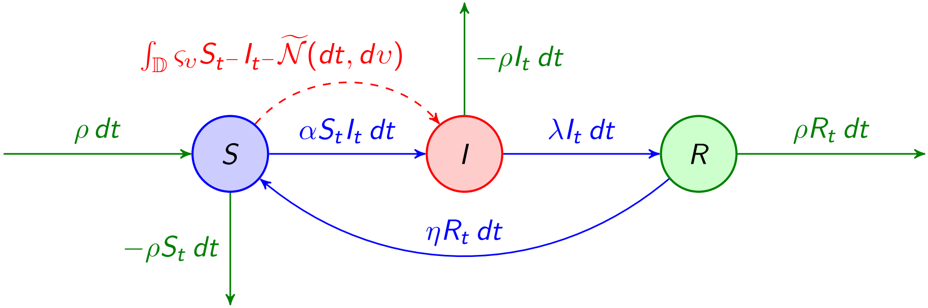
$$\begin{cases} dS_t = [\rho(1-S_t) + \eta R_t - \alpha S_t I_t]dt - \int_{\mathbb{D}} \varsigma_\upsilon S_{t^-} I_{t^-} \widetilde{\mathcal{N}}(dt, d\upsilon) \\ dI_t = [\alpha S_t I_t - (\rho+\lambda) I_t]dt + \int_{\mathbb{D}} \varsigma_\upsilon S_{t^-} I_{t^-} \widetilde{\mathcal{N}}(dt, d\upsilon) \\ dR_t = [\lambda I_t - (\rho+\eta) R_t]dt \end{cases} \tag{1}$$

Jump Process Definitions

- $\mathcal{N}(dt, d\upsilon)$: Random Poisson measure counting jumps
  - •• $\mathcal{N}((t, t+dt] \times A)$: Number of jumps in $A \subseteq \mathbb{D}$ during $(t, t+dt]$
- $\pi(d\upsilon)$: Lévy measure (jump intensity) and $\pi(A)$: Jump intensity in $A \subseteq \mathbb{D}$
- $\widetilde{\mathcal{N}}(dt, d\upsilon)$: Compensated Poisson measure $\widetilde{\mathcal{N}}(dt, d\upsilon) = \mathcal{N}(dt, d\upsilon) - \pi(d\upsilon)dt$

Model Parameters

- ς_υ: Jump amplitude ($|\varsigma_\upsilon| < 1$) and $\mathbb{D}$: Jump domain



- Deterministic transitions
- Stochastic jumps
- Demographic dynamics

Assumptions and Conditions

- Jump control: $\sup_{t \geq 0} \int_{\mathbb{D}} \ln \left(1 + \varsigma(\upsilon)\right) \pi(d\upsilon) < \infty$
- Population positivity: $S_t + I_t + R_t = 1 \quad \text{with} \quad S_t, I_t, R_t \geq 0 \quad \forall t \geq 0$

Meyer's Angle Bracket

For a local martingale $(M_t)_{t\geq 0}$: $\langle M\rangle_t$ is the unique predictable increasing process such that $M_t^2 - \langle M\rangle_t$ is a local martingale

Theorem (Asymptotic Behavior[6])

For any local martingale M_t starting at 0:

$$\varphi_{M_t} := \int_0^t (1+s)^{-2} d\langle M\rangle_s$$

If $\mathbb{P}(\lim_{t\to\infty} \varphi_{M_t} < \infty) = 1$, *then:*

$$\mathbb{P}\left(\lim_{t\to\infty} \frac{M_t}{t} = 0\right) = 1$$

This lemma will be crucial for analyzing the long-term behavior of our system.

Theorem (Existence and Uniqueness)

Let $\upsilon \in \mathbb{D}$, $(S, I) \in (0,1)^2$, and define $\Psi(\upsilon, S, I) = [1 - \varsigma_\upsilon I]\,[1 + \varsigma_\upsilon S]$, if

$$\sup_{0<S,I<1} \int_{\mathbb{D}} \ln \left[\Psi^{-1}(\upsilon, S, I)\right] \pi(d\upsilon) < \infty. \tag{2}$$

Then for each initial value $(S_0, I_0, R_0) \in \Delta$, there exists a unique solution $(S_t, I_t, R_t) \in \Delta$ for equation (1).

Proof Idea.

Define $N = S + I + R$ so $dN_t = -\rho(N_t - 1)dt \Rightarrow N_t = 1 + (N_0 - 1)e^{-\rho t}$ and show that $N_t = 1$ almost surely by integration. By local Lipschitz continuity of the coefficients, there exists a maximal local solution. $\square$

Applying Itô's formula to $\Sigma_t = -\ln(S_t I_t R_t)$,

$$d\Sigma_s \leq \Big[3\rho + \lambda + \eta + \alpha + \pi(\mathbb{D}) + \underbrace{\sup_{0<S,I<1} \int_{\mathbb{D}} \ln\left[\Psi^{-1}(\upsilon, S, I)\right] \pi(d\upsilon)\Big] ds}_{\text{jump terms}}$$

$$- \underbrace{\int_{\mathbb{D}} \ln\left[\,(1 + \varsigma_\upsilon S_s)\,(1 - \varsigma_\upsilon I_s)\,\right] \widetilde{\mathcal{N}}(ds, d\upsilon)}_{\text{martingale}}. \tag{3}$$

We obtain an upper bound. If $\tau_e < \infty$, we reach a contradiction. Thus $\tau_e = \infty$ and the solution is global.

Theorem (Exponential Extinction Criterion)

Let $(S_0, I_0, R_0) \in \Delta$ and assume that (2) holds. Also assume that

$$\sup_{0<y<1} \int_{\mathbb{D}} \ln^2 [1 + \varsigma_\upsilon y] \, \pi(d\upsilon) < \infty. \tag{4}$$

We define the thresholds

$$\mathcal{T}^3 = \alpha \left[\rho + \frac{1}{4} \int_{\mathbb{D}} \varsigma_\upsilon^2 \pi(d\upsilon) \right]^{-1}, \tag{5}$$

and

$$\mathcal{T}^4 = \frac{1}{2} \int_{\mathbb{D}} \varsigma_\upsilon^2 \pi(d\upsilon). \tag{6}$$

If $\mathcal{T}^3 < 1$ or $\alpha \geq \mathcal{T}^4$, then system (1) exhibits extinction with an exponential decay rate.

Proof Idea.

Let $\Sigma_t = \ln(Z_t)$ with $Z_t = I_t + R_t$. By Itô's formula:

$$d\Sigma_t \leq \underbrace{\left[-\rho + \alpha\frac{S_t I_t}{Z_t} - \frac{1}{4}\int_{\mathbb{D}} \varsigma_v^2 \pi(dv)\left(\frac{S_t I_t}{Z_t}\right)^2\right] dt}_{\text{deterministic terms}} + \underbrace{\int_{\mathbb{D}} \ln\left(1 + \varsigma_v \frac{S_t I_t}{Z_t}\right)\widetilde{\mathcal{N}}(dt, dv)}_{\text{local martingale}}$$

$$\leq \underbrace{\left[-\rho + \alpha\delta - \frac{1}{4}\int_{\mathbb{D}} \varsigma_v^2 \pi(dv)\delta^2\right] dt}_{\text{deterministic terms}} + \underbrace{\int_{\mathbb{D}} \ln\left(1 + \varsigma_v\delta\right)\widetilde{\mathcal{N}}(dt, dv)}_{\text{local martingale}}, \tag{7}$$

where $\delta = \frac{S_t I_t}{Z_t}$ and M_t is a local martingale. Integrating and using $\limsup_{t\to\infty} \frac{M_t}{t} = 0$ a.s., the result follows. □

Public Health Implications

- Strategies targeting the reduction of the transmission coefficient α
- Importance of controlling rare but significant amplitude events

$\Rightarrow$ Prevention policies adapted to non-linear dynamics

The system is governed by the following equations and parameters:

$$\begin{cases} H(S) = -(\rho+\lambda) + \alpha S - \left[\frac{1}{4}\int_{\mathbb{D}} \varsigma_{\upsilon}^{2}\pi(d\upsilon)\right] S^2 \\ \mathcal{T}^1 = \dfrac{\alpha}{\rho+\lambda+\frac{1}{4}\int_{\mathbb{D}} \varsigma_{\upsilon}^{2}\pi(d\upsilon)} \\ \Pi(S) = -(\rho+\lambda) + \alpha S - \left[\frac{1}{2}\int_{\mathbb{D}} \varsigma_{\upsilon}^{2}\pi(d\upsilon)\right] S^2 \\ \mathcal{T}^2 = \dfrac{\alpha}{\rho+\lambda+\frac{1}{2}\int_{\mathbb{D}} \varsigma_{\upsilon}^{2}\pi(d\upsilon)} \end{cases} \tag{8}$$

Remark

Threshold comparison: $\mathcal{T}^1 > \mathcal{T}^2$ shows that $\mathcal{T}^1$ is the more conservative threshold.

Sensitivity: $H(S) > \Pi(S)$ indicates that $H(S)$ has greater sensitivity than $\Pi(S)$.

Theorem

Under assumptions (2), $|\varsigma_\upsilon| < 1$ and

$$\sup_{0<y<1} \int_{\mathbb{D}} \ln^2\left(1 + \varsigma_\upsilon y\right) \pi(d\upsilon) < \infty, \tag{9}$$

for $(S_0, I_0, R_0) \in \Delta$, if $\mathcal{T}^1 > 1$ and $\mathcal{T}^2 > 1$ for all $\upsilon \in \mathbb{D}$, then:

(i) $\limsup_{t\to\infty} S_t \geq \varrho \quad a.s.$

(ii) $\liminf_{t\to\infty} I_t \leq \dfrac{(\rho+\eta)(1-\varrho)}{\rho+\eta+\lambda} \quad a.s.$

(iii) $\liminf_{t\to\infty} R_t \leq \dfrac{\lambda(1-\varrho)}{\rho+\eta+\lambda} \quad a.s.$

(iv) $\liminf_{t\to\infty} S_t \leq \varrho' \quad a.s.$

(v) $\limsup_{t\to\infty} I_t \geq \dfrac{(\rho+\eta)(1-\varrho')}{\rho+\eta+\lambda} \quad a.s.$

(vi) $\limsup_{t\to\infty} R_t \geq \dfrac{\lambda(1-\varrho')}{\rho+\eta+\lambda} \quad a.s.$

where ϱ and ϱ' denote the positive roots on the interval $(0,1)$ of the equations $H(S) = 0$ and $\Pi(S) = 0$ respectively.

Important Remark

Since $-1 < \varsigma_v < 1$ for all $v \in \mathbb{D}$, it follows that for all $S \in (0,1)$, $\Pi(S) < H(S)$, and consequently $\varrho < \varrho'$.

- (i) Use Itô's formula for $\ln(I_t)$

$$\ln(I_t) = \underbrace{\ln(I_0)}_{\substack{\text{Initial}\\ \text{value}}} - \underbrace{\int_0^t [(\rho+\lambda) - \alpha S_s]\, ds}_{\substack{\text{Deterministic}\\ \text{decay}}} + \underbrace{\int_{\mathbb{D}} \int_0^t \left[\ln\left(1+\varsigma_\upsilon S_s\right) - \varsigma_\upsilon S_s\right] \pi(d\upsilon)}_{\substack{\text{Jump correction}\\ \text{(compensator drift)}}}$$

$$+ \underbrace{\int_{\mathbb{D}} \int_0^t \ln\left[1+\varsigma_\upsilon S_s\right] \widetilde{\mathcal{N}}(ds, d\upsilon)}_{\substack{\text{Compensated jump process}\\ \text{(martingale)}}}. \tag{10}$$

- Show that $H(S) < 0$ for $S < \varrho$ leads to $I_t \to 0$ and $S_t \to 1$, hence contradiction
- (iv) Same approach but with a lower bound and $\Pi(S) > 0$ for $S > \varrho'$
- (ii) and (v) Combination of (i) and (iv) with Fatou's lemma on R_t
- (iii) and (vi) Direct consequences of the other points

Discretization of the Stochastic SIRS System

For fixed $\Delta t > 0$ and $t_k = k\Delta t$, the scheme is written as:

$$\begin{cases} S_{k+1} = S_k + [\rho(1 - S_k) + \eta R_k - \alpha S_k I_k]\,\Delta t - \sum_{i=1}^{N_k} \varsigma_{\upsilon_i} S_k I_k \\ I_{k+1} = I_k + [\alpha S_k I_k - (\rho + \lambda) I_k]\,\Delta t + \sum_{i=1}^{N_k} \varsigma_{\upsilon_i} S_k I_k \\ R_{k+1} = R_k + [\lambda I_k - (\rho + \eta) R_k]\,\Delta t \end{cases}$$

- **Jump process**: $N_k \sim \mathcal{P}(\pi(\mathbb{D})\Delta t)$ (Poisson law)
- **Amplitudes**: $\varsigma_{\upsilon_i} \sim \pi(d\upsilon)$ (Lévy measure on $\mathbb{D}$)
- **Constraint**: $S_k + I_k + R_k = 1$ (population conservation)

Scheme Convergence

Strong order 0.5 for jumps (Platen & Bruti-Liberati theorem, see [4])

Algorithm (Pseudocode)

```
Initialize $S_0, I_0, R_0, \Delta t, T$
for $k = 0$ to $N - 1$ do
    $N_k \sim \text{Poisson}(\pi(\mathbb{D})\Delta t)$
    Generate $\{\varsigma_{\upsilon_i}\}_{i=1}^{N_k}$ according to $\pi(d\upsilon)$
    Update via Euler-Maruyama scheme
    Normalization: $S_{k+1} + I_{k+1} + R_{k+1} = 1$
end for
```

Recommended Parameters

- Δt: 0.01 day (numerical stability)
- $\pi(\mathbb{D})$: Lévy measure intensity (typically 0.1)
- ς_υ: amplitudes on domain $\mathbb{D}$

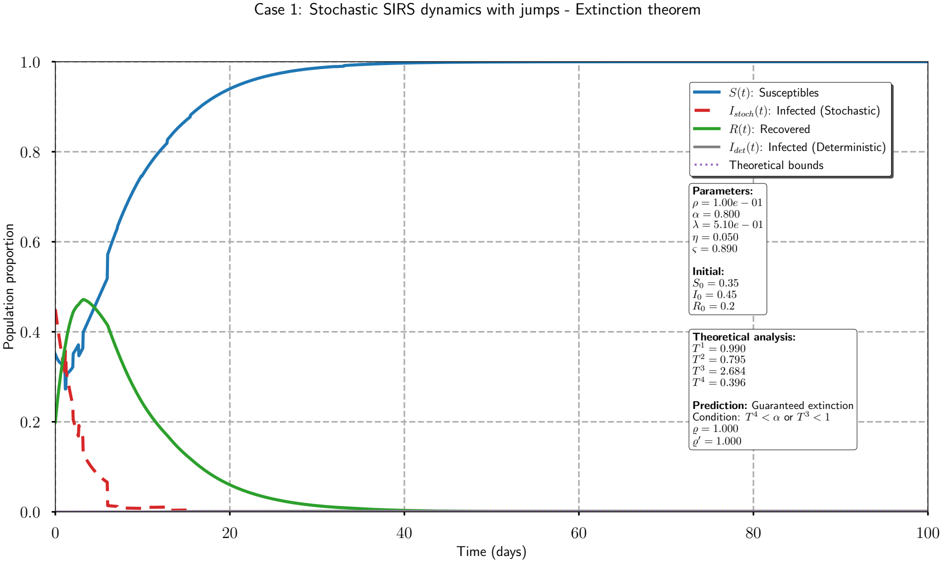


Figure 1: Guaranteed extinction according to the theorem: $\mathcal{T}^4 = 0.396 < \alpha = 0.800$ (first condition satisfied). Condition $\mathcal{T}^3 = 2.684 < 1$ is not verified.

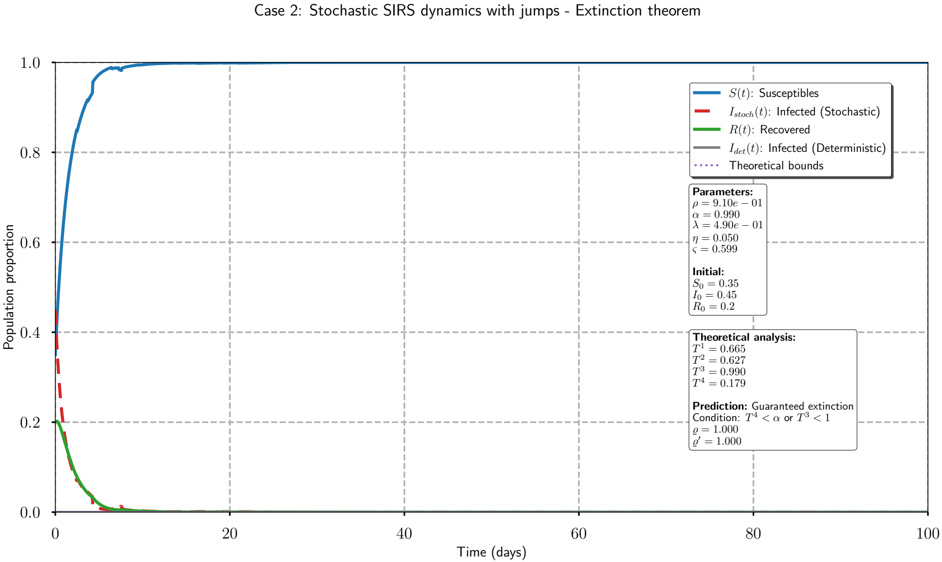


Figure 2: Exponential extinction conforming to the theorem: although $\mathcal{T}^4 = 0.179 < \alpha = 0.99$ is satisfied, the alternative condition $\mathcal{T}^3 = 0.952 < 1$ is also verified.

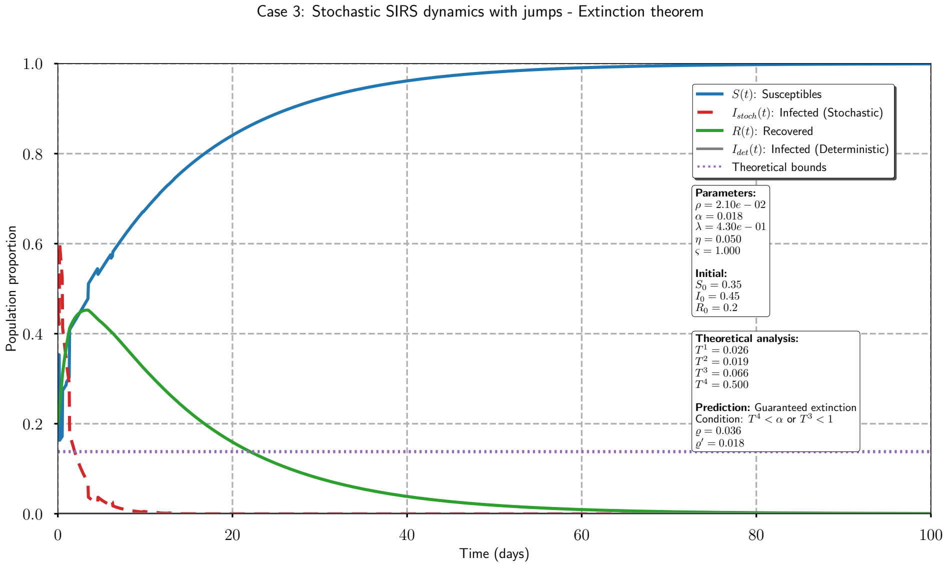


Figure 3: Exponential extinction: the first condition $\mathcal{T}^4 = 0.500 < \alpha = 0.018$ is not satisfied, while the condition $\mathcal{T}^3 = 0.066 < 1$ is verified.

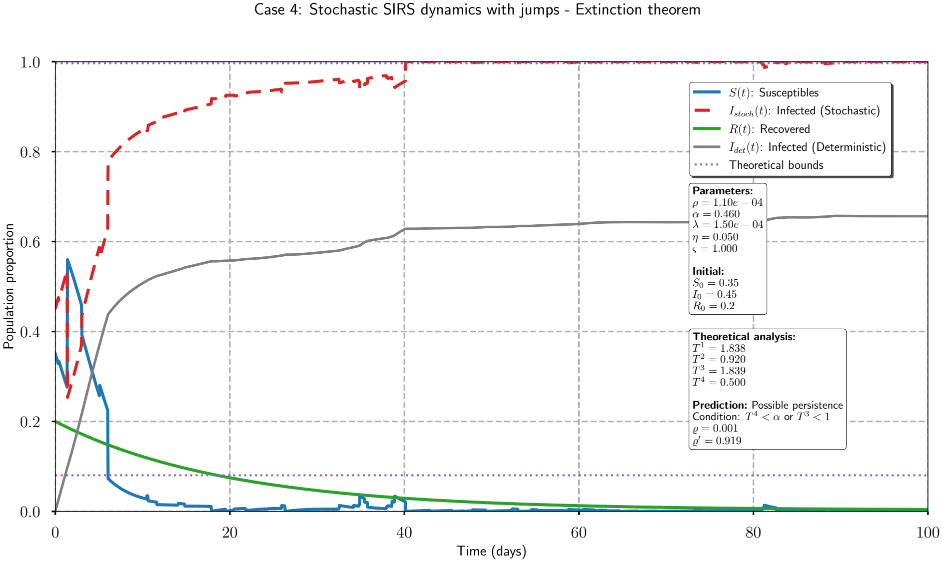


Figure 4: Possible persistence according to predictions: $\mathcal{T}^4 = 0.500 > \alpha = 0.460$ AND $\mathcal{T}^3 = 1.839 > 1$ (neither alternative condition satisfied).

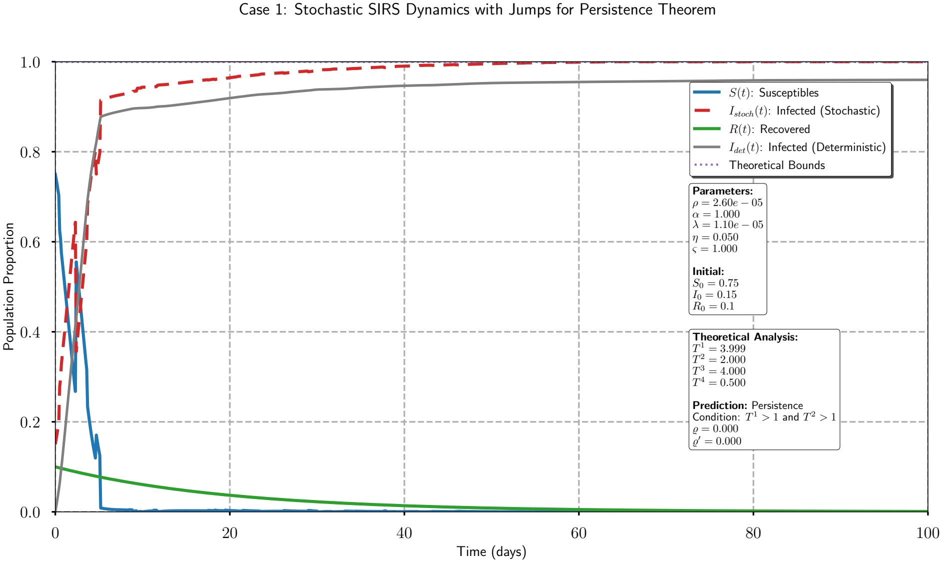


Figure 5: Marked persistence ($\mathcal{T}^1 = 3.999 > 1$, $\mathcal{T}^2 = 2.000 > 1$). Parameters: $\rho = 2.60 \times 10^{-5}$, $\alpha = 1.0$, $\lambda = 1.10 \times 10^{-5}$. Stable dynamics with predominance of infected.

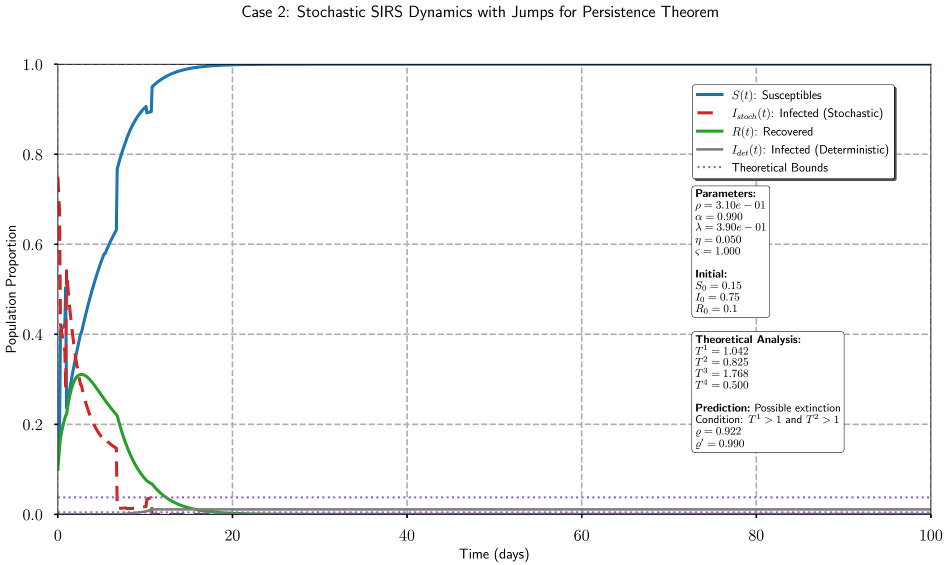


Figure 6: Progressive extinction ($\mathcal{T}^1 = 1.042 > 1$, $\mathcal{T}^2 = 0.825 \leq 1$). Parameters: $\rho = 0.31$, $\alpha = 0.99$, $\lambda = 0.39$. Sharp drop in infected before $t = 20$.

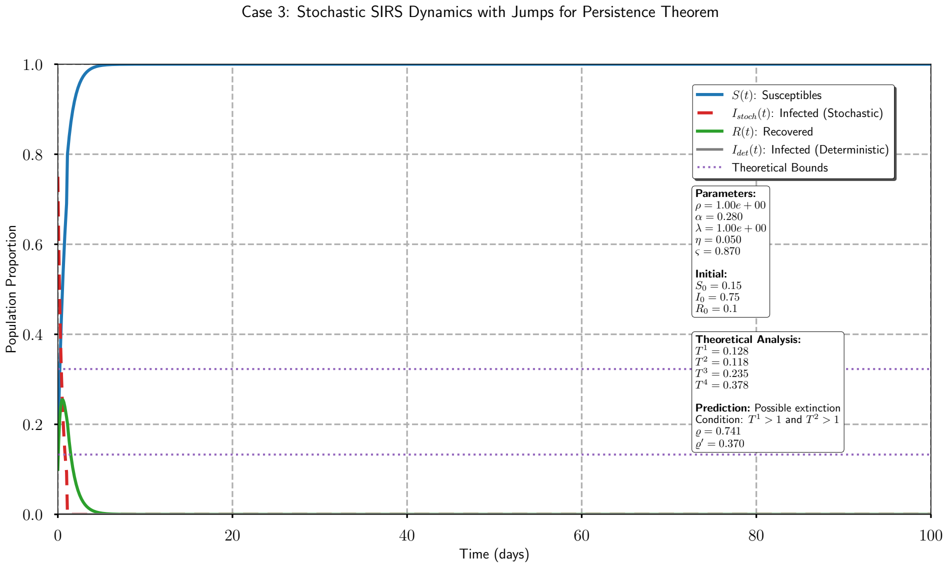


Figure 7: Rapid extinction ($\mathcal{T}^1 = 0.128 \ll 1$, $\mathcal{T}^2 = 0.118 \ll 1$). Parameters: $\rho = 1.0$, $\alpha = 0.28$, $\lambda = 1.0$. Exponential decay of infected.

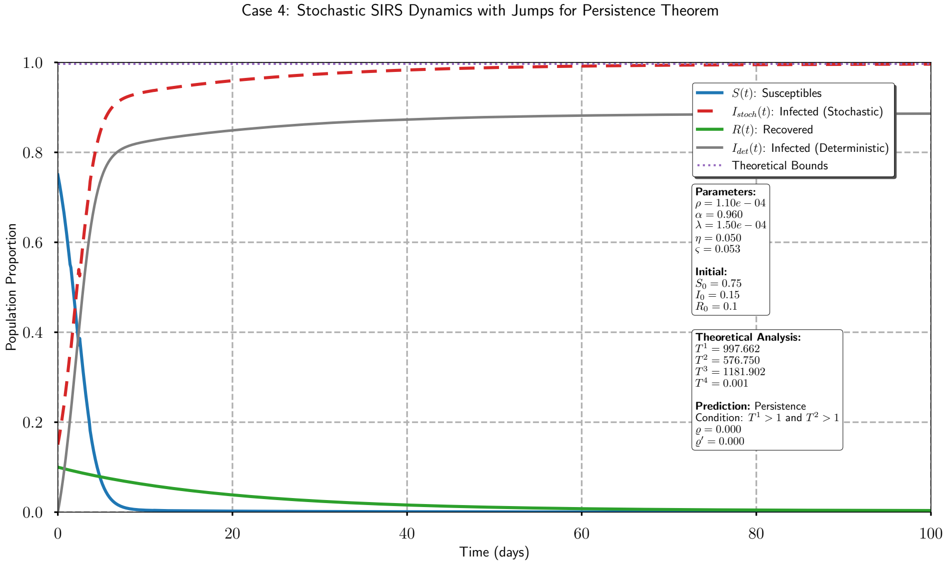


Figure 8: Persistence ($\mathcal{T}^1 = 997.662 \gg 1$, $\mathcal{T}^2 = 576.750 \gg 1$). Parameters: $\rho = 1.10 \times 10^{-4}$, $\alpha = 0.96$, $\lambda = 1.50 \times 10^{-4}$.

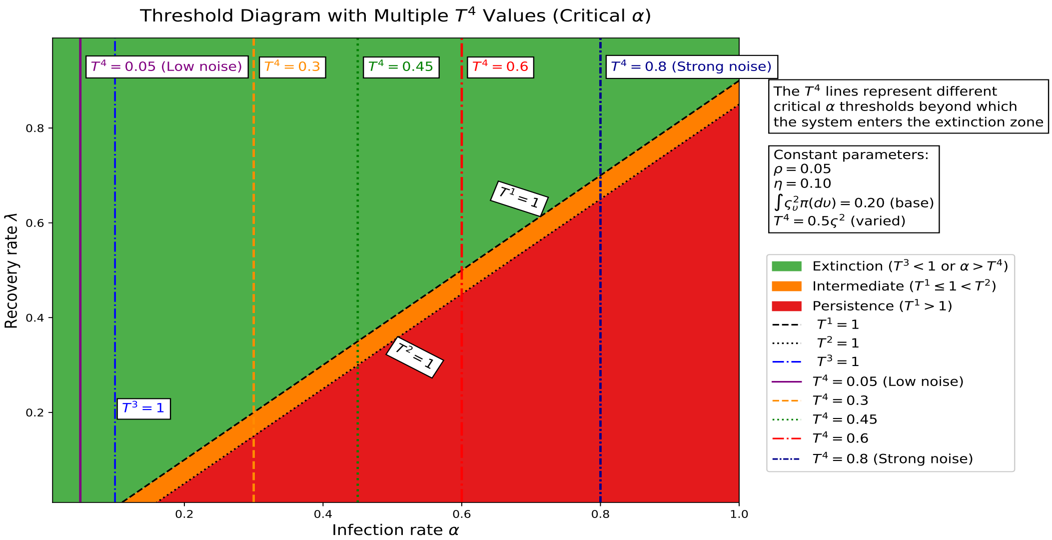


Figure 9: Threshold diagram for the stochastic SIRS model: extinction (green), intermediate (orange), persistence (red). Critical lines T^1, T^2, T^3, and $T^4 \in [0.05, 0.8]$.

Extinction Conditions

- Exponential extinction when:
  - •• $\mathcal{T}^3 < 1$ (where $\mathcal{T}^3 = \frac{\alpha}{\rho + \frac{1}{4}\mathbb{E}[\varsigma^2]}$)
  - •• $\alpha \geq \mathcal{T}^4$ (where $\mathcal{T}^4 = \frac{1}{2}\mathbb{E}[\varsigma^2]$)
- Visualization in the diagram:
  - •• Green zone ($\alpha > \mathcal{T}^4$ or left of $\mathcal{T}^3 = 1$)
  - •• Vertical lines $\mathcal{T}^4$ (0.05 to 0.8) as critical thresholds

Key Parameters

- Controlled by α (infection rate) and $\mathbb{E}[\varsigma^2]$ (jump intensity)
- Fixed parameters:
  - •• $\rho = 0.05$ (demographic rate)
  - •• $\eta = 0.1$ (immunity loss)
  - •• $\mathbb{E}[\varsigma^2] = 0.2$ (baseline)

Persistence Conditions

- Stochastic persistence when:
  - •• $\mathcal{T}^1 > 1$ and $\mathcal{T}^2 > 1$
  - •• Red zone in the diagram
- Noise effects:
  - •• $\mathbb{E}[\varsigma^2] \uparrow \Rightarrow$ reduced persistence zone
  - •• Stricter thresholds than deterministic model

Control Strategies

- Reducing α: sanitary measures
- Controlling $\mathbb{E}[\varsigma^2]$: limit super-spreading events
- Increasing λ (recovery rate) and η (vaccination)

Adaptive Strategies Based on Epidemiological Thresholds

- Continuous monitoring of critical thresholds:
  - $\mathcal{T}^1$ (conservative threshold)
  - $\mathcal{T}^2$ (standard threshold)
  - $\mathcal{T}^3$, $\mathcal{T}^4$ (extinction criteria)
- Persistence case Figure 5:
  - Maintaining healthcare capacity
  - Control strategies for stable endemicity
  - ($\mathcal{T}^1 > 1$ and $\mathcal{T}^2 > 1$)
- Extreme case Figure 8:
  - Emergency plan activation
  - Managing unpredictable oscillations
  - ($\mathcal{T}^1 \gg 1$ and $\mathcal{T}^2 \gg 1$)
- Borderline cases Figures 6 and 7:
  - Early detection of residual outbreaks
  - Targeted interventions
  - ($\mathcal{T}^1 \approx 1$ or $\mathcal{T}^2 < 1$)

Theoretical reminder: Persistence is guaranteed when $\mathcal{T}^1 > 1$ and $\mathcal{T}^2 > 1$ (Theorem 4), while exponential extinction occurs if $\mathcal{T}^3 < 1$ or $\alpha \geq \mathcal{T}^4$ (Theorem 3).

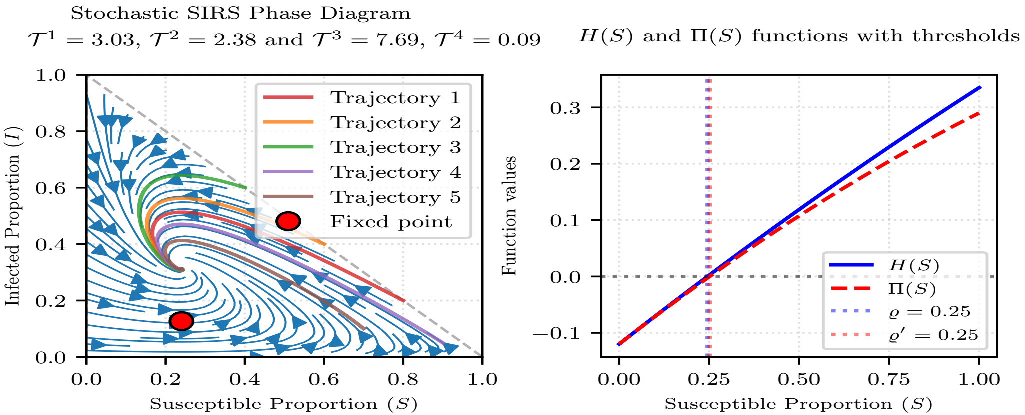


Figure 10: **Trajectories**: Numerical solutions showing the evolution of populations $S(t)$ and $I(t)$ under different initial conditions. **Fixed point**: Equilibrium where $\frac{dS}{dt} = \frac{dI}{dt} = 0$ (disease-free or endemic). In the stochastic version, the trajectories fluctuate around the theoretical fixed point. The convergence/divergence reveals system stability.

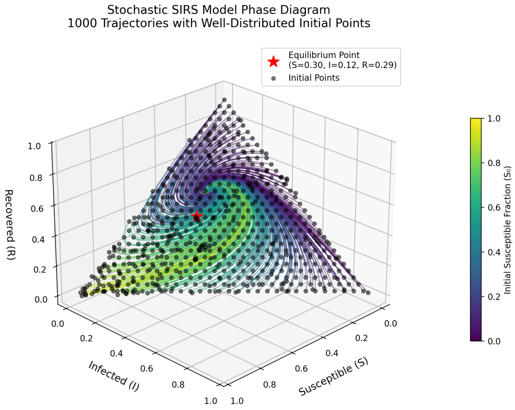


Figure 11: The diagram illustrates trajectories of the stochastic SIRS model with jumps, showing varied dynamics according to initial conditions. Despite random perturbations, all curves converge toward a stable epidemic equilibrium with constant proportions of susceptible, infected, and recovered individuals.

Public Health

- Better preparation for rare events
- Optimization of intervention strategies
- Enhanced epidemiological surveillance

Future Research

- Extension to multi-patch models
- Integration of real-world data
- Optimization of control policies

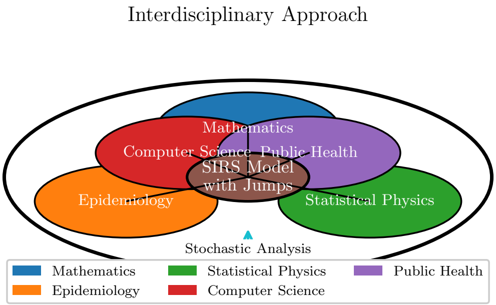


Figure 12: Interdisciplinary approach

Key Results

- Deterministic dynamics show smoothed behavior (thin line)
- A significant gap between stochastic and deterministic trajectories is systematically observed
- Theoretical thresholds $\mathcal{T}^1$ to $\mathcal{T}^4$ precisely characterize epidemic dynamics
- Stochastic jumps (ς_υ) introduce critical variations (Figure 8)
- A hybrid approach combining deterministic and stochastic models proves essential

Major Implications

Random jump processes radically alter
epidemic dynamics
and must be systematically integrated into predictive models

Fundamental Question

HOW CAN AN EPIDEMIC COME TO AN END?

Analysis reveals that:

Conditions $\mathcal{T}^3 < 1$ or $\alpha \geq \mathcal{T}^4$

govern the system's exponential extinction

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Thank you for your attention!